INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

UNIT-I: LAPLACE TRANSFORMS

Objectives:

- **To know the properties of Laplace transforms**
- **To know the Transform of one variable function to another variable function.**
- **To find the Laplace Transform of standard functions**

Syllabus: Laplace transform of standard functions- Properties: Shifting Theorems, change of scale, derivatives, integrals, multiplication and division – Unit step function – Dirac Delta function, Evaluation of improper integrals.

Course Outcomes:

The students is able to

- **Calculate the Laplace transform of standard functions both from the definition and by using formulas**
- **Select and use the appropriate shift theorems in finding Laplace transforms.**
- **Evaluation of Improper integrals.**

Introduction:

The Laplace Transformation

Pierre-Simon Laplace (1749-1827)

Laplace was a French **mathematician**, **astronomer**, and **physicist** who applied the Newtonian theory of gravitation to the solar system (an important problem of his day). He played a leading role in the development of the **metric system**.

The **Laplace Transform** is widely used in **engineering applications** (mechanical and electronic), especially where the driving force is discontinuous. It is also used in process control.

Laplace Transform (**LT**) is a powerful technique to replace the operations of calculus by operations of algebra.

Definition**:** Let f be a function defined for t ≥ 0.We define Laplace transform of f ,denoted by

F(s) or L{f(t)) or $\overline{f}$ (s) as F(s) = L{f(t)}= $\int_0^\infty e^{-st} f(t)dt$ for those s for which the integral exists is

called the Laplace Transform or one sided Laplace Transform.

Sufficient conditions for the existence of L.T:

1) f is piecewise continuous on the interval 0 ≤ t ≤ A for any A > 0.

2) f is of exponential order i.e., If $f(t)$ is defined for all $t > 0$ and there exists constants α and M such that $|f(t)| \leq Me^{\alpha t}$ for all t.

- *Note (1):* One sided LTs are unilateral whereas two sided LTs are bilateral Laplace Transforms.
- *Note (2):* A two sided LT obtained by setting the other limit of integral as $-\infty$.

Laplace transforms of some elementary functions:

Let $f(t) = 1$ then $L\{f(t)\} = L(1) = \frac{1}{s}, \quad s > 0$

1. Let $f(t) = e^{at}$ then $L\{f(t)\} = L(e^{at}) = \frac{1}{s-a}, \quad s > a$
2. Let $f(t) = e^{-at}$ then $L\{f(t)\} = L(e^{-at}) = \frac{1}{s+a}, \quad s > -a$.
3. Let $f(t) = t^n$ then $L\{f(t)\} = L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$.
4. Let $f(t) = \sin at$ then $L\{f(t)\} = L(\sin at) = \frac{a}{s^2 + a^2}$, $s > 0$.
5. Let $f(t) = \cos at$ then $L\{f(t)\} = L(\sin at) = \frac{s}{s^2 + a^2}$, $s > 0$.
6. Let $f(t) = \sinh at$ then $L\{f(t)\} = L(\sinh at) = \frac{a}{s^2 - a^2}$, $s > |a|$.

7. Let $f(t) = \cosh at$ then $L\{f(t)\} = L(\sin at) = \frac{s}{s^2 - a^2}$, $s > |a|$.

Properties of Laplace transform:

1. Laplace transform operator L is linear. Laplace transform of a linear combination (sum) of functions is the linear combination (sum) of Laplace transforms of the functions.
2. Change of scale property: When the argument t of f is multiplied by a constant k, s is replaced by s/k in $\bar{f}(s)$ or $F(s)$ and multiplied by $1/k$.
3. First shift theorem proves that multiplication of $f(t)$ by e^{at} amounts to replacement of s by $s-a$ in $\bar{f}(s)$.
4. Laplace transform of a derivative f' amounts to multiplication of $\bar{f}(s)$ by s (approximately but for the constant $-f(0)$).
5. Laplace transform of integral of f amounts to division of $\bar{f}(s)$ by s.
6. Laplace transform of multiplication of $f(t)$ by t^n amounts to differentiation of $\bar{f}(s)$ for n times w.r.t. s (with $(-1)^n$ as sign).
7. Division of $f(t)$ by t amounts to integration of $\bar{f}(s)$ between the limits s to ∞.
8. Second shift theorem proves that the L.T. of shifted function $f(t-a)u(t-a)$ is obtained by multiplying $\bar{f}(s)$ by e^{-at}.

Problems:

1) If f(t) = t^3+4t^2+5, then L[f(t)] = $\frac{\Gamma(4)}{s^4} + 4\frac{\Gamma(3)}{s^3} + 5\frac{\Gamma(2)}{s^2} = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s^2}$

2) Find Laplace transform of $\sin t \cos 2t$.

Solution: Let $f(t) = \sin t \cos 2t$

$$= \frac{1}{2}(\sin 3t - \sin t)$$

Apply LT on both sides, we have

$$L(\sin t \cos 2t) = L\left[\frac{1}{2}(\sin 3t - \sin t)\right] = \frac{1}{2}L(\sin 3t) - \frac{1}{2}L(\sin t)$$ (Using linearity property of LT)

$$= \frac{1}{2}\left(\frac{3}{s^2+9}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right).$$

3) Find the LT of $e^{-4t}\sin 3t$.

Solution: Let $f(t) = \sin 3t$

By the definition of LT, $L\{\sin 3t\} = \frac{3}{s^2+a^2}$

Hence by first shifting theorem, $L\{e^{-4t}\sin 3t\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$.

Laplace transforms of derivatives:

Statement: Let $f(t)$ be a real continuous function which is of exponential order and $f'(t)$ is sectionally continuous and is of exponential order. Then $L\{f'(t)\} = s\bar{f}(s) - f(0)$ Where $\bar{f}(s) = L\{f(t)\}$.

In general,

$$L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0).$$

Laplace transforms of integrals:

Statement: Suppose $f(t)$ is a real function and $g(t) = \int_0^t f(u)du$ is a real function such that both $f(t)$, $g(t)$ satisfy the conditions of existence of Laplace transform then

$$L\{g(t)\} = L\left[\int_0^t f(u)du\right] = \frac{\bar{f}(s)}{s} \quad \text{Where } \bar{f}(s) = L\{f(t)\}.$$

Laplace transform of the function $f(t)$ multiplied by t^n:

Statement: If $f(t)$ is sectionally continuous and is of exponential order and if $L\{f(t)\} = \bar{f}(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$ where $n = 1, 2, \ldots$.

Laplace transform of the function $f(t)$ divided by t^n:

If $L\{f(t)\} = \bar{f}(s)$ then $L\left(\frac{f(t)}{t}\right) = \int_0^\infty \bar{f}(s)ds$ provided $f(t)$ satisfy the condition of existence of LT and the right hand side integral exists.

4) Problem: Find the Laplace transform of $f(t) = t\cosh at$, using LT of derivatives.

Solution: We are given $f(t) = t\cosh at$.

It is known that $f'(t) = a\cosh at + at\sinh at$ and

$$f''(t) = 2a\sinh at + a^2 t\cosh at$$

By applying LT on both sides, $L\{f''(t)\} = 2aL\{\sinh at\} + a^2 L\{t\cosh at\}$

By the LT of derivatives, $s^2 L\{f(t)\} - sf(0) - f'(0) = 2a\,\frac{a}{s^2 - a^2} + a^2 L\{t\cosh at\}$

Since $f(0) = 0$ and $f'(0) = 1$, on simplification, we have

$$L\{t\cosh at\} = \frac{2a^2}{(s^2 - a^2)^2}.$$

5) ***Problem:*** Find $L\left(\int_0^t ue^{-u}\sin 4u\,du\right)$.

Solution: Let $f(t) = \sin 4u$

By LT, $L\{\sin 4u\} = \frac{4}{s^2 + 4^2} = \frac{4}{s^2 + 16}$

By first shifting theorem, $L\{e^{-u}\sin 4u\} = \frac{4}{(s+1)^2 + 16} = \frac{4}{s^2 + 2s + 17}$

Then by LT of $t^n f(t)$, $L\{ue^{-u}\sin 4u\} = -\frac{d}{ds}\left(\frac{4}{s^2 + 2s + 17}\right) = \frac{4}{(s^2 + 2s + 17)} = \bar{f}(s)$.

Therefore, the LT of integrals, we have

$$L\left(\int_0^t ue^{-u}\sin 4u\,du\right) = \frac{\bar{f}(s)}{s} = \frac{4}{s(s^2 + 2s + 17)}.$$

6) ***Problem:*** Find $L\left(\frac{\sin at\cos bt}{t}\right)$.

Solution: Let $f(t) = \sin at\cos bt$

$$= \frac{1}{2}\left[\sin(a+b)t + \sin(a-b)t\right]$$

By applying LT on both sides,

$$L\{\sin at \cos bt\} = \frac{1}{2}\left[L\{\sin(a+b)t\} + L\{\sin(a-b)t\}\right]$$

$$= \frac{1}{2}.\frac{(a+b)}{s^2+(a+b)^2} + \frac{1}{2}.\frac{(a-b)}{s^2+(a-b)^2} = \bar{f}(s)$$

Now, by the LT of $\frac{f(t)}{t}$, $L\left\{\frac{\sin at \cos bt}{t}\right\} = \frac{1}{2}\int_s^\infty \frac{(a+b)}{k^2+(a+b)^2}\,ds + \frac{1}{2}\int_s^\infty \frac{(a-b)}{k^2+(a-b)^2}\,ds$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a+b}\right)\right]_s^\infty + \frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a-b}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a+b}\right)\right] + \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a-b}\right)\right]$$

$$= \frac{1}{2}\cot^{-1}\left(\frac{s}{a+b}\right) + \frac{1}{2}\cot^{-1}\left(\frac{s}{a-b}\right).$$

Unit Step function:

Definition: Unit step function is defined as $U(t-a) = 0,\ t < a$

$= 1,\ t > a$ i.e. this function jumps by1 at

$t = a$.

This function is also known as Heaviside unit function.

Laplace transform of Unit step function $U(t-a)$ is given by

$$L\{U(t-a)\} = \int_0^\infty e^{-st}U(t-a)dt = \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt = \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$

Unit impulse function:

Definition: The unit impulse function denoted by $\delta(t-a)$ and is defined by

$$\delta(t-a) = \infty,\ t = a$$

$$= 0,\ t \neq a$$

So that $\int_0^{\infty} \delta(t-a)dt = 1 \quad (a \geq 0)$.

If a moving object collide with another object then for a short period of time large force is acting on the other body. To explain such mechanism we make use of unit impulse function, which is also called Dirac Delta function.

Evaluation of improper integrals by Laplace transforms:

Problem: Evaluate the integral, $\int_0^{\infty} \frac{\cos at - \cos bt}{t} dt$.

Solution: Let $I = \int_0^{\infty} \frac{\cos at - \cos bt}{t} dt$.

$$= \int_0^{\infty} \frac{\cos at}{t} dt - \int_0^{\infty} \frac{\cos bt}{t} dt$$

Clearly the given integral is in the form $\int_0^{\infty} e^{-st} \frac{f(t)}{t} dt$ with $f_1(t) = \cos at$ and $f_1(t) = \cos bt$

We observe that $\int_0^{\infty} e^{-st} \frac{\cos at}{t} dt = \int_s^{\infty} L(\cos at) ds = \int_s^{\infty} \frac{s}{s^2 + a^2} ds$ and

$$\int_0^{\infty} e^{-st} \frac{\cos bt}{t} dt = \int_s^{\infty} L(\cos bt) ds = \int_s^{\infty} \frac{s}{s^2 + b^2} ds$$

$$\therefore \int_0^{\infty} e^{-st} \left(\frac{\cos at - \cos bt}{t} \right) dt = \int_0^{\infty} \frac{s}{s^2 + a^2} ds - \int_0^{\infty} \frac{s}{s^2 + b^2} ds = \int_0^{\infty} \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds$$

It is clear that the above integral reduces to I when $s = 0$.

Therefore,

$$I = \int_0^{\infty} \frac{\cos at - \cos bt}{t} dt = \int_0^{\infty} \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds = \left[\frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right]_0^{\infty}$$

$$= \frac{1}{2} \left[\log\left(\frac{s^2 + a^2}{s^2 + b^2} \right) \right]_0^{\infty} = \frac{1}{2} \left[\log 1 - \log\left(\frac{a^2}{b^2} \right) \right] = \frac{1}{2} \log\left(\frac{a^2}{b^2} \right).$$

Assignment/Tutorial Questions

SECTION-A

1. The Laplace transform of $f(t)=\sin^2 2t$ is____________________.
2. If $f(t)=e^{3t}(\sin 2t+\cos 3t)$ then $L\{f(t)\}=$____________________.
3. If $f(t)=\dfrac{e^{2t}-e^{3t}}{t}$ then $L\{f(t)\}=$____________________.
4. If $f(t)=t\sin t$ then $L\{f(t)\}=$____________________.
5. The value of $\int_0^\infty e^{-3t}t\,dt$ is____________________.
6. $L\{e^{at}t^n\}=$___________________________.
7. The Laplace transform of $\dfrac{(1-e^t)}{t}$ is ___________________________.
8. If $L\{f(t)\}=\bar{f}(s)=\dfrac{s}{s^2+1}$, $f(0)=0$ then $L\{f'(t)\}=$________________________.
9. Find the Laplace transform of $t^{5/2}$

   (a) $\dfrac{15\sqrt{\pi}}{8\, s^{\frac{5}{2}}}$ (b) $\dfrac{15\sqrt{\pi}}{8\, s^{\frac{7}{2}}}$ (c) $\dfrac{9\sqrt{\pi}}{4\, s^{\frac{7}{2}}}$ (d) $\dfrac{15\sqrt{\pi}}{4\, s^{\frac{7}{2}}}$
10. Laplace transform of f(t) is given by

    a) $F(s)=\int_{-\infty}^{\infty} f(t)e^{-st}\,dt$

    b) $F(t)=\int_{-\infty}^{\infty} f(t)e^{-t}\,dt$

    a) $f(s)=\int_{-\infty}^{\infty} f(t)e^{-st}\,dt$

    b) $f(t)=\int_{-\infty}^{\infty} f(t)e^{-t}\,dt$
11. Laplace transform of sin(at)u(t) is

    a) s/a^2+s^2

    b) a/a^2+s^2

    c) s^2/a^2+s^2

    d) a^2/a^2+s^2
12. Find the laplace transform of $y(t)=e^{|t-1|}$ u(t).

    a) $\frac{2s}{1-s^2}e^{s}$

    b) $\frac{2s}{1+s^2}e^{-s}$

    c) $\frac{2s}{1+s^2}e^{s}$

    d) $\frac{2s}{1-s^2}e^{-s}$

13. Find the Laplace transform of e^t Sin(t).

a) $\frac{a}{a^2+(s+1)^2}$

b) $\frac{a}{a^2+(s-1)^2}$

c) $\frac{s+1}{a^2+(s+1)^2}$

d) $\frac{s+1}{a^2+(s+1)^2}$

SECTION-B

1. Find L[tcosat] by multiplication t property.
2. Find L[cos(at+b)]
3. Find $L[sin^2(2t)]$
4. Find $L[sin2tcos3t]$
5. Find the Laplace transform of $f(t) = \begin{cases} e^t, 0 < t < 1 \\ 0 \quad , t > 1 \end{cases}$
6. Find the Laplace transform of $(\sqrt{t} + \frac{1}{\sqrt{t}})^3$
7. Define Unit-step function and also write its Laplace transform.
8. Define Dirac Delta function.
9. Evaluate $L[t^2 e^{-t} cos^2 t]$
10. Evaluate $L[\frac{cosat - cosbt}{t}]$
11. Evaluate $L\,[\int_0^t \frac{e^{-t} sint}{t} dt]$
12. Evaluate L[tsint] and hence find $L[\int_0^t \int_0^t tsintdt\,dt]$
13. Derive the Laplace transform of Unit Step function and hence find $L[e^{t-3} u(t-3)]]$
14. Evaluate $\int_0^\infty e^{-t} \frac{sin^2 t}{t} dt$
15. Evaluate $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt$, using Laplace transform.

SECTION-C

GATE PREVIOUS QUESTIONS

1. The Laplace Transform of cos(ωt) is $\frac{s}{s^2+\omega^2}$ then L(e^{-2t} cos4t) is **(GATE-2010)**

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$ (c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

2. The L.T of f(t) = $\frac{1}{s^2(s+1)}$ then f(t) is **(GATE-2010)**

(a) $t-1+e^{-t}$ (b) $t+1+e^{-t}$ (c) $-1+e^{-t}$ (d) $2t+e^{t}$

3. If L.T of sin wt is $\frac{s}{s^2+w^2}$ then L.T of e^{-2t}.sint is **(GATE-2014)**

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$ (c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

4. If F(s) is the L.T of f(t) then. L.T of $\int_0^t f(\tau)d\tau$ is **(GATE-2007)**

(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s)$ -f(0) (c)sF(s)-f(0) (d) $\int F(s)ds.$

5. L.T of functions t.u(t) and u(t).sint are respectively. **(GATE-1987)**

(a) $\frac{1}{s^2}, \frac{s}{s^2+1}$ (b) $\frac{1}{s}, \frac{1}{s^2+1}$ (c) $\frac{1}{s^2}, \frac{1}{s^2+1}$ (d) $s, \frac{s}{s^2+1}$

6. The L.T of i(t) is given by I(s) = $\frac{2}{s(1+s)}$ as $t \to \infty$ the value of i(t) tends to

(a) 0 (b) 1 (c) 2 (d) ∞

7. The unilateral Laplace transform of f (t)=$\frac{1}{s^2+s+1}$ is **(GATE-2012)**

(a) $\frac{-s}{(s^2+s+1)^2}$ (b) $\frac{s}{(s^2+s+1)^2}$ (c) $\frac{-(2s+1)}{(s^2+s+1)^2}$ (d) $\frac{2s+1}{(s^2+s+1)^2}$

INVERSE LAPLACE TRANSFORMS

Objectives:

- To understand the properties of Inverse Laplace transforms
- To solve Integral equations by using convolution theorem.
- To convert differential equations into algebraic equations using Laplace Transforms and inverse Laplace transforms.

Syllabus:

Inverse Laplace Transforms – by partial fractions - Convolution theorem (without proof).

Application: Solution of ordinary differential equations.

Subject Outcomes/Unit Outcomes:

After learning this unit, students will be able to:

- Find inverse Laplace Transforms of the transformation $\bar{f}(s)$ to obtain $f(t)$.
- Apply convolution theorem to find the inverse Laplace
- Use the method of Laplace transforms to solve systems of linear ordinary differential equations.

Definition: Suppose $f(t)$ is a piecewise continuous function and is of exponential order. Let $L\{f(t)\}=\int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$. The inverse Laplace Transform (ILT) of $\bar{f}(s)$ is defined as $L^{-1}\{\bar{f}(s)\} = f(t),$ where L^{-1}inverse operator of is L and vice-versa.

Inverse Laplace transforms of some elementary functions:

(1). $L^{-1}\left\{\frac{1}{s}\right\}=1$ (2). $L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$ (3). $L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\}=t^n$ (4). $L^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at$

(5). $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at$ (6). $L^{-1}\left\{\frac{a}{s^2-a^2}\right\}=\sinh at$ (7). $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh at$, etc.

Properties of Inverse Laplace transform:

Linear property:

If $L^{-1}\{\bar{f}(s)\}=f(t), L^{-1}\{\bar{g}(s)\}=g(t)$, then $L^{-1}\{a\,\bar{f}(s)+b\,\bar{g}(s)\}=a\,f(t)+b\,g(t)$

Shifting Property:

If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\{\bar{f}(s-a)\}=e^{at}f(t),\ s>a$.

Change of scale property:

If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\{\bar{f}(as)\}=\frac{1}{a}\bar{f}\left(\frac{t}{a}\right)$ *and* $L^{-1}\left\{\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)\right\}=f(at)$

Problem: let $\bar{f}(s) = \dfrac{4s+4}{4s^2-9}$. Then by linearity property of inverse Laplace transforms (ILT),

$$L^{-1}\left\{\frac{4s+4}{4s^2-9}\right\} = L^{-1}\left\{\frac{4s}{4s^2-9}\right\} + L^{-1}\left\{\frac{4}{4s^2-9}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2-(3/2)^2}\right\} + L^{-1}\left\{\frac{1}{s^2-(3/2)^2}\right\} = \cosh\frac{3}{2}t + \frac{2}{3}\sinh\frac{3}{2}t$$

Problem: Find the ILT of $\dfrac{4}{(s+1)(s+2)}$.

Solution: Let $\bar{f}(s) = \dfrac{4}{(s+1)(s+2)}$

By applying partial fractions, we can rewrite $\bar{f}(s)$ as

$$\bar{f}(s) = \frac{4}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{As+2A+Bs+B}{(s+1)(s+2)}$$

Comparing like terms in the numerator, we obtain $A = 4$ and $B = -4$.

Therefore, $\bar{f}(s) = \dfrac{4}{(s+1)(s+2)} = \dfrac{4}{(s+1)} - \dfrac{4}{(s+2)}$

By applying linearity property, we have

$$L^{-1}\{\bar{f}(s)\} = 4L^{-1}\left\{\frac{1}{s+1}\right\} - 4L^{-1}\left\{\frac{1}{s+2}\right\} = 4e^{-t} - 4e^{-2t}.$$

Problem: Find the ILT of $\dfrac{s+1}{s^2+s+1}$.

Solution: Consider $\bar{f}(s) = \dfrac{s+1}{s^2+s+1}$

$$= \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

By the linearity property of ILT, we have

$$L^{-1}\left(\frac{s+1}{s^2+s+1}\right) = L^{-1}\left(\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right) + L^{-1}\left(\frac{1/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t = e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right].$$

Inverse Laplace Transforms of Derivatives:

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\left(\frac{d^n(\bar{f}(s))}{ds^n}\right)=(-1)^n t^n f(t)$.

Inverse Laplace Transforms of Integrals:

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\left(\int_s^\infty \bar{f}(s)ds\right)=\frac{f(t)}{t}$.

Inverse Laplace Transform of type $s\bar{f}(s)$: (Multiplication by s)

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ and $f(0)=0$ then $L^{-1}\left(s\bar{f}(s)\right)=f'(t)$

Inverse Laplace Transform of type $\frac{\bar{f}(s)}{s}$: (Division by s)

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\left(\frac{\bar{f}(s)}{s}\right)=\int_0^t f(t)dt$

Similarly, $L^{-1}\left(\frac{\bar{f}(s)}{s^2}\right)=\int_0^t\int_0^t f(t)dt$ and hence in general, $L^{-1}\left(\frac{\bar{f}(s)}{s^n}\right)=\int_0^t\int_0^t\ldots\int_0^t f(t)dt\,dt\ldots dt$ (n-folded integral).

Problem: Evaluate $L^{-1}\left\{\frac{s}{\left(s^2+2^2\right)^2}\right\}$ using derivative property of ILT.

Solution: We know that $L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$, then by derivative property of ILT,

we have $L^{-1}\left[\frac{-2s}{\left(s^2+a^2\right)^2}\right]=-\frac{t}{a}\sin at$, $\therefore L^{-1}\left\{\frac{s}{(s^2+2^2)^2}\right\}=\frac{t}{4}\sin 2t$.

Convolution Theorem:-

This is used to find inverse Laplace transforms of product of transforms.

Definition: The convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$f(t)*g(t)=\int_0^t f(r)g(t-r)dr$$, provided the integral exists.

Note: the operation of convolution between two functions yields another function.

Convolution Theorem:-

If $L^{-1}\{\bar{f}(s)\}=f(t)$ and $L^{-1}\{\bar{g}(s)\}=g(t)$ then $L^{-1}\{\bar{f}(s)\bar{g}(s)\}=f(t)*g(t)$.

Example: Using convolution theorem find the inverse Laplace transform of $\frac{s^2}{(s^2+4)(s^2+9)}$.

Solution: We are given $f(t) = \dfrac{s^2}{(s^2+4)(s^2+9)}$

The given function $f(t)$ can be rewritten as,

$$f(t) = \frac{s^2}{(s^2+4)(s^2+9)} = \frac{s}{(s^2+4)} . \frac{s}{(s^2+9)}$$

By applying inverse Laplace transform, we have,

$$L^{-1}\{f(t)\} = L^{-1}\left\{\frac{s^2}{(s^2+4)} . \frac{s^2}{(s^2+9)}\right\}$$

Hence by convolution theorem,

$$L^{-1}\left\{\frac{s^2}{(s^2+4)} . \frac{s^2}{(s^2+9)}\right\} = (\cos 2t) * (\cos 3t) \qquad \text{since, } L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t \text{ and } L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$$

$$= \int_0^t [\cos 2u \cos 3(t-u)]du = \int_0^t \frac{1}{2}[\cos(3t-u) + \cos(5u-3t)]du$$

$$= \frac{1}{2}\left[\frac{\sin(3t-u)}{(-1)}\right]_0^t + \frac{1}{2}\left[\frac{\sin(5u-3t)}{5}\right]_0^t = \frac{-1}{2}[\sin 2t - \sin 3t] + \frac{1}{10}[\sin 2t + \sin 3t]$$

$$= \sin 2t\left(-\frac{1}{2} + \frac{1}{10}\right) + \sin 3t\left(\frac{1}{2} + \frac{1}{10}\right) = \frac{1}{5}(3\sin 3t - 2\sin 2t).$$

Solution of Ordinary differential equation (An application):

Problem: Solve the differential equation $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 2y = 0;$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$.

Solution: We are given the linear non-homogeneous differential equation with constant coefficients:

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0 \text{ where } y = y(t) \text{ or } f(t)$$

Applying Laplace transform on both sides,

$$L\left(\frac{d^3y}{dt^3}\right) + 2L\left(\frac{d^2y}{dt^2}\right) - L\left(\frac{dy}{dt}\right) - 2L(y) = L(0)$$

$$\Rightarrow [s^3 f(s) - s^2 f(0) - sy'(0) - y''(0)] + 2[s^2 f(s) - sy(0) - y'(0)] - [sf(s) - y(0)] - 2\bar{f}(s) = 0$$

$$\Rightarrow \bar{f}(s)[s^3 + 2s^2 - s - 2] - y(0)[s^2 + 2s - 1] - y'(0)(s+2) - y''(0) = 0$$

Substituting $y(0) = y'(0) = 0$ and $y''(0) = 6$, we get,

$$\bar{f}(s)(s^3 + 2s^2 - s - 2) - 6 = 0$$

$$\Rightarrow \bar{f}(s) = \frac{6}{(s^3 + 2s^2 - s - 2)}$$

Now by applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s)) = L^{-1}\left(\frac{6}{s^3+2s^2-s-2}\right) = L^{-1}\left(\frac{6}{s^2(s+2)-(s+2)}\right)$$

$$f(t) = L^{-1}\left(\frac{6}{(s+2)(s+1)(s-1)}\right)$$

Consider $\bar{f}(s) = \dfrac{6}{(s-1)(s+1)(s+2)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s+1)} + \dfrac{C}{(s+2)}$

On simplification we obtain $A = 1, \quad B = -3, \quad C = 2$

$$\therefore \quad L^{-1}(\bar{f}(s)) = f(t) = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{3}{s+1}\right) + L^{-1}\left(\frac{2}{s+2}\right)$$

$$= e^{t} - 3e^{-t} + 2e^{-2t}$$

Hence, the solution of the given differential equation is $y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$.

Problem: Solve the differential equation $t\dfrac{d^2y}{dt^2} + (1-2t)\dfrac{dy}{dt} - 2y = 0$ where $y(0) = 1, \; y'(0) = 2$.

Solution: We are given the linear differential equation with variable coefficients:

$$t\frac{d^2y}{dt^2} + (1-2t)\frac{dy}{dt} - 2y = 0$$

Applying Laplace transform on both sides,

$$L\left(t\frac{d^2y}{dt^2}\right) + L\left((1-2t)\frac{dy}{dt}\right) - 2L(y) = 0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2 f(s) - sf(0) - f'(0)\right) + \left(sf(s) - f(0)\right) + 2\frac{d}{ds}\left(sf(s) - f(0)\right) - 2f(s) = 0$$

$$\Rightarrow \bar{f}'(s)(2s - s^2) - s\bar{f}(s) = 0$$

$$\Rightarrow \frac{\bar{f}'(s)}{\bar{f}(s)} = -\frac{1}{s-2}$$

Integrating on both sides, we have,

$$\log \bar{f}(s) = -\log(s-2) + \log c$$

$$\Rightarrow \bar{f}(s) = \frac{c}{s-2}$$

By applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s)) = L^{-1}\left(\frac{c}{s-2}\right)$$

$$\Rightarrow f(t) = ce^{2t}$$

By using the initial condition, we have $c = 1$.

Therefore, the particular solution of the differential equation is $f(t) = e^{2t}$.

Assignment/Tutorial Questions

<u>SECTION-A</u>

1. $L^{-1}\left(\dfrac{1}{s^2+a^2}\right) =$

   (a) $\sin at$ (b) $\cos at$ (c) $\dfrac{1}{a}\sin at$ (d) $\dfrac{1}{a}\cos at$

2. $L^{-1}\left(\dfrac{1}{3s-6}\right) =$

   (a) e^{6t} (b) $\dfrac{1}{3}e^{2t}$ (c) e^{2t} (d) does not exist

3. $L^{-1}\left(\dfrac{1}{(s+a)(s+b)}\right) =$

   (a) $\dfrac{e^{at}-e^{bt}}{b-a}$ (b) $\dfrac{e^{-at}+e^{-bt}}{b-a}$ (c) $\dfrac{e^{-at}-e^{-bt}}{b-a}$ (d) $\dfrac{e^{at}+e^{bt}}{b-a}$

4. $L^{-1}\left(\dfrac{s+2}{(s-2)^2}\right) =$

   (a) $e^{2t}(1+2t)$ (b) $te^{2t}(1+2t)$ (c) $(1+2t)$ (d) $t(1+2t)$

5. $L^{-1}\left(\dfrac{s+2}{s^2-2s+5}\right) =$

   (a) $\cos 2t + \dfrac{3}{2}\sin 2t$ (b) $\sin 2t + \dfrac{3}{2}\cos 2t$ (c) $e^t \cos 2t + \dfrac{3}{2}e^t \sin 2t$ (d) $\cos 2t$

6. $L^{-1}\left(\dfrac{1}{1-e^{-st}}\int_0 e^{-at} f(u)du\right) =$

   (a) $f(t)$ (b) $e^{st}f(t)$ (c) $e^{-st}f(t)$ (d) none of the above

7. $L^{-1}\left(\int_s^{\infty} \bar{f}(s)ds\right) =$

   (a) $\dfrac{f(t)}{t}$ (b) $\int_0^t f(t)dt$ (c) $\int_0^t \dfrac{f(t)}{t}dt$ (d) $f(t)$

8. Time domain function of $\frac{s}{s^2+a^2}$ is given by
   a) Cos(at)
   b) Sin(at)
   c) Cos(at)Sin(at)
   d) None of the above
9. If F(s)=L[f(t)], then the formula for $L^{-1}[\int_s^{\infty} F(s)ds]$ is ________________
10. If F(s)=L[f(t)], then the formulae for (i) $L^{-1}[F'(s)]$ is ___________________

11. As per the convolution theorem, $L^{-1}\{\bar{f}(s)\bar{g}(s)\} =$ ______________________

12. $L^{-1}[\frac{s}{(s+3)^2+4}]$= __________________

13. $L^{-1}[\frac{s}{((s)^2+a^2)^2}]$= ____________________

$SECTION - B$

1. **Find the inverse Laplace transform of $\frac{s+2}{s^2-4s+13}$**
2. Find the inverse Laplace transform of $\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}$
3. Find the inverse Laplace transform of $\frac{3s+7}{(s^2-2s-3)}$
4. Find the inverse Laplace transform of $\frac{1}{2}log[\frac{s^2+a^2}{s^2+b^2}]$
5. Using convolution theorem to evaluate $L^{-1}[\frac{s^2}{(s^2+a^2)(s^2+b^2)}]$
6. Using convolution theorem to evaluate $L^{-1}[\frac{s^2}{(s^2+a^2)^2}]$
7. Using convolution theorem, evaluate $L^{-1}[\frac{1}{s^2(s+1)^2}]$
8. Find the inverse Laplace theorem of $\frac{1}{s(s+a)(s+b)}$.
9. Solve the differential equation $(D^2+2D+5)y = e^t \sin t;\ \ y(0)=0,\ y'(0)=1.$
10. Apply "Method of Laplace transforms" ,
    Solve the differential equation $(D^2+2D+5)y = e^t \sin t;\ \ y(0)=0,\ y'(0)=1.$
11. Apply Laplace transform to the initial value problem $y'' + y' - 2y = sint$, $y(0) = 0, y'(0) = 0$.
12. Apply "Method of Laplace transforms" , Solve $x'' + 2x' + 5x = e^t sint, x(0) = 0, x'(0) = 1$.
13. Apply "Method of Laplace transforms" , Solve $x'' - 3x' + 2x = 1 - e^{2t}, x(0) = 1, x'(0) = 0$.
14. Using Laplace transform, solve $x'' + 9x = \cos 2t$, if $x(0) = 1, x'(\frac{\pi}{2}) = -1$.
15. Solve, by Laplace transform method, the following initial value problem:
    $$(D^2+1)x = tcos2t, such\ that\ x = Dx = 0\ at\ t = 0$$

<u>GATE PREVIOUS QUESTIONS</u>

7. The function f(t) satisfies the differential equation $\frac{d^2f}{dt^2} + f = 0$ and the auxiliary conditions,f(0)=0, $\frac{df}{dt}(0) = 4$. The Laplace transform of f(t) is given by **(GATE-2009)**

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$ (c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

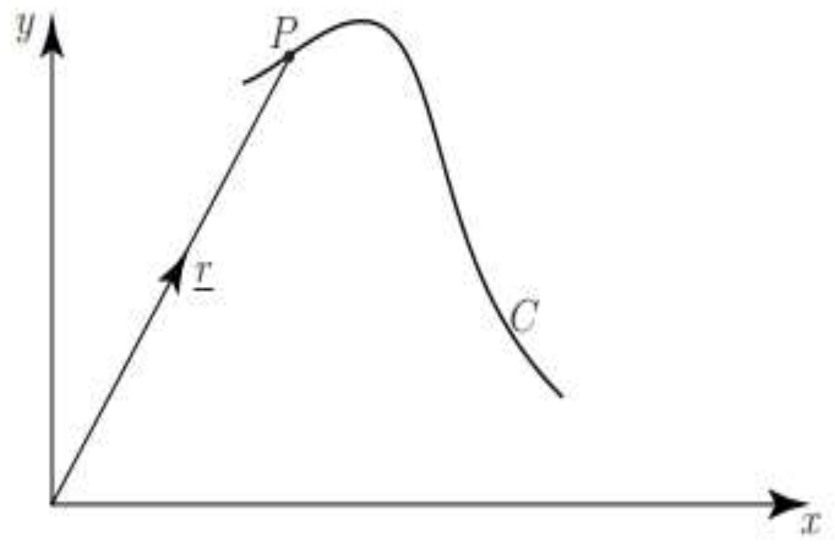
8. The inverse Laplace transform of the function $F(s) = \frac{1}{s(s+1)}$ is given by **(GATE-2007)**

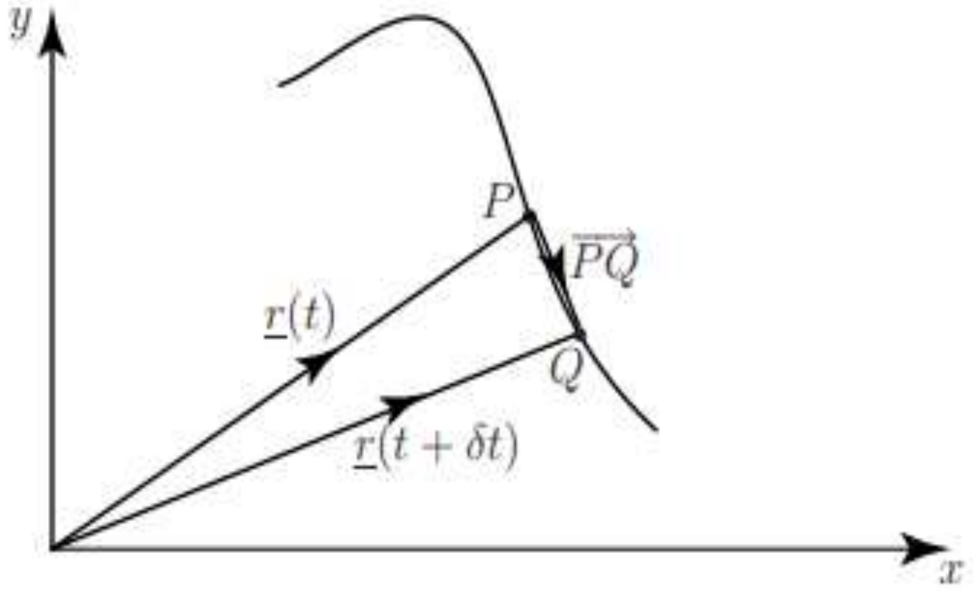
(a) f(t)=sint (b) f(t) = e^{-t} sint (c) e^{-t} (d) 1- e^{-t}

9. The inverse Laplace transform of $F(s) = s+1/(s^2+4)$ is **(GATE-2011)**
(a) cos2t+ sin2t (b) cos2t-(1/2) sin2t (c) cos2t+(1/2)sin2t (d) cos2t-sin2t

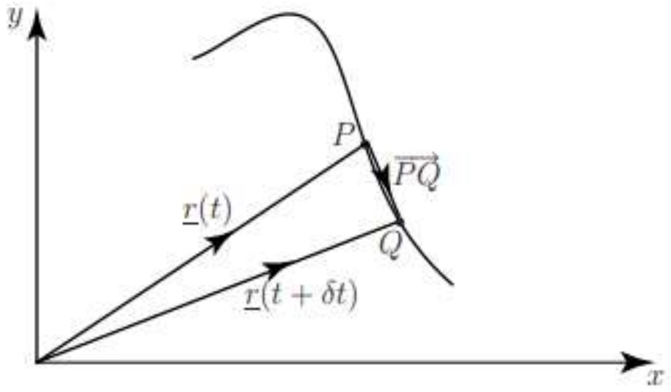
Vector Differentiation

If $\underline{r}$ represents the **position vector** of an object which is moving along a curve C, then the position vector will be dependent upon the time, t. We write $\underline{r} = \underline{r}(t)$ to show the dependence upon time. Suppose that the object is at the point P, with position vector $\underline{r}$ at time t and at the point Q, with position vector $\underline{r}(t+\delta t)$, at the later time $t+\delta t$,





Then $\overrightarrow{PQ}$ represents the **displacement vector** of the object during the interval of time δt. The len of the displacement vector represents the distance travelled, and its direction gives the direction motion. The average velocity during the time from t to $t + \delta t$ is defined as the displacement vec divided by the time interval δt, that is,



$$\text{average velocity} = \frac{\overrightarrow{PQ}}{\delta t} = \frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t}$$

If we now take the limit as the interval of time δt tends to zero then the expression on the right hand side is the **derivative** of $\underline{r}$ with respect to t. Not surprisingly we refer to this derivative as **the instantaneous velocity**, $\underline{v}$. By its very construction we see that the velocity vector is always tangential to the curve as the object moves along it. We have:

$$\underline{v} = \lim_{\delta t \to 0} \frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t} = \frac{d\underline{r}}{dt}$$

★ $$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

then the velocity vector is

$$\underline{v} = \dot{\underline{r}}(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}$$

The magnitude of the velocity vector gives the speed of the object.

We can define the acceleration vector in a similar way, as the rate of change (i.e. the derivative) of the velocity with respect to the time:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

➡ **Example :** If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, find

(a) $\frac{d\underline{w}}{dt}$ (b) $\left|\frac{d\underline{w}}{dt}\right|$ (c) $\frac{d^2\underline{w}}{dt^2}$

Solution

(a) If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, then differentiation with respect to t yields: $\frac{d\underline{w}}{dt} = 6t\underline{i} - 2\sin 2t\underline{j}$

(b) $\left|\frac{d\underline{w}}{dt}\right| = \sqrt{(6t)^2 + (-2\sin 2t)^2} = \sqrt{36t^2 + 4\sin^2 2t}$

(c) $\frac{d^2\underline{w}}{dt^2} = 6\underline{i} - 4\cos 2t\underline{j}$

DIFFERENTIATION FORMULAS. If **A**, **B** and **C** are differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u, then

1. $$\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$$

2. $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

3. $$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

4. $$\frac{d}{du}(\phi\mathbf{A}) = \phi\frac{d\mathbf{A}}{du} + \frac{d\phi}{du}\mathbf{A}$$

5. $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}\times\frac{d\mathbf{C}}{du} + \mathbf{A}\cdot\frac{d\mathbf{B}}{du}\times\mathbf{C} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$$

6. $$\frac{d}{du}\{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\} = \mathbf{A}\times(\mathbf{B}\times\frac{d\mathbf{C}}{du}) + \mathbf{A}\times(\frac{d\mathbf{B}}{du}\times\mathbf{C}) + \frac{d\mathbf{A}}{du}\times(\mathbf{B}\times\mathbf{C})$$

Example : If $\underline{w} = t^3\underline{i} - 7t\underline{k}$ and $\underline{z} = (2+t)\underline{i} + t^2\underline{j} - 2\underline{k}$

(a) find $\underline{w} \cdot \underline{z}$, (b) find $\frac{d\underline{w}}{dt}$, (c) find $\frac{d\underline{z}}{dt}$, (d) show that $\frac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z}$

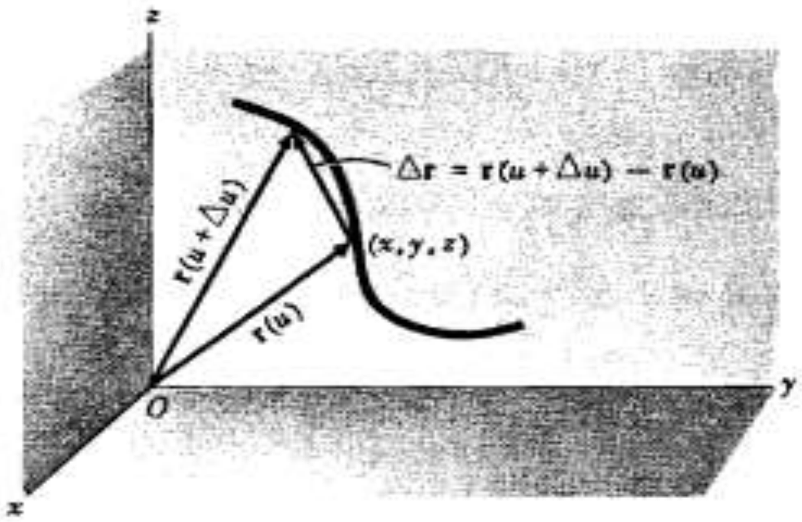
SPACE CURVES. If in particular $\mathbf{R}(u)$ is the position vector $\mathbf{r}(u)$ joining the origin O of a coordinate system and any point (x, y, z), then

$$\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$$

As u changes, the terminal point of $\mathbf{r}$ describes a *space curve* having parametric equations

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

Then $\dfrac{\Delta \mathbf{r}}{\Delta u} = \dfrac{\mathbf{r}(u+\Delta u) - \mathbf{r}(u)}{\Delta u}$ is a vector in the direction of $\Delta \mathbf{r}$ (see adjacent figure). If $\lim\limits_{\Delta u \to 0} \dfrac{\Delta \mathbf{r}}{\Delta u} = \dfrac{d\mathbf{r}}{du}$ exists, the limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and is given by



$$\frac{d\mathbf{r}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

If u is the time t, $\dfrac{d\mathbf{r}}{dt}$ represents the *velocity* $\mathbf{v}$ with which the terminal point of $\mathbf{r}$ describes the curve. Similarly, $\dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$ represents its *acceleration* $\mathbf{a}$ along the curve.

. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Partial Differentiation

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \quad \frac{\partial^2 \mathbf{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{A}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{A}}{\partial z}\right)$$

$$\frac{\partial^2 \mathbf{A}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \quad \frac{\partial^3 \mathbf{A}}{\partial x\, \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \mathbf{A}}{\partial z^2}\right)$$

1. $\frac{\partial}{\partial x}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B}$

2. $\frac{\partial}{\partial x}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B}$

3. $\frac{\partial^2}{\partial y\, \partial x}(\mathbf{A} \cdot \mathbf{B}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{A} \cdot \mathbf{B})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B}\right\}$

$$= \mathbf{A} \cdot \frac{\partial^2 \mathbf{B}}{\partial y\, \partial x} + \frac{\partial \mathbf{A}}{\partial y} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x} \cdot \mathbf{B}, \quad \text{etc.}$$

Jerrold E. Marsden and Anthony J. Tromba

Vector Calculus

Fifth Edition

Chapter 2: Differentiation

2.6 Gradients and Directional Derivatives

2.6 Gradients and Directional Derivatives

Key Points in this Section.

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1. The***gradient*** of a differentiable function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

2. The ***directional derivative*** of f in the direction of a *unit* vector $\mathbf{v}$ at the point $\mathbf{x}$ is

$$\left.\frac{d}{dt}f(\mathbf{x} + t\mathbf{v})\right|_{t=0} = \nabla f(\mathbf{x}) \cdot \mathbf{v}$$

3. The direction in which f is *increasing the fastest* at $\mathbf{x}$ is the direction parallel to $\nabla f(\mathbf{x})$. The direction of fastest *decrease* is parallel to $-\nabla f(\mathbf{x})$.

4. For $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ a C^1 function, with $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$, the vector $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level set $f(x, y, z) = f(x_0, y_0, z_0)$. Thus, the ***tangent plane*** to this level set is

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

5. The gravitational force field

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{n}$$

(the inverse square law), where $\mathbf{n} = \mathbf{r}/r$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\|$, is a gradient. Namely,

$$\mathbf{F} = -\nabla V,$$

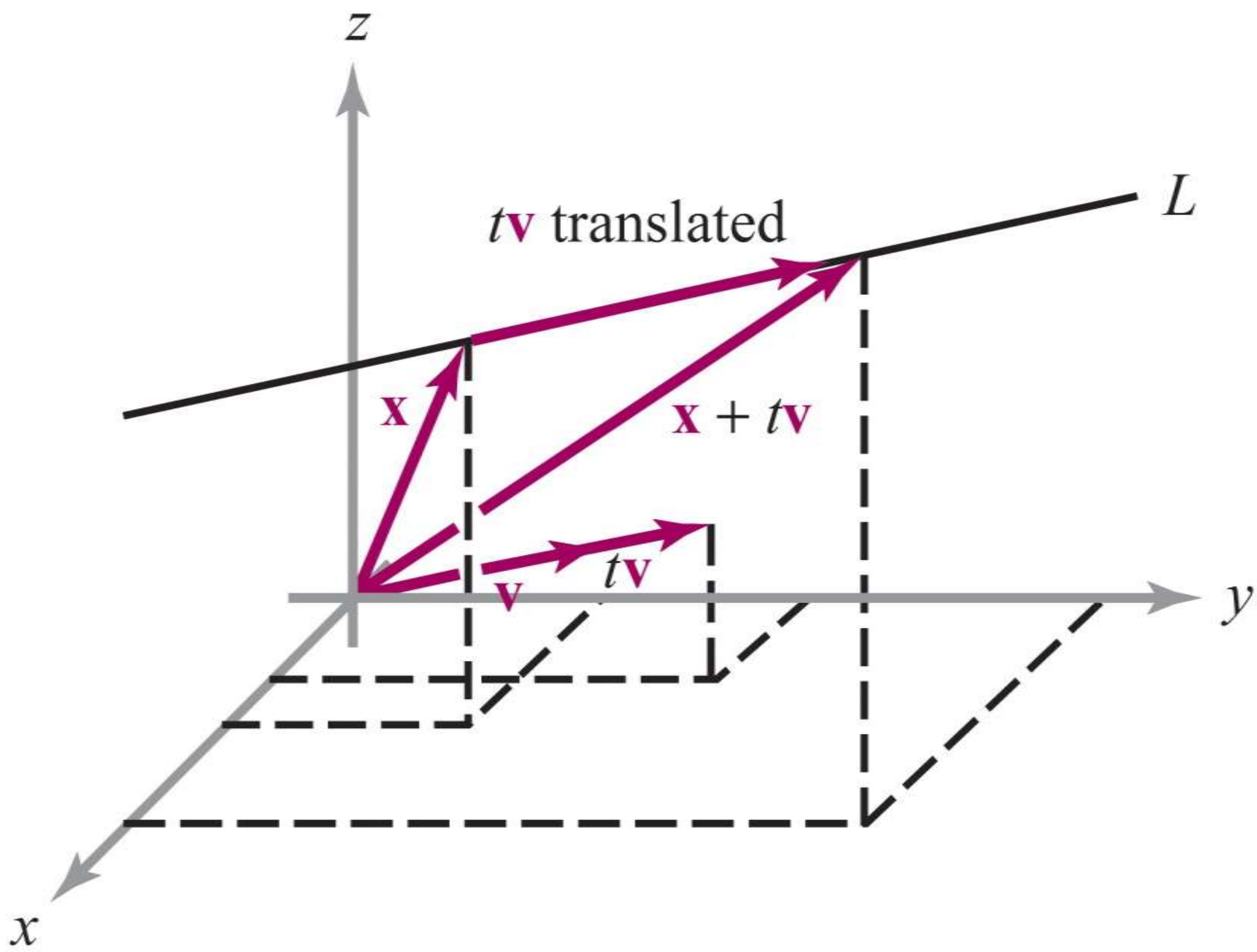
where

$$V = -\frac{GMm}{r}.$$

DEFINITION: The Gradient If $f\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable, the ***gradient*** of f at (x, y, z) is the vector in space given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

This vector is also denoted $\nabla f(x, y, z)$. Thus, ∇f is just the matrix of the derivative $\mathbf{D}f$, written as a vector.


z
L
tv translated
x
x + tv
tv
v
y
x

DEFINITION: Directional Derivatives If $f\colon \mathbb{R}^3 \to \mathbb{R}$, the ***directional derivative*** of f at $\mathbf{x}$ along the vector $\mathbf{v}$ is given by

$$\frac{d}{dt} f(\mathbf{x} + t\mathbf{v})\Big|_{t=0}$$

if this exists.

In the definition of a directional derivative, we normally choose $\mathbf{v}$ to be a *unit* vector. In this case we are moving in the direction $\mathbf{v}$ with unit speed and we refer to $\nabla f(\mathbf{x}) \cdot \mathbf{v}$ as the ***directional derivative of f in the direction $\mathbf{v}$***.

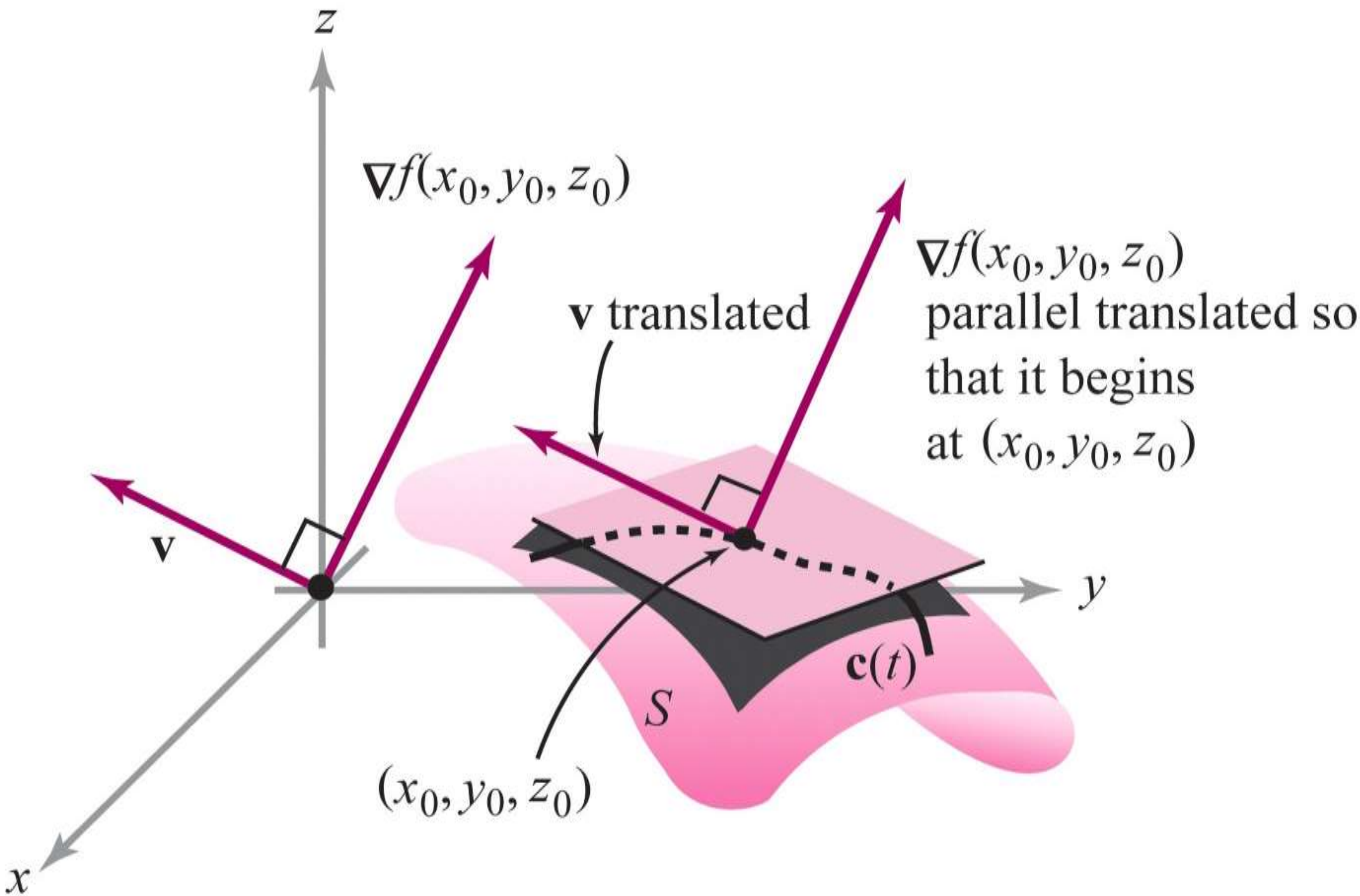
THEOREM 12 If $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then all directional derivatives exist. The directional derivative at $\mathbf{x}$ in the direction $\mathbf{v}$ is given by

$$\mathbf{D}f(\mathbf{x})\mathbf{v} = \operatorname{grad} f(\mathbf{x}) \cdot \mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v} = \left[\frac{\partial f}{\partial x}(\mathbf{x})\right] v_1 + \left[\frac{\partial f}{\partial y}(\mathbf{x})\right] v_2 + \left[\frac{\partial f}{\partial z}(\mathbf{x})\right] v_3,$$

where $\mathbf{v} = (v_1, v_2, v_3)$.

THEOREM 13 Assume $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then $\nabla f(\mathbf{x})$ points in the direction along which f is increasing the fastest.

THEOREM 14: The Gradient is Normal to Level Surfaces Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x, y, z) = k$, for k a constant. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface in the following sense: If $\mathbf{v}$ is the tangent vector at $t = 0$ of a path $\mathbf{c}(t)$ in S with $\mathbf{c}(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \cdot \mathbf{v} = 0$ (see Figure 2.6.2).

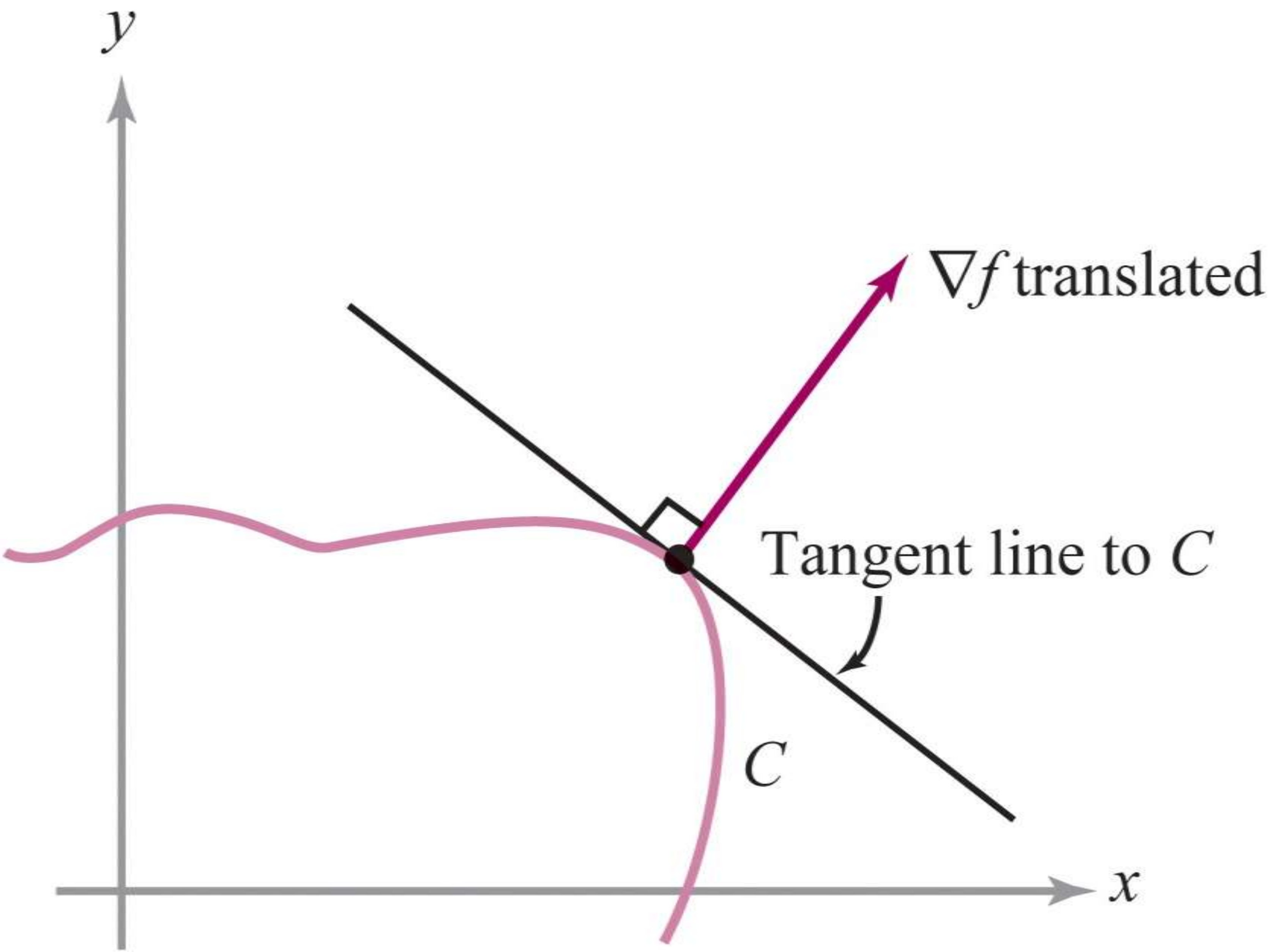


z
$\nabla f(x_0, y_0, z_0)$
$\nabla f(x_0, y_0, z_0)$
parallel translated so
that it begins
at (x_0, y_0, z_0)
v translated
v
y
$\mathbf{c}(t)$
S
(x_0, y_0, z_0)
x

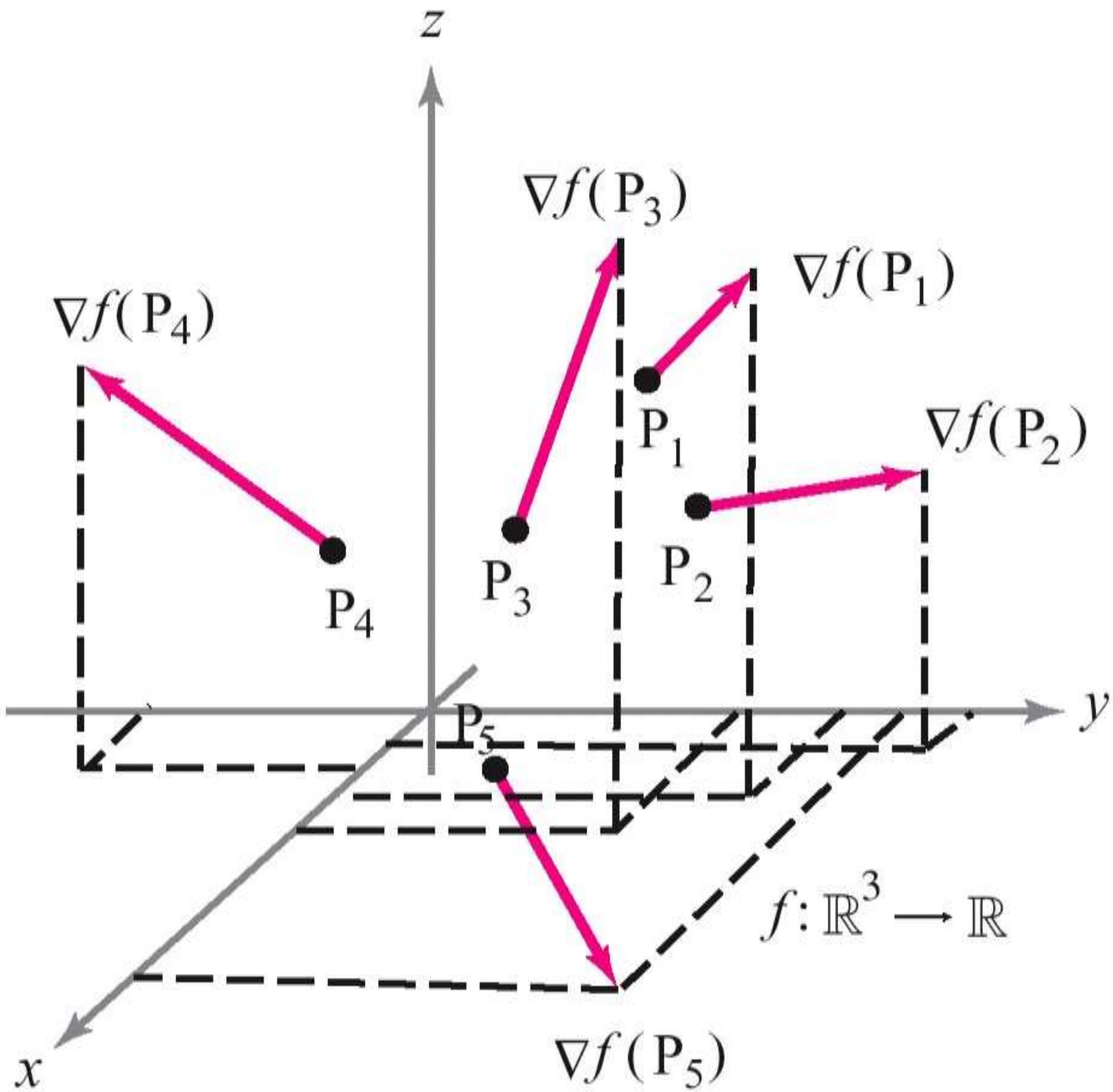
DEFINITION: Tangent Planes to Level Surfaces Let S be the surface consisting of those (x, y, z) such that $f(x, y, z) = k$, for k a constant. The ***tangent plane*** of S at a point (x_0, y_0, z_0) of S is defined by the equation

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0 \qquad (1)$$

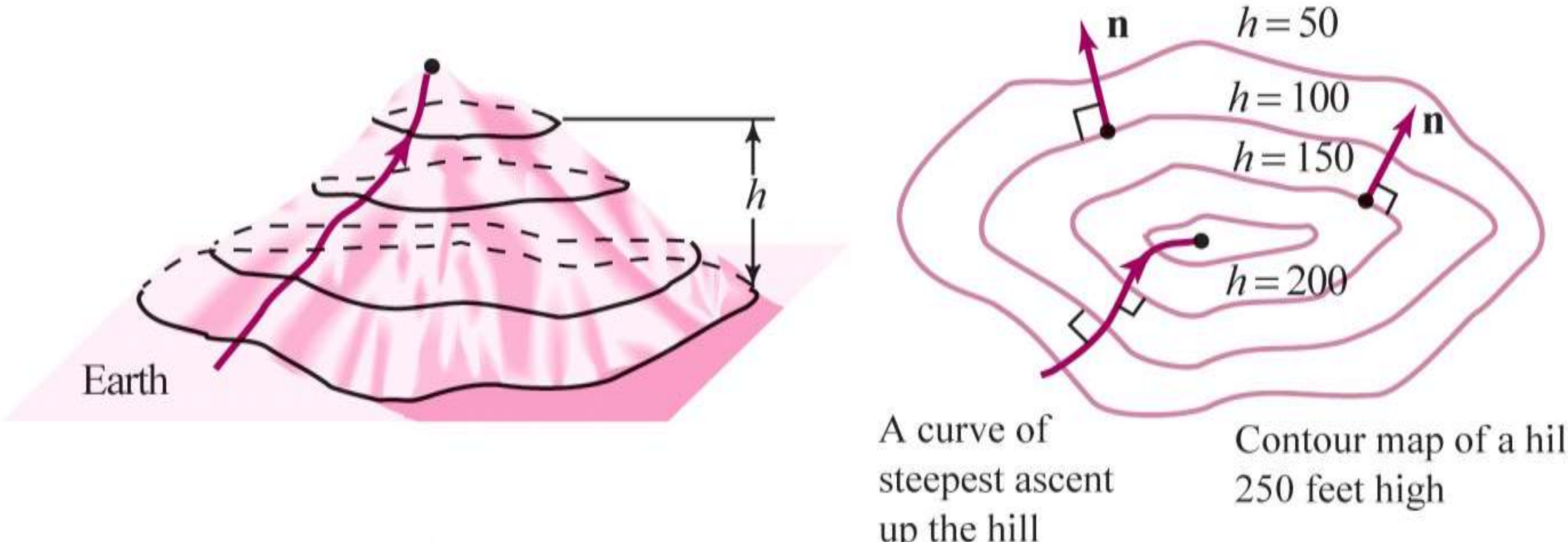
if $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$. That is, the tangent plane is the set of points (x, y, z) that satisfy equation (1).



y
∇f translated
Tangent line to C
C
x

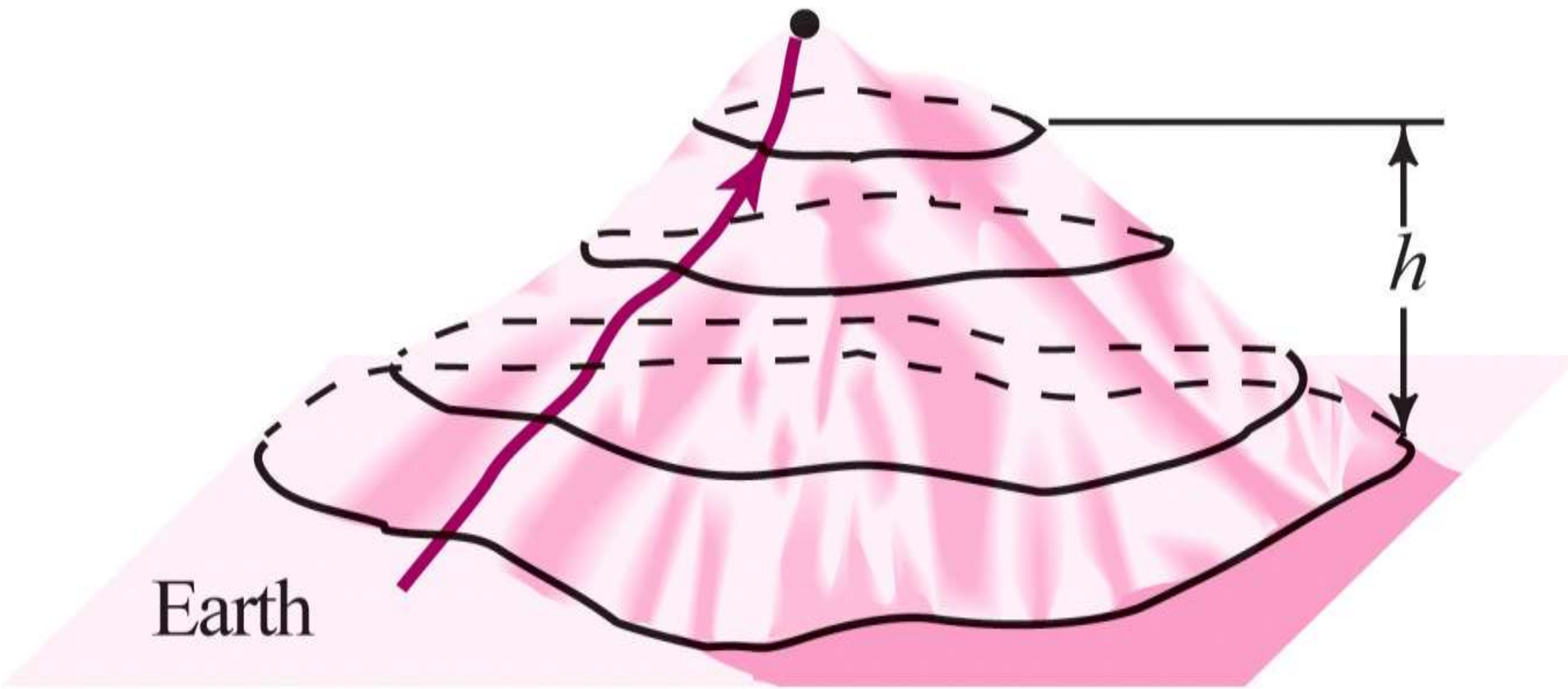


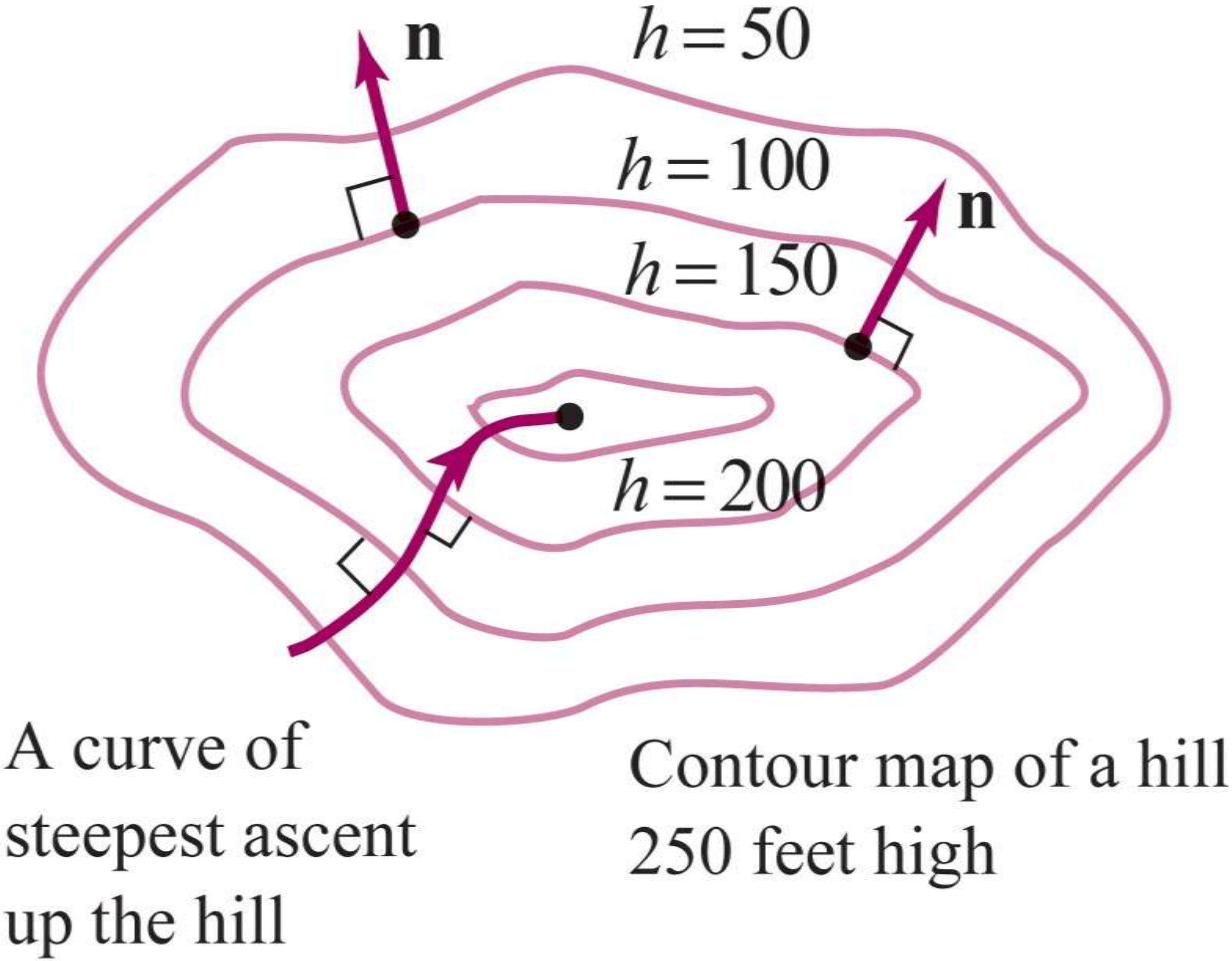
z
∇f(P3)
∇f(P1)
∇f(P4)
P1
∇f(P2)
P4
P3
P2
y
P5
f: ℝ3 → ℝ
x
∇f (P5)

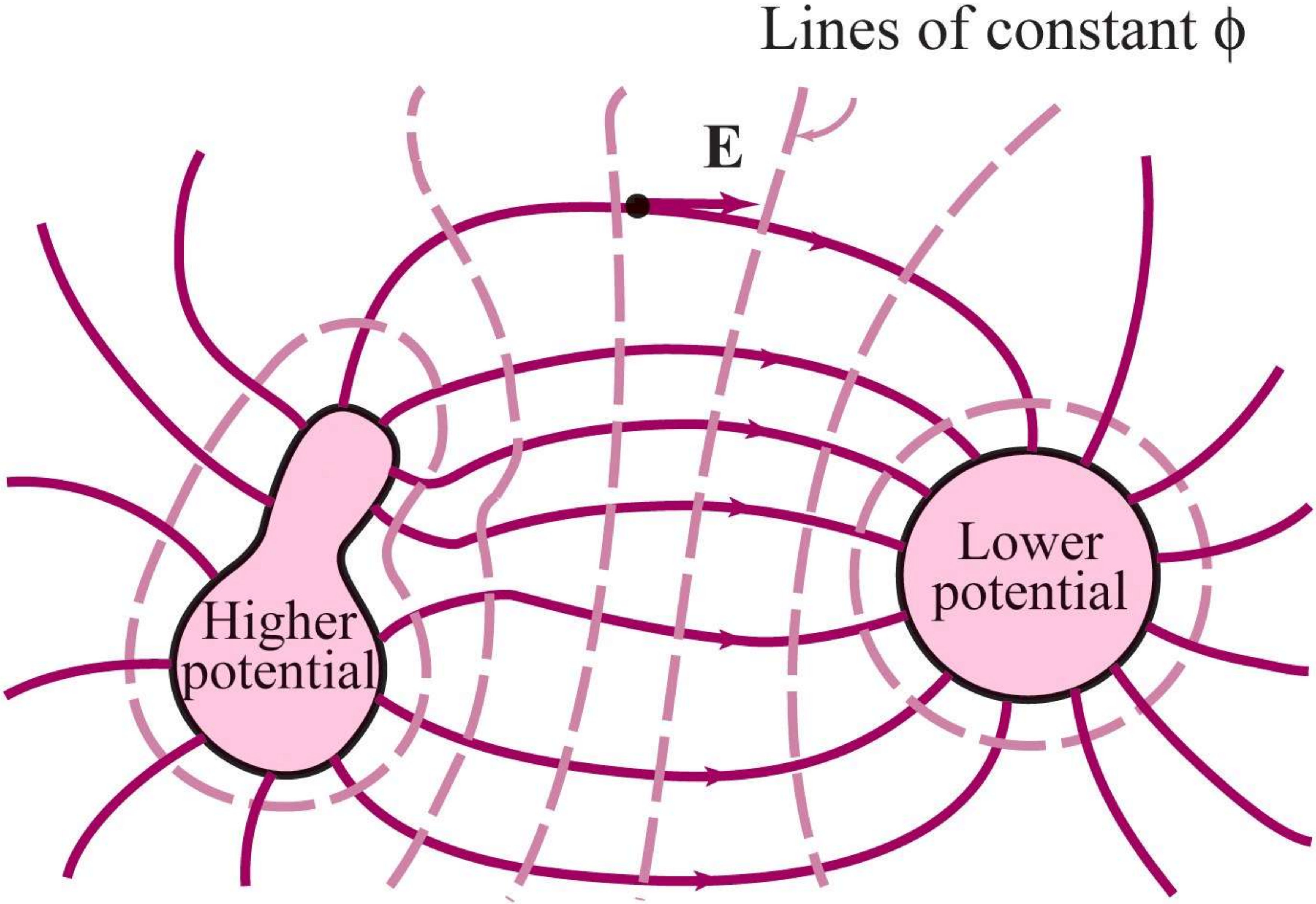


Earth
h
n
n
h = 50
h = 100
h = 150
h = 200
A curve of steepest ascent up the hill
Contour map of a hill 250 feet high

(a) (b)


h
Earth


n
$h = 50$
$h = 100$
n
$h = 150$
$h = 200$
A curve of steepest ascent up the hill
Contour map of a hill 250 feet high


Lines of constant ϕ
E
Higher potential
Lower potential

16 Vector Calculus

16.9 The Divergence Theorem

The Divergence Theorem

We write Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where *C* is the positively oriented boundary curve of the plane region *D*.

If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

1 $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

where *S* is the boundary surface of the solid region *E*.

The Divergence Theorem

It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem.

Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function (div **F** in this case) over a region to the integral of the original function **F** over the boundary of the region.

We state the Divergence Theorem for regions *E* that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.)

The Divergence Theorem

The boundary of *E* is a closed surface, and we use the convention, that the positive orientation is outward; that is, the unit normal vector **n** is directed outward from *E*.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of **F** across the boundary surface of *E* is equal to the triple integral of the divergence of **F** over *E*.

Example 1

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

First we compute the divergence of **F**:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leq 1$.

Example 1 – *Solution*

cont'd

Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \operatorname{div} \mathbf{F} \, dV$$

$$= \iiint_B 1 \, dV$$

$$= V(B)$$

$$= \tfrac{4}{3}\pi(1)^3$$

$$= \frac{4\pi}{3}$$

The Divergence Theorem

Let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2.

Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See Figure 3.)

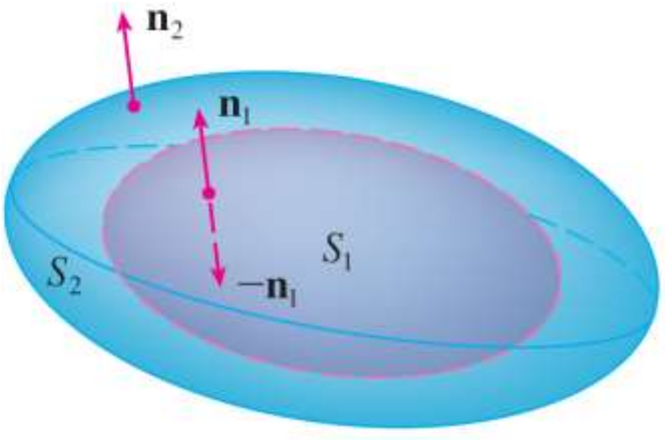


Figure 3

The Divergence Theorem

Applying the Divergence Theorem to *S*, we get

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$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS$$

$$= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

Example 3

We considered the electric field:

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector.

Use the Divergence Theorem to show that the electric flux of **E** through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$$

Example 3 – *Solution*

The difficulty is that we don't have an explicit equation for S_2 because it is *any* closed surface enclosing the origin. The simplest such surface would be a sphere, so

we let S_1 be a small sphere with radius a and center the origin. You can verify that div **E** = 0.

Therefore Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_{E} \operatorname{div} \mathbf{E} \, dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} \, dS$$

Example 3 – *Solution*

cont'd

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at **x** is $\mathbf{x}/|\mathbf{x}|$.

Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\varepsilon Q}{|\mathbf{x}|^4}\mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$.

Example 3 – *Solution*

cont'd

Thus we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\varepsilon Q}{a^2} \iint_{S_1} dS = \frac{\varepsilon Q}{a^2} A(S_1) = \frac{\varepsilon Q}{a^2} 4\pi a^2 = 4\pi\varepsilon Q$$

This shows that the electric flux of **E** is $4\pi\varepsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss's Law for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.]

The Divergence Theorem

Another application of the Divergence Theorem occurs in fluid flow. Let **v**(x, y, z) be the velocity field of a fluid with constant density ρ. Then **F** = ρ**v** is the rate of flow per unit area.

The Divergence Theorem

If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ for all points in B_a since div $\mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F} \, dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0) \, dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

8

$$\operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

The Divergence Theorem

Equation 8 says that div $\mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence.*)

If div $\mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**.

If div $\mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

The Divergence Theorem

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1.

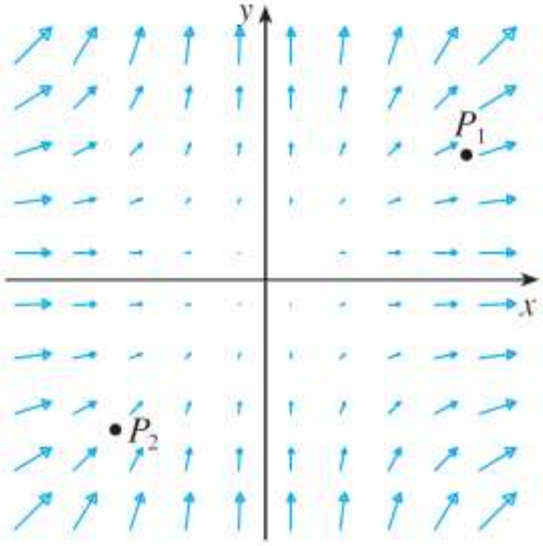


Figure 4

The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

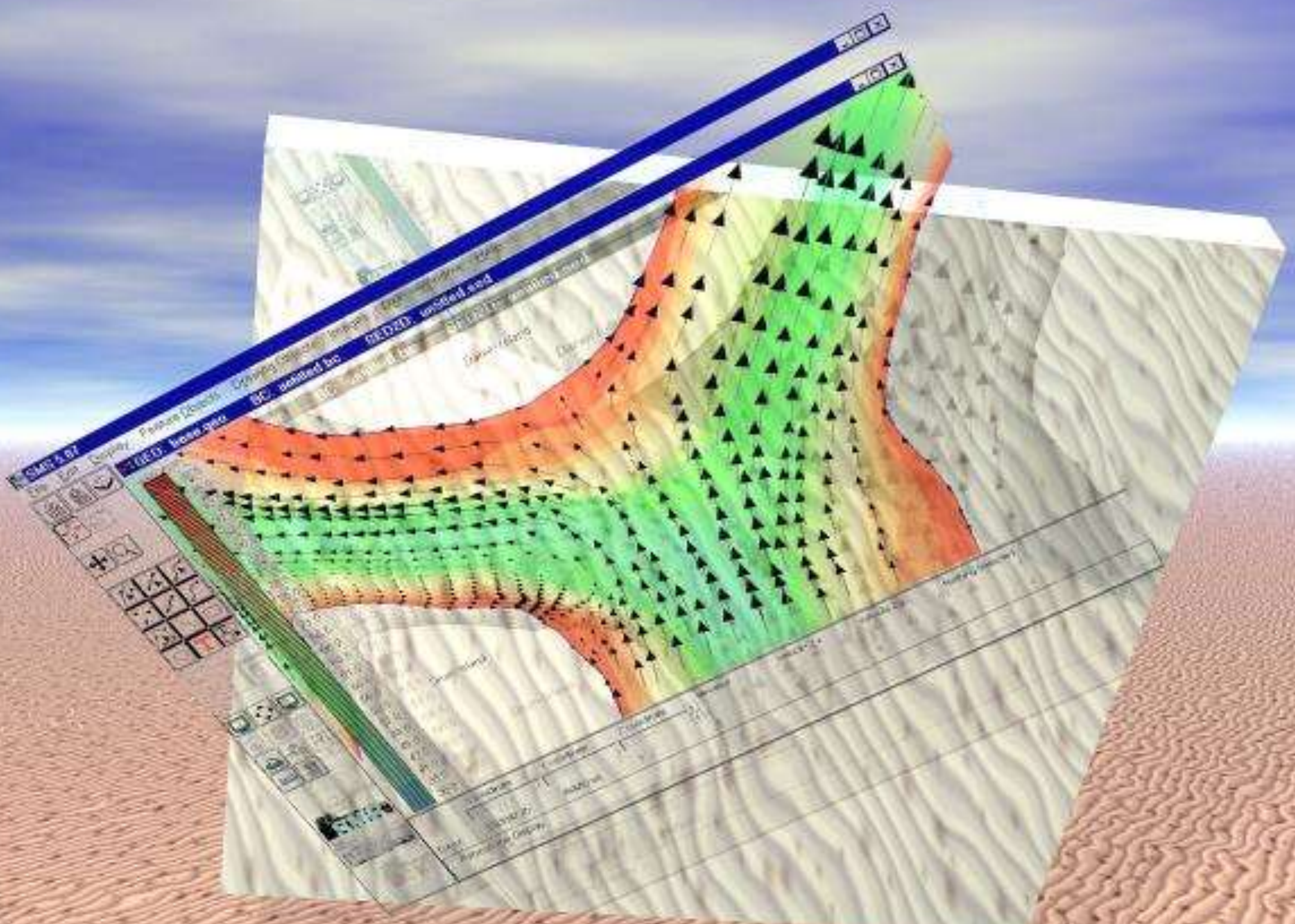
The Divergence Theorem

Thus the net flow is outward near P_1, so div $\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows.

Here the net flow is inward, so div $\mathbf{F}(P_2) < 0$ and P_2 is a sink.

We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have div $\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

Green's Theorem in the Plane



Green's Theorem

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\ dy = \oint_C (F_1\ dx + F_2\ dy)$$

here we integrate along the entire boundary C of R such that R is on the left as we advance in the direction of integration.

Example

Evaluate

$$\oint_C [(e^{x^3} + y)\ dx + (x^2 + \sin^{-1} y^2)\ dy]$$

for C the rectangle with vertices $(1, 2)$, $(4, 2)$, $(4, 3)$, and $(1, 3)$.

Example

Verify Green's Theorem for

$$\oint_C (1 + 10xy + y^2)\ dx + (6xy + 5x^2)\ dy$$

where C is the square with vertices $(0,0), (a,0), (a,a), (0,a)$

Area Formulas

$$A = \iint_R dx\ dy$$

$$= \oint_C x\ dy$$

$$= -\oint_C y\ dx$$

$$= \frac{1}{2}\oint_C (x\ dy - y\ dx)$$

$$= \frac{1}{2}\oint_C r^2\ d\theta$$

Example

Show that the area of the region Ω enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.