UNIT 1

Introduction to Control Systems

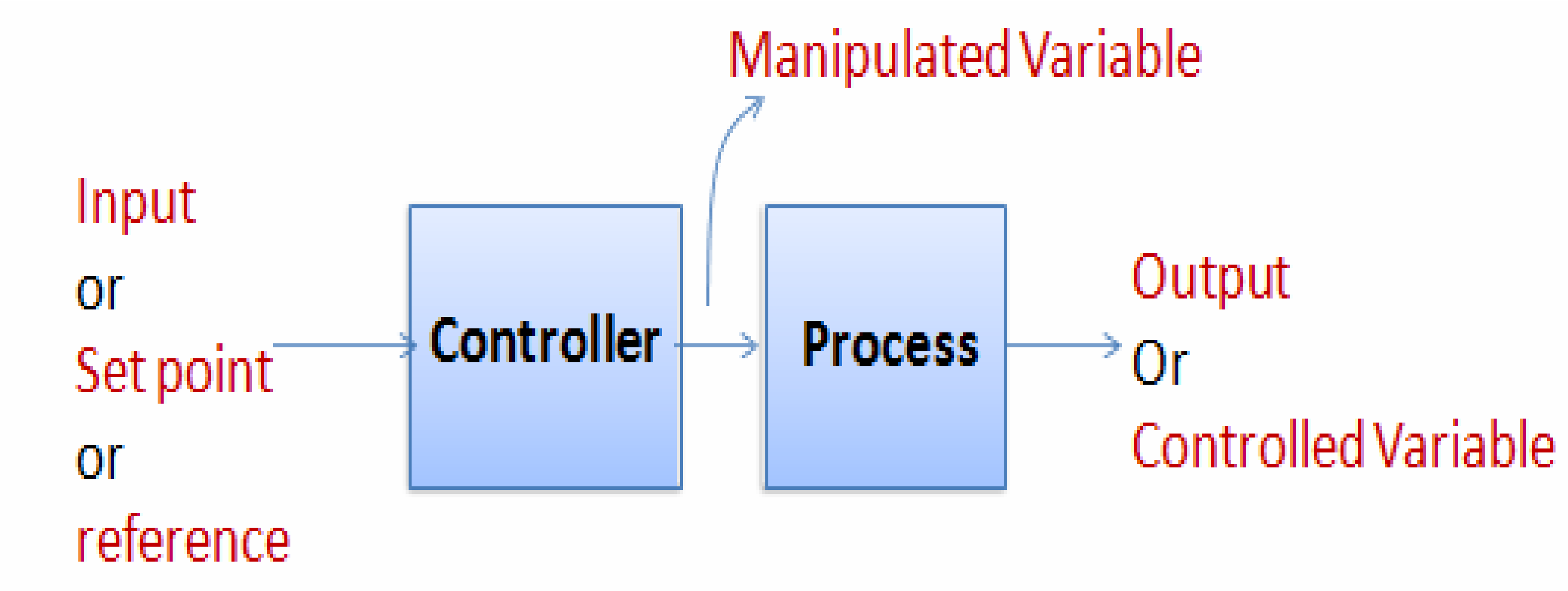
System – An interconnection of elements and devices for a desired purpose.

Control System – An interconnection of components forming a system configuration that will provide a desired response.

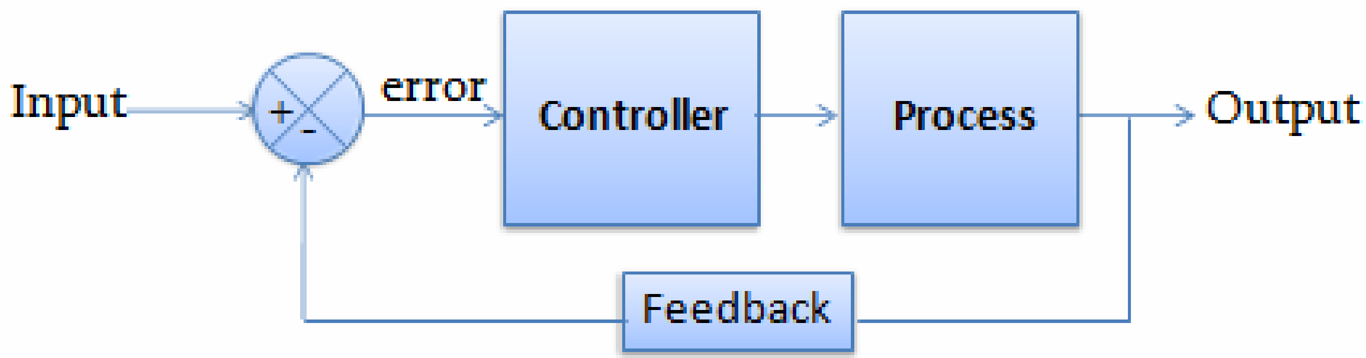
What is Control System?

- A system that provides an output or response for a given input

Open loop system



GENERAL BLOCK DIAGRAM OF CLOSED LOOP CONTROL SYSTEM



Basic Components of Control System

- Plant
- Feedback
- Controller
- Error detector

- **Plant:** The portion of a system which is to be controlled or regulated is called as plant or process. It is a unit where actual processing is performed and if we observe in the above figure, the input of the plant is the controlled signal generated by a controller.
- **Feedback:** It is a controlled action in which the output is sampled and a proportional signal is given to the input for automatic correction of any changes in the desired output.

  The output is given as feedback to the input for correction i.e. information about output is given to input for correcting the changes in output due to disturbances. The feedback signal is fed to the error detector.

- **Error Detector:** The function of error detector is to compare the reference input with the feedback signal. It produces an error signal which is a difference of two inputs which are reference signal and a feedback signal. The error signal is fed to the controller for necessary controlled action.
- **Controller:** the element of a system within itself or external to the system which controls the plant is called as a controller. The error signal will be a weak signal and so it has to be amplified and then modified for better control action.

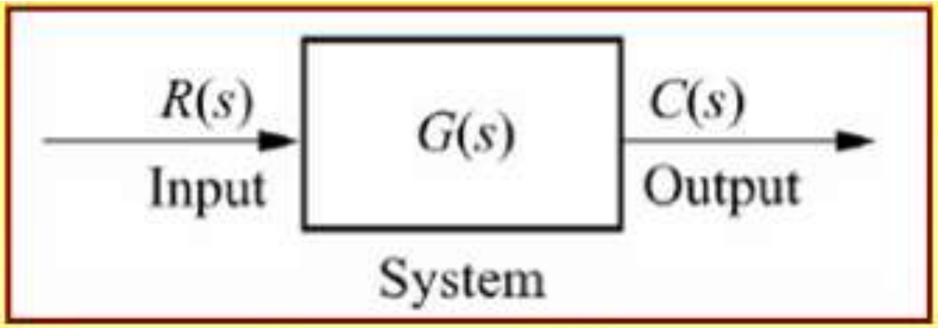
  In most of the systems, the controller itself amplifies the error signal and integrates or differentiates to generate a control signal.

Feedback and its effect

- Feedback system is a system that maintains a relationship between the output and some reference input by comparing them and using the difference as a means of control.
- Feedback is used to reduce the error between reference and the system output.
- The feedback has effects on performance characteristics as
  - Stability
  - overall gain
  - noise (external disturbance).

Classification of control system

- Single input single output (SISO)
- Multiple input multiple output (MIMO)
- Linear
- Non-linear
- Time-variant
- Time-invariant
- Analog
- Digital
- Process Control
- Sequential Control



- **Transfer function** is the ratio of the output over the input variables.

$$G(s) = \frac{C(s)}{R(s)}$$

- The output signal can then be derived as; C = GR

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Subject name :**Control Systems**

Unit No :**1**

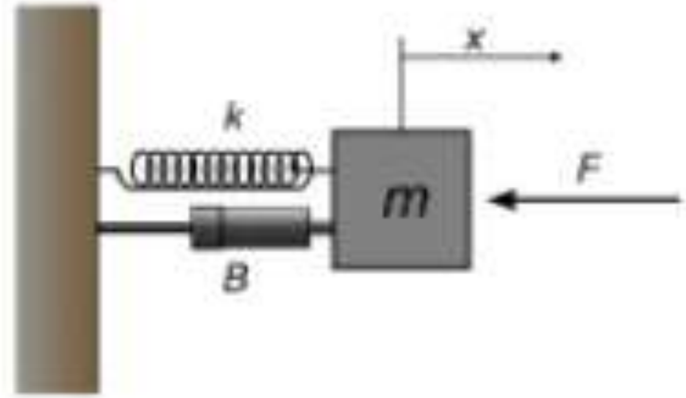
Topic :**Mechanical Translational Systems**

Faculty Name :**Dr. Lekshmi Sree B**

Mathematical Modeling of Mechanical Translational Systems

Basic Types of Mechanical Systems

- Translational
  - Linear Motion



- Rotational
  - Rotational Motion

Basic Elements of Translational Mechanical Systems

Translational Spring

i)

Translational Mass

ii)

Translational Damper

iii)

Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)

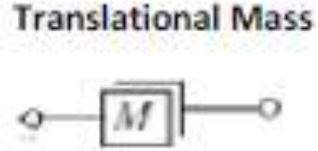
Circuit Symbols

Translational Spring

Translational Mass

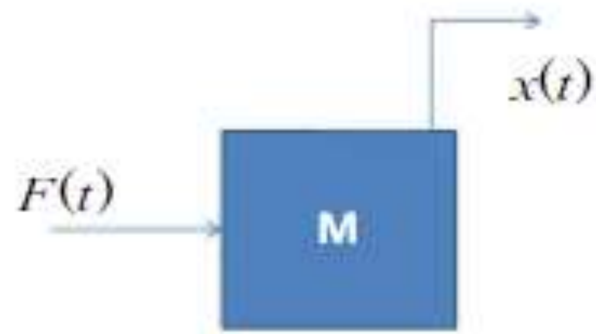
- Translational Mass is an inertia element.

ii)



- A mechanical system without mass does not exist.

- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation b/w force and displacements is given by Newton's law.



$$F = M\ddot{x}$$

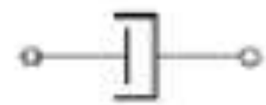
Translational Damper

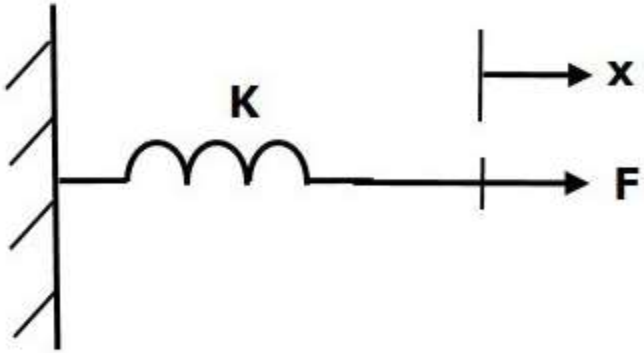
- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

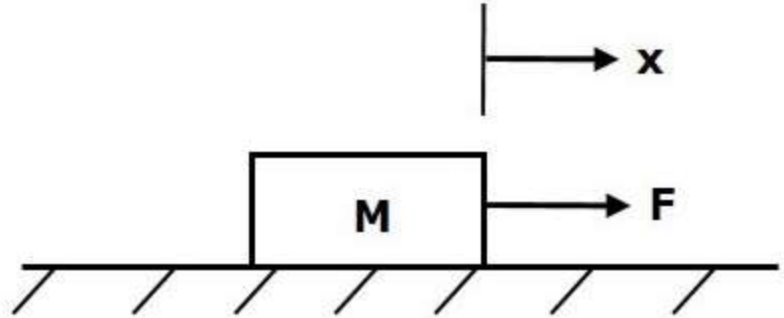
- All the materials exhibit the property of damping to some extent.

- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

iii) Translational Damper





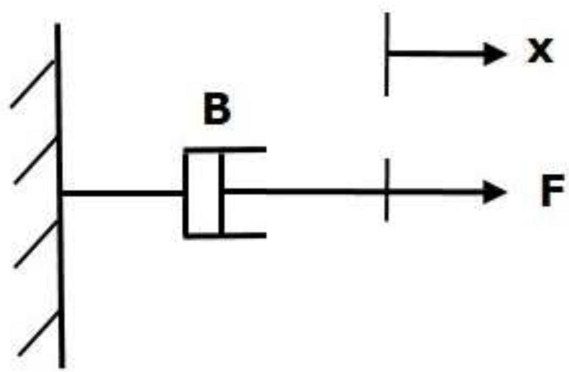


$$F_m \propto a$$

$$F \propto x$$

$$\Rightarrow F_m = Ma = M\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\Rightarrow F_k = Kx$$

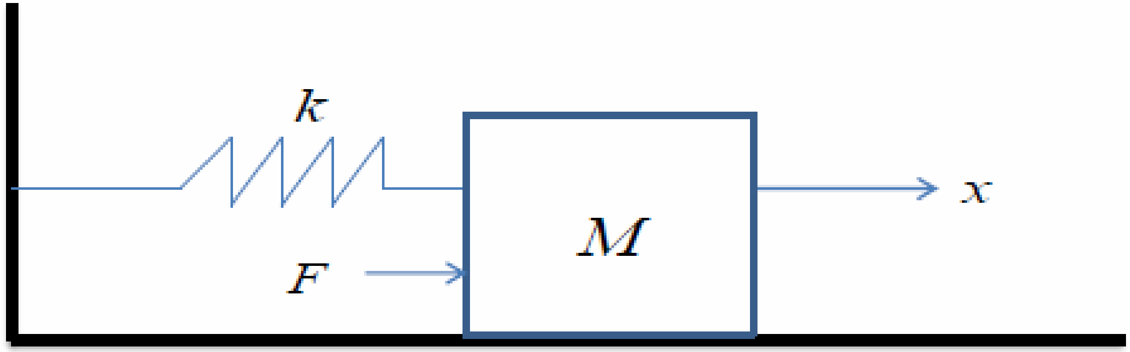


$$F_b \propto \nu$$

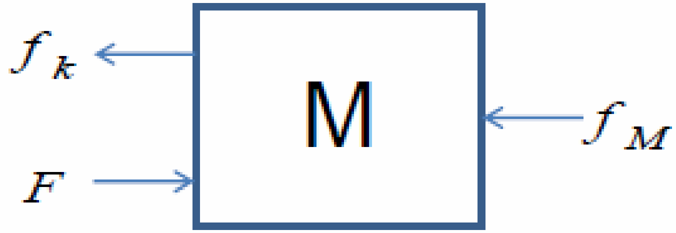
$$\Rightarrow F_b = B\nu = B\frac{\mathrm{d}x}{\mathrm{d}t}$$

EXAMPLE 1

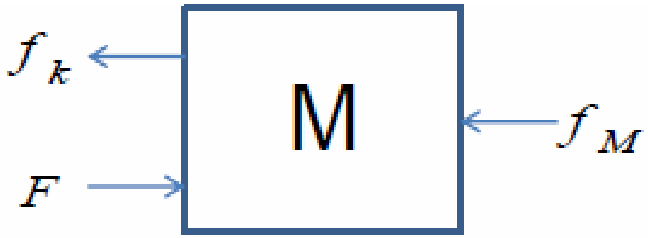
- Consider the following system (friction is negligible)



- Free Body Diagram



- Where f_k and f_M are force applied by the spring and inertial force respectively.



$$F = f_k + f_M$$

- Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

- Taking the Laplace Transform of both sides and ignoring initial conditions we get

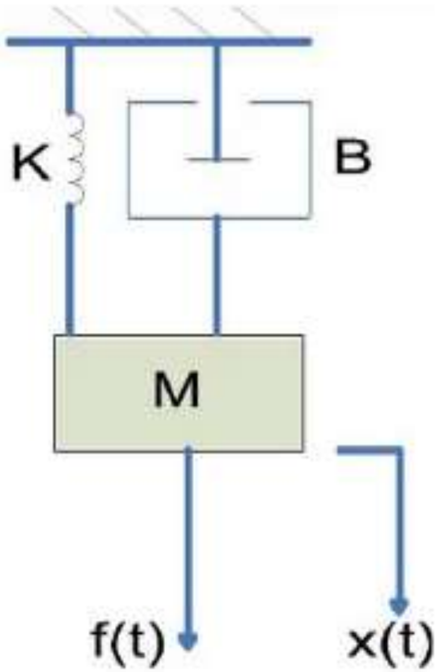
$$F(s) = Ms^2 X(s) + kX(s)$$

- The transfer function of the system is

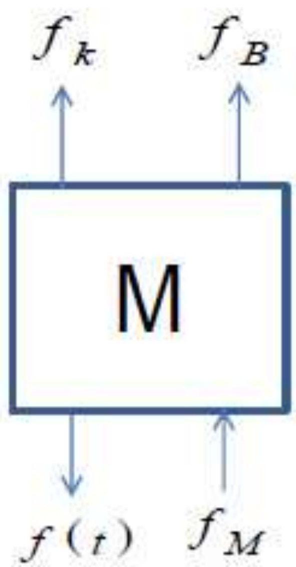
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + k}$$

EXAMPLE 2

- Find the transfer function of the mechanical translational system given in Figure.



Free Body Diagram



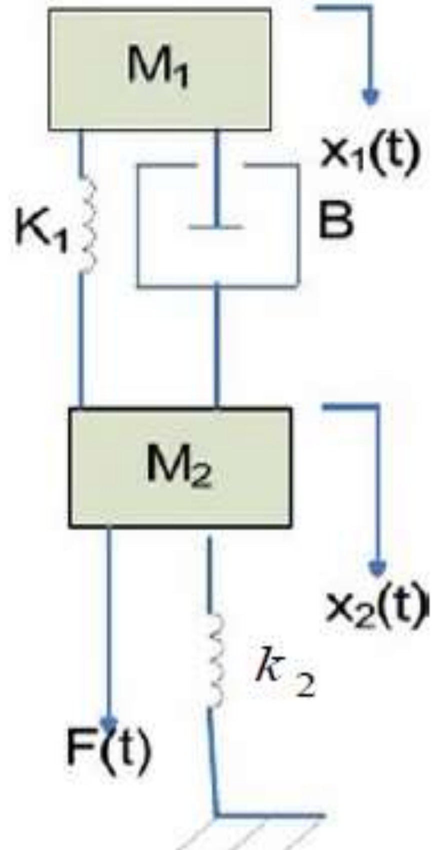
$$f(t) = f_k + f_M + f_B$$

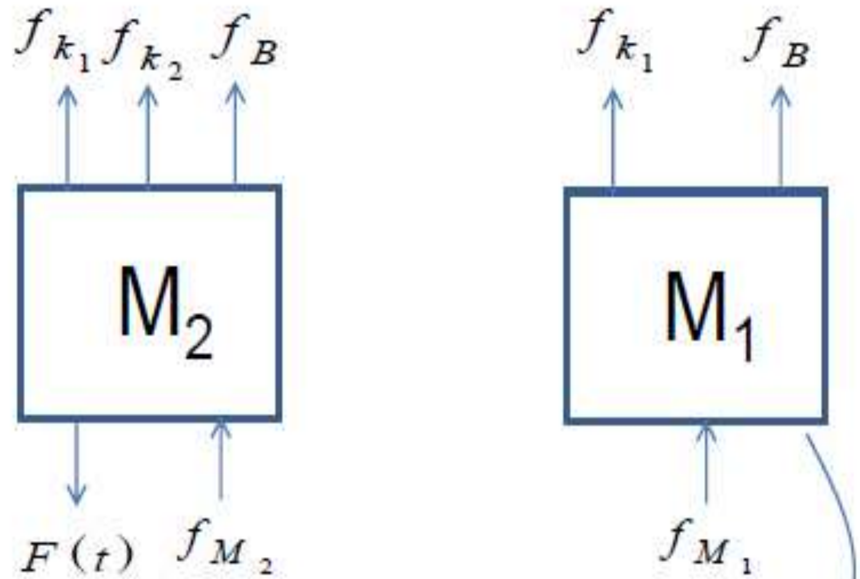
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

❑Find the transfer function $X_2(s)/F(s)$ of the following system.

EXAMPLE 3

Free Body Diagram





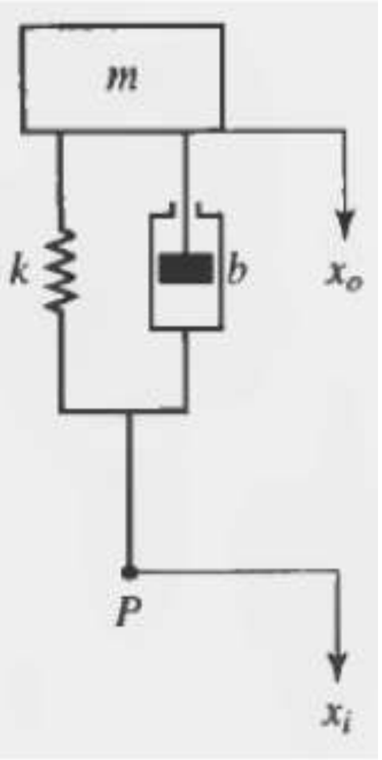
$$F(t) = f_{k_1} + f_{k_2} + f_{M_2} + f_B$$

$$0 = f_{k_1} + f_{M_1} + f_B$$

EXAMPLE 4

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0 \qquad (eq.1)$$

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i \qquad (eq.\ 2)$$



- Taking Laplace Transform of the equation (2)

$$ms^2X_o(s) + bsX_o(s) + kX_o(s) = bsX_i(s) + kX_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

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Subject name :Control Systems

Unit No :1

Topic :Mechanical Rotational Systems

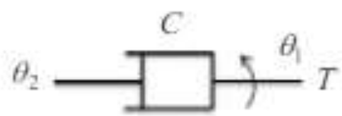
Faculty Name :Dr. Lekshmi Sree B

BASIC ELEMENTS OF ROTATIONAL SYSTEM

Rotational Spring

$$T = k(\theta_1 - \theta_2)$$

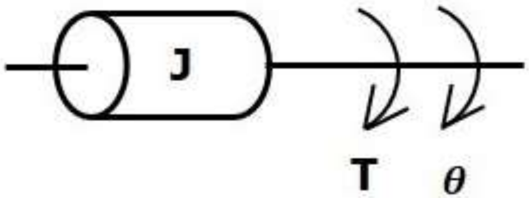
Rotational Damper

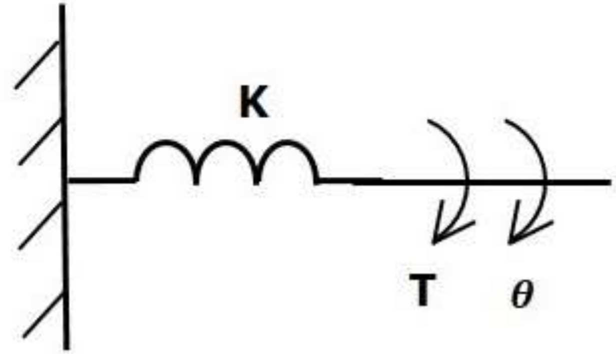


$$T = C(\dot{\theta}_1 - \dot{\theta}_2)$$

Moment of Inertia

$$T = J\ddot{\theta}$$



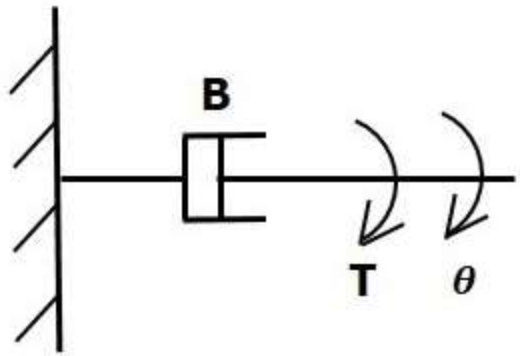


$$T_j \propto \alpha$$

$$\Rightarrow T_j = J\alpha = J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$T_k \propto \theta$$

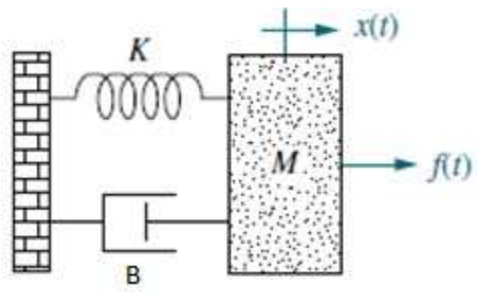
$$\Rightarrow T_k = K\theta$$



$$T_b \propto \omega$$

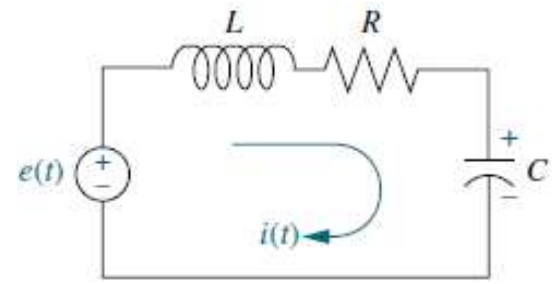
$$\Rightarrow T_b = B\omega = B\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Series (Force voltage) Analogy



(a)

1. Mass= M-> inductor =L
2. Damper=B-> resistor =R
3.Spring=K-> capacitor=1/C
4. Applied force=f(t)-> voltage source=e(t)
5. Velocity =v(t)-> mesh current=i(t)



(b)

Equation of motion of the above translational mechanical system is;

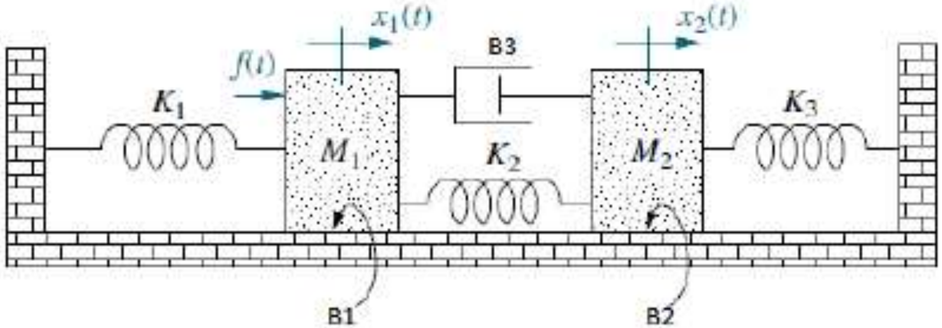
$$(Ms^2 + Bs + K)X(s) = F(s)$$

Kirchhoff's mesh equation for the above simple series RLC network is;

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$

Mechanical System to a Series Analog

- Draw a series analog for the mechanical system.

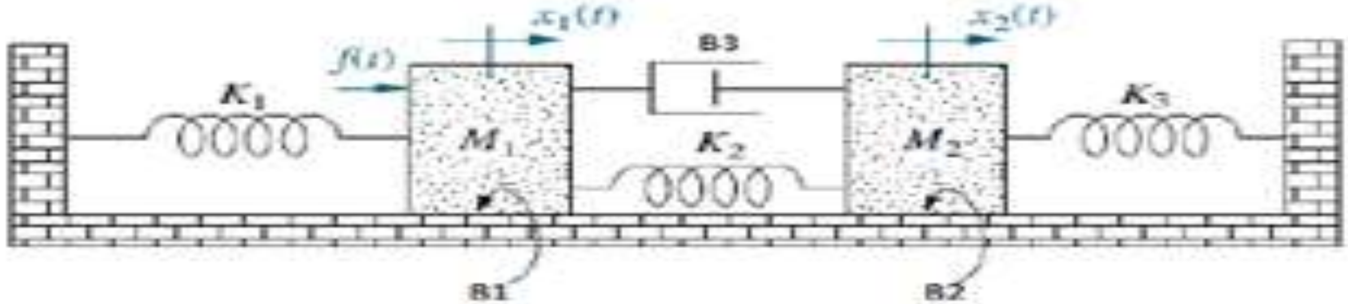


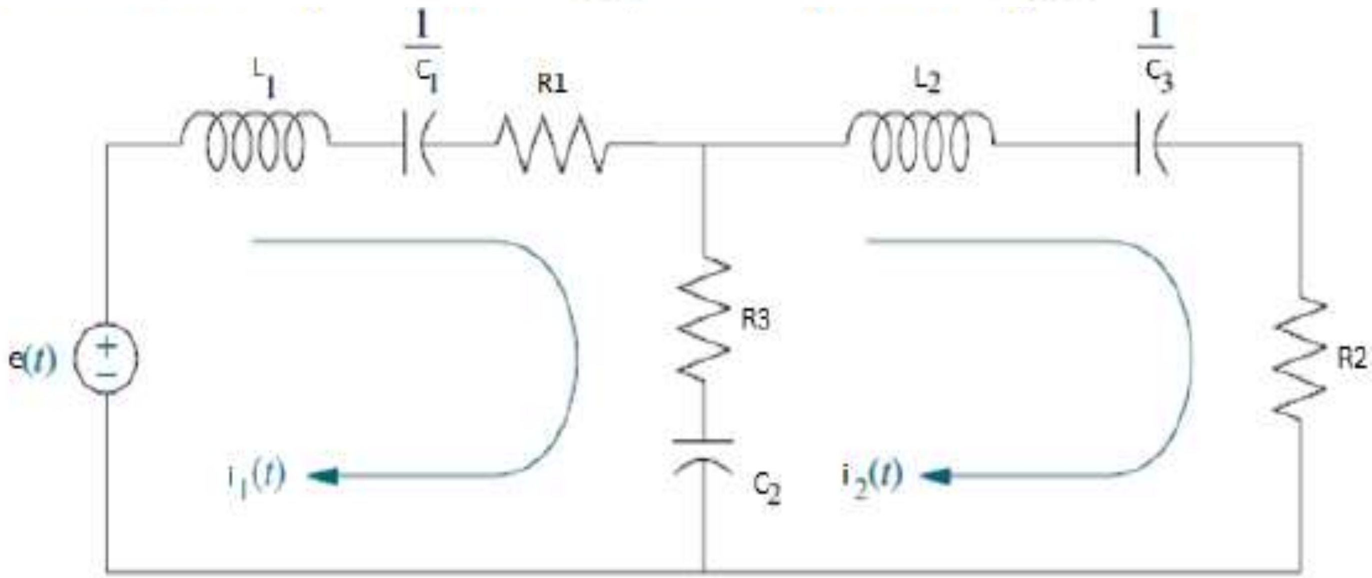
- The equations of motion in the Laplace transform domain are;

$$(M_1 s^2 + (B_1 + B_3)s + (K_1 + K_2)X_1(s) - (B_3 s + K_2)X_2(s) = F(s) \rightarrow (1)$$

$$-(B_3 s + K_2)X_1(s) + [M_2 s^2 + (B_2 + B_3)s + (K_2 + K_3)]X_2(s) = 0 \rightarrow (2)$$

- Coefficients represent sums of electrical impedance.
- Mechanical impedances associated withM1 form the first mesh,
- Where as impedances between the two masses are common to the two loops.

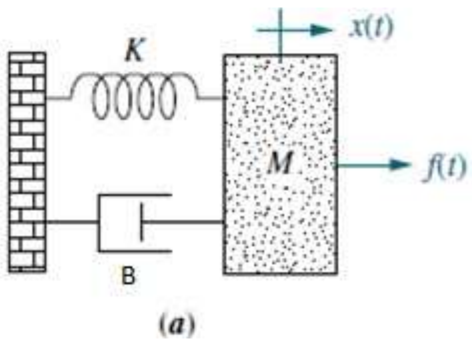




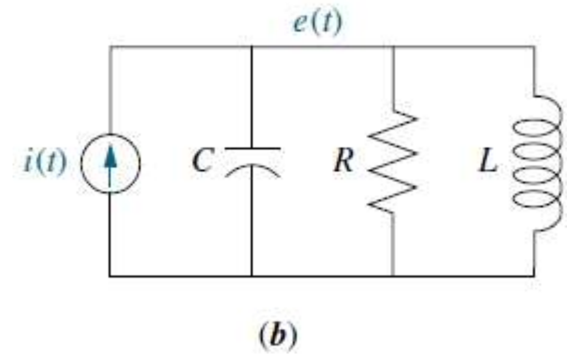
$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R_3(i_1 - i_2) + \frac{1}{C_2}\int (i_1 - i_2)dt = e(t) \rightarrow (1)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_3}\int i_2 dt + R_3(i_2 - i_1) + \frac{1}{C_2}\int (i_2 - i_1)dt = 0 \rightarrow (2)$$

Parallel (Force current) Analogy



1. Mass= M-> capacitor =C
2. Damper=B-> resistor =1/R
3.Spring=K-> inductor=1/L
4. Applied force=f(t)->



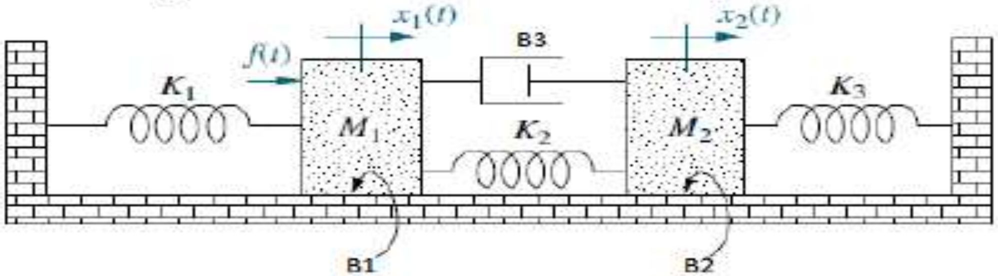
- Kirchhoff's nodal equation for the simple parallel RLC network shown above is;

$$(Ms^2 + Bs + K)X(s) = F(s)$$

$$\left(Cs + R + \frac{1}{Ls}\right)V(s) = I(s)$$

Mechanical System to a Parallel Analog

- Draw a parallel analog for the mechanical system.

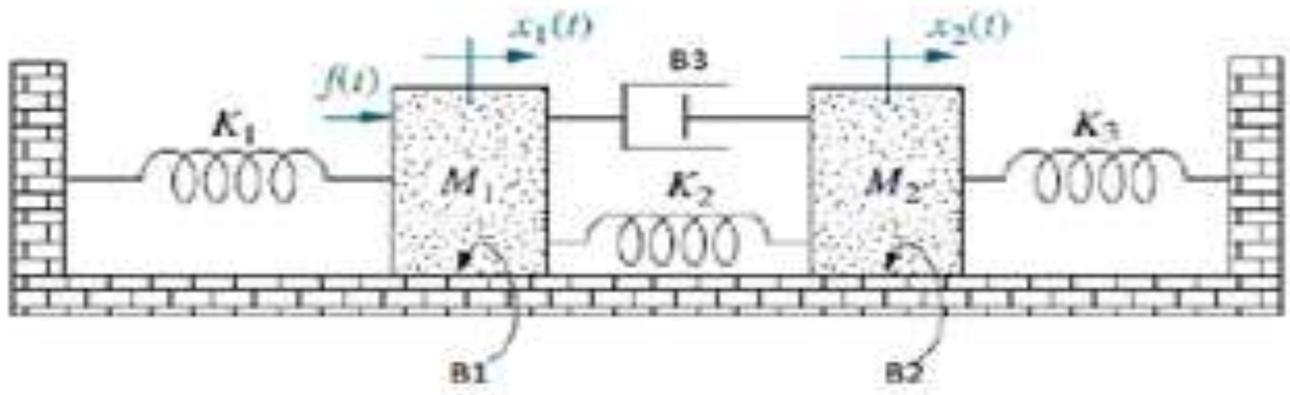


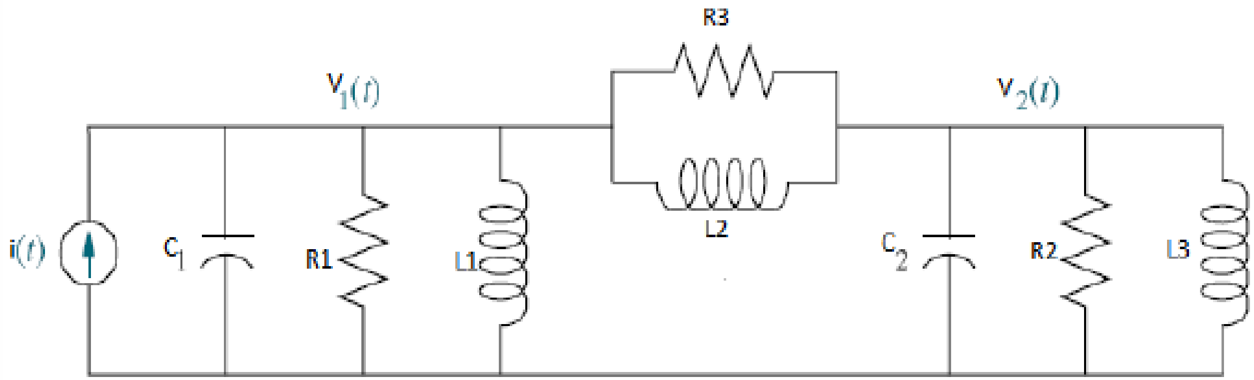
- Equations of motion after conversion to velocity are;

$$\left[(M_1 s + (B_1 + B_3) + \frac{(K_1 + K_2)}{s}\right] v_1(s) - \left(B_3 + \frac{K_2}{s}\right) v_2(s) = F(s) \rightarrow (1)$$

$$-\left(B_3 + \frac{K_2}{s}\right) v_1(s) + \left[M_2 s + (B_2 + B_3) + \frac{(K_2 + K_3)}{s}\right] v_2(s) = 0 \rightarrow (2)$$

- The Equation (1) and (2) are also analogous to electrical node equations.
- Coefficients represent sums of electrical admittances.
- Admittances associated with M1 form the elements connected to the first node





$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L_1} \int V_1 dt + \frac{1}{R_3}(V_1 - V_2) + \frac{1}{L_2} \int (V_1 - V_2) dt = i(t) \rightarrow (1)$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_3} \int V_2 dt + \frac{1}{R_3}(V_2 - V_1) + \frac{1}{L_2} \int (V_2 - V_1) dt = 0 \rightarrow (2)$$

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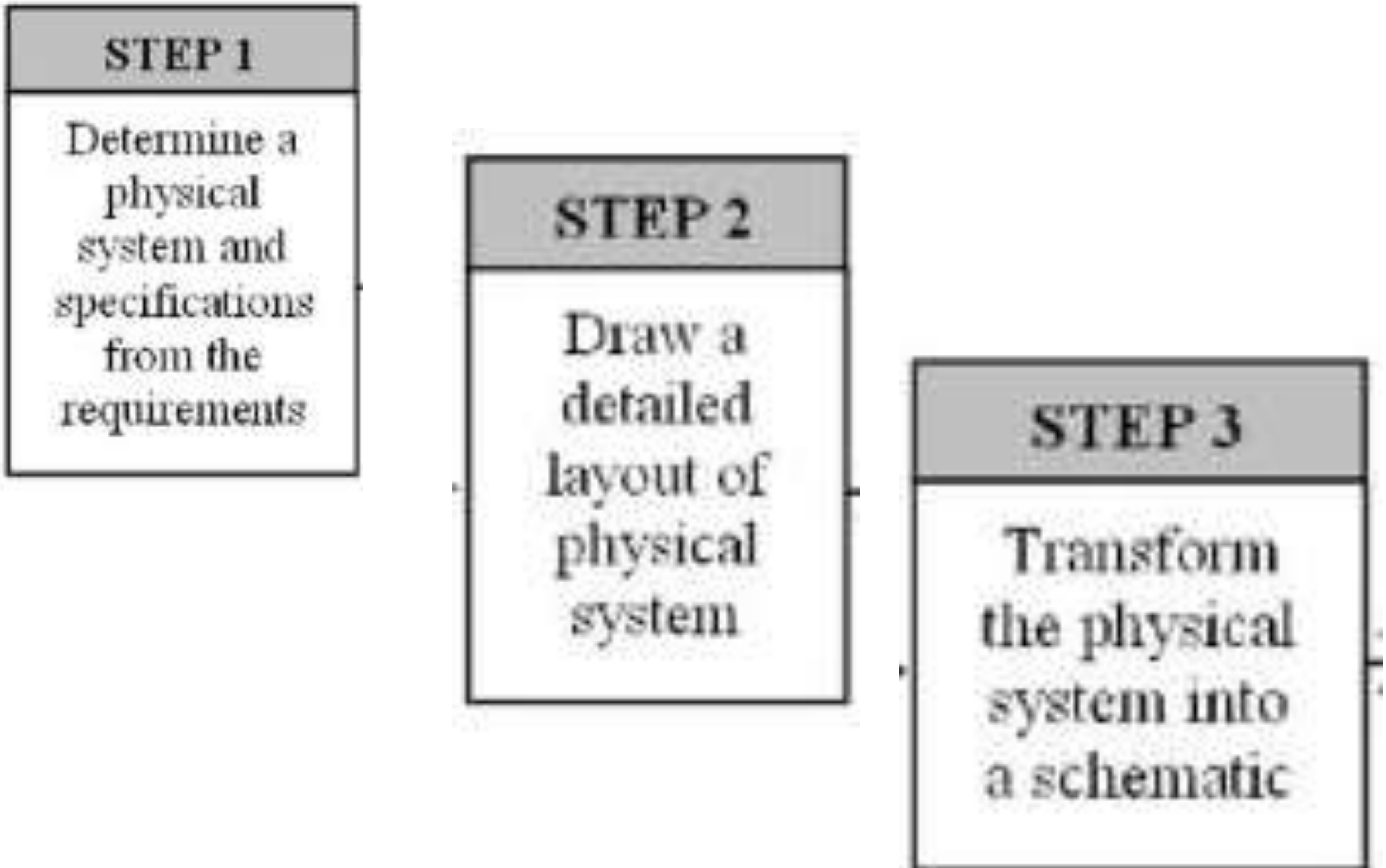
Subject name :**Control Systems**

Unit No :**1**

Topic :**Block Diagram Reduction Techniques**

Faculty Name :**Dr. Lekshmi Sree B**

DESIGN OF CONTROL SYSTEM



STEP 4

Use schematic to obtain a block diagram, signal flow diagram, or state-space representation

STEP 5

If multiple blocks, reduce the block diagram to a single block or closed-loop system

STEP 6

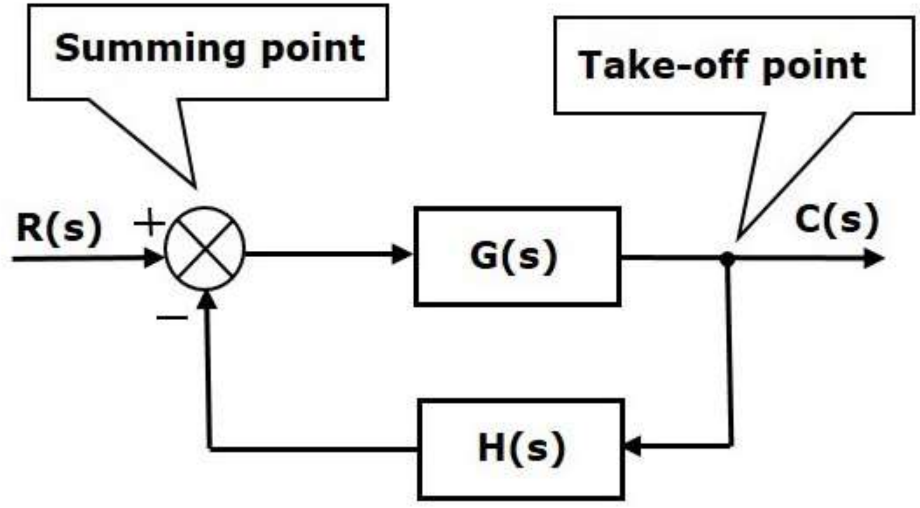
Analyze, design, and test to see that requirements and specifications are met

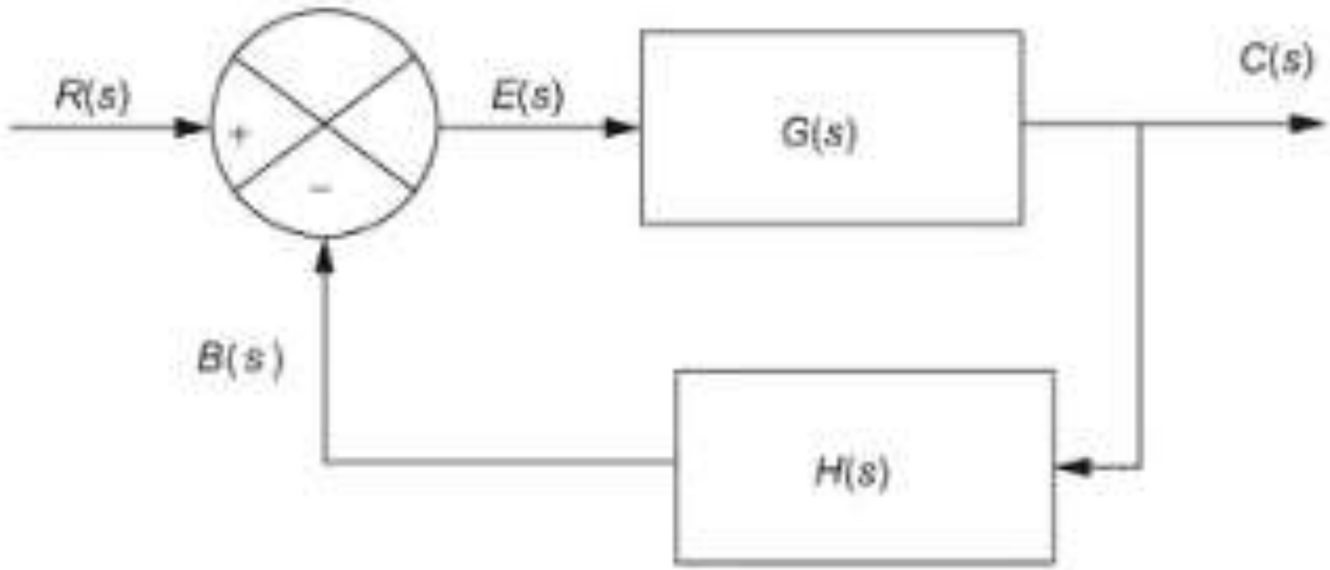
BLOCK DIAGRAM

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point.





$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

Reduction of Complicated Block Diagrams

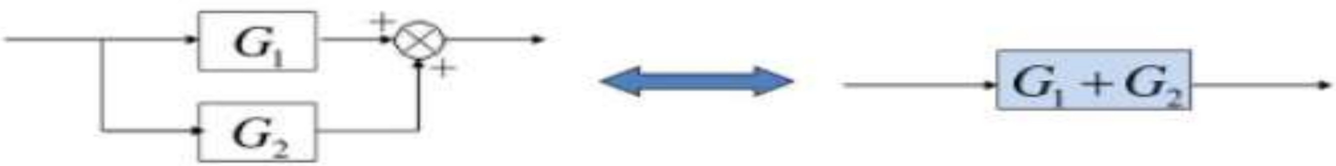
- The block diagram of a practical control system is often quite complicated.

- It may include several feedback or feedforward loops, and multiple inputs.

- By means of systematic block diagram reduction, every multiple loop linear feedback system may be reduced to canonical form.

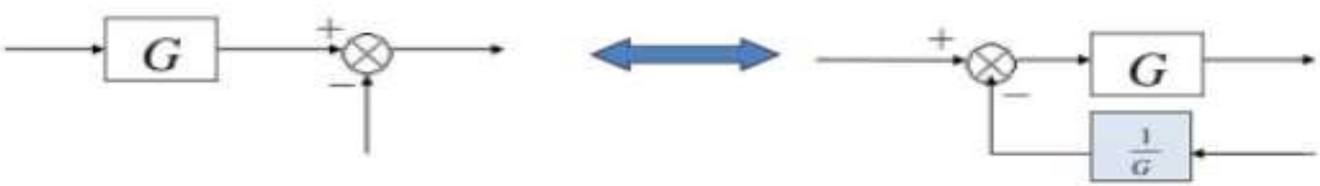
1. Combining blocks in cascade

2. Combining blocks in parallel



3. Moving a summing point behind a block

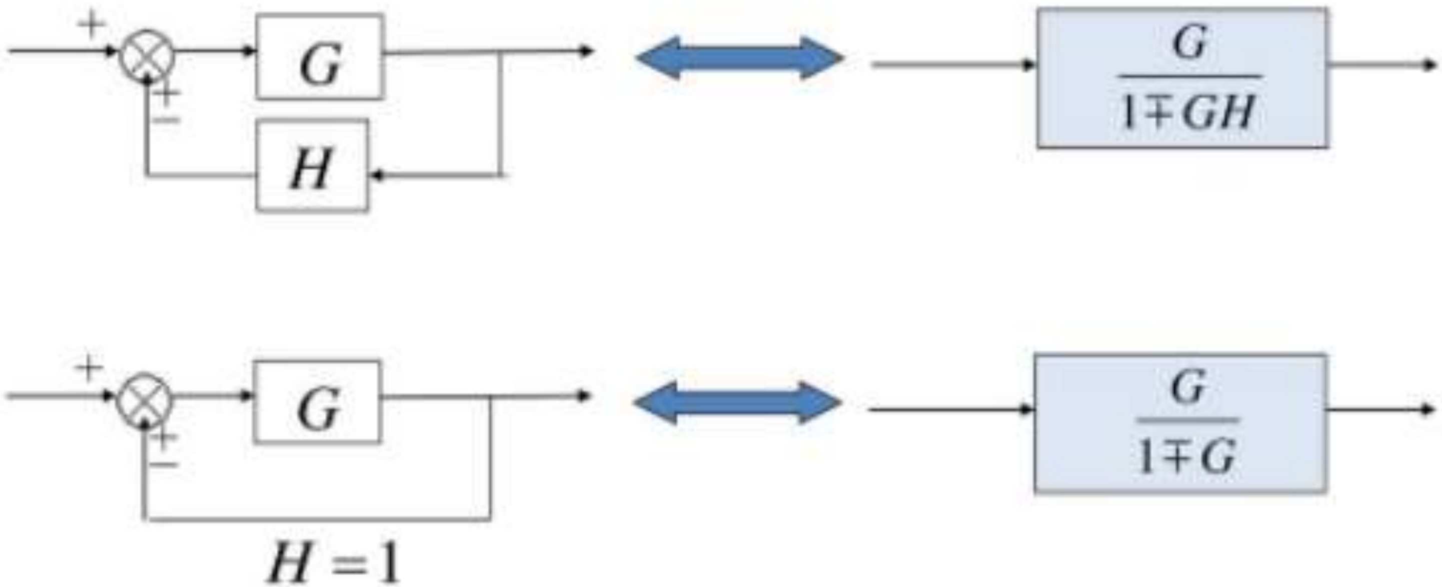
4. Moving a summing point ahead of a block



5. Moving a pickoff point behind a block

6. Moving a pickoff point ahead of a block

7. Eliminating a feedback loop



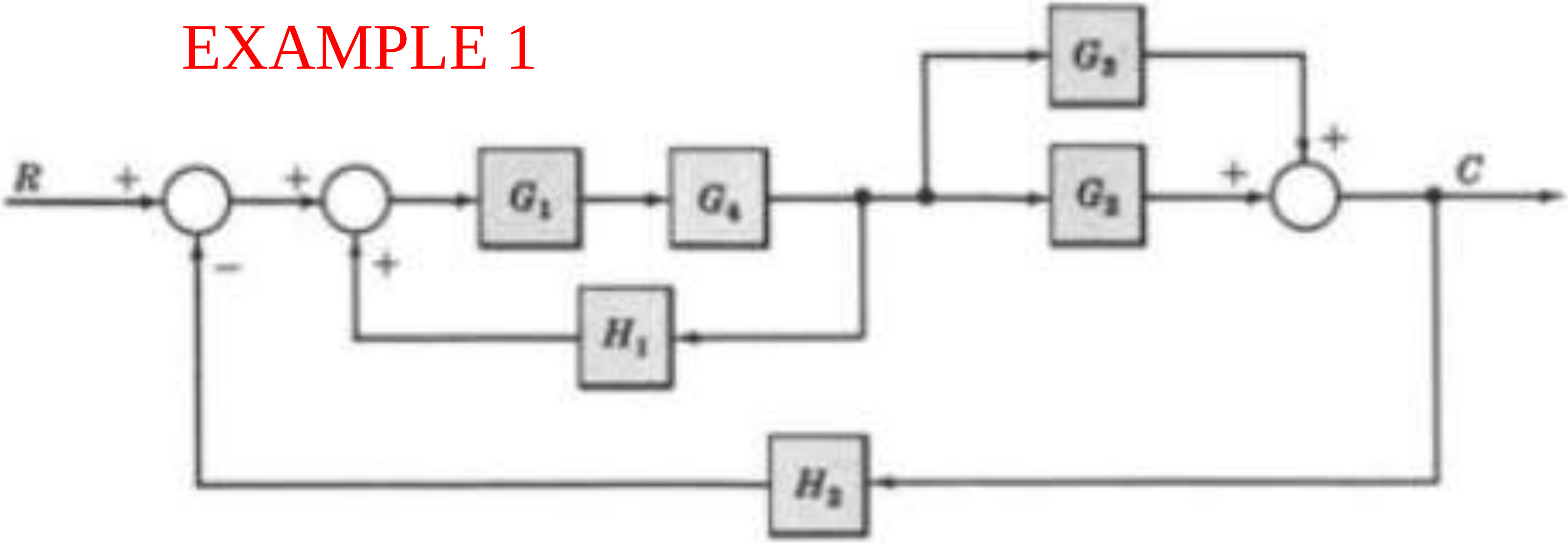
8. Swap with two neighboring summing points

Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

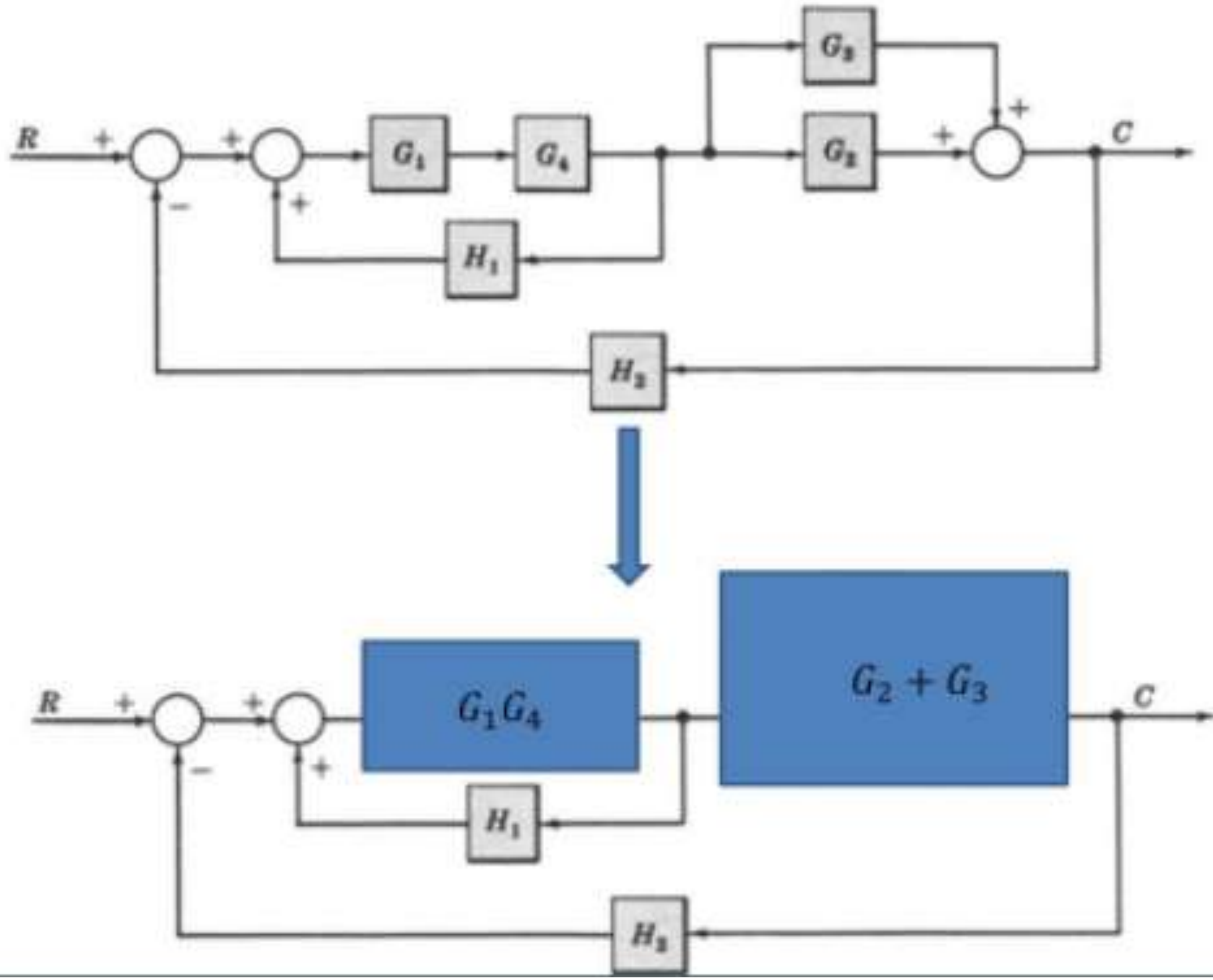
- **Rule 1** – Check for the blocks connected in series and simplify.
- **Rule 2** – Check for the blocks connected in parallel and simplify.
- **Rule 3** – Check for the blocks connected in feedback loop and simplify.
- **Rule 4** – If there is difficulty with take-off point while simplifying, shift it towards right.
- **Rule 5** – If there is difficulty with summing point while simplifying, shift it towards left.
- **Rule 6** – Repeat the above steps till you get the simplified form, i.e., single block.

EXAMPLE 1



- Combine all cascade block using rule-1

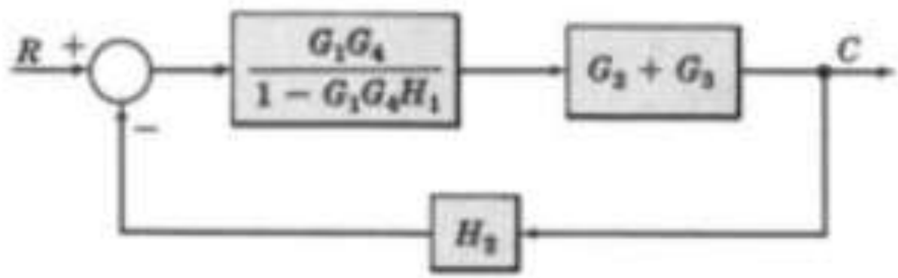
- Combine all parallel block using rule-2



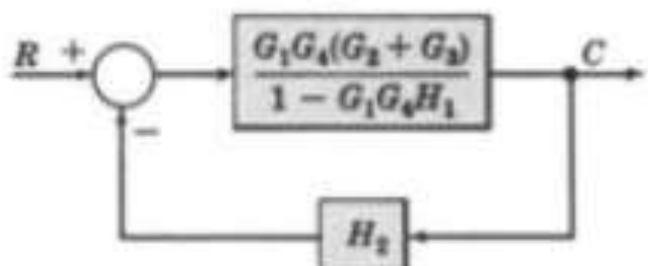
R
G_1
G_4
G_3
G_2
C
H_1
H_2
R
$G_1 G_4$
$G_2 + G_3$
C
H_1
H_2

- Eliminate all minor feedback loops using rule-7

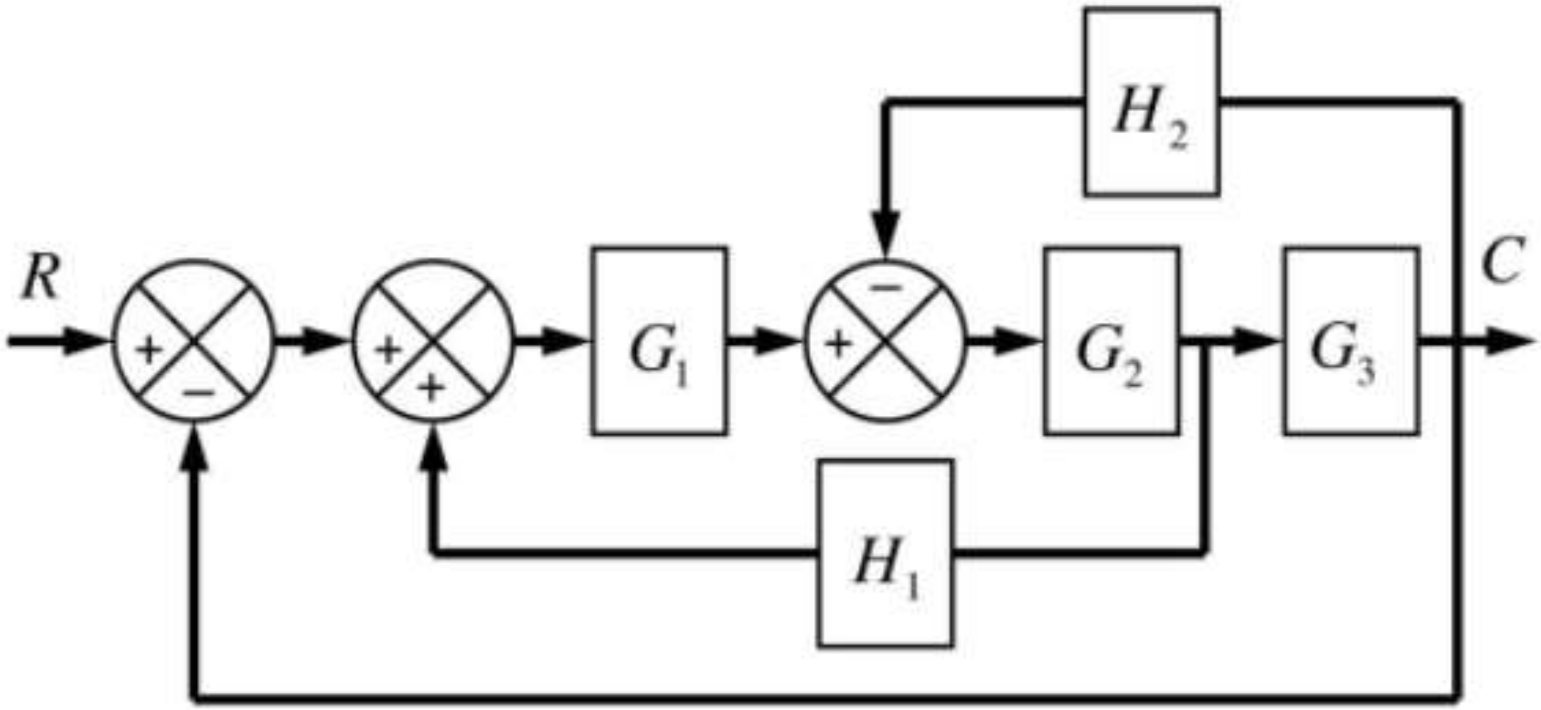
- After the elimination of minor feedback loop the block diagram is reduced to as shown

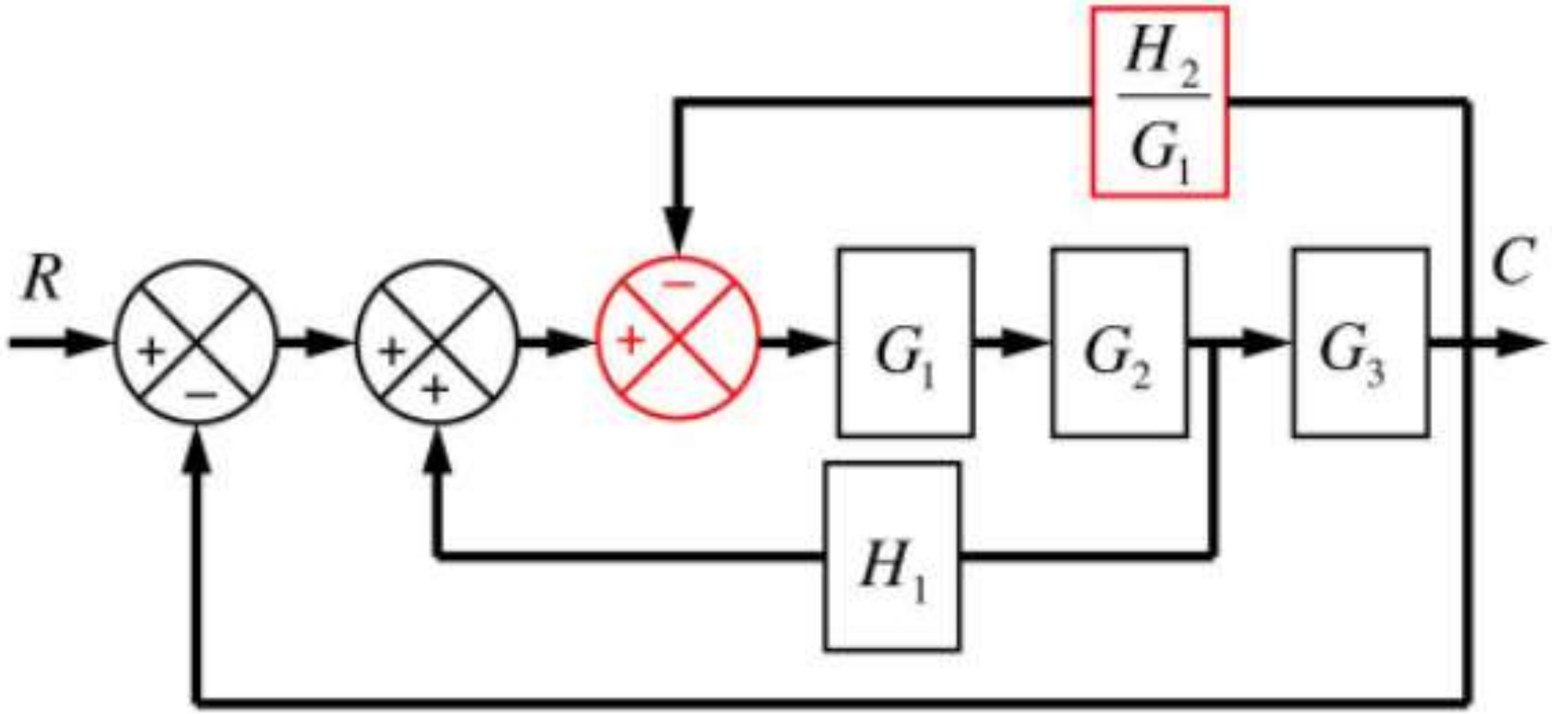


- Again blocks are in cascade are removed using rule-1

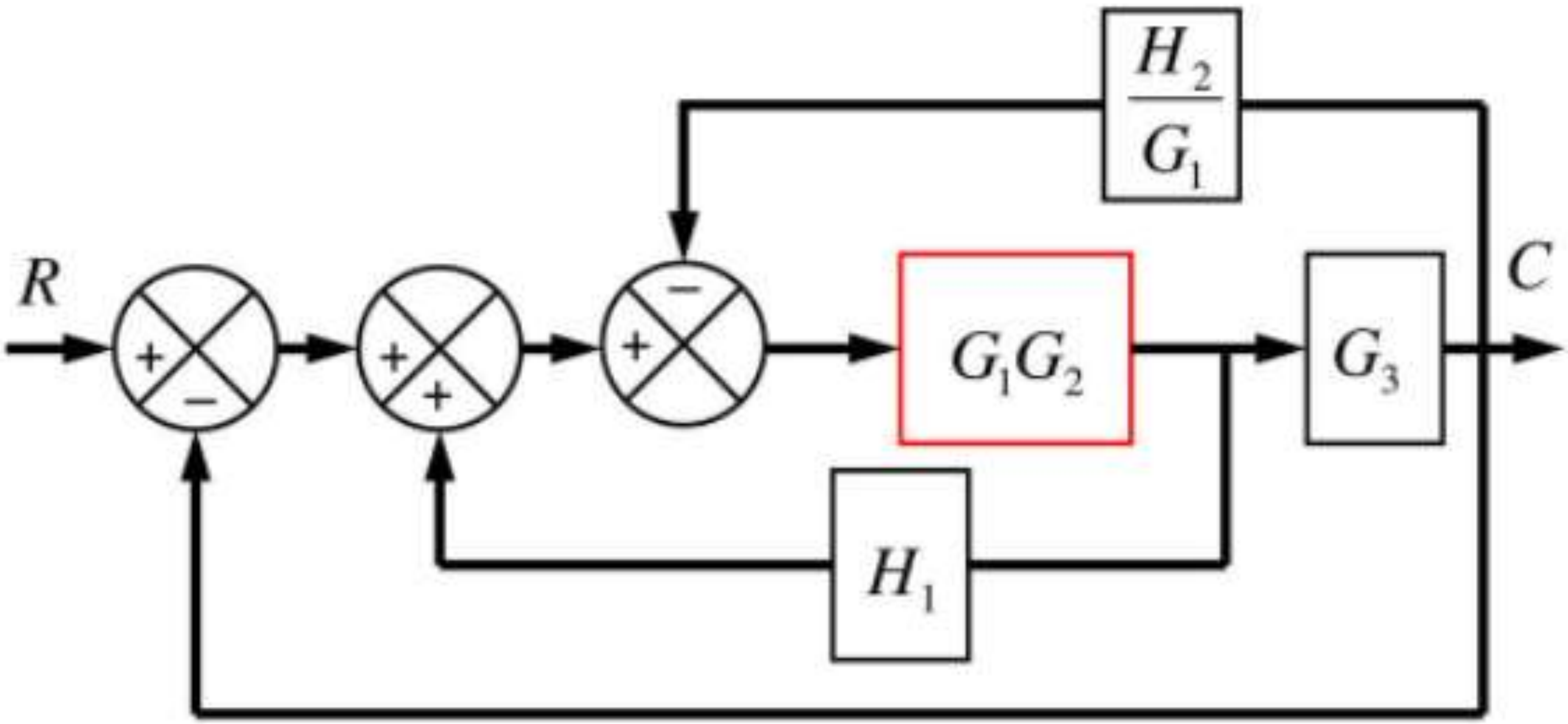


EXAMPLE 2

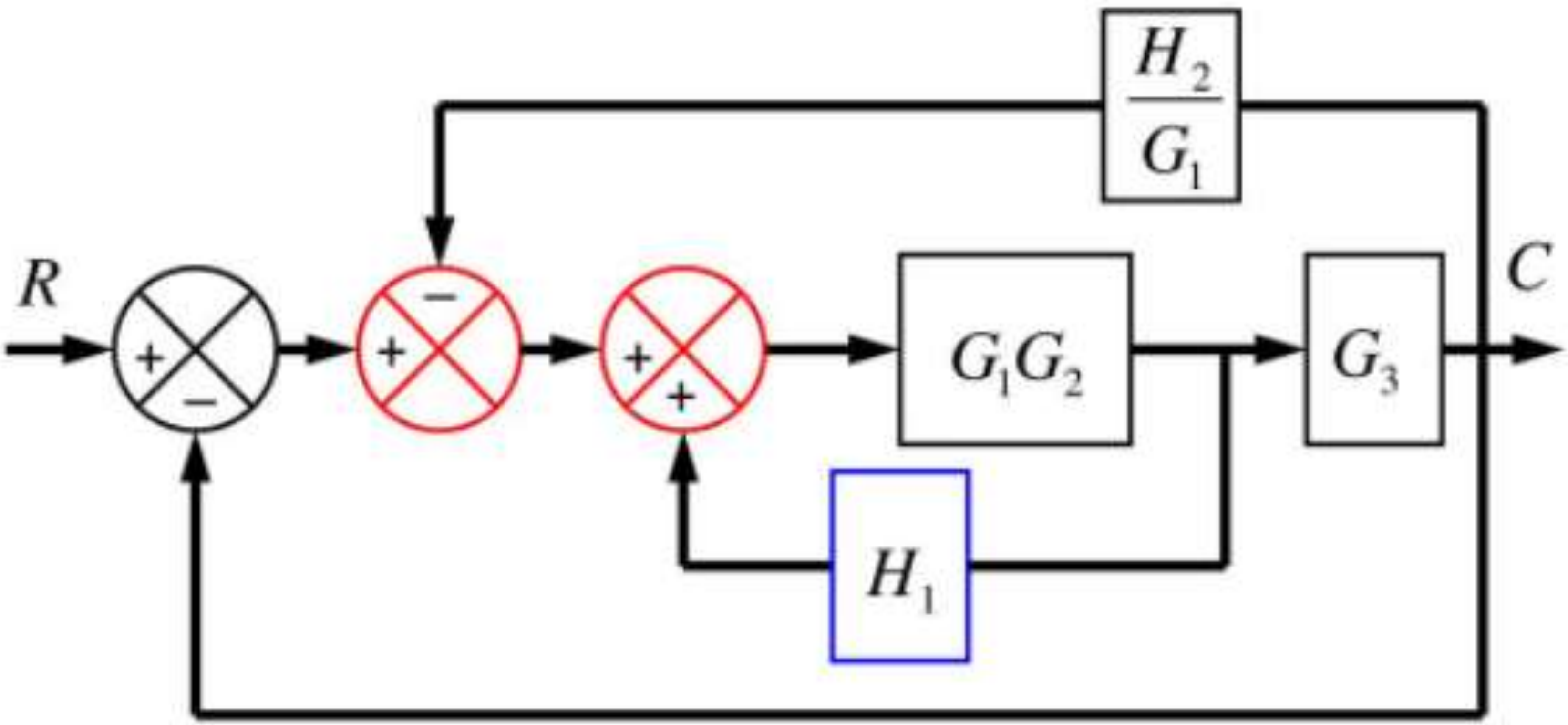




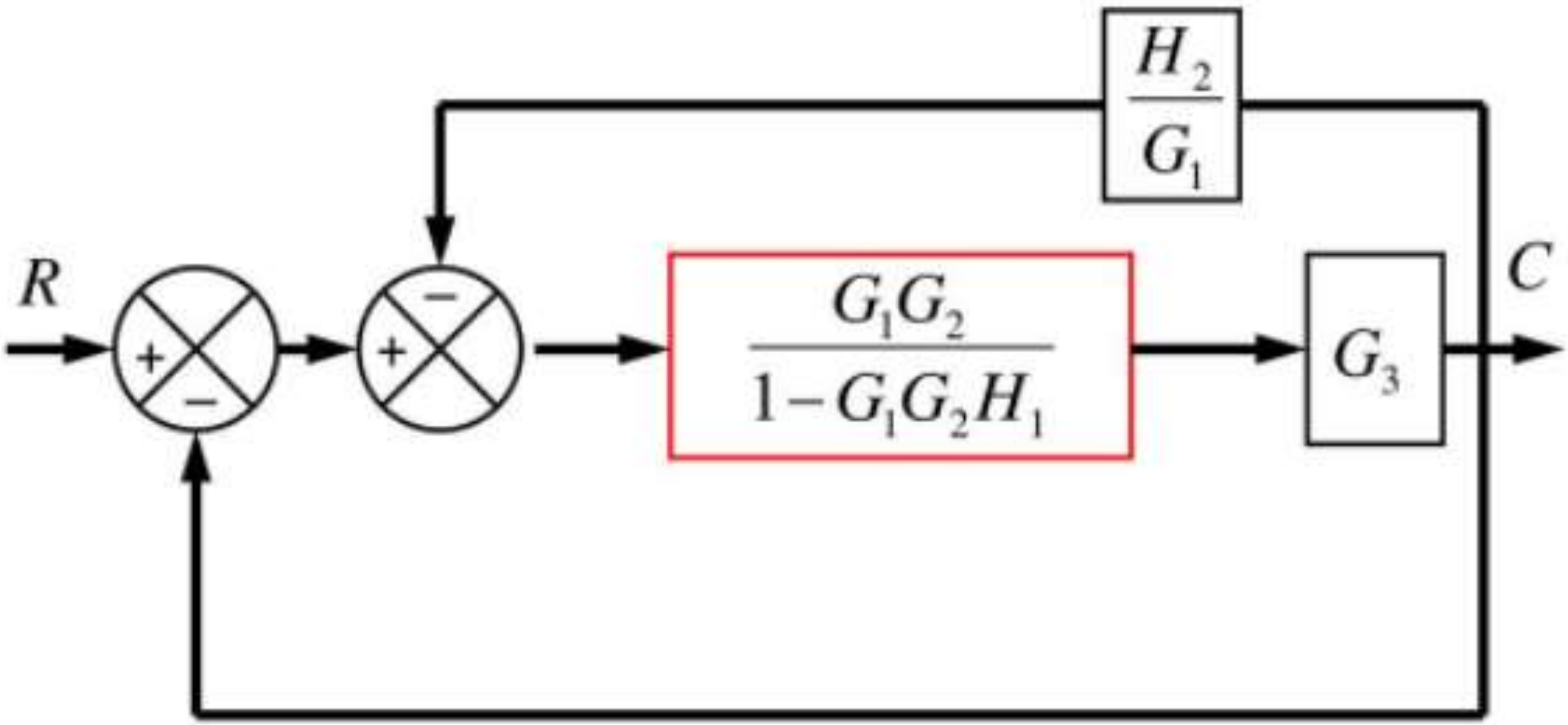
R
C
$\frac{H_2}{G_1}$
G_1
G_2
G_3
H_1
+
−
+
+
+
−



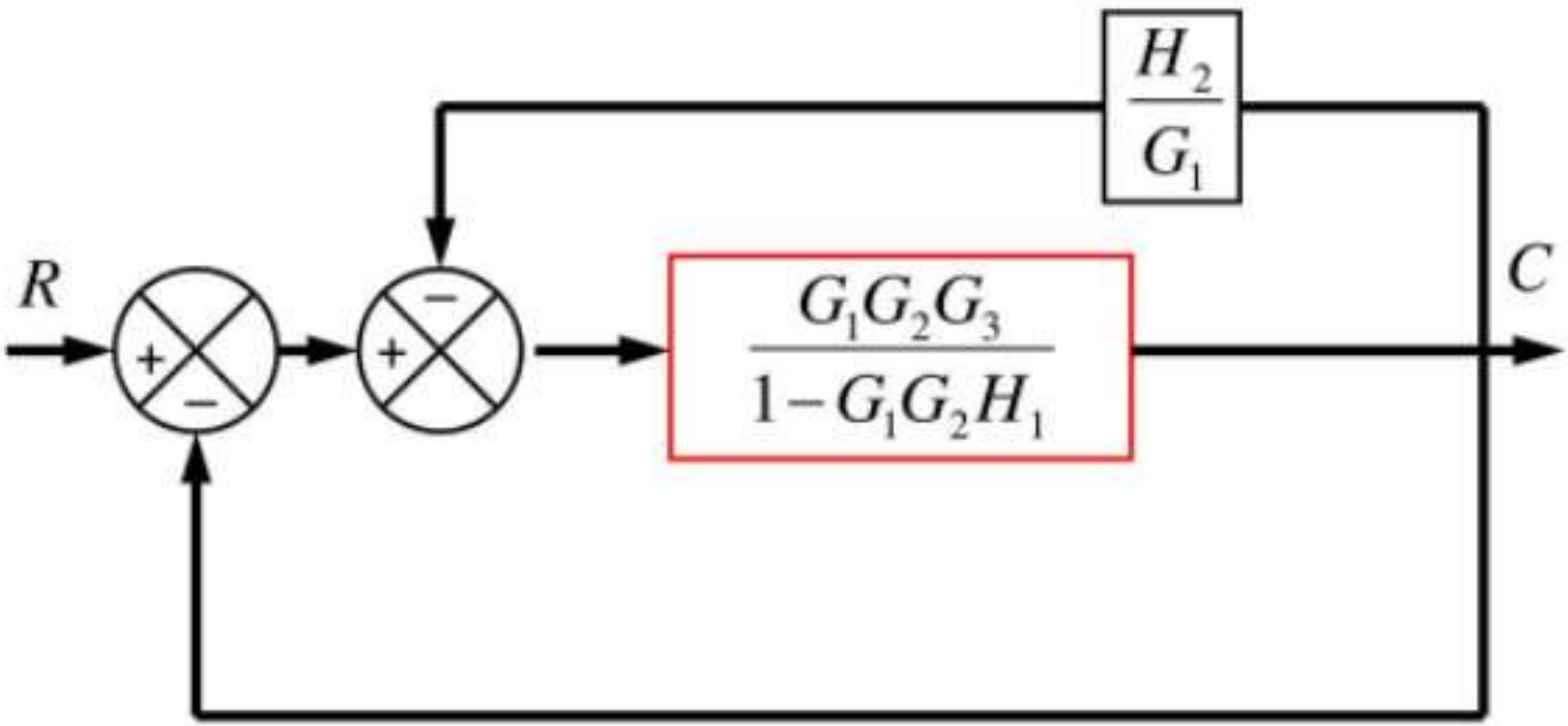
R
C
$\frac{H_2}{G_1}$
G_1G_2
G_3
H_1
+
−
+
+
+
−



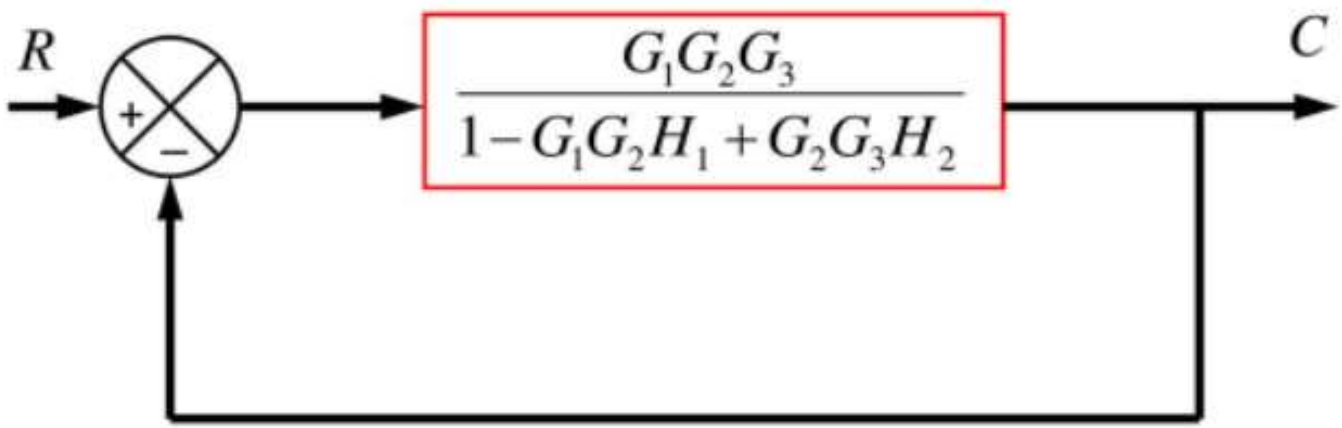
R
+
−
−
+
+
+
H_2 / G_1
G_1G_2
G_3
C
H_1

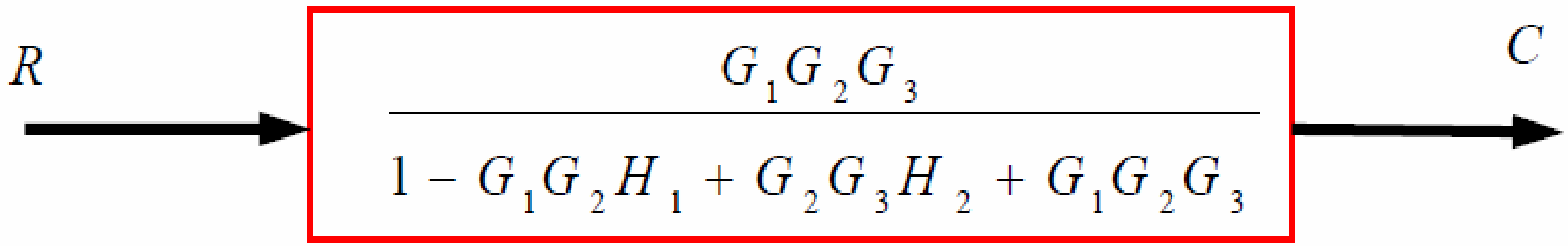


R
$+$
$-$
$+$
$-$
$\frac{H_2}{G_1}$
$\frac{G_1G_2}{1-G_1G_2H_1}$
G_3
C



R
C
+
−
+
−
$\frac{H_2}{G_1}$
$\frac{G_1G_2G_3}{1-G_1G_2H_1}$


R
+
−
$\frac{G_1G_2G_3}{1-G_1G_2H_1+G_2G_3H_2}$
C


R
$\dfrac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$
C

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Subject name :Control Systems

Unit No :1

Topic :Signal Flow Graph

Faculty Name :Dr. Lekshmi Sree B

SIGNAL FLOW GRAPH

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let us discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

Basic Elements of Signal Flow Graph

- Nodes and branches are the basic elements of signal flow graph.
- Node
- **Node** is a point which represents either a variable or a signal. There are three types of nodes — input node, output node and mixed node.
- **Input Node** − It is a node, which has only outgoing branches.
- **Output Node** − It is a node, which has only incoming branches.
- **Mixed Node** − It is a node, which has both incoming and outgoing branches.

Mason's gain formula is

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{N} P_i \Delta_i}{\Delta}$$

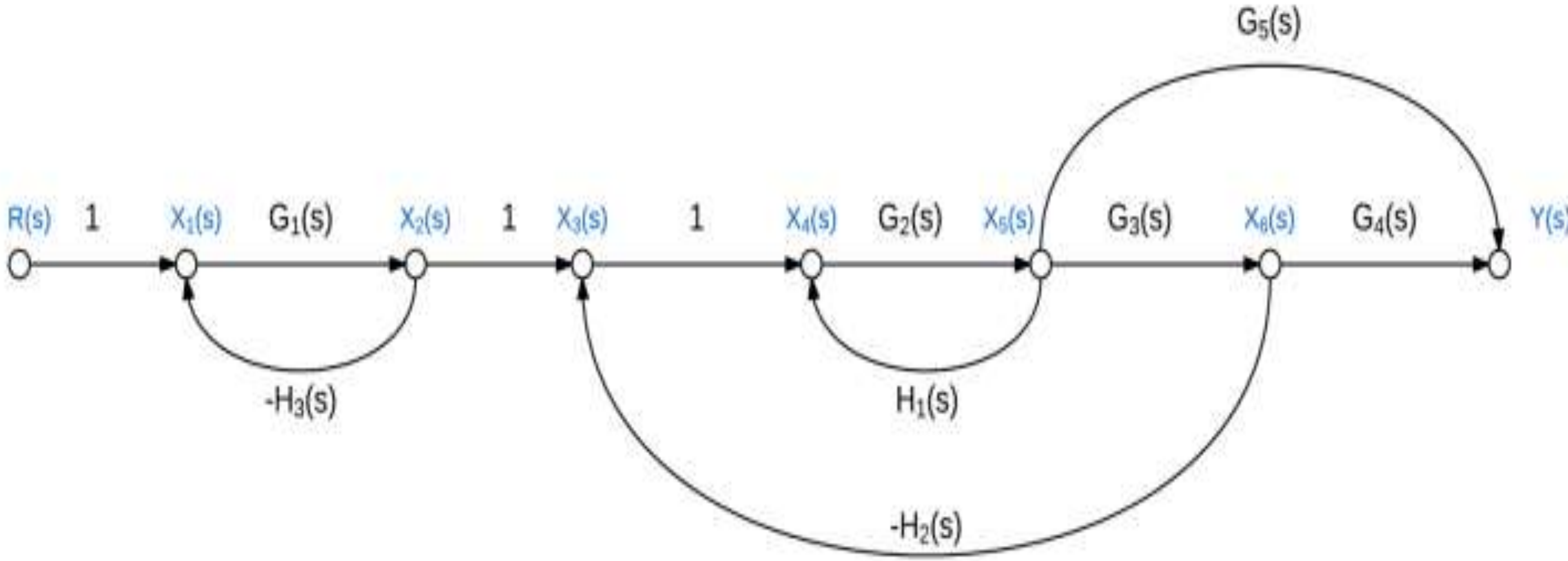
Where,

- **C(s)** is the output node
- **R(s)** is the input node
- **T** is the transfer function or gain between $R(s)$ and $C(s)$
- $\mathbf{P_i}$ is the i^{th} forward path gain

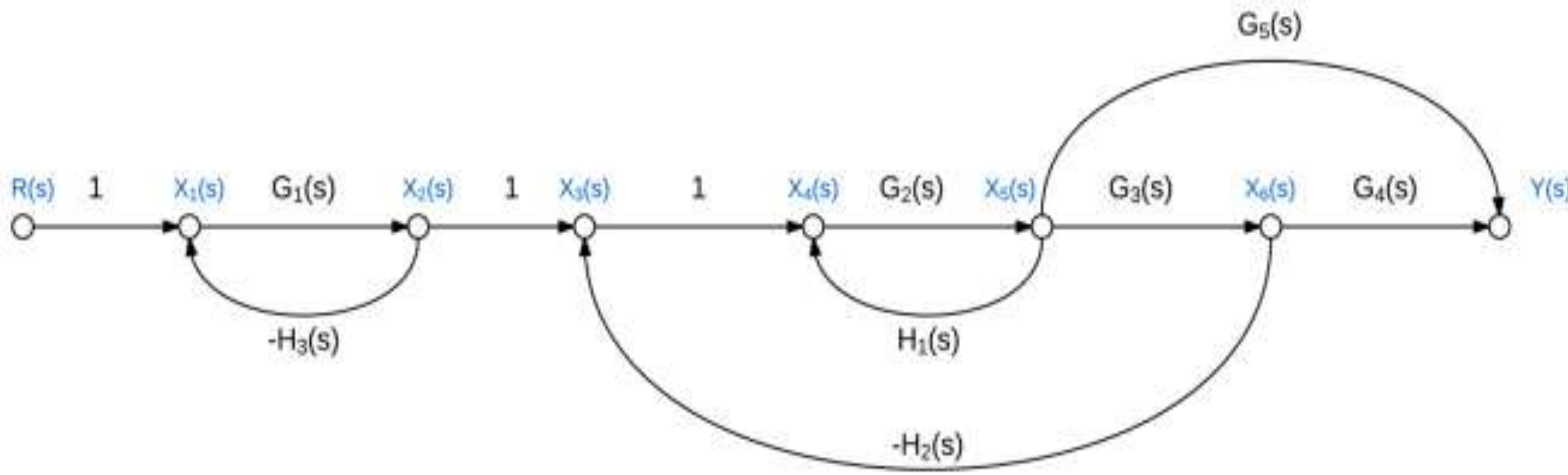
$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

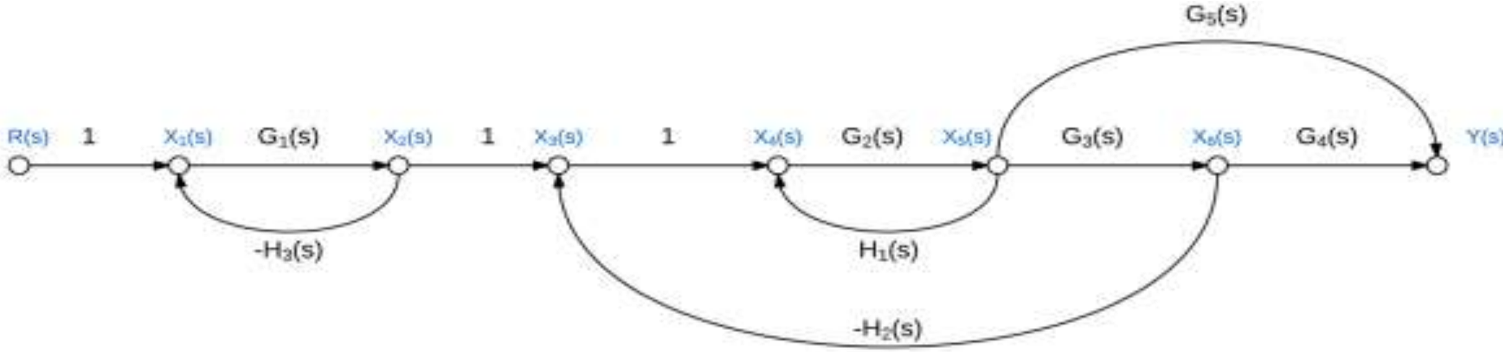


- ***Forward path gain*** – gain along any path from the input to the output
  - Not passing through any node more than once
- Here, there are two forward paths with the following gains:
  1. $G_1G_2G_3G_4$
  2. $G_1G_2G_5$



- ***Loop gain*** – total gain (product of individual gains) around any path in the signal flow graph
  - Beginning and ending at the same node
  - Not passing through any node more than once
- Here, there are three loops with the following gains:

1. $-G_1H_3$
2. G_2H_1
3. $-G_2G_3H_2$



- ***Non-touching loops*** – loops that do not have any nodes in common
- Here,
  1. $-G_1H_3$ does not touch G_2H_1
  2. $-G_1H_3$ does not touch $-G_2G_3H_2$

Here, there are only two *pairs* of non-touching loops

1. $[-G_1H_3] \cdot [G_2H_1]$
2. $[-G_1H_3] \cdot [-G_2G_3H_2]$

$$\Delta = 1 - \Sigma(\text{loop gains})$$
$$+\Sigma(\text{non-touching loop gains taken two-at-a-time})$$
$$-\Sigma(\text{non-touching loop gains taken three-at-a-time})$$
$$+\Sigma(\text{non-touching loop gains taken four-at-a-time})$$

- # of forward paths:

  $P = 2$

- Forward path gains:

  $T_1 = G_1 G_2 G_3 G_4$

  $T_2 = G_1 G_2 G_5$

- Σ(loop gains):

  $-G_1 H_3 + G_2 H_1 - G_2 G_3 H_2$

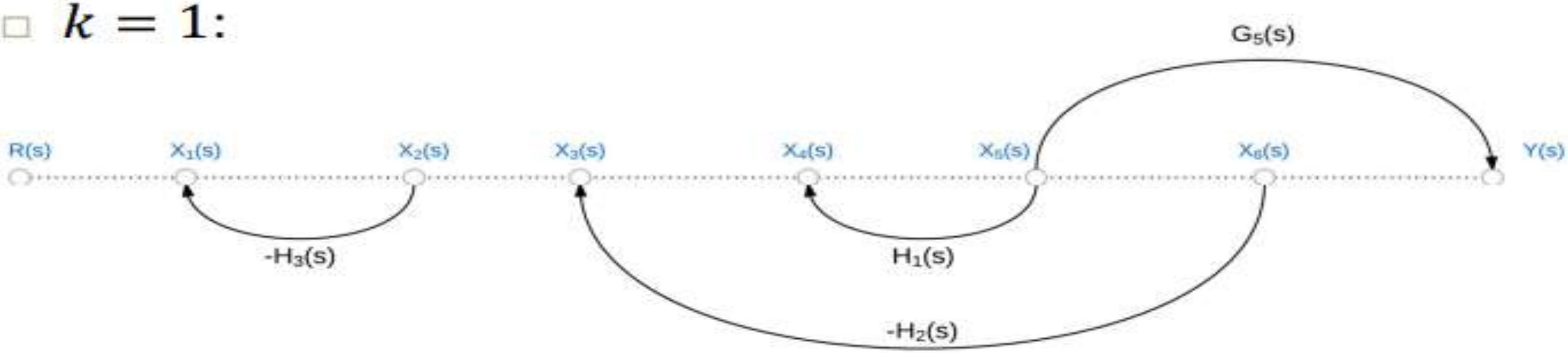
- Σ(NTLGs taken two-at-a-time):

  $(-G_1 H_3 G_2 H_1) + (G_1 H_3 G_2 G_3 H_2)$

- Δ:

  $$\Delta = 1 - \ (-G_1 H_3 + G_2 H_1 - G_2 G_3 H_2)$$
  $$+(-G_1 H_3 G_2 H_1 + G_1 H_3 G_2 G_3 H_2)$$

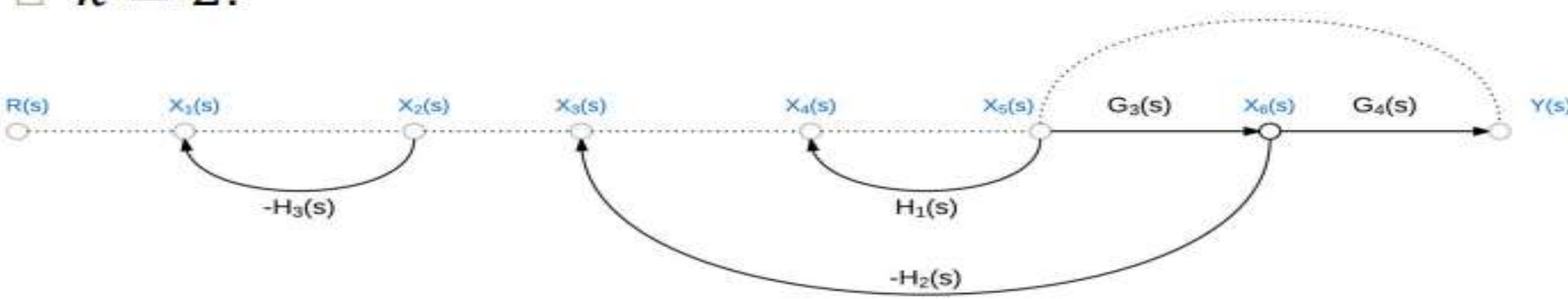
- $k = 1$:



- With forward path 1 removed, there are no loops, so

$$\Delta_1 = 1 - 0$$
$$\Delta_1 = 1$$

- $k = 2$:



- Similarly, removing forward path 2 leaves no loops, so

$$\Delta_2 = 1 - 0$$
$$\Delta_2 = 1$$

- For our example:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1}^{P} T_k \Delta_k$$

$$P = 2$$

$$T_1 = G_1 G_2 G_3 G_4$$

$$T_2 = G_1 G_2 G_5$$

$$\Delta = 1 + G_1 H_3 - G_2 H_1 + G_2 G_3 H_2 - G_1 H_3 G_2 H_1 + G_1 H_3 G_2 G_3 H_2$$

$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

- The closed-loop transfer function:

$$T(s) = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

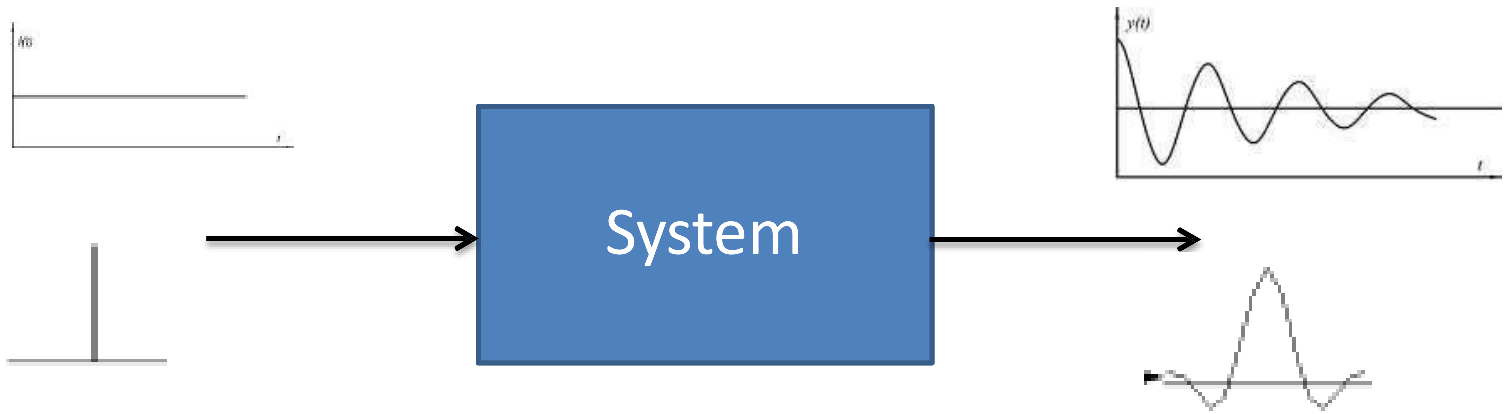
$$T(s) = \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_5}{1 + G_1 H_3 - G_2 H_1 + G_2 G_3 H_2 - G_1 H_3 G_2 H_1 + G_1 H_3 G_2 G_3 H_2}$$

Subject name :Control Systems - UNIT 2

Topic :Introduction to Time Response

Introduction

- Time response of a dynamic system response to an input expressed as a function of time.



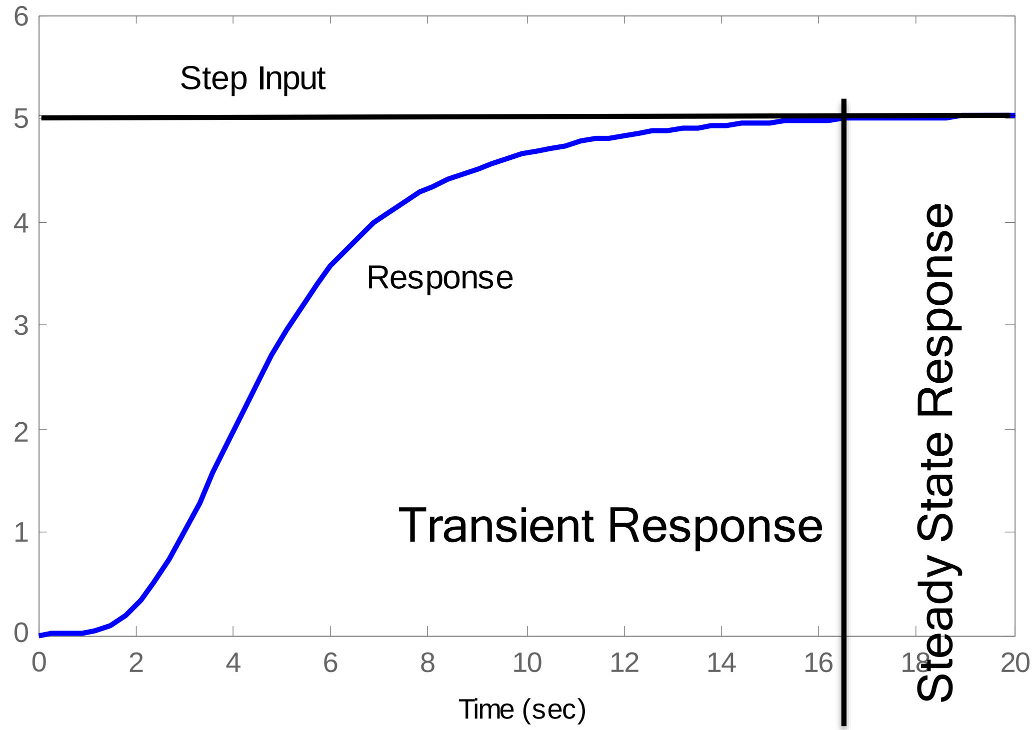
- The time response of any system has two components
    - Transient response
    - Steady-state response

Introduction (Contd..)

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.



Introduction (Contd..)

- Transient response depend upon the system poles only and not on the type of input.

- It is therefore sufficient to analyze the transient response using a step input.

- The steady-state response depends on system dynamics and the input quantity.

- It is then examined using different test signals by final value theorem.

Introduction (Contd..)

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction (Contd..)

- The first order system given below.

$$G(s) = \frac{10}{3s + 1}$$

- D.C gain is 10 and time constant is 3 seconds.

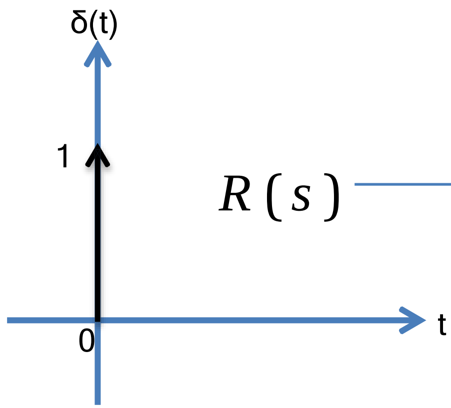
- For the following system

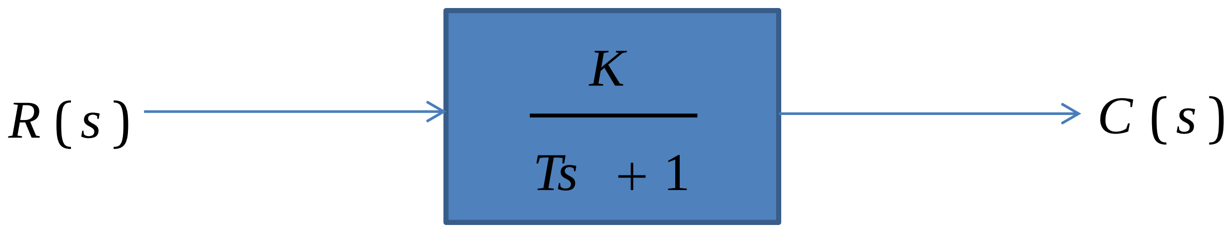
$$G(s) = \frac{3}{s + 5} = \frac{3/5}{1/5s + 1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Impulse Response of 1st Order System

- Consider the following 1st order system





$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts + 1}$$

- Re-arrange following equation as

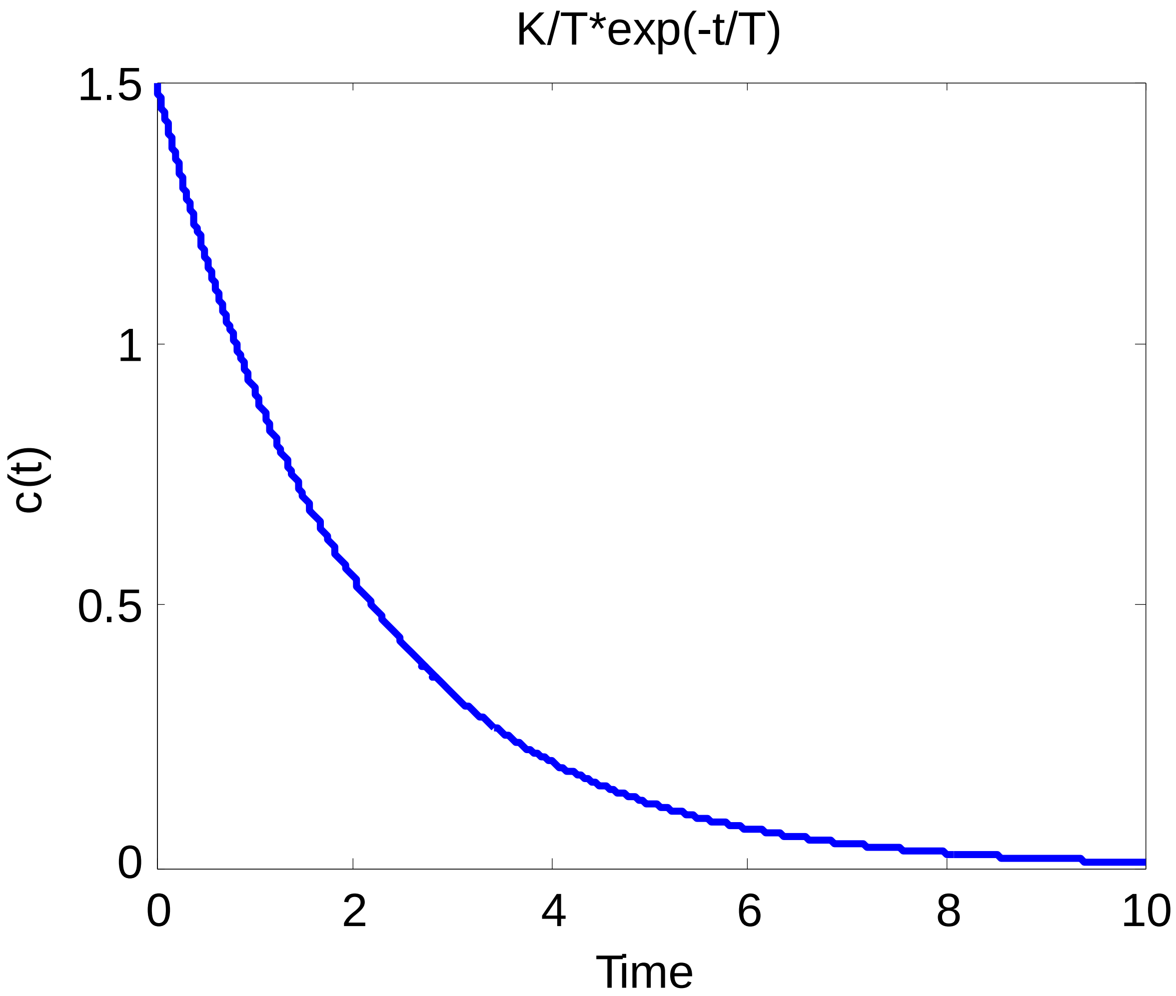
$$C(s) = \frac{K/T}{s + 1/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T}$$

Impulse Response of 1st Order System

- If K=3 and T=2s then $$c(t) = \frac{K}{T} e^{-t/T}$$



Step Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

$$c(t) = K\left(1 - e^{-t/T}\right)$$

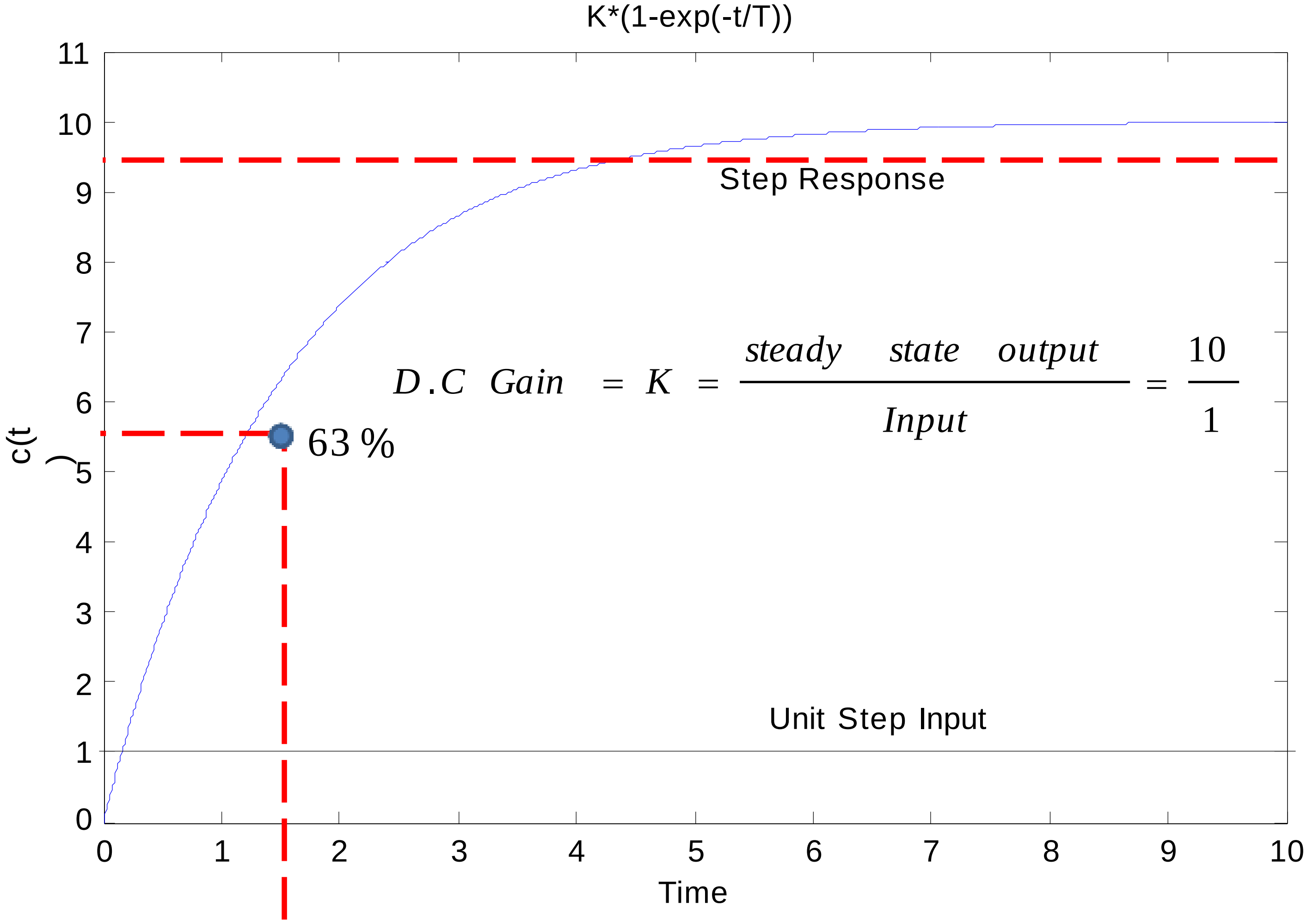
- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632\,K$$

Step Response of 1st Order System

- If K=10 and T=1.5s then

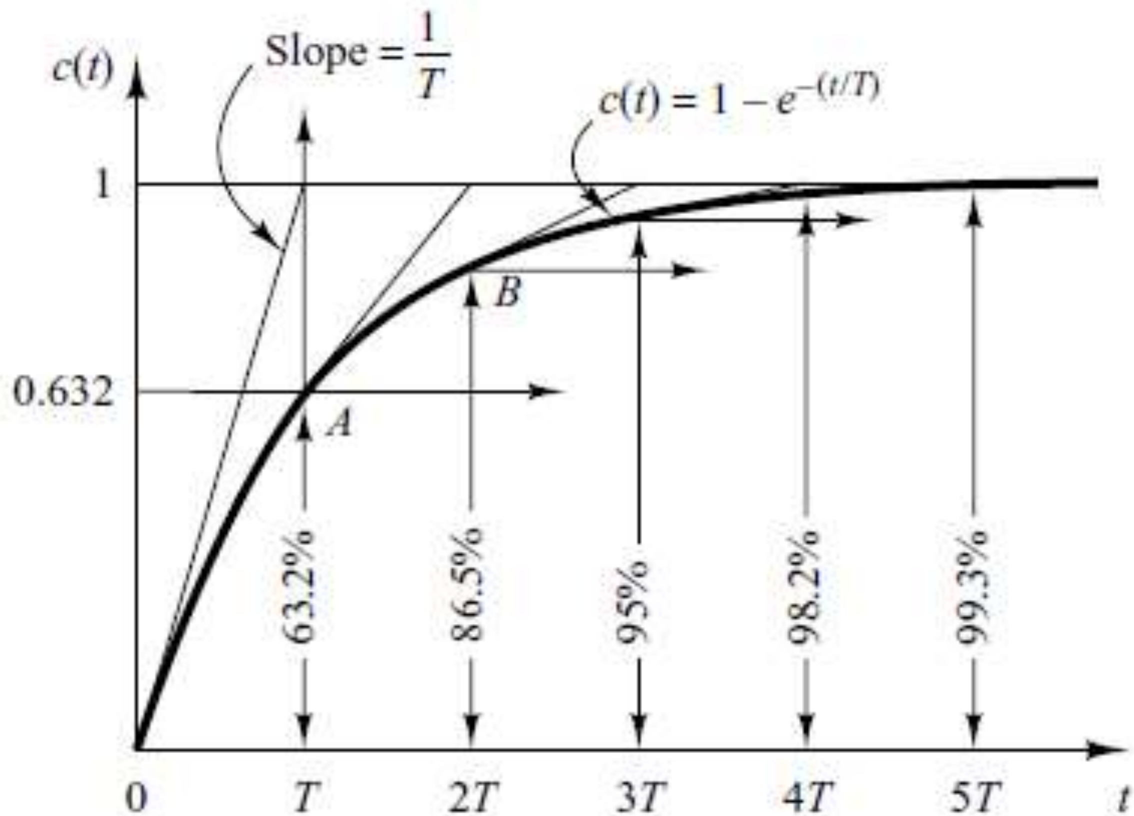
$$c(t) = K\left(1 - e^{-t/T}\right)$$



$$D.C\ Gain = K = \frac{steady\ \ state\ \ output}{Input} = \frac{10}{1}$$

Step Response of 1st Order System

- System takes five time constants to reach its final value.



Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

- Usually, the input signals to control systems are not known fully ahead of time.

- It is therefore difficult to express the actual input signals mathematically by simple equations.

Standard Test Signals

- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.

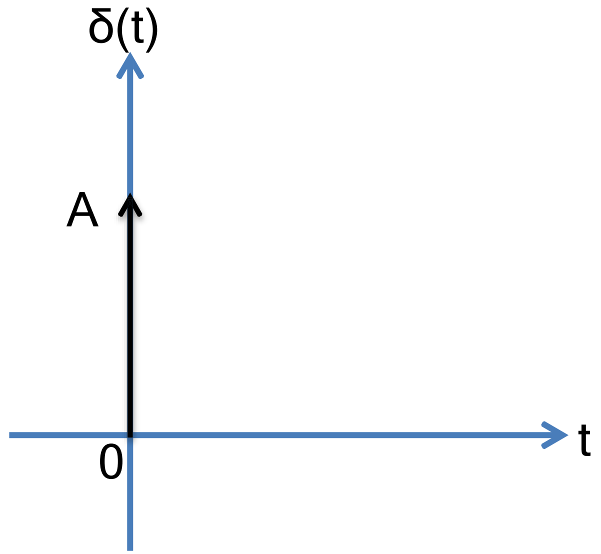
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.

- The other standard signal of great importance is a sinusoidal signal.

Standard Test Signals

- **Impulse signal**
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$



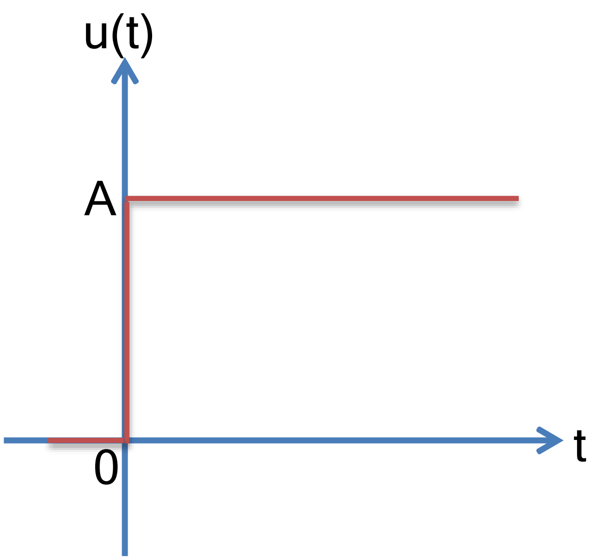
  - If A=1, the impulse signal is called unit impulse signal.

Standard Test Signals

- **Step signal**
  - The step signal imitate the sudden change characteristic of actual input signal.

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If A=1, the step signal is called unit step signal

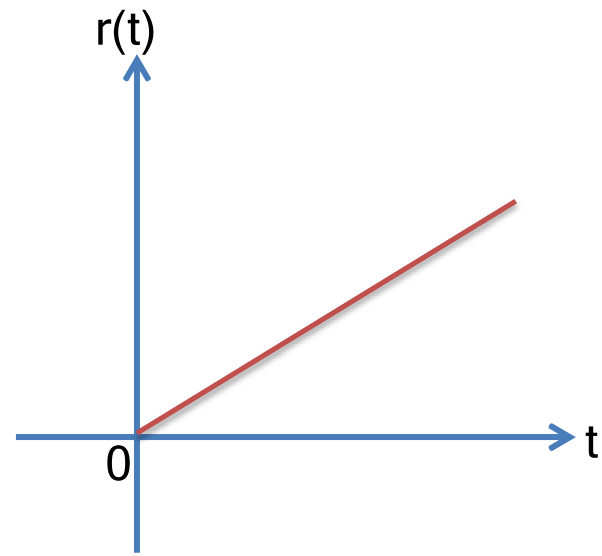


Standard Test Signals

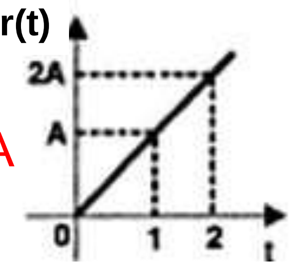
- **Ramp signal**
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

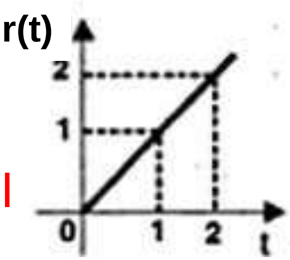
  - If *A=1*, the ramp signal is called unit ramp signal



ramp signal with slope A



unit ramp signal

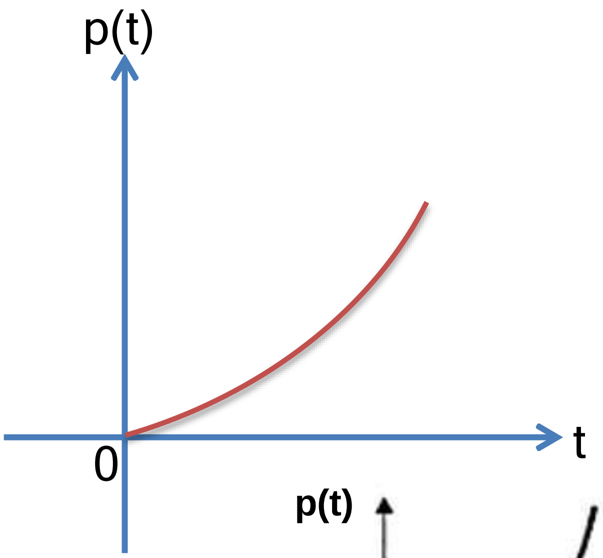


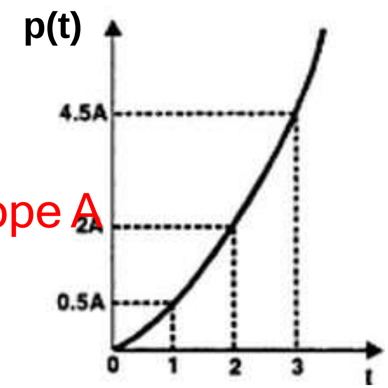
Standard Test Signals

- **Parabolic signal**
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

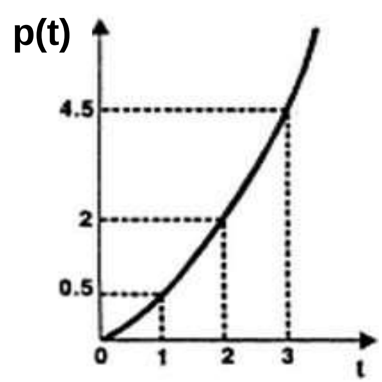
$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If $A=1$, the parabolic signal is called unit parabolic signal.





parabolic signal with slope A



Unit parabolic signal

Relation between standard Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$\int$ $\qquad$ $\frac{d}{dt}$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ $\qquad$ $\frac{d}{dt}$

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ $\qquad$ $\frac{d}{dt}$

- Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

- Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

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Subject name :Control Systems

Unit No :2

Topic :Second Order Systems

Faculty Name :S LAKSHMI DEVI

Second Order System

- We have already discussed transient response of 1st order systems.

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response

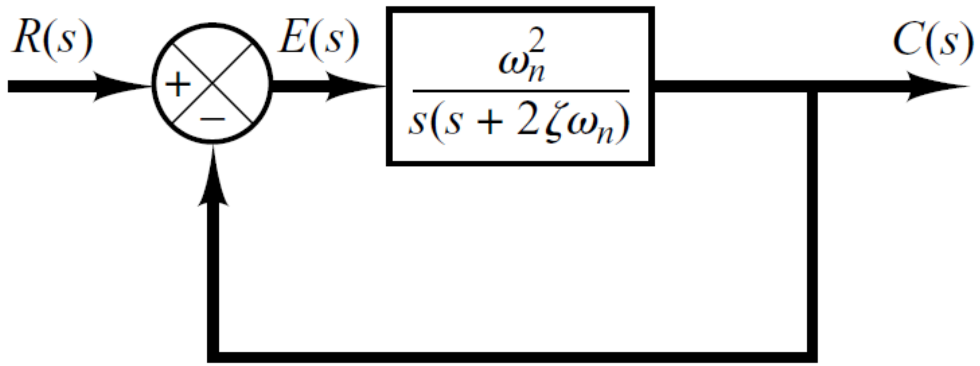
- Whereas, changes in the parameters of a second-order system can change the *form of* the response.

- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.

Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



ω_n $\longrightarrow$ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

ζ $\longrightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

Example

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \qquad \Rightarrow \omega_n = 2$$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

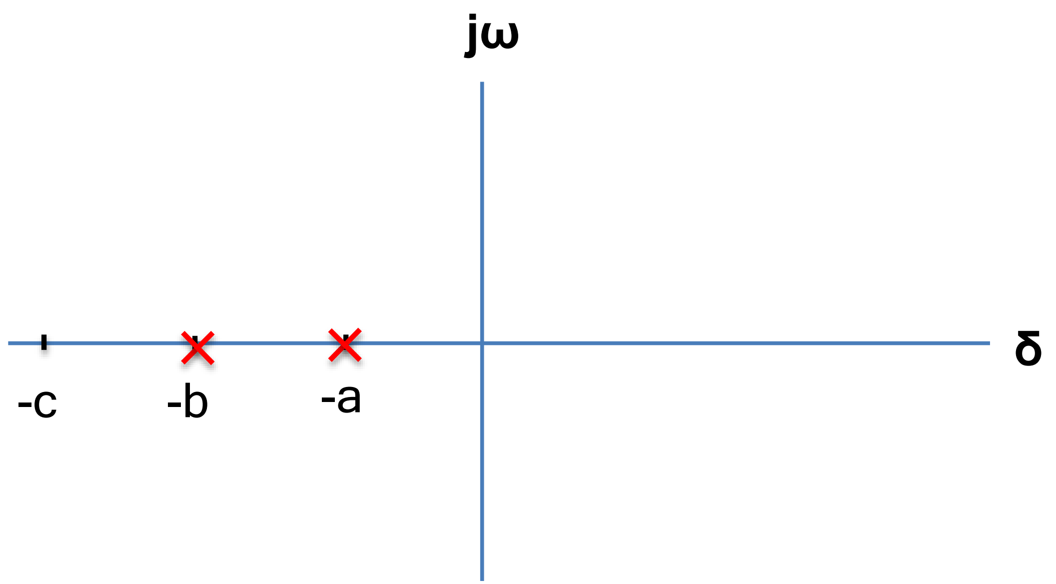
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of , a second-order system can be set into one of the four categories

1. *Overdamped* - when the system has two real distinct poles (ζ>1).

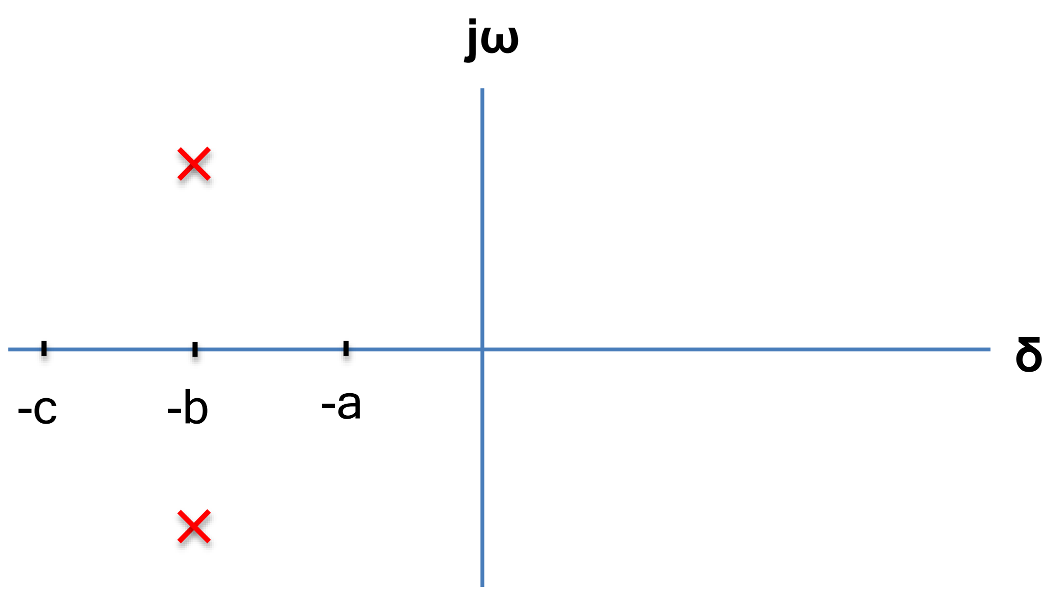


Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)

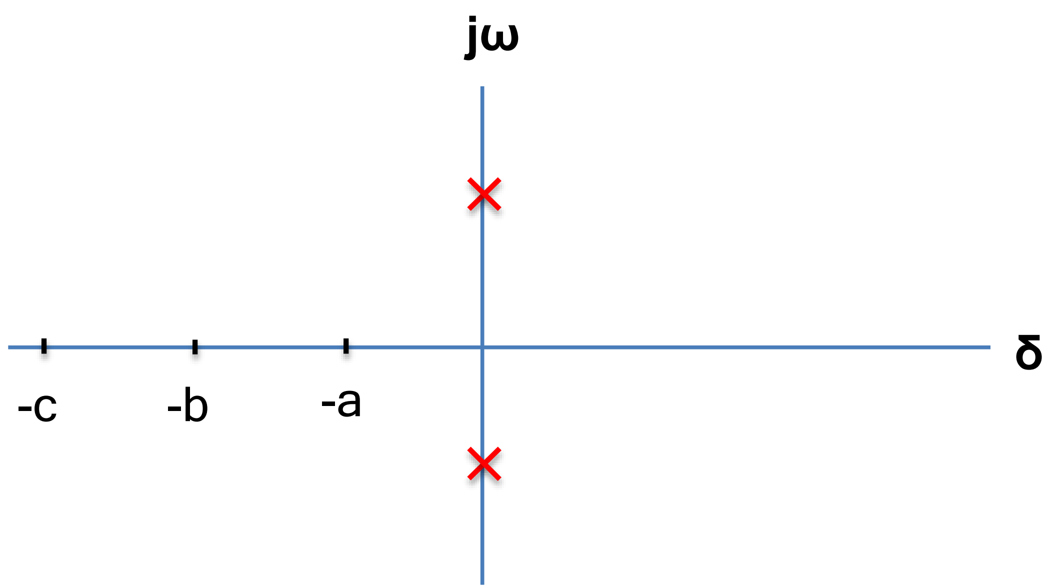


Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ, a second-order system can be set into one of the four categories

3. *Undamped* - when the system has two imaginary poles $\zeta=0$).

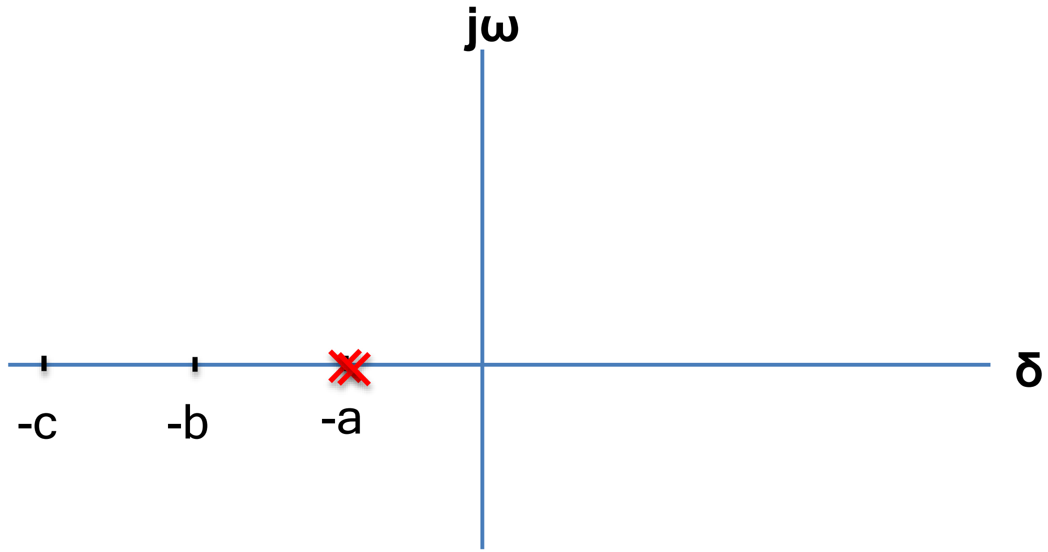


Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ, a second-order system can be set into one of the four categories

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).



Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2\left(1 - \zeta^2\right)$$

$$\left(s + 2\zeta\omega_n\right)^2$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$, is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1 - \zeta^2}}\omega_n\sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1 - \zeta^2}}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

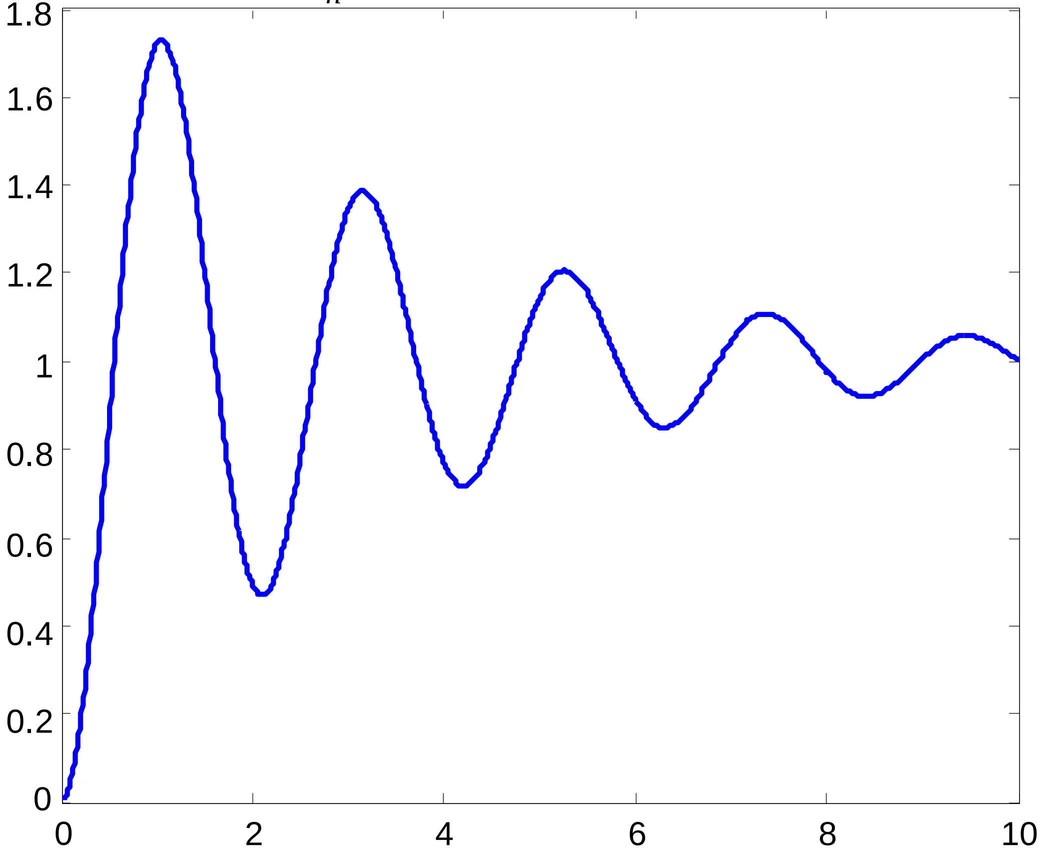
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

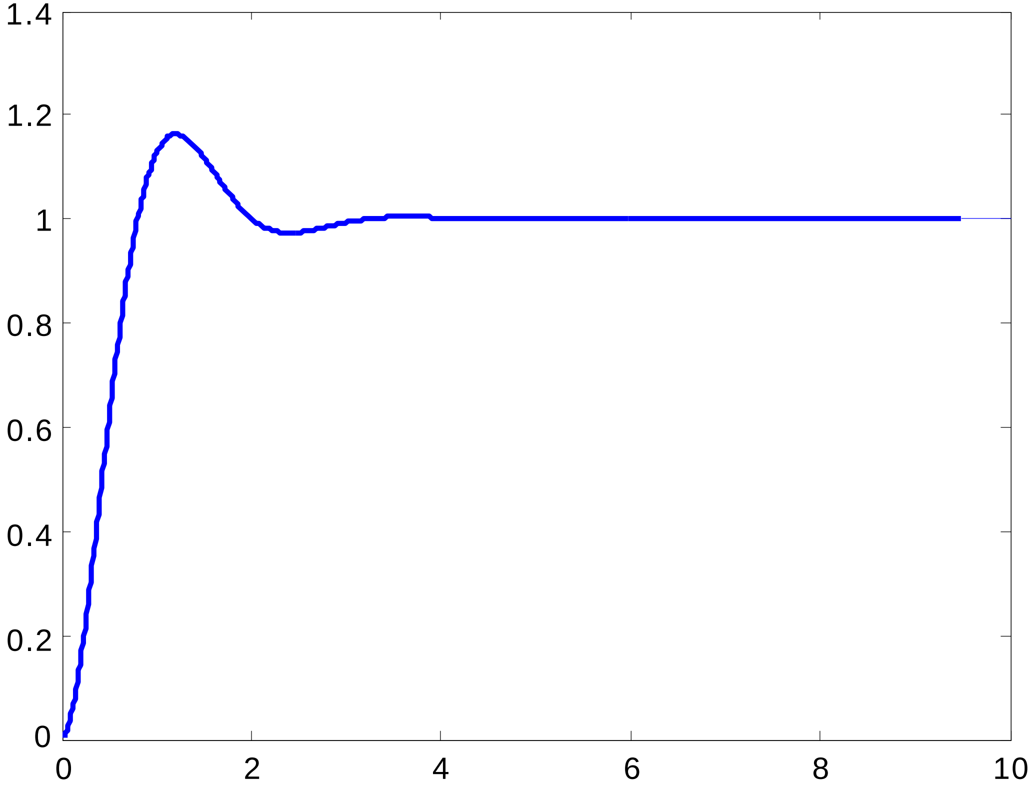
if $\zeta = 0.1$ and $\omega_n = 3$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right]$$

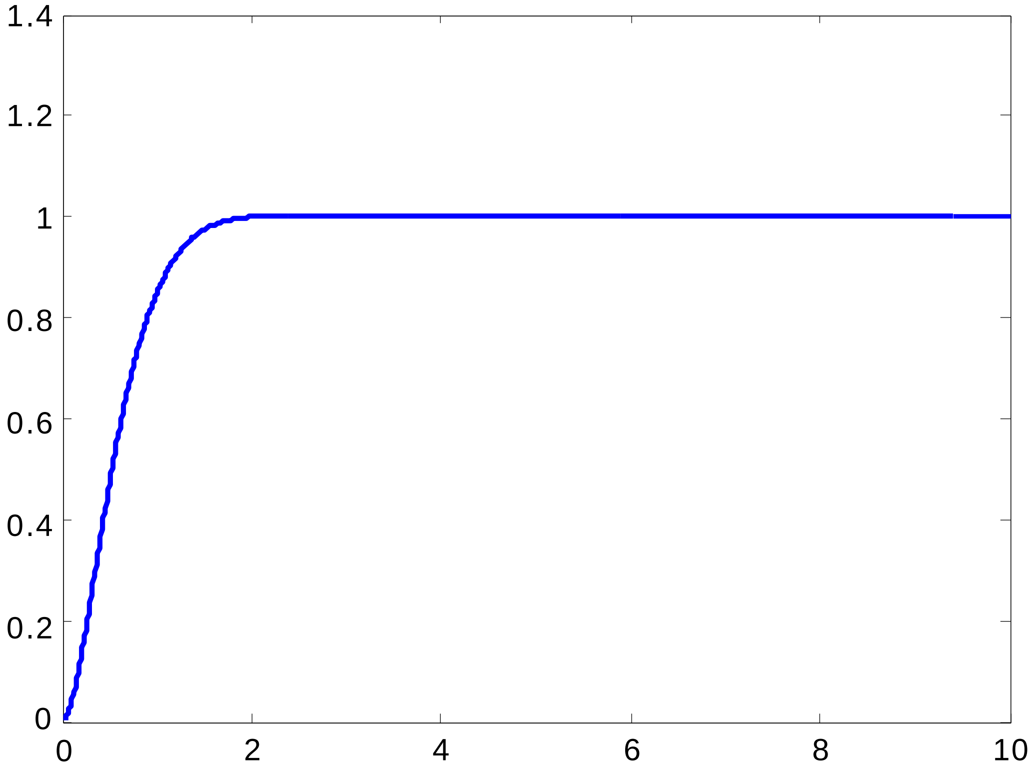
if $\zeta = 0.5$ and $\omega_n = 3$



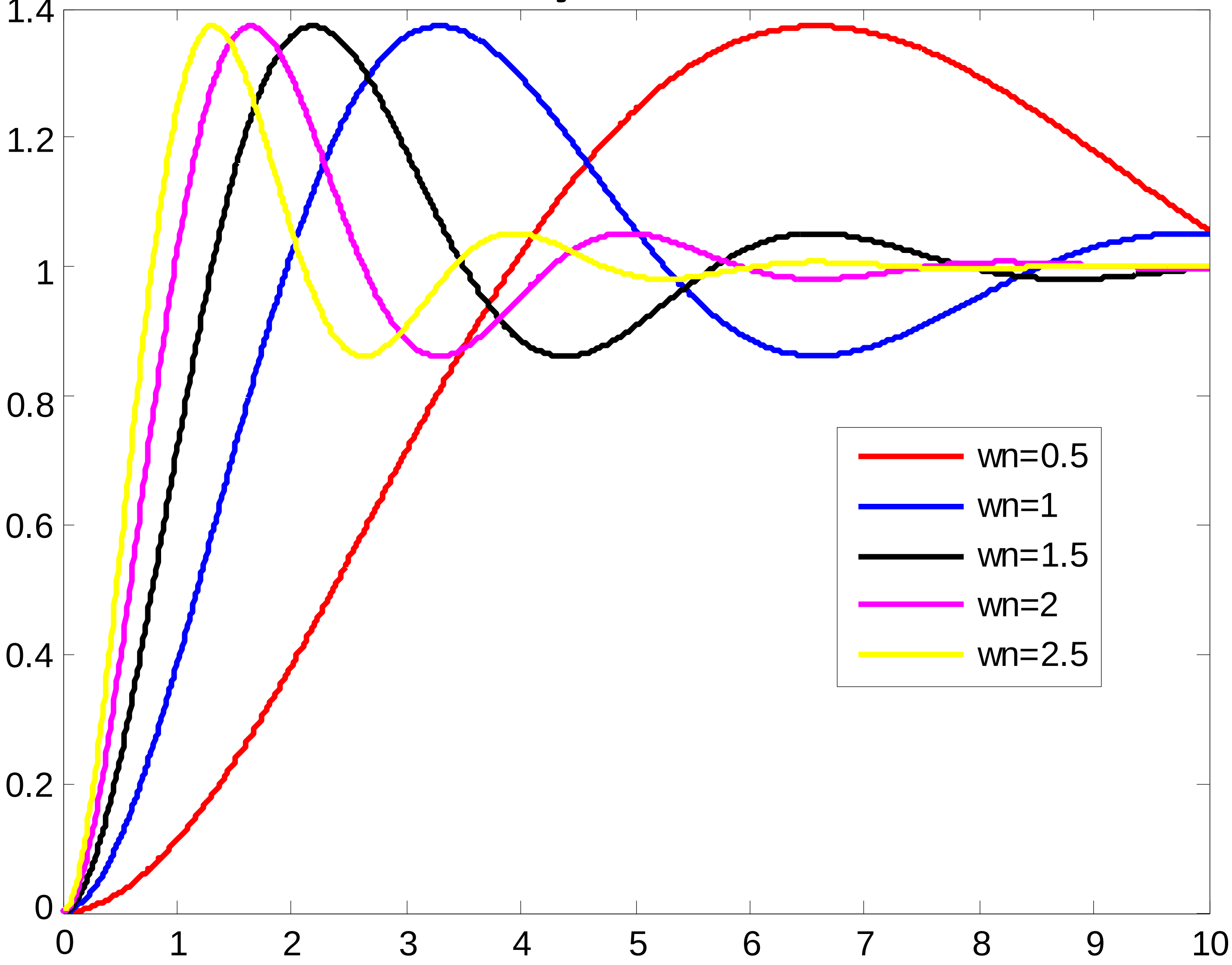
Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$\text{if} \quad \zeta = 0.9 \quad \text{and} \quad \omega_n = 3$$



Step Response of underdamped System



NARSIMHA REDDY ENGINEERING COLLEGE

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Subject name :Control Systems

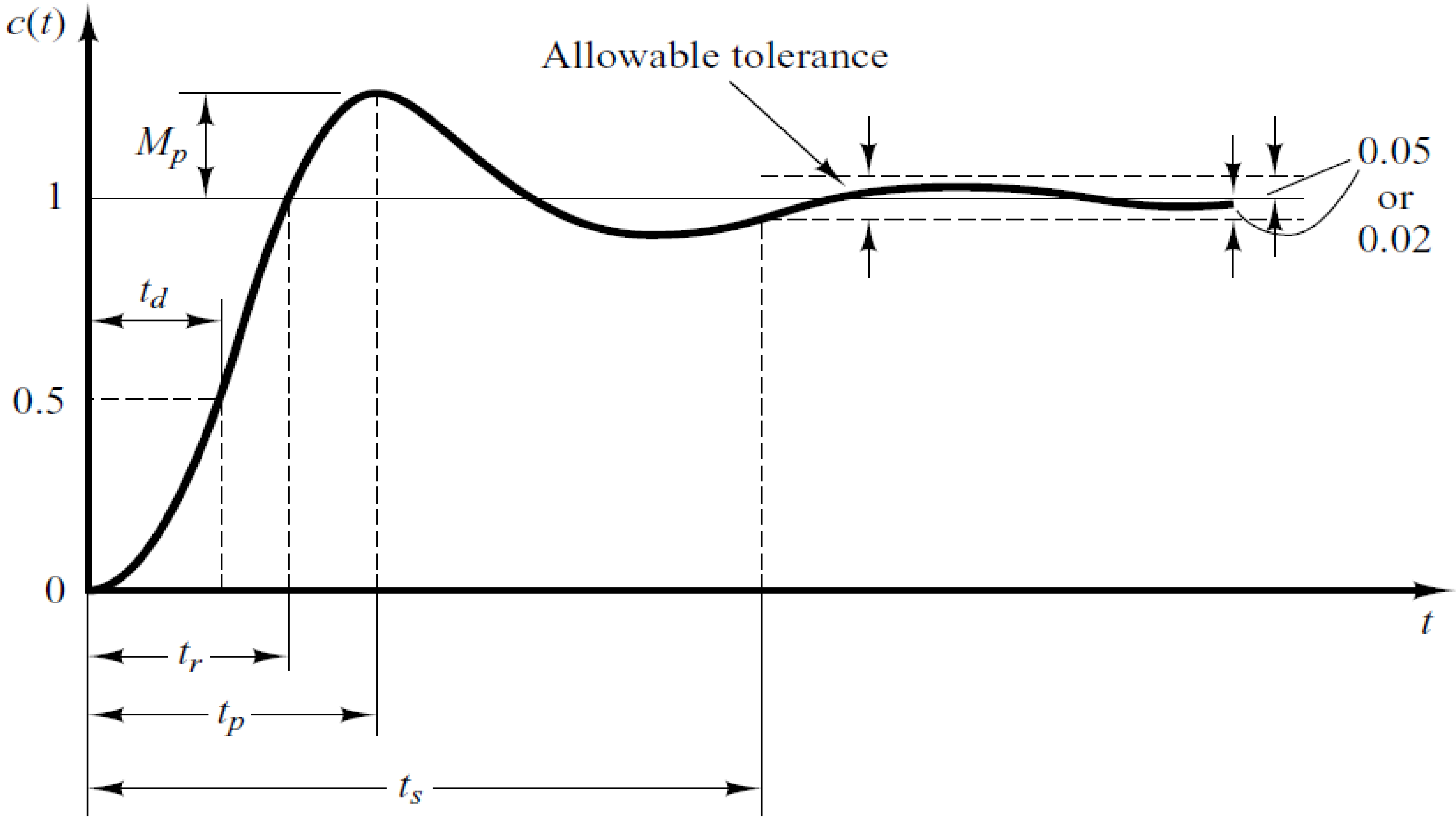
Unit No :2

Topic :Time Domain Specifications

Faculty Name :S LASKSHMI DEVI

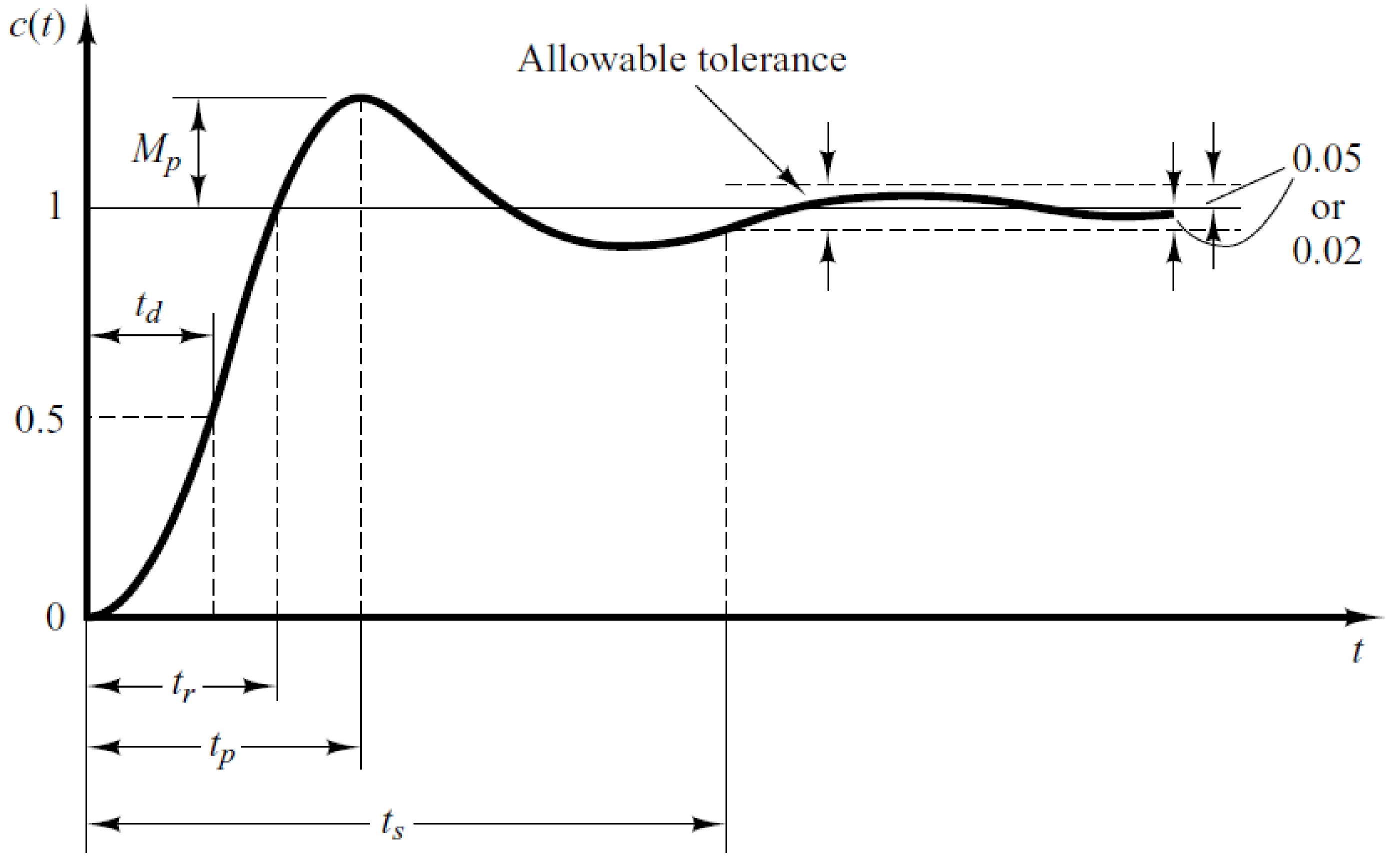
Underdamped System

For $0<\zeta<1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input is as follows.
Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



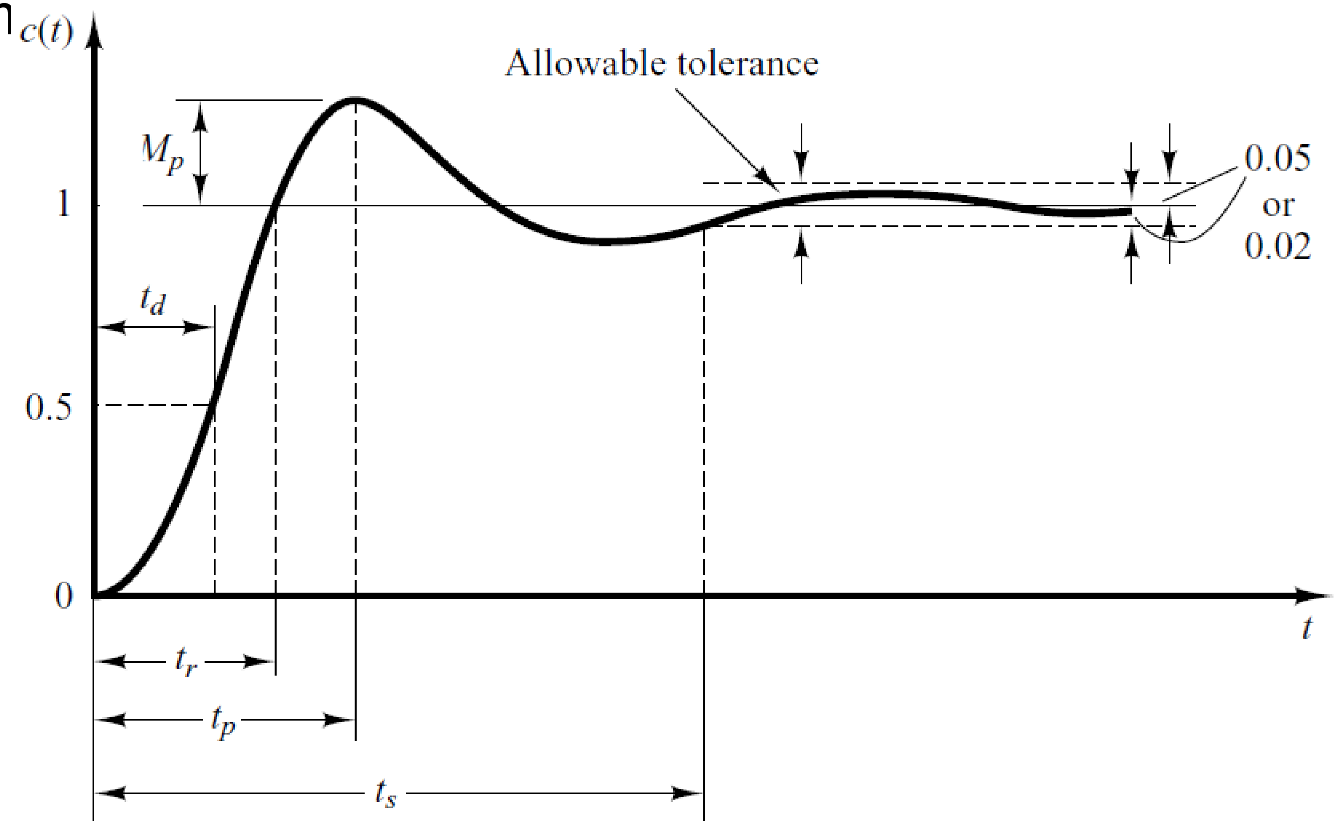
Delay Time

- The delay (t_d) time is the time required for the response to reach half the final value the very first time.



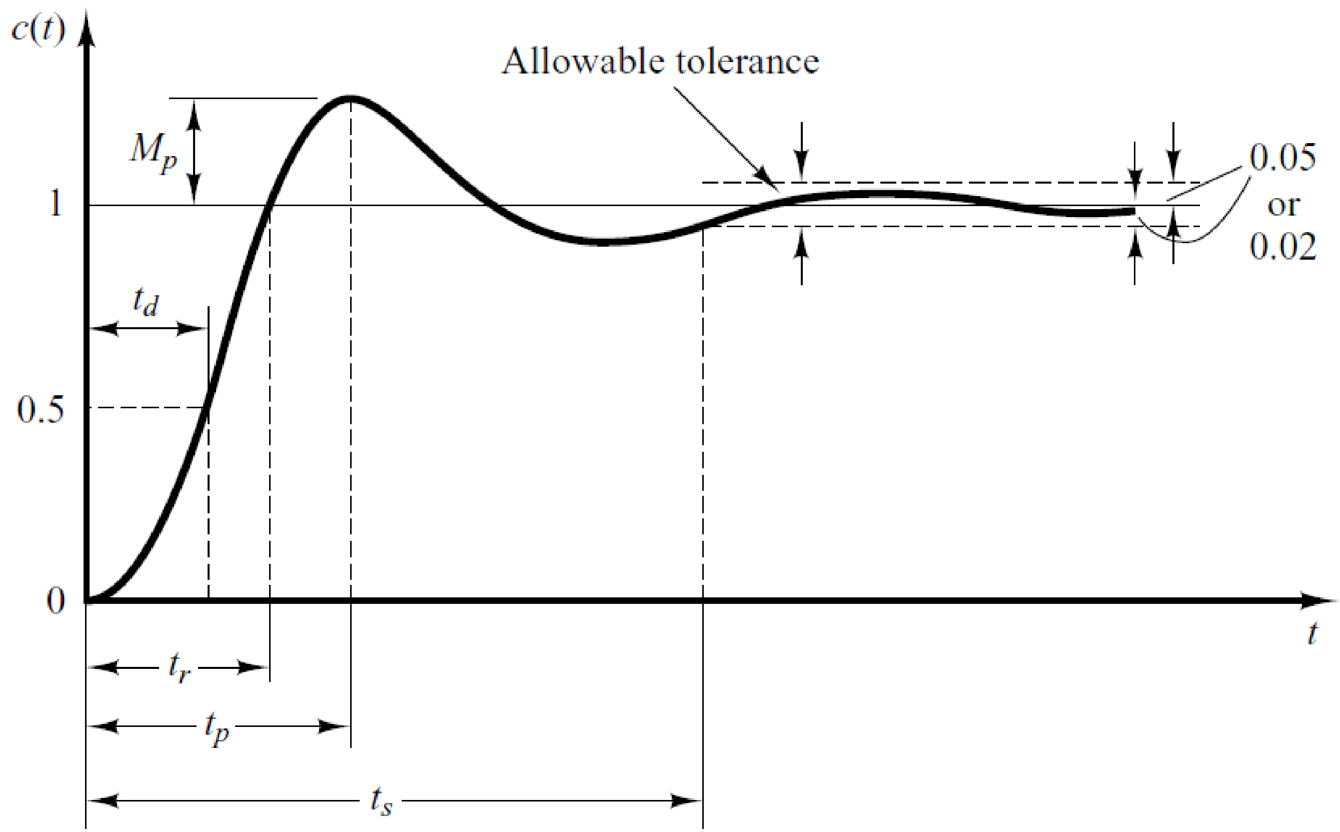
Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is com



Peak Time

- The peak time is the time required for the response to reach the first peak of the overshoot.



Time Domain Specifications (Rise Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

Put $t = t_r$ in above equation

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

Where $c(t_r) = 1$

$$0 = -e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

$$-e^{-\zeta\omega_n t_r} \neq 0 \qquad\qquad 0 = \left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

Time Domain Specifications (Rise Time)

$$\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right] = 0$$

above equation can be re - writen as

$$\sin\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}\cos\omega_d t_r$$

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

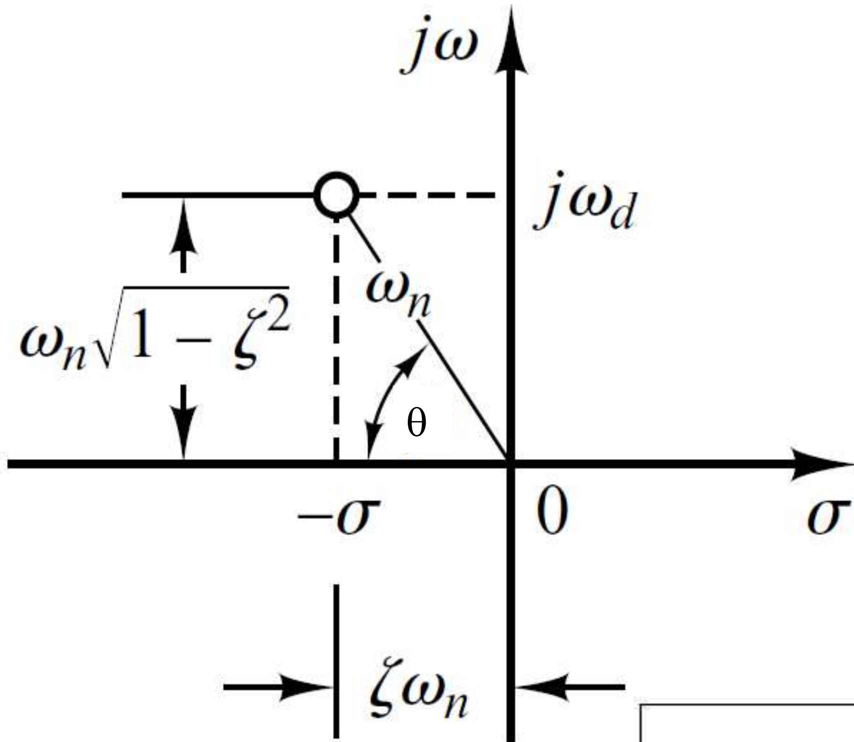
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Time Domain Specifications (Rise Time)

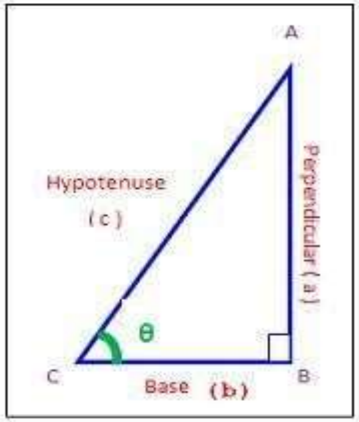
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta}\right)$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$



$$\theta = \tan^{-1}\frac{a}{b}$$



Time Domain Specifications (Peak Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- In order to find peak time let us differentiate above equation w.r.t *t*.

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] - e^{-\zeta\omega_n t}\left[-\omega_d\sin\omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\omega_n\cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\omega_{\cancel{n}}\cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\zeta\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\cos\omega_d t\right]$$

Time Domain Specifications (Peak Time)

$$0 = e^{-\zeta\omega_n t}\left[\zeta\cancel{\omega_n}\cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\zeta\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\cos\omega_d t\right]$$

$$e^{-\zeta\omega_n t}\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t\right] = 0$$

$$e^{-\zeta\omega_n t} \neq 0 \qquad \left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t\right] = 0$$

$$\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right] = 0$$

Time Domain Specifications (Peak Time)

$$\sin \omega_d t \left[\frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} + \omega_d \right] = 0$$

$$\left[\frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} + \omega_d \right] \neq 0 \qquad \sin \omega_d t = 0$$

$$\omega_d t = \sin^{-1} 0$$

$$t = \frac{0, \pi, 2\pi, \cdot}{\omega_d}$$

- Since for underdamped stable systems first peak is maximum peak therefore,

$$t_p = \frac{\pi}{\omega_d}$$

Maximum Overshoot

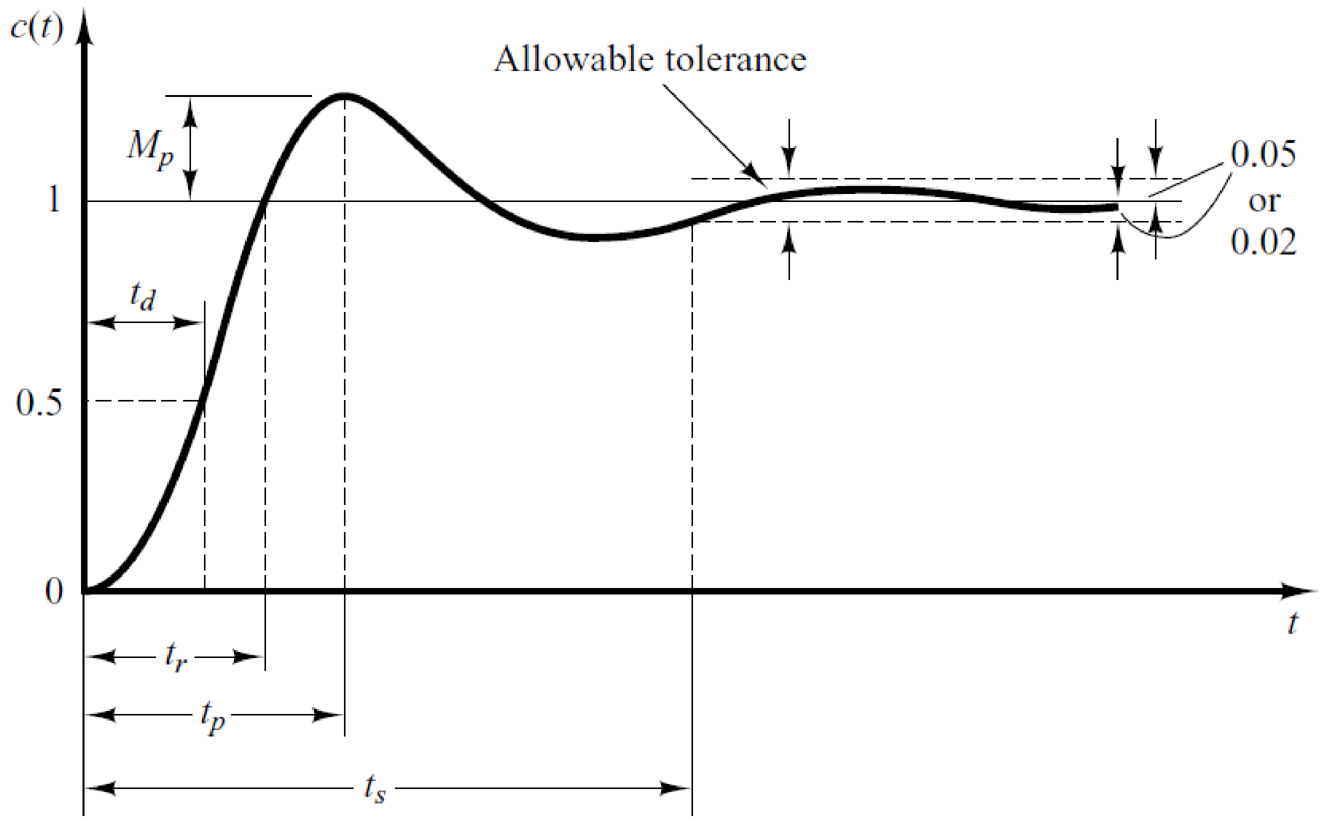
- The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Settling Time

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



Time Domain Specifications

ot)

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$c(t_p) = 1 - e^{-\zeta\omega_n t_p}\left[\cos \omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_p\right]$$

$$c(\infty) = 1$$

$$M_p = \left[\cancel{1} - e^{-\zeta\omega_n t_p}\left(\cos \omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_p\right) - \cancel{1}\right] \times 100$$

Put $t_p = \dfrac{\pi}{\omega_d}$ in above equation

$$M_p = \left[- e^{-\zeta\omega_n \frac{\pi}{\omega_d}}\left(\cos \omega_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d \frac{\pi}{\omega_d}\right)\right] \times 100$$

Time Domain Specifications (Maximum Overshoot)

$$M_p = \left[-e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \left(\cos \cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} \right) \right] \times 100$$

Put $\omega_d = \omega_n \sqrt{1-\zeta^2}$ in above equation

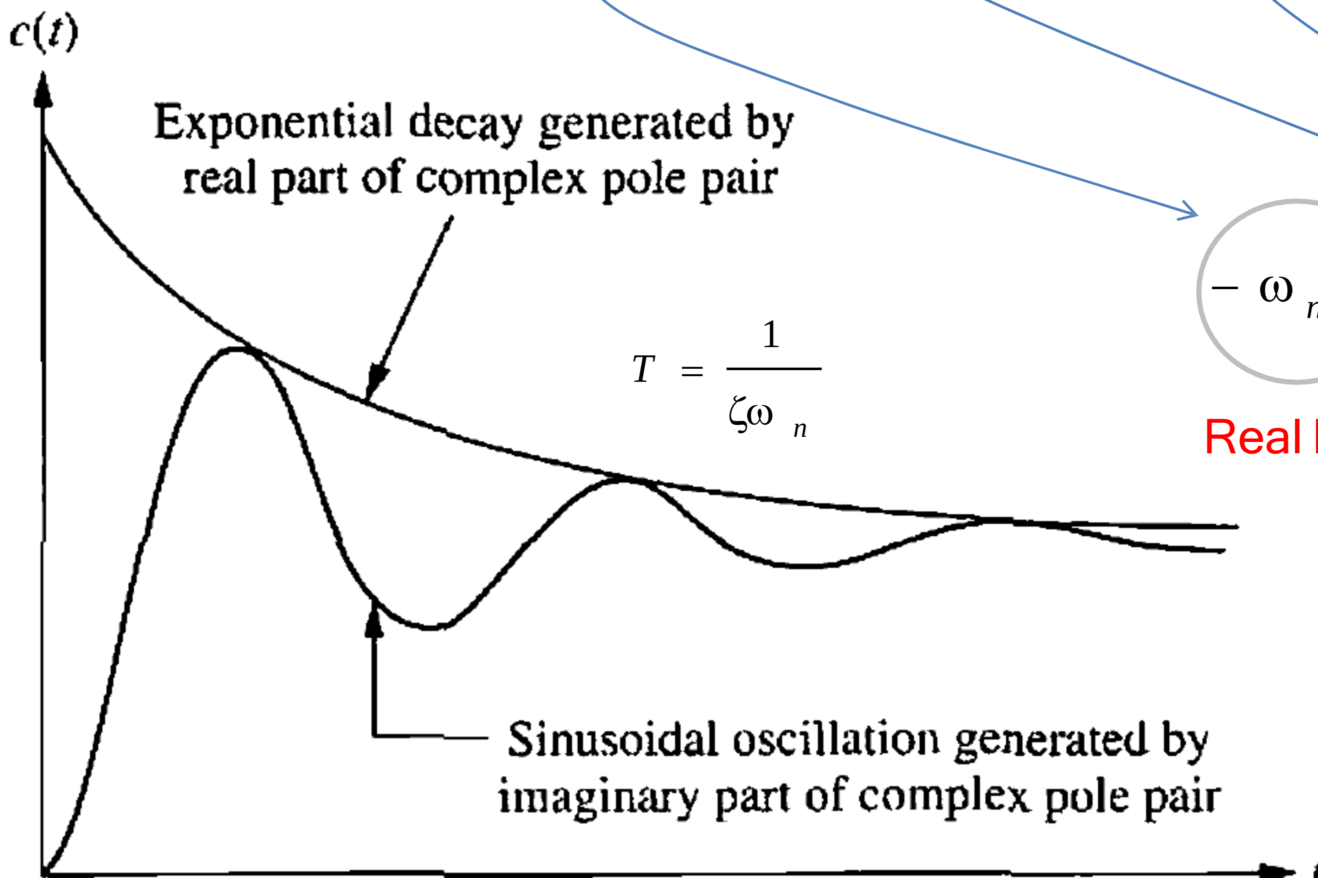
$$M_p = \left[-e^{-\zeta\cancel{\omega_n} \frac{\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}}} \left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \right] \times 100$$

$$M_p = \left[-e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} (-1+0) \right] \times 100$$

$$\boxed{M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100}$$

Time Domain Specifications (Settling Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right]$$



$$-\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1}$$

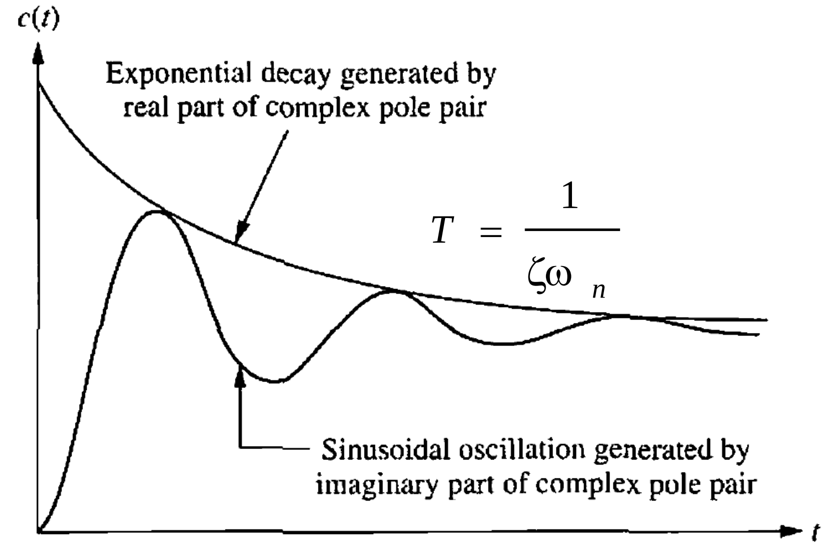
Real Part

Imaginary Part

Time Domain Specifications (Settling Time)

- Settling time (2%) criterion
  - Time consumed in exponential decay up to 98% of the input.

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$



- Settling time (5%) criterion
  - Time consumed in exponential decay up to 95% of the input.

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Summary of Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

NARSIMHA REDDY ENGINEERING COLLEGE
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Subject name :Control Systems

Unit No :2

Topic :Steady State Error and Static Error

Faculty Name :S LAKSHMI DEVI

Steady State Error

- If the output of a control system at steady state does not exactly match with the input, the system is said to have steady state error

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.

- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

- It involves the term s^N in the denominator, representing N poles at the origin.

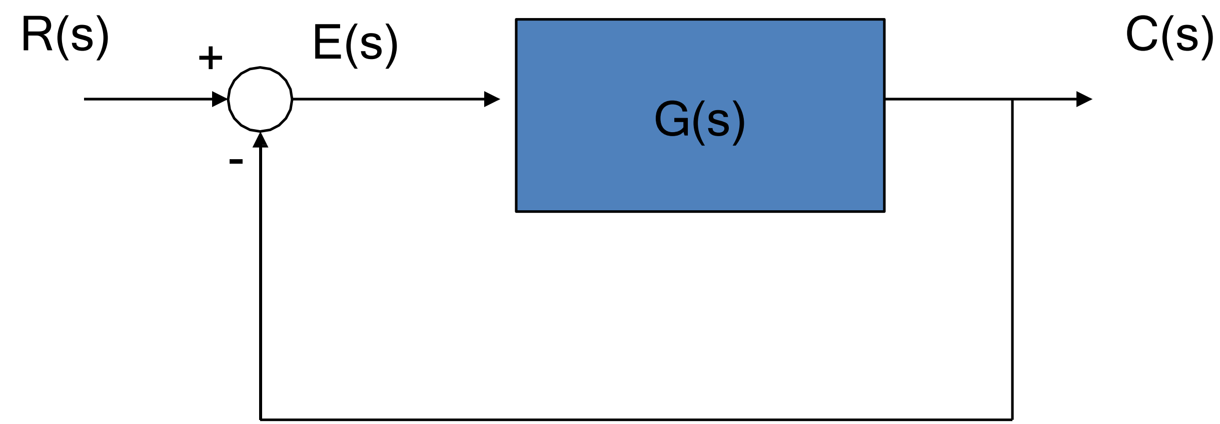
- A system is called type 0, type 1, type 2, ... , if N=0, N=1, N=2, ... , respectively.

- As the type number is increased, accuracy is improved.

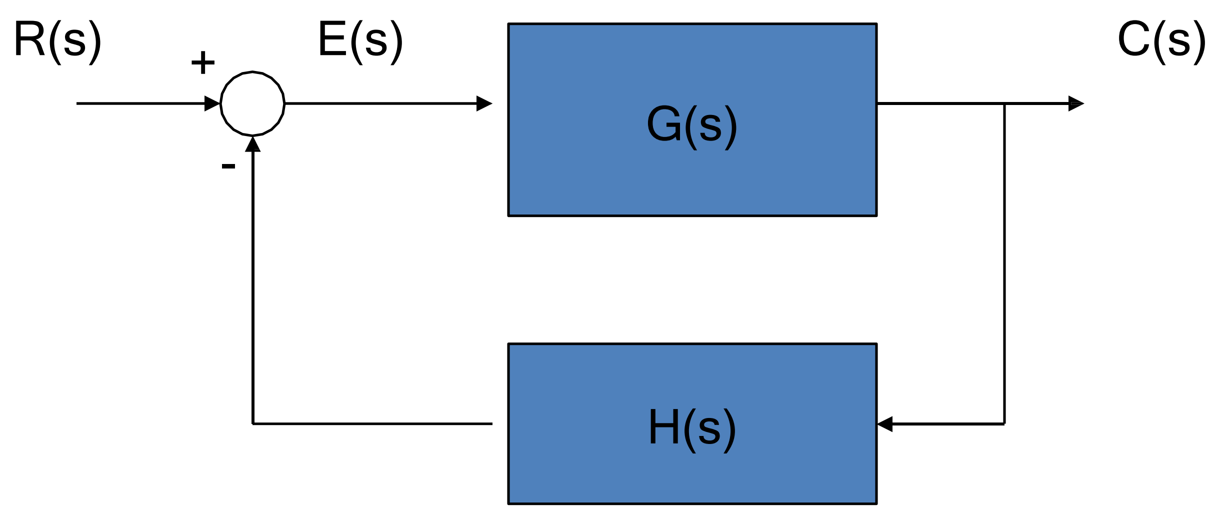
- However, increasing the type number aggravates the stability problem.

- A compromise between steady-state accuracy and relative stability is always necessary.

Steady-state error analysis



Unity feedback
H(s)=1



Non-unity feedback
H(s)≠1

Steady-state error analysis

For unity feedback system:

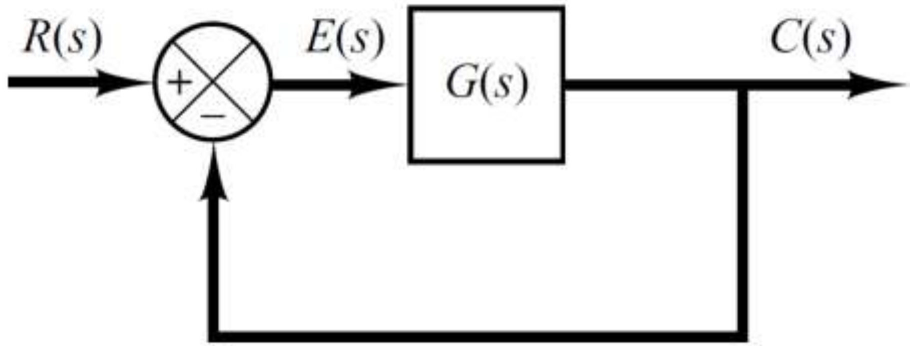
$$E(s) = R(s) - C(s) \quad \rightarrow \quad \text{System error}$$

For a non-unity feedback system:

$$E(s) = R(s) - H(s)C(s) \quad \rightarrow \quad \text{Actuating error}$$

Steady State Error of Unity Feedback Systems

- Consider the system shown in following figure.



- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

Steady State Error of Unity Feedback Systems

- Steady state error is defined as the error between the input signal and the output signal when t-> infinity

- The transfer function between the error signal E(s) and the input signal R(s) is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

- The final-value theorem provides a convenient way to find the steady-state performance of a stable system.

- Since E(s) is

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

- The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

- In a given system, the output may be the position, velocity, pressure, temperature etc…

- Therefore, we can say the output as “position,” and the rate of change of the output as “velocity,” and so on.

- This means that in a temperature control system “position” represents the output temperature, “velocity” represents the rate of change of the output temperature, and so on.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{\cancel{s}}{1 + G(s)} \frac{1}{\cancel{s}}$$

$$= \frac{1}{1 + G(0)}$$

- The static position error constant K_p is defined by

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

- Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Position Error Constant (K_p)

- For a Type 0 system

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For Type 1 or higher order systems

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

- For a unit step input the steady state error **e_{ss}** is

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

Static Velocity Error Constant (K_v)

- The steady-state error of the system for a unit-ramp input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{sG(s)}$$

- The static velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

- Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

Static Velocity Error Constant (K_v)

- For a Type 0 system

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 2 or higher order systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

Static Velocity Error Constant (K_v)

- For a ramp input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

Static Acceleration Error Constant (K_a)

- The steady-state error of the system for parabolic input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- The static acceleration error constant K_a is defined by

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- Thus, the steady-state error in terms of the static acceleration error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

Static Acceleration Error Constant (K_a)

- For a Type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For type 2 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 3 or higher order systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Static Acceleration Error Constant (K_a)

- For a parabolic input the steady state error e_{ss} is

$$e_{ss} = \infty, \quad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 2 systems}$$

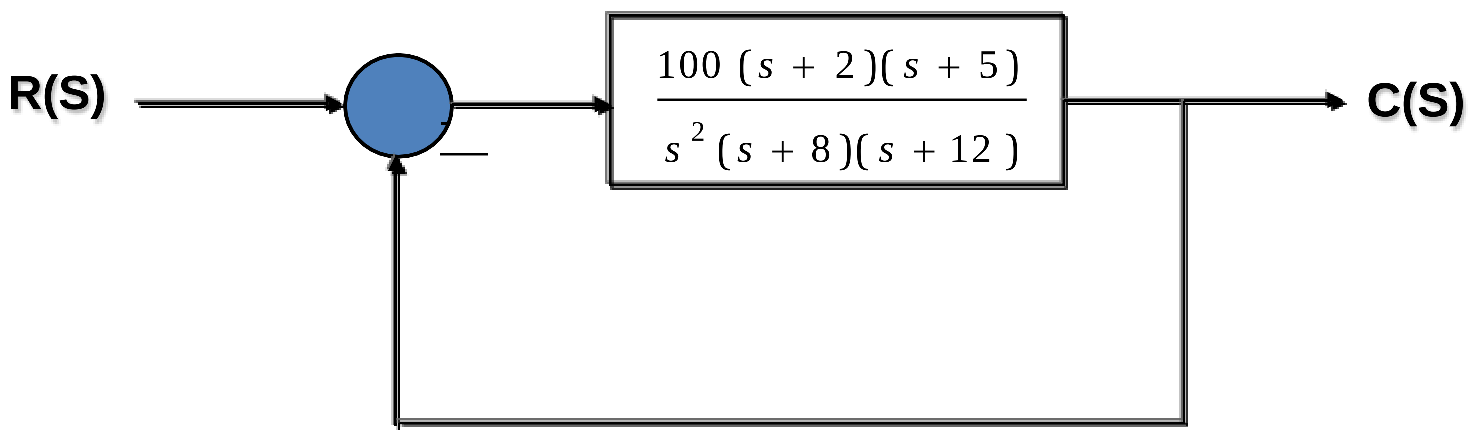
$$e_{ss} = 0, \quad \text{for type 3 or higher systems}$$

Summary

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Example

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Example

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s \to 0} G(s)$$

$$K_p = \lim_{s \to 0} \left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_p = \infty$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \lim_{s \to 0} \left(\frac{100\, s(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_v = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

$$K_a = \lim_{s \to 0} \left(\frac{100\, s^2(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_a = \left(\frac{100(0+2)(0+5)}{(0+8)(0+12)} \right) = 10.4$$

Example

$K_p = \infty$ $K_v = \infty$ $K_a = 10.4$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

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Subject name :Control Systems

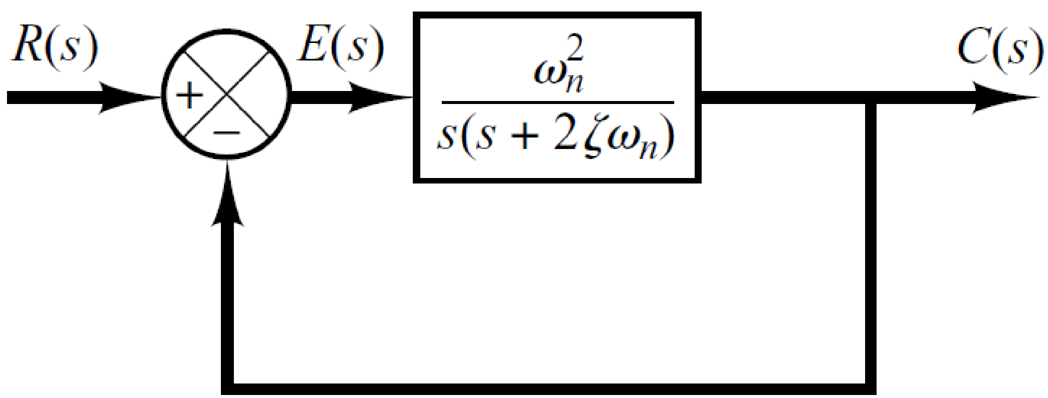
Unit No :2

Topic :Problems

Faculty Name :S LAKSHMI DEVI

Example

- Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 4% criterion t_s when the system is subjected to a unit-step input.



Example

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example

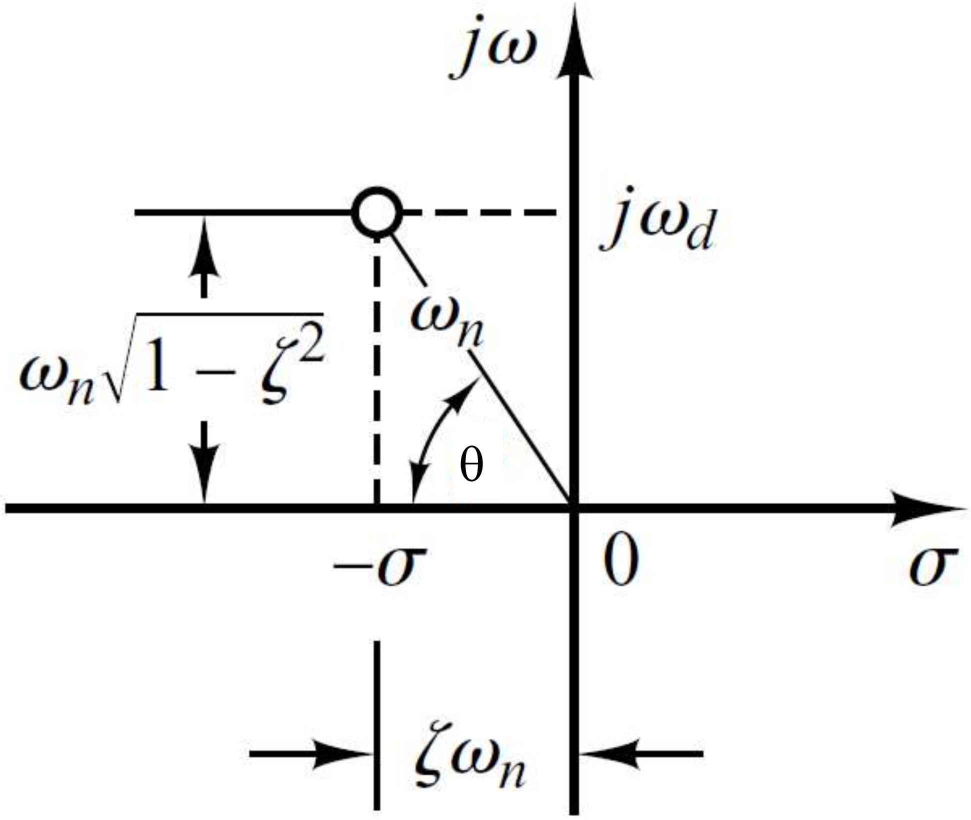
Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \frac{3.141 - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$\theta = \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{\zeta\omega_n}\right) = 0.93 \text{ rad}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1 - 0.6^2}} = 0.55\, s$$



Example

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{3.141}{4} = 0.785\ s$$

Settling Time (2%)

$$t_s = \frac{4}{\zeta\omega_n}$$

$$t_s = \frac{4}{0.6 \times 5} = 1.33\ s$$

Settling Time (4%)

$$t_s = \frac{3}{\zeta\omega_n}$$

$$t_s = \frac{3}{0.6 \times 5} = 1s$$

Example

Maximum Overshoot

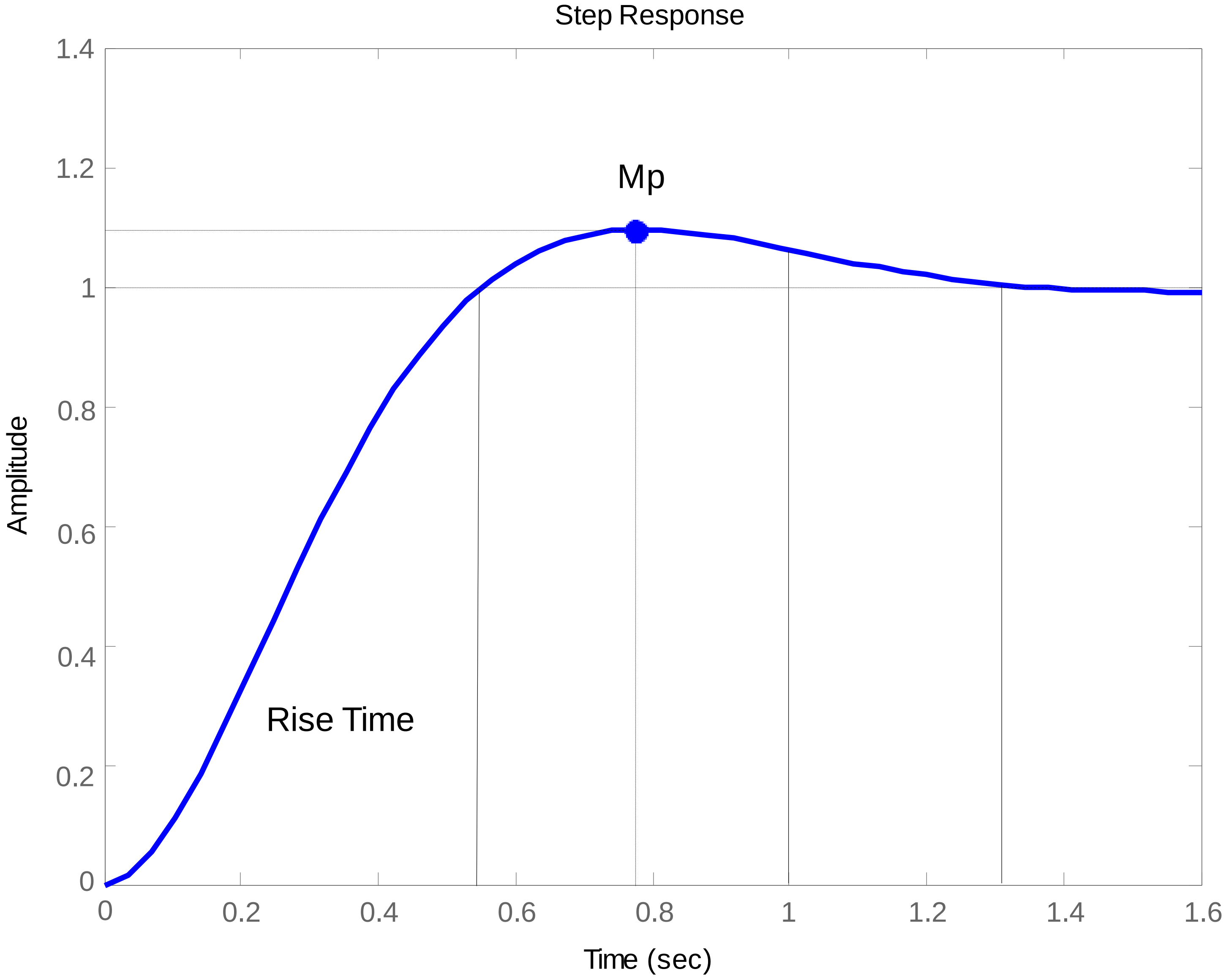
$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100$$

$$M_p = 0.095 \times 100$$

$$M_p = 9.5\%$$

Example



Step Response
Mp
Rise Time
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Example

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T=2
  - D.C Gain K=6
  - Transfer Function
  - Step Response

Example

Transfer Function

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Example

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out C if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$

$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Example

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input , $R(s) = \dfrac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

Time domain specification	Formula
Delay time	$t_d = \frac{1+0.7\delta}{\omega_n}$
Rise time	$t_r = \frac{\pi-\theta}{\omega_d}$
Peak time	$t_p = \frac{\pi}{\omega_d}$
% Peak overshoot	$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}\right) \times 100\%$

UNDERDAMPED

Example : Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

$$find \quad T_s, \%OS, T_p$$

Solution:

$$\omega_n = 10 \qquad \xi = 0.75$$

$$T_s = 0.533s, \ \%OS = 2.838\%, \ T_p = 0.475s$$

UNDERDAMPED

Example : *Find the natural frequency and damping ratio for the system with transfer function*

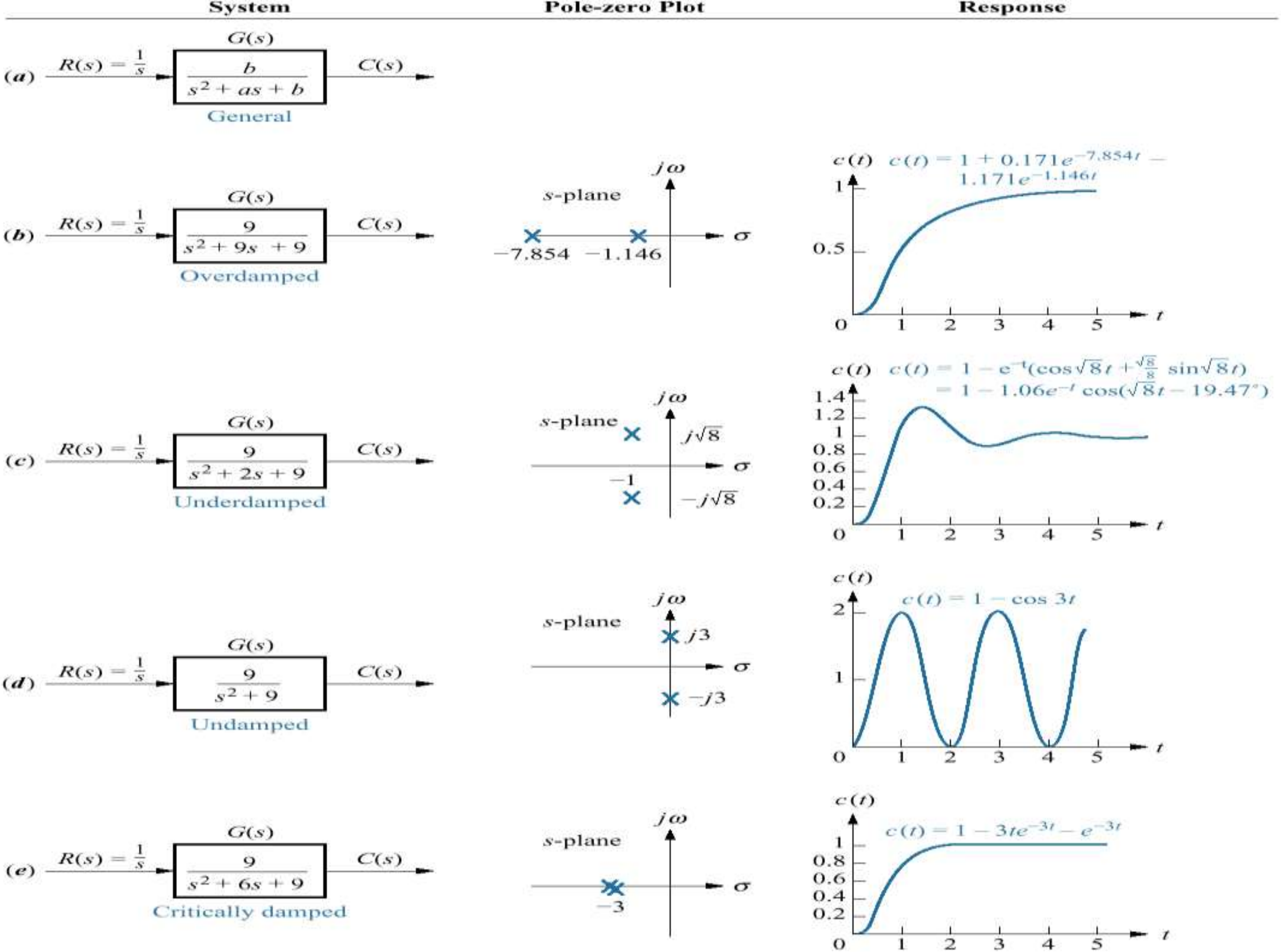
$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Solution:

Compare with general TF

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- $\omega_n = 6$
- $\xi = 0.35$


System
Pole-zero Plot
Response
(a)
R(s) = 1/s
G(s)
b / (s² + as + b)
C(s)
General
(b)
R(s) = 1/s
G(s)
9 / (s² + 9s + 9)
C(s)
Overdamped
jω
s-plane
σ
−7.854
−1.146
c(t)
c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
1
0.5
0
1
2
3
4
5
t
(c)
R(s) = 1/s
G(s)
9 / (s² + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
σ
−1
−j√8
c(t)
c(t) = 1 − e^{−t}(cos √8 t + √8/8 sin √8 t)
= 1 − 1.06e^{−t} cos(√8 t − 19.47°)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t
(d)
R(s) = 1/s
G(s)
9 / (s² + 9)
C(s)
Undamped
jω
s-plane
j3
σ
−j3
c(t)
c(t) = 1 − cos 3t
2
1
0
1
2
3
4
5
t
(e)
R(s) = 1/s
G(s)
9 / (s² + 6s + 9)
C(s)
Critically damped
jω
s-plane
σ
−3
c(t)
c(t) = 1 − 3te^{−3t} − e^{−3t}
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t

Subject name :Control Systems

Unit No :3

Topic :Stability Analysis

Concept of Stability

- In order to know the location of the poles, we need to find the roots of the closed-loop characteristic equation.

- It turned out, however, that in order to judge a system's stability we don't need to know the actual location of the poles, just their sign. that is whether the poles are in the right-half or left-half plane.

- The Hurwitz criterion can be used to indicate that a characteristic polynomial with negative or missing coefficients is unstable.

- The Routh-Hurwitz Criterion is called a necessary and sufficient test of stability because a polynomial that satisfies the criterion is guaranteed to stable. The criterion can also tell us how many poles are in the right-half plane or on the imaginary axis.

Before discussing the Routh-Hurwitz Criterion in detail, firstly we will study the stable, unstable and marginally stable system.

- **Stable System**: If all the roots of the characteristic equation lie on the **left** half of the 'S' plane then the system is said to be a stable system.
- **Marginally Stable System**: If all the roots of the system lie on the imaginary axis of the 'S' plane then the system is said to be marginally stable.
- **Unstable System**: If all the roots of the system lie on the **right** half of the 'S' plane then the system is said to be an unstable system.

Statement of Routh-Hurwitz Criterion

Routh Hurwitz criterion states that any system can be stable if and only if all the roots of the first column have the same sign and if it does not has the same sign or there is a sign change then the number of sign changes in the first column is equal to the number of roots of the characteristic equation in the right half of the s-plane i.e. equals to the number of roots with positive real parts.

Routh-Hurwitz Stability Criterion

1. All the coefficients of the equation should have the same sign.
2. There should be no missing term.

- These requirements are necessary but not sufficient. That is we know the system is unstable if they are not satisfied; yet if they are satisfied, we must proceed further to ascertain the stability of the system.

Routh-Hurwitz Stability Criterion (Contd..)

- The Routh-Hurwitz criterion applies to a polynomial (characteristic equation) of the form:

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots\dots + a_1 s + a_0$$

$$\text{assume } a_0 \neq 0$$

- The Routh-Hurwitz array:

$$\begin{array}{llllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \dots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \dots \\
. & . & . & & & \\
. & . & . & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}$$

Routh-Hurwitz Stability Criterion (Contd..)

- Columns of s are only for accounting.
- The b row is calculated from the two rows above it.
- The c row is calculated from the two rows directly above it.
- Etc…
- The equations for the coefficients of the array are:

$$b_1 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \qquad b_2 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \ldots\ldots.$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}, \ldots\ldots$$

- Note: the determinant in the expression for the ith coefficient in a row is formed from the first column and the (i+1)th column of the two preceding rows.

Routh-Hurwitz Stability Criterion

- The number of polynomial roots in the right half plane is equal to the number of sign changes in the first column of the array.
- Example:

$$P(s) = s^3 + s^2 + 2s + 8 = (s+2)(s^2 - s + 4)$$

The Routh array is :

s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

- Since there are two sign changes on the first column, there are two roots of the polynomial in the right half plane: system is unstable.

- Note: The Routh-Hurwitz criterion shows only the stability of the system, it does not give the locations of the roots, therefore no information about the transient response of a stable system is derived from the R-H criterion.

Advantages of Routh- Hurwitz Criterion

- We can find the stability of the system without solving the equation.
- We can easily determine the relative stability of the system.
- By this method, we can determine the range of K for stability.
- By this method, we can also determine the point of intersection for root locus with an imaginary axis.

Limitations of Routh- Hurwitz Criterion

- This criterion is applicable only for a linear system.
- It does not provide the exact location of poles on the right and left half of the S plane.
- In case of the characteristic equation, it is valid only for real coefficients.

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Subject name :Control Systems

Unit No :3

Topic :Routh-Hurwitz Stability Criterion

Faculty Name :S LAKSHMI DEVI

Example: Check the stability of the system whose characteristic equation is given by

$$s^4 + 2s^3+6s^2+4s+1 = 0$$

s^4	1	6	1
s^3	2	4	
s^2	4	1	
s^1	3.5		
s^0	1		

Since all the coefficients in the first column are of **the same sign,** i.e., positive, the given equation has no roots with positive real parts; therefore, the system is said to be **stable.**

Example 1: The characteristic equation of a system is given below. Determine the stability of the system.

$s^4 + 4s + 16s^2 + 10s^2 + 6 = 0$

Applying Routh Hurwitz Criteria and forming Routh array, we get

$$
\begin{array}{llll}
s^4 & 1 & 16 & 5 \\
s^3 & 4 & 10 & 0\,(for\ missing\ term) \\
s^2 & \frac{16\times4-10\times1}{4} = 13.5 & \frac{5\times4-0\times1}{4} = 5 & \\
s^1 & \frac{13.5\times10-5\times4}{13.5} = 8.52 & 0 & \\
s^0 & 5 & &
\end{array}
$$

Example 4: The characteristic equation of a system is given as

$$s^5 + s^4 + 3s^3 + 3s^2 + 2s + 5 = 0$$

Applying Routh Hurwitz Criteria and forming Routh Array, we get

$$\begin{array}{llll}
s^5 & 1 & 3 & 2 \\
s^4 & 1 & 3 & 5 \\
s^3 & 0 \rightarrow \epsilon & -3 & 0 \\
s^2 & \frac{3\epsilon+3}{\epsilon} & 5 & 0 \\
s^1 & \frac{\frac{-3(3\epsilon+3)}{\epsilon}-5\epsilon}{\frac{3\epsilon+3}{\epsilon}} & & \\
s^0 & 5 & &
\end{array}$$

Since there is a sign change at s1 row, hence the system is **unstable** and having **two poles** in right half of s-plane due to two sign changes.

Routh-Hurwitz Stability Criterion (Contd..)

- **Case 1:** none of the elements in the first column of the array is zero. This is the simplest case. Follow the algorithm as shown in the previous slides.

- **Case 2:** The first element in a row is zero, with at least one nonzero element in the same row. In this case, replace the first element which is zero by a small number x. All the elements that follow will be functions of x. After all the elements are calculated, the signs of the elements in the first column are determined by letting ε approach zero.

Routh-Hurwitz Stability Criterion (Contd..)

- **Case 3:** All elements in a row are zero.
- Example: $P(s) = s^2 + 1$

$$\begin{array}{lll} s^2 & 1 & 1 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

- Here the array cannot be completed because of the zero element in the first column.
- Another example: $P(s) = s^3 + s^2 + 2s + 2$

The array is :

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

Routh-Hurwitz Stability Criterion (Contd..)

- Case 3 polynomial contains an even polynomial as a factor. It is called the auxiliary polynomial. In the first example, the auxiliary polynomial is $s^2 + 1$
- And in the second example, auxiliary polynomial is $s^2 + 2$
- Case 3 polynomial may be analyzed as follows:
- Suppose that the row of zeros is the s^i row, then the auxiliary polynomial is differentiated with respect to s, and the coefficients of the resulting polynomial used to replace the zeros in the s^i row. The calculation of the array then continues as in the case 1.

Routh-Hurwitz Stability Criterion (Contd..)

- Example: $$P(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

The Routh array is :

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 2 & \\ s^2 & 1 & 2 & \\ s^1 & 0 & & \\ s^0 & & & \end{array}$$

- Since the S1 row contains zeros, the auxiliary polynomial is obtained from the s2 row:

$$P_{aux}(s) = s^2 + 2$$

- The derivative is 2s, therefore 2 replaces 0 in the s1 row, and the routh array is then completed.

Routh-Hurwitz Stability Criterion (Contd..)

- ❑ Example: $P(s) = s^4 + s^3 + 3s^2 + 2s + 2$

The Routh array now becomes :

$$\begin{array}{llll} s^4 & 1 & 3 & 2 \\ s^3 & 1 & 2 & \\ s^2 & 1 & 2 & \\ s^1 & \boxed{2} & & \\ s^0 & 2 & & \end{array}$$

- ❑ Hence there are no roots in the right half plane.

❑Note: When there is a row of zeros in the routh array, the systems is not stable. That is it will have roots either on the imaginary axis (as in this example), or it has roots on the right half plane.

Determination of range of gain K using RH Criterion

- Example: $$P(s) = s^3 + 5s^2 + (9-K)s + K$$

  The Routh array is :

  $$\begin{array}{lll} s^3 & 1 & 9-K \\ s^2 & 5 & K \\ s^1 & 9-1.2K & \\ s^0 & K & \end{array}$$

- For the system to be stable there should not be any sign changes in the elements of 1st column
- Hence choose the value of K so that 1st column elements are positive
- From s0 row, system to be stable K>0
- From s1 row

  $$9 - 1.2K > 0$$

  $$9 > 1.2K$$

  $$K < 7.5$$

- Hence the range of K is 0<K<7.5

Problems on RH Criterion

- **Example-1:** $P(s) = s^3 + 10s^2 + 31s + 1030$

  The Routh array is :

$$
\begin{array}{lll}
s^3 & 1 & 31 \\
s^2 & 1 & 103 \text{ (by dividing with 10)} \\
s^1 & -72 & \\
s^0 & 103 &
\end{array}
$$

- 1st Column of routh array has two sign changes (from 1 to -72 and from -72 to 103). Hence the system is unstable with two poles in the right-half plane.

Problems on RH Criterion (Contd..)

- **Example 2:**
- Construct a Routh table and determine the number of roots with positive real parts for the equation;

$$2s^3 + 4s^2 + 4s + 12 = 0$$

- Solution:

Since there are two changes of sign in the first column of Routh table, the equation above have two roots at right side (positive real parts).

Problems on RH Criterion (Contd..)

- **Example 3:**
- The characteristic equation of a given system is:

$$s^4 + 6s^3 + 11s^2 + 6s + K = 0$$

What restrictions must be placed upon the parameter K in order to ensure that the system is stable?

- Solution:

  For the system to be stable, 60 – 6K < 0, or k < 10, and K > 0.

  Thus 0 < K < 10

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Subject name :Control Systems

Unit No :3

Topic :Root Locus Technique

Faculty Name :S LAKSHMI DEVI

Basic concepts of root locus

- In the previous sections, we have studied that the stability of a system. It depends on the **location of the roots** of the characteristic equation. We can also say that the stability of the system depends on the location of **closed-loop poles.** Such knowledge of the movement of the poles in the s-plane when the parameters are varied is important. The minor changes in the parameters can greatly help in the system designing. The nature of the system's transient response is closely related to the location of the poles in the s-plane.
- We have also studied the Routh Hurwitz criteria that describe the stability of the algebraic equation. If any of the term in the first column of the Roth table possesses a sign change, the system tends to become unstable.

INTRODUCTION

Root Locus Technique:

The root locus method was introduced by **W.R Evans** in 1948.

- It is a graphical method for determining the location of the poles of a given closed loop system for some parameter values of the system. The parameter can be the system gain or time constant.
- Time constant being the design value of an open loop system is normally not varied.
- It is a time domain method.

INTRODUCTION (Contd)..

- We know that

  for a unity feedback system the characteristic equation is given by

  $$1 + G(S) = 0, \text{ and}$$

- For a non-unity feedback system the characteristic equation is given by

  $$1 + G(S)\,H(S) = 0$$

  where,

- $G(S)$: open loop transfer function of the system that is to be controlled for desired time domain specifications, and
- $H(S)$: feedback element (normally a transducer)

INTRODUCTION (Contd)..

- We know that for a closed loop system to be stable, its closed loop poles (roots of characteristic equation) should lie in the left half of the S-plane.
- We also know that a closed loop system is limitedly stable (on the verge of instability) if any of its roots lie on the imaginary axis of the S-plane and it is unstable if its poles lie in the right half of the S-plane.
- Using this method, we can exactly position the location of closed loop poles for a given value of system gain ‘K’ whereas Routh’s method does not facilitate this.
- Using Routh’s method we cannot determine relative stability of a system whereas this method allows us to do that.

Illustration by Example

- We know that for a second order closed loop system the general form is given by

$$M(S) = \omega_n^2 / (S^2 + 2\xi\omega_n S + \omega_n^2) = N(S)/D(S)$$

- Let

$$G(S) = K/S(S+1)\ ;\ M(S) = G(S)/1+G(S) = K/(S^2 + S + K)$$

$$M(S) = N(S)/D(S)$$

- For a unity feedback system, the characteristics equation is:

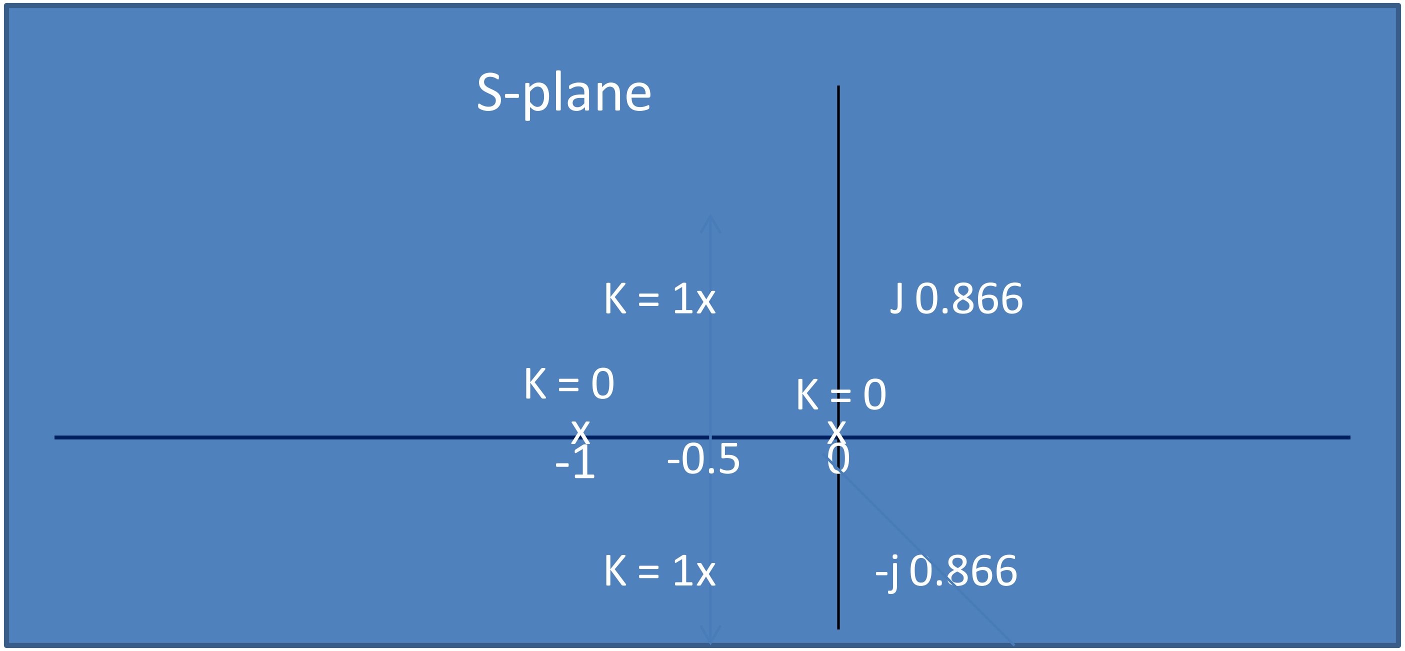
$$Q(S) = 1+G(S) = 0 \Rightarrow 1 + K/S(S+1) = 0$$

$$\Rightarrow S^2 + S + K = 0$$

- For K = 0; the roots of Q(S) are at S=0 & S=-1; which are the poles of the system.

Illustration by Example (Contd)..

- Looking at $Q(S) = S^2 + S + K = 0$ we conclude that,
- As we vary K from '0' to any higher value, the location of the roots of Q(S) will change (shift) in the S-plane.
- Thus the roots will chalk out a locus in the S-plane for a given range of 'K'. This is called Root Locus.



Why Requirement of Root Locus Method ?

- We know that we are interested in finding the roots of a characteristic equation for a range of a parameter of the system which generally is system gain ‘K’. Generally speaking we may be interested in determining the location of closed loop poles for a range of ‘K’

$$0 \leq K \leq \infty$$

- Now it is easy to factorize a second and third order characteristic equation for various values of ‘K’, but for higher order polynomials it is very difficult (near impossible) to factorize for determining their roots.

- Therefore we need a method to do so & that method is Root Locus.

The Method (Contd)..

- Before going ahead with the method, it is necessary to define what is called 'rational transfer function'.

- A rational transfer function is the one which has equal number of poles and zeros; that is Np = Nz

  Np: number of poles   Nz: number of zeros

- Consider the following transfer functions:

- $G_1(S)\ H_1(S)$ or $G_1(S) = K\ (S+1)/(S+2)$ -------- 1
- $G_2(S) = K\ (S+1)(S+2)/(S+3)(S+4)$ ---- 2
- $G_3(S) = K\ (S+1)/(S+2)(S+3)$ ------- 3
- $G_4(S) = K\ (S+1)/(S+2)(S+3)(S+4)$ ---- 4

The Method (Contd)..

- For, $G_1(S)$ = K (S+1)/(S+2), there is a finite pole at S = -2 & a finite zero at S = -1; Np= Nz = 1; hence it is a rational function

- $G_2(S)$ also has equal number of poles and zeros; Np = Nz = 2;

- $G_3(S)$ has 2 finite poles & 1 finite zero; Np ≠ Nz

- $G_4(S)$ has 3 finite poles and 1 finite zero; Np ≠ Nz

- Does it mean that $G_3(S)$ & $G_4(S)$ are not rational functions!!

- They both are, indeed, rational functions; the need is to find out the location of remaining zeros so that Np = Nz.

- In order to resolve the issue of 'how many zeros' a transfer function has, we need to understand what is zero of a transfer function.

- Let $G(S) = K (S+1)/(S+2)(S+3)$

- We all understand 'G(S)' as 'frequency dependent gain' offered by the system.

- Now, if we substitute $S = -1$ in $G(S)$, its value = '0'; it means that gain offered at $S= -1$ equals '0'. Therefore $S = -1$ is a zero of the transfer function, $G(S)$

- Pole of a transfer function is a singularity because gain offered by $G(S)$ at its pole = ∞. For example, $S = -2$ & -3 causes gain of $G(S)= \infty$

- Therefore, we say If the number of zeros are not equal to the number of finite poles of G(S), then number of zeros = Np – Nz shall lie at ∞.
- Let

  $G(S) = K\,(S+1)/(S+2)(S+3)$
- $\lim_{S \to \infty} G(S) \equiv \lim_{S \to \infty} K/S = 0$; the power of S is '1' therefore there is one zero at ∞. Thus we have one finite zero and another zero at ∞. Hence Np = Nz
- For, $G(S) = K\,(S+1)/(S+2)(S+3)(S+4)$
- we have one finite zero at S = -1 and two zeros at ∞
- Therefore both are rational functions

- Let

$$G(S)\,H(S) = K \frac{\prod_{j=1}^{m} (S + Zj)}{S^r \prod_{i=1}^{i=n} (S + Pi)}$$

- where, K: gain in the system

  r: number of poles at the origin of S-plane

  n & m: number of poles and zeros in the S-plane

The Method (Contd)..

$$|G(S)H(S)| = K\frac{\prod_{j=1}^{m} |(S + Zj)|}{|S^r| \prod_{i=1}^{i=n} |(S + Pi)|} = 1.0$$

$$K \prod_{j=1}^{j=m} |(S + Zj)| = |S^r| \prod_{i=1}^{I=n} |(S + Pi);$$ for K =0 we get poles of G(S)H(S)

The Method (Contd)..

$$\prod_{j=1}^{j=m} |(S + Z_j)| = |S^r| \prod_{i=1}^{i=n} |(S + P_i)/K;$$

- For $K \longrightarrow \infty$; we get zeros of G(S)H(S)
- We draw root locus for $0 \leq K \leq \infty$

  Therefore,
- Starting points of root locus are poles of G(S)H(S), K=0
- End points of root locus are zeros of G(S)H(S), $K = \infty$

The Method (Contd)..The Angle Criteria

❑ The Angle Criteria:

$$G(S)H(S) = K \frac{\prod_{j=1}^{m} (S + Zj)}{\prod_{i=1}^{n} (S + Pi)}$$

The angle criteria is in degrees given by:

$$\sum_{j=1}^{m} \arg(S + Zj) - \sum_{i=1}^{n} \arg(S + Pj) = +/- (2q + 1)180; \quad q = 0,1,2,....$$

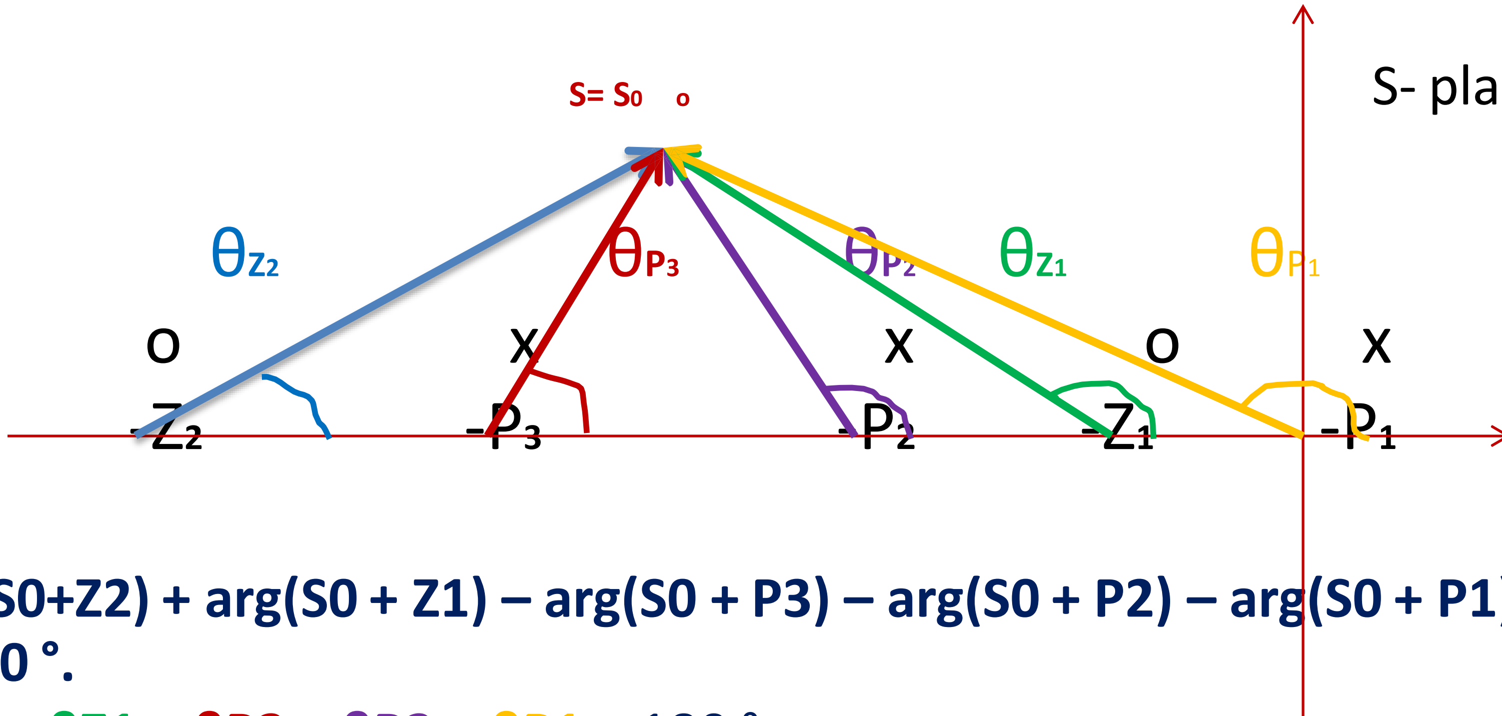
Implement Angle Criteria

- Since root locus is drawn satisfying angle criteria, now we explain how it is done.

1. Plot location of poles & zeros of G(S)H(S) in the S-plane
2. Choose any point S = S**0** in the S-plane.
3. From each pole & zero draw vectors to the chosen point, S**0**
4. Measure the angle subtended by each pole & zero at S**0**, in the CCW direction.
5. Remember that angle subtended by a pole is negative & that by a zero is positive
6. Algebraically add all the angles. If they sum up to 180 degrees, then S = S**0** is a point on the root locus.

Graphical Implementation of Angle Criteria

❑ Graphical Illustration for Angle Criteria:



arg(S0+Z2) + arg(S0 + Z1) – arg(S0 + P3) – arg(S0 + P2) – arg(S0 + P1) = +/_ 180 °.

θZ2 + **θZ1** – **θP3** – **θP2** – **θP1** = 180 °

❑ If the above angle condition is satisfied then **S0** is on the locus.

Magnitude Criteria

- From the magnitude criteria, we calculate the value of gain 'K' at the point $S = S0$ which lies on the root locus (that is $S=S0$ satisfies angle criteria).

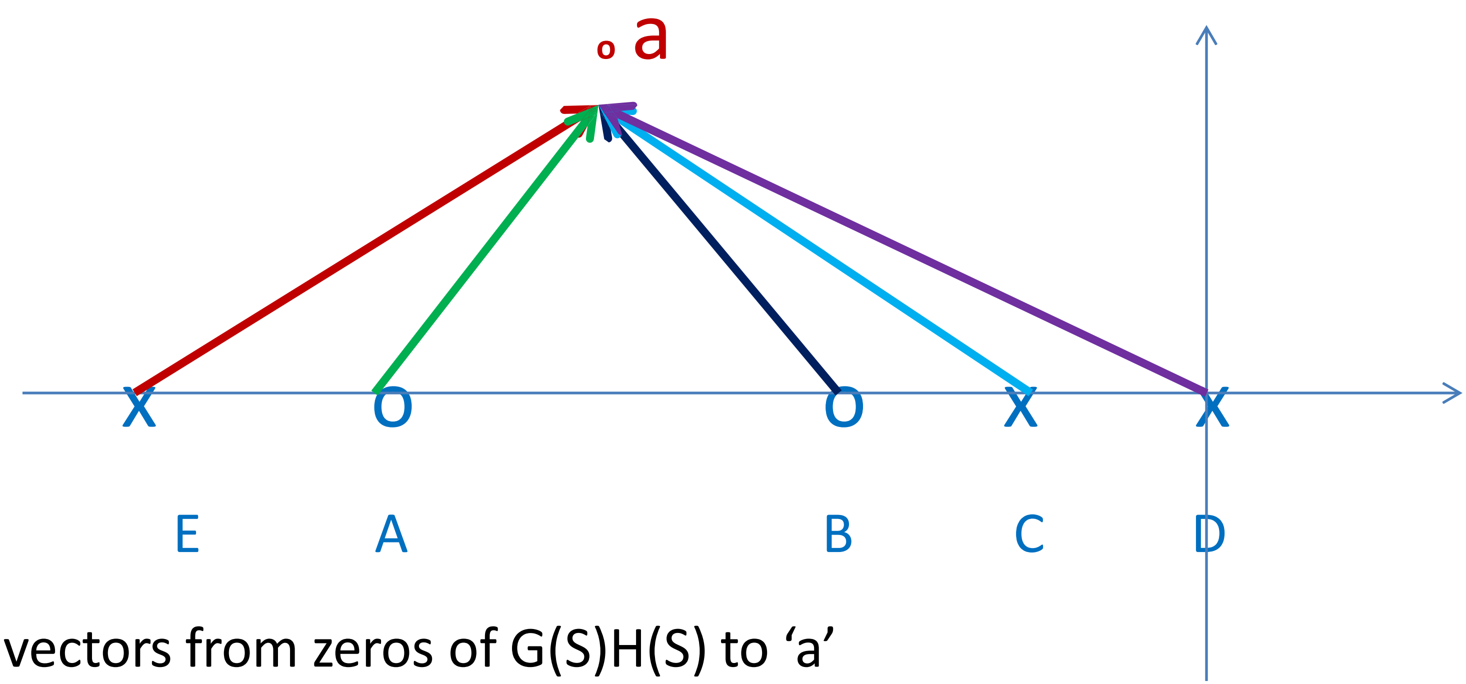
$$K \frac{\prod_{j=1}^{m} |(S_o + Z_j)|}{\prod_{i=1}^{n} |(S_o + P_i)|} = 1 \quad \text{or,} \quad K = \frac{\prod_{i=1}^{n} |(S_o + P_i)|}{\prod_{j=1}^{m} |(S_o + Z_j)|}$$

- K = product of vector lengths from poles of G(S)H(S) to S**0**/product of vector lengths from zeros of G(S)H(S) to S**0**.

Graphical Implementation of Magnitude Criteria

- **Graphical method for determination of ‘K’:**

Ea Ca Da : vectors from poles of G(S)H(S) to point ‘a’: S = S**0**



Aa Ba : vectors from zeros of G(S)H(S) to ‘a’

Gain K = (Ea)(Ca)(Da)/(Aa)(Ba)

We measure vector lengths, as per scale, and then calculate K

NARSIMHA REDDY ENGINEERING COLLEGE
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Subject name :Control Systems

Unit No :3

Topic :Construction Rules for Root Locus

Faculty Name :S LAKSHMI DEVI

Construction Rules for Root Locus

- **Rule 1:**

  Root Locus is symmetrical about real axis of S-plane, because roots are either real or complex conjugate.

- **Rule 2:**

  As ‘K’ increases from ‘0’ to ‘∞’, the open loop poles of G(S)H(S) move (branch out) towards the zeros of G(S)H(S); some of the zeros may be at ‘∞’.

  The number of branches terminating on ‘∞’ equals Np – Nz; that is the difference between number of finite poles & zeros of G(S)H(S).

Construction Rules for Root Locus

❑ **Rule 3:**

A point S = S**0** on the real axis shall lie on the root locus iff the total number of open loop poles & zeros of G(S)H(S) to the right of S**0** is odd. (Loci lie in the region 2, 4 & 6)

- ❑ The number of poles + zeros to the right of region ‘6’ = 1(odd)
- ❑ The number of poles + zeros to the right of region ‘5’ = 2(even)
- ❑ The number of poles + zeros to the right of region ‘4’ = 3(odd)
- ❑ The number of poles + zeros to the right of region ‘3’ = 4(even)
- ❑ The number of poles + zeros to the right of region ‘2’ = 5(odd)
- ❑ The number of poles + zeros to the right of region ‘1’ = 6(even)

Construction Rules for Root Locus

- **Rule 3 (contd)..**

- The poles are K= 0 points & the zeros are K = ∞ points. As we are interested in the range of K, $0 \leq K \leq \infty$, therefore the poles will start moving towards their respective zeros, in the region on the real axis, and terminate at zeros (K = ∞)

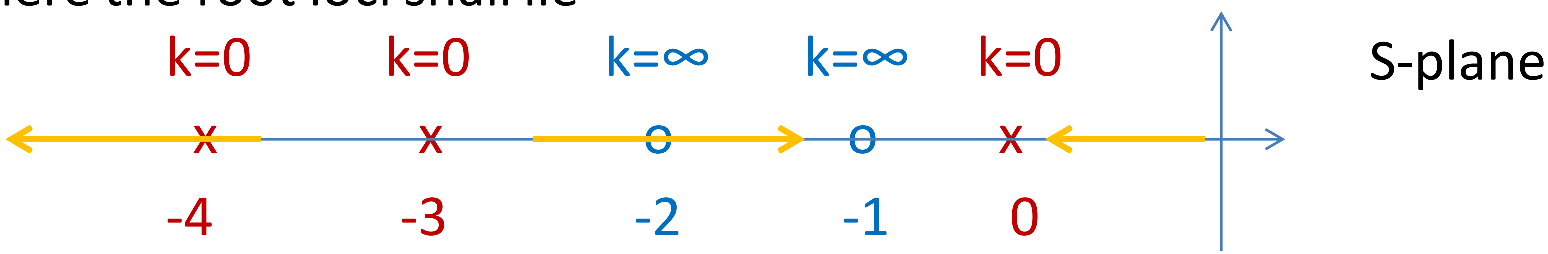
- Therefore, we can say that the loci of closed loop poles start at K = 0 (the location of the poles of G(S)H(S)) and terminate at K =∞ (the location of the zeros of G(S)H(S))

Construction Rules for Root Locus

❑ Rule 3 (contd): Example for implementation

Let G(S)H(S) = K(S+1)(S+2)/s(S+3)(S+4)

1. Draw pole zero locations in the S-plane
2. Use angle criteria to mark the regions on the real axis of the S-plane where the root loci shall lie



❑ The regions where the loci shall lie are highlighted in yellow where the total angle subtended by poles & zeros = 180°

Construction rules for Root Locus

Rule 3 (contd): Example for implementation

In the considered example:

1. No. of open loop poles = 3; root loci branches = 3 because each pole is a starting point.
2. Root Loci will start from S =0, -3 & -4 (K = 0 points)
3. As K increases, the loci moves from the poles to respective zeros (K = ∞ points)
4. The arrows show the direction of movement of poles
5. Np = 3 Nz = 2; no. of poles for which the loci shall terminate at ∞ = Np – Nz = 1
6. We observe that pole at S = -4 terminates at ∞

Construction rules for Root Locus

❑ Rule 4: (Angle of Asymptotes)

The (Np – Nz) branches of the root locus asymptotically tend to ∞. The angles of asymptotes are given by:

$$\phi \mathbf{q} = (2q+1)\,180°/(Np - Nz);\; q = 0,1,2, \ldots, (Np\text{-}Nz\text{-}1)$$

1. $G(S) = K\,(S+1)(S+2)/S(S+3)(S+4)$

Np = no. of poles = 3; Nz = no. of zeros = 2; Np-Nz = 1

q = 0; ϕ = 180°

2. $G(S) = K(S+2)/(S+1)(S+3)(S+5)(S+6)$

Np = no. of poles = 4; Nz = no. of zeros = 1; Np-Nz = 3

q = 0,1,2; ϕ**0** = 60°, ϕ**1** = 180° , ϕ**2** = 300°

Construction rules for Root Locus

Rule 5: (Centroid)

If no. of asymptotes are more than 1, they cross the real axis of the S-plane. Their point of intersection on the real axis is known as Centroid.

Centroid **σA** is given by:

$$\sigma_A = \frac{\sum_{i=1}^{n} P_i - \sum_{j=1}^{m} Z_j}{(N_p - N_z)} = \frac{(\text{Sum of real parts of poles - Sum of real parts of zeros})}{(\text{No. of poles} - \text{No. of zeros})}$$

Construction rules for Root Locus An Example

- **Example:**

  Determine 1) no. of loci on the real axis and their regions, 2) no. of asymptotes, 3) angle of asymptotes, 4) Centroid for a unity feedback system whose open-loop transfer function is given as: $G(S) = K/S(S+1)(S+2)$

- Solution Steps:

- Draw pole zero locations in the S-plane
- Determine no. of finite poles, Np, and zeros, Nz & Np-Nz
- Mark regions on the real axis where loci lie
- Find no. of asymptotes = Np – Nz & their respective angles
- If (Np-Nz) > 1 determine value of centroid
- Sketch root loci (free hand)

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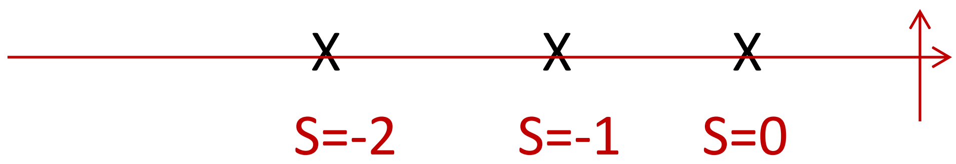
Construction rules for Root Locus An Example

1. 

(poles are K=0 pts.)

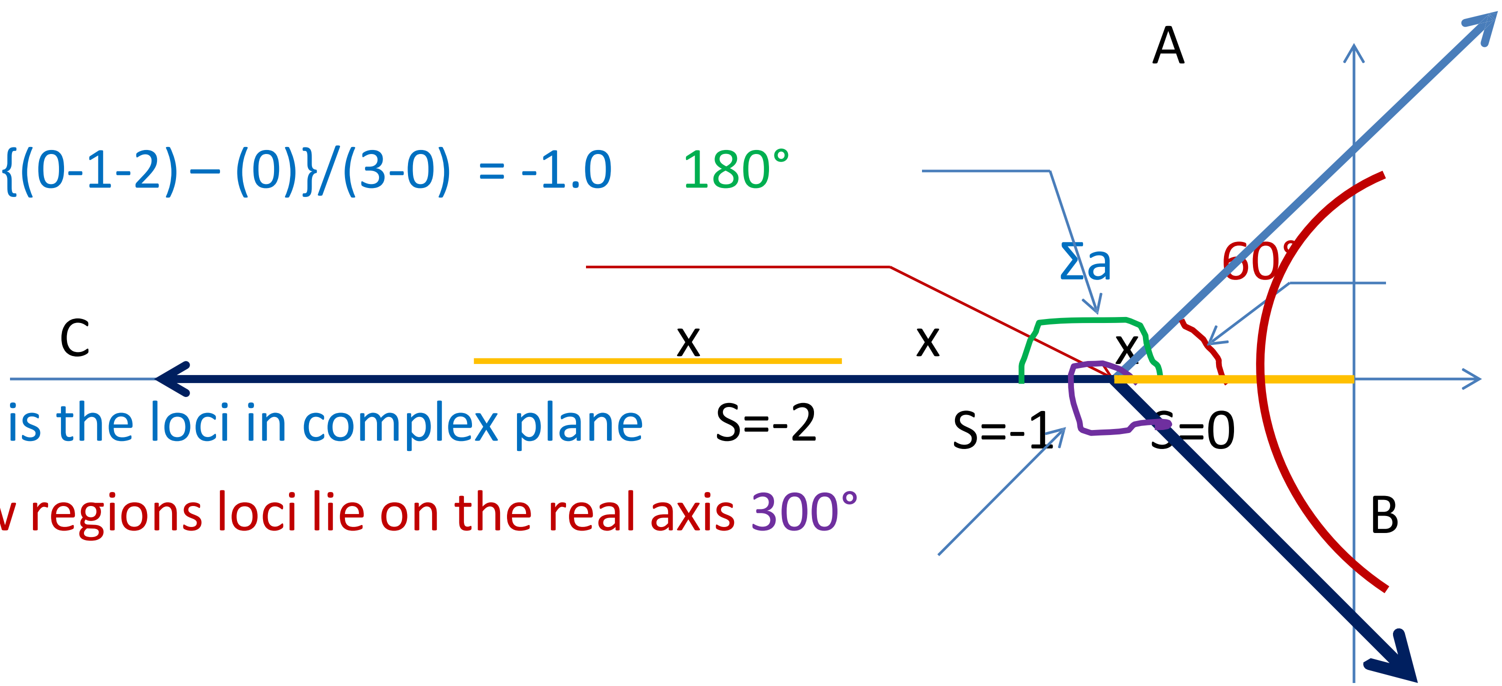
- Np = 3 Nz = 0 (no finite zero ; therefore all zeros at ∞)
- Np-Nz = 3
- Loci on the real axis will lie between S= 0 & S= -1; it will also lie in the region after S = -2 because total no. of poles & zeros to the right of the regions = odd.
- No. of asymptotes = Np-Nz = 3 & angles of asymptotes are given by $\phi \mathbf{q} = (2q+1)\,180°/(Np - Nz)$; $q = 0,1,2$; $\phi \mathbf{0} = 60°$, $\phi \mathbf{1} = 180°$, $\phi \mathbf{2} = 300°$
- Since (Np-Nz)>1 = 3 we will determine Centroid

Construction rules for Root Locus An Example

- **Centroid is given by:**

$$\sigma A = \frac{\text{(sum of real parts of poles – sum of real parts of zeros)}}{\text{(no. of finite poles – no. of finite zeros)}}$$

σA = {(0-1-2) – (0)}/(3-0) = -1.0



red loci is the loci in complex plane

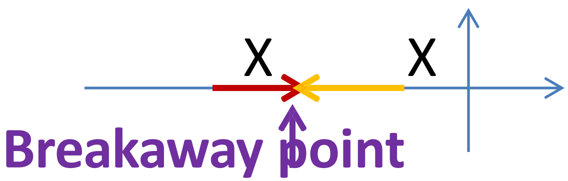
in yellow regions loci lie on the real axis

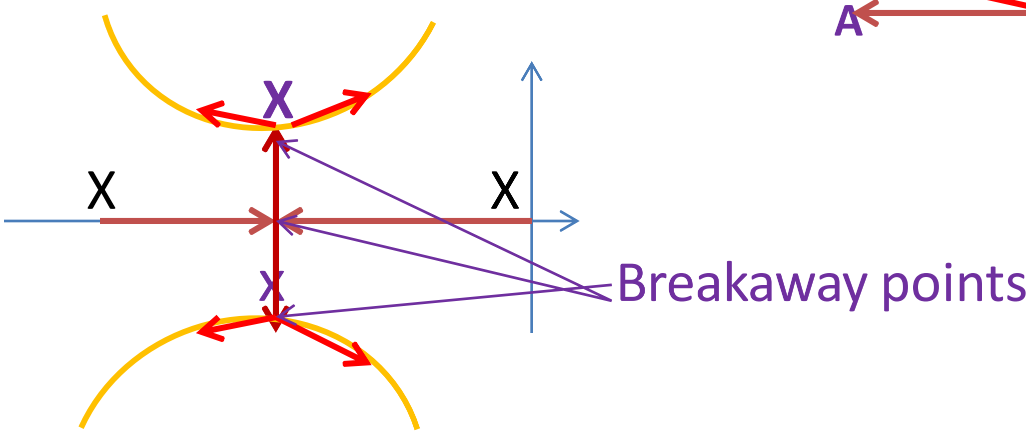
Construction Rules for Root Locus (Breakaway points)

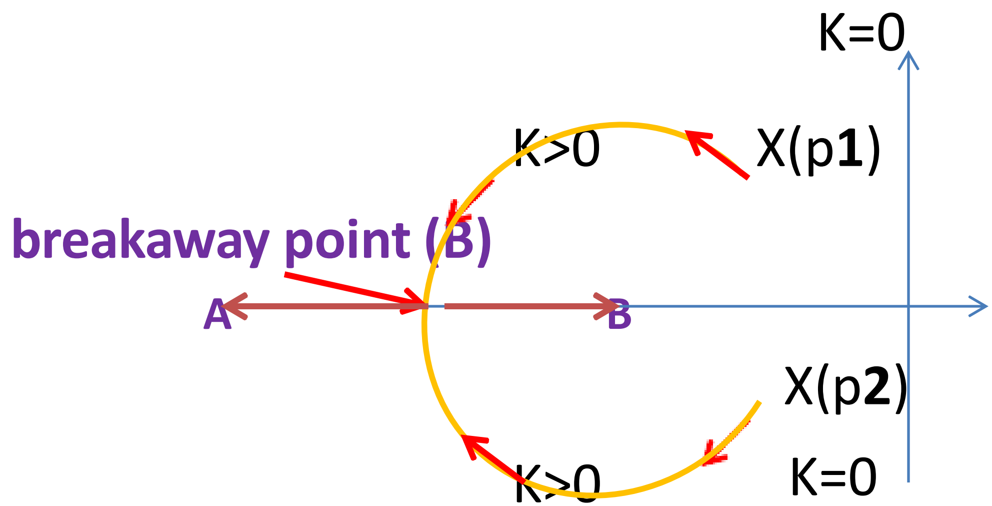
❑ Breakaway Points:

Multiple roots of the characteristic equation occur at these points. These are obtained using the formula dK / dS = 0. These points also satisfy the angle criteria.

Examples:







Construction Rules for Root Locus (Breakaway points Example)

- Example: Calculation for Breakaway points

$G(S) = K/S(S+1)(S+2)$

$1 + G(S)H(S) = 0 \Rightarrow K/S(S+1)(S+2) = -1$

$\Rightarrow K = -(S^3 + 3\,S^2 + 2S)$

$\Rightarrow dK/dS = -(3\,S^2 + 6S + 2) = 0$

We find the roots of the polynomial

$$3\,S^2 + 6S + 2 = 0$$

We get $S1 = -0.423$ & $S2 = -1.577$

We know that for the given G(S), the loci on the real axis will lie between '0' & '-1'; therefore the breakaway point is = -0.423. S2 = -1.577 is not a breakaway point because between S=-1 & -2 no loci exists on the real axis of the S-plane.

Construction Rules for Root Locus (Breakaway points Example)

- **Example:**

  $G(S)H(S) = K/S(S+4)(S^2 + 4S + 20) = K/S(S+4)(S+2+j4)(S+2-j4)$

  To determine the breakaway points: $dK/dS = 0$. Substitute in $1+G(S)H(S) = 0$ to get $K = -S(S+4)(S^2 + 4S + 20)$

  $$dK/dS = S^3 + 6\,S^2 + 18S + 20 = 0$$

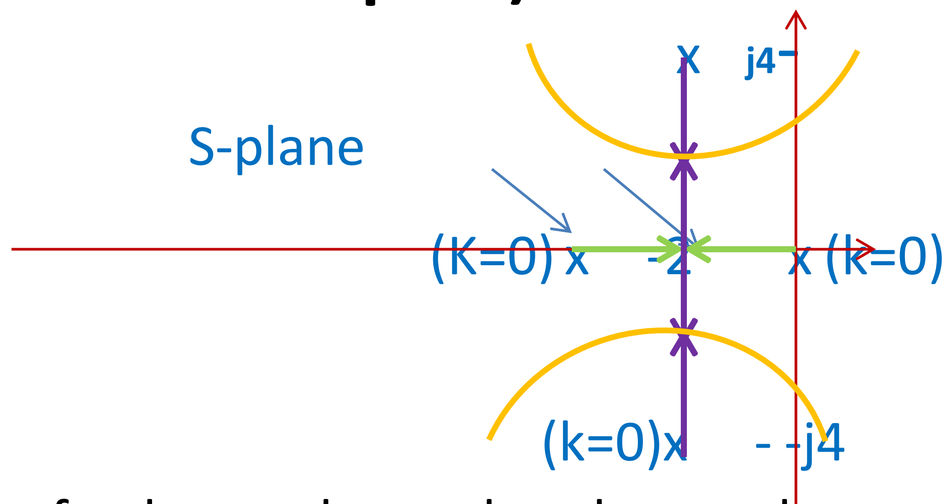
  Factorize $dK/dS=0$, we get $S = -2$; $S = -2 +/-j\,2.45$

- Now we find out that out of the roots of $dK/dS = 0$ which qualify to be breakaway points. To do this, we first draw the pole – zero locations of $G(S)H(S)$ in the S-plane

(next slide)

Construction Rules for Root Locus (Breakaway points Example)

- **Example (contd):**

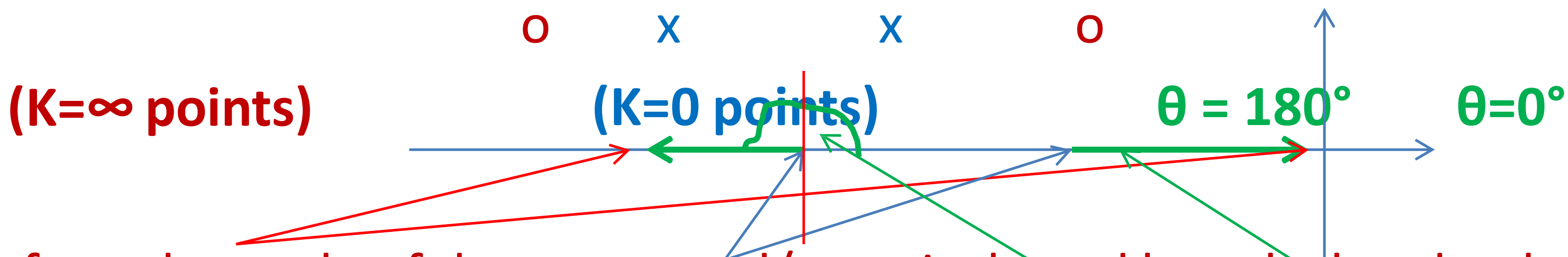


- Having plotted the location of poles, we know that the root locus on the real axis will lie between S = 0 (K=0) & S=-4(K=0).
- Now, one root of dK/dS = 0 lies at S = -2; therefore S=-2 is a breakaway point. Since, -2 is also real part of the complex pole (-2 +/- j4), therefore S= -2 +/- j2.45 (root of dK/dS =0) is also a breakaway point.

Construction Rules for Root Locus (Angle of Departure/ Arrival)

❑ Angle of Departure/Arrival:

For poles on the real axis: (either 0° or 180°)



Therefore, the angle of departure and/or arrival need be calculated only for complex poles & zeros.

Method:

1. choose a point S**0** very close to the pole ‘p’
2. Graphically determine the angle contributions due to other poles & zeros at the point S**0**.
3. determine angle of departure θ**p** from the pole ‘p’.

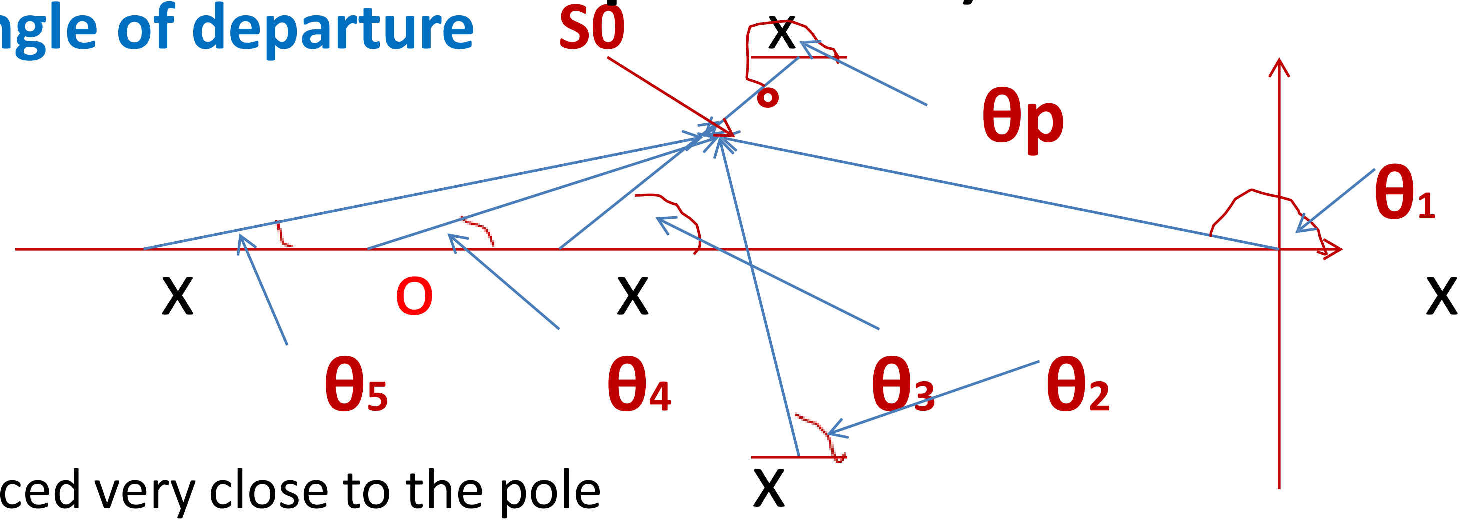
Construction Rules for Root Locus (Angle of Departure/ Arrival)

- Draw the pole-zero locations of G(S)H(S)
- Draw a point S**0** in the S-plane very close to the pole/zero for which departure angle is to be determined.
- Draw vectors to S**0** from each pole & zero of G(S)H(S).
- Calculate total angle, ϕ, subtended at S**0**.
- Angle of departure/**arrival** is given by $\phi - \theta p$/ $\phi + \theta z = (2q + 1)\ 180°$, or we have $\theta p = +/- (2q + 1)\ 180° + \phi$;

  $\theta z = +/- (2q+1)180° - \phi$
- $\theta p/\theta z$:the angle of departure/arrival for the pole/zero; θp is subtracted from ϕ because it is angle subtended by a pole.

Construction Rules for Root Locus (Determining Angle of Departure)

- **θp: angle of departure**



S0 is placed very close to the pole for which angle of departure is to be calculated. For the sake of clarity, here, it is shown some distance from the pole.

- Angle subtended by other poles & zeros at S0, ϕ, is given by:

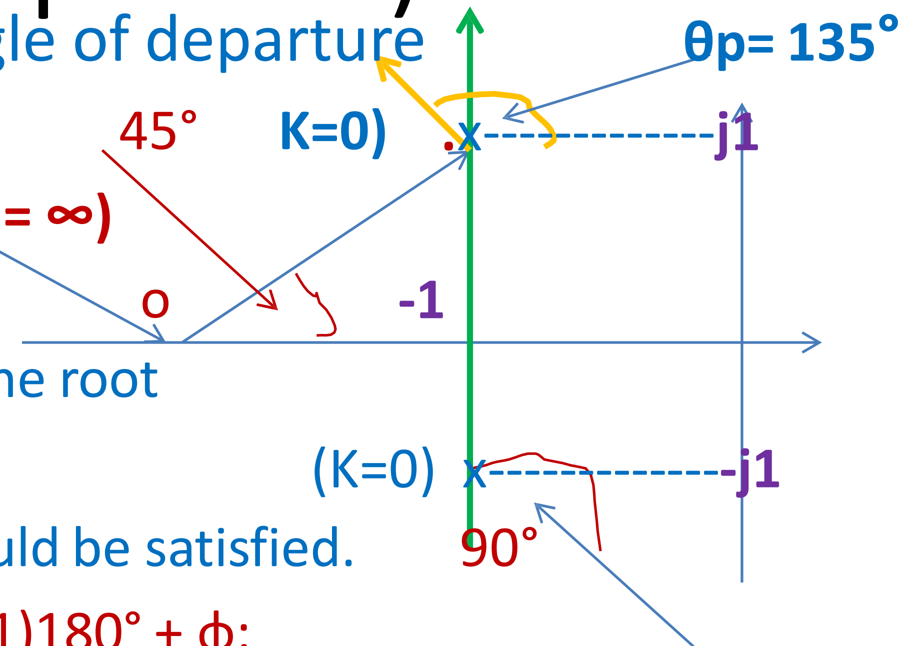
$$\phi = \theta 4 - (\theta 1 + \theta 2 + \theta 3 + \theta 5)$$

$\phi - \theta p = +/- (2q+1)180°$; $q = 0, 1, 2, \ldots$; $\theta p = +/-(2q+1)180° + \phi$

- Angle of arrival at a zero is calculated in a similar way.

Construction Rules for Root Locus (Example: Angle of Departure)

- Example: Calculation of angle of departure



Poles are at -1 +/- j1

Zero at S= -2

The K = 0 points are also points on the root locus; therefore at open loop pole location too, the angle criterion should be satisfied.

The angle $\phi = (45° – 90°)$; $\theta p = (2q+1)180° + \phi$;

$\theta p = 180° + (45° - 90°) = 135°$ is the angle of departure

Construction Rules for Root Locus (Example: Angle of Arrival)

❑ **Example: Angle of Arrival (at zero located at -1+j1)**

tan θ**1** = ½ = 0.5

θ**1**= 26.56°

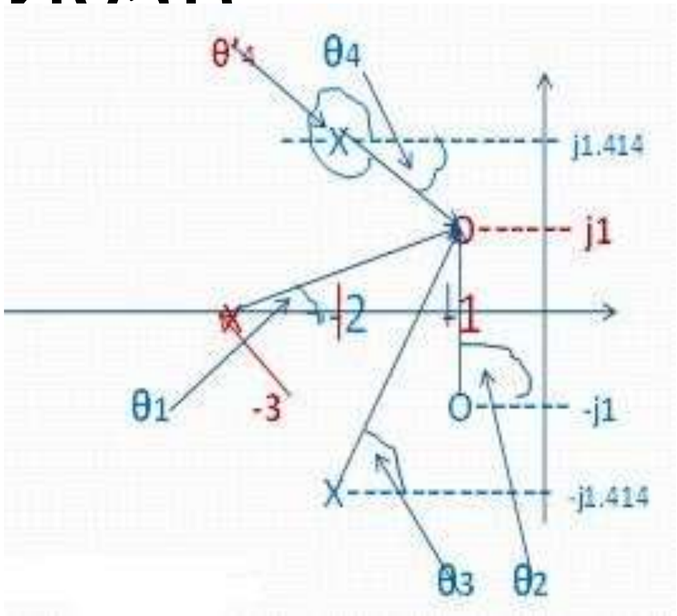
tan θ**3** = 2.414/1

θ**3** = 67.49°

θ**2** = 90°

tan θ**4** = -0.414/1

θ**4** = -22.49° θ’4 = 360-22.49= 337.5°



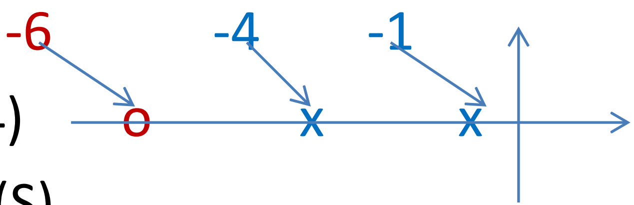
The total angle,ϕ, subtended at the zero= θ**2** – θ**3** – θ**1** + θ**4** = 18.44°.

Therefore angle of arrival θz = 180° - ϕ = 161.6°

Graphical determination of 'K' for specified damping ratio

❑ Example:

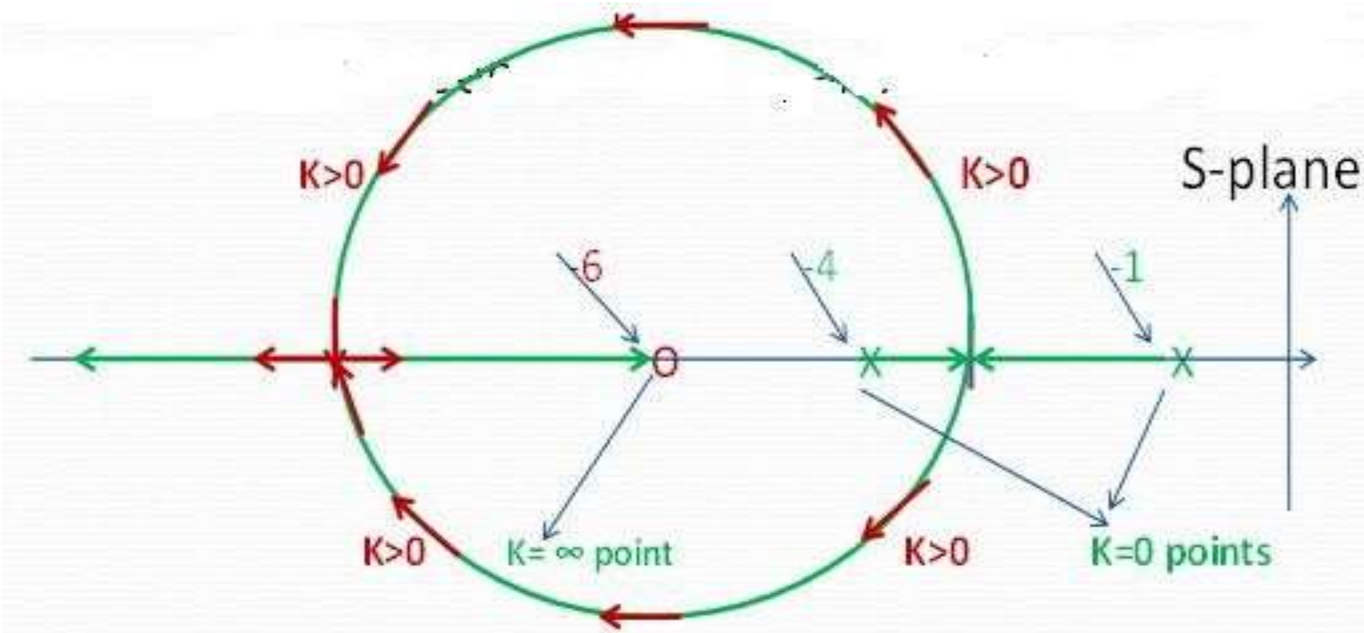
$G(S) = K\ (S+6)/(S+1)(S+4)$



1. K=0 points: S = -1 & S= -4 are poles of G(S)
2. $K = \infty$ points: S = -6 are zeros of G(S)
3. Loci on the real axis lies between S = -1 & S= -4; and between S = -6 & ∞
4. Since one zero is at ∞, therefore one closed loop pole will approach this zero asymptotically
5. Angle of asymptote: $\phi = 180°(2q+1)/Np\text{-}Nz = 180°$; $q = 0$
6. Since there is only one asymptote, there is no centroid

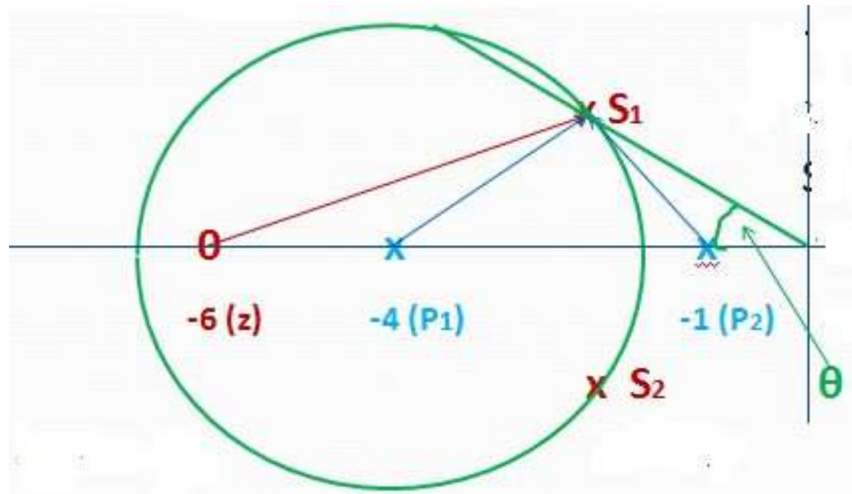
Graphical determination of K for specified damping ratio (contd)..

- Breakaway points: $1 + G(S) = 0$; $1 + K (S+6)/(S+1)(S+4) = 0$; therefore, $K = -(S+1)(S+4)/(S+6)$
- $dK/dS = 0$; ⇒ $S^2 + 12 S + 26 = 0$ ⇒ $S_1 = -9.16$, $S_2 = -2.84$
- Both S1 & S2 are breakaway points because the root loci on the real axis lies between S = -1 & -4; and between S = -6 & ∞



Graphical determination of K for specified damping ratio (contd)..

- Let us fix the location of closed poles at S1 & S2. Now we want to find K which yields S1 & S2. Let



  S1 = -2 + j 1.5

  ξ = Cos(θ)

- Draw vectors from each pole & zero of G(S) to S**1** or S**2** as shown.
- Then **K = product of the length of vectors from poles/ product of length of vectors from zeros**

  K = |S1 + 4||S1 + 1|/|S1 + 6| **=** |-2+j1.5+4||-2+j1.5+1|/|-2+j1.5+6| = 1.05

  ξ = Cos(45°) = 0.707

Effect of adding Zeros on Stability of a Closed loop system

- $G(S) = K/(S+1)(S+2)(S+3)$
- The root loci is obtained as: As the root loci cross in to RH of S-plane,
- The closed loop system becomes unstable for a value of $K>K_1$. Let us now add a zero.
- Let us now add a zero at S= -4 the loci will be
- We observe that addition of a zero has stabilized the closed loop system for all values of K; $0 \leq K \leq \infty$

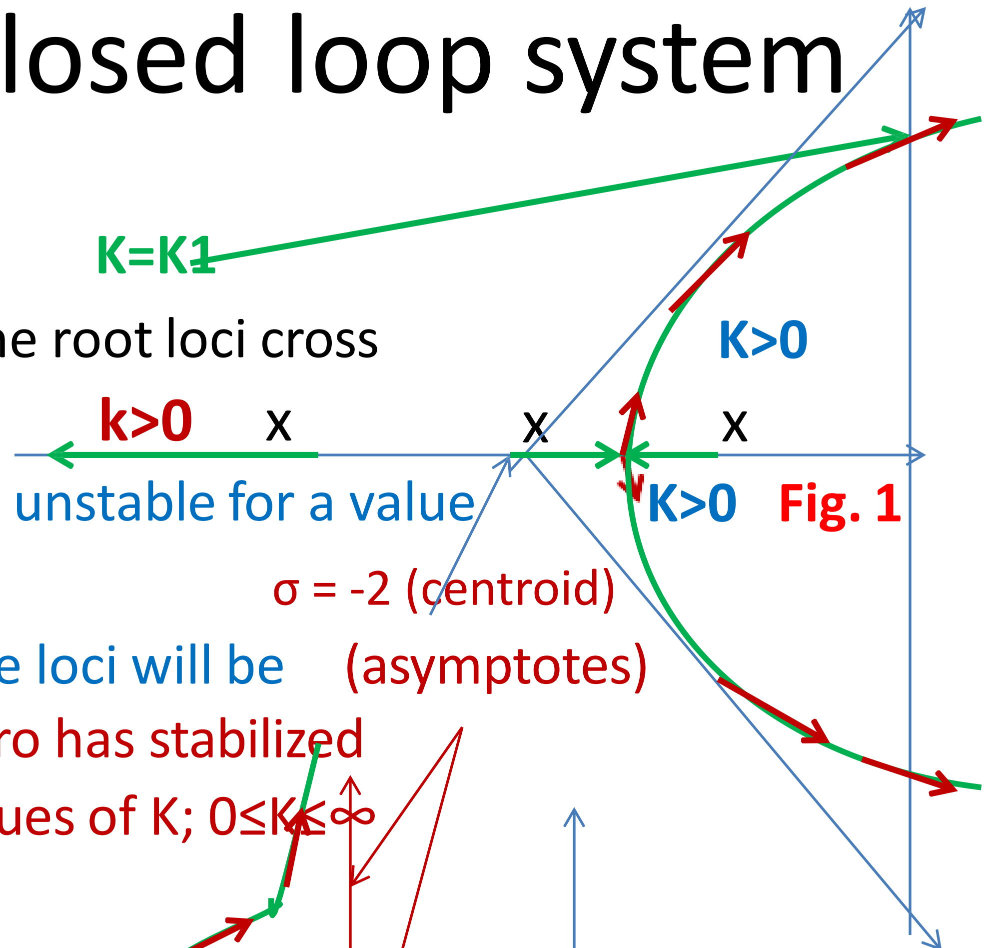
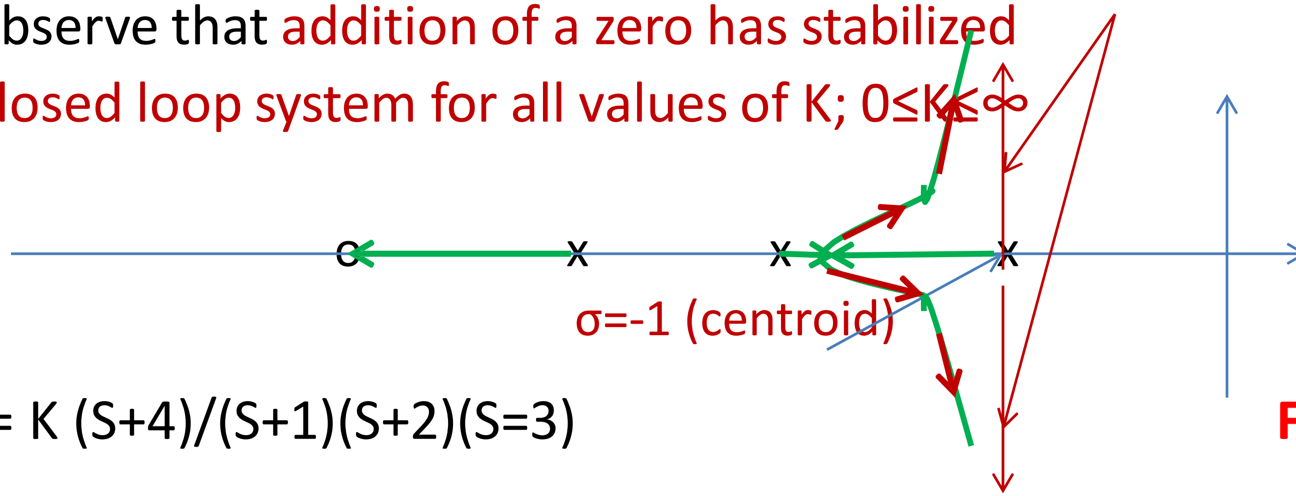


Fig. 1



$G(S) = K\,(S+4)/(S+1)(S+2)(S=3)$

Fig. 2

Effect of adding Zeros on Stability of a Closed loop system (contd)..

- Let us now add a zero at S = -2.5 G(S) = K (S+2.5)/(S+1)(S+2)(S+3) σ = -1.75

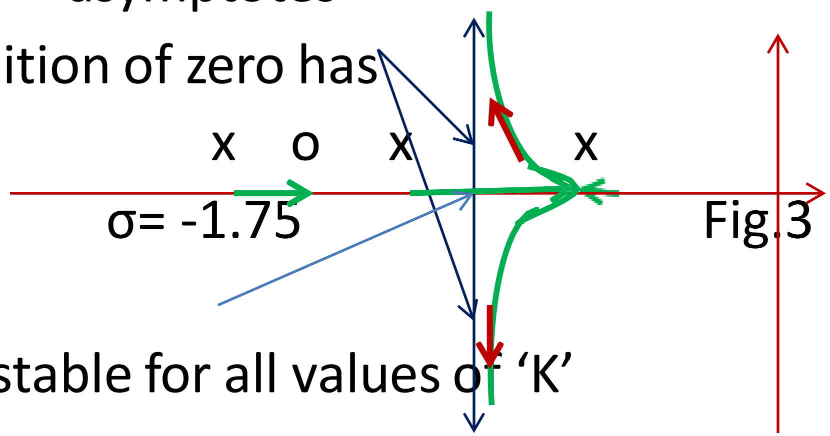


Fig.3

Looking at Figs. 1, 2 & 3 we see that addition of zero has

1. Reduced no. of asymptotes thereby preventing the locus from moving in to RH of the S-plane.
2. Therefore the CL system has become stable for all values of ‘K’
3. The location of zero also affects the locus.
4. Shifting zero location from S= -4 to -2.5 has moved centroid from -1 to -1.75 thereby shifting the starting point of asymptotes to further away from the Imaginary axis of the S-plane. In Fig.2 the breakaway point is to the left of σ; in Fig.3 it is to the right of σ.
5. Thus the system has become relatively more stable

Effect of adding Poles on Stability of a Closed loop system

- Adding a pole:

$G(S) = K/(S=1)(S+2)$

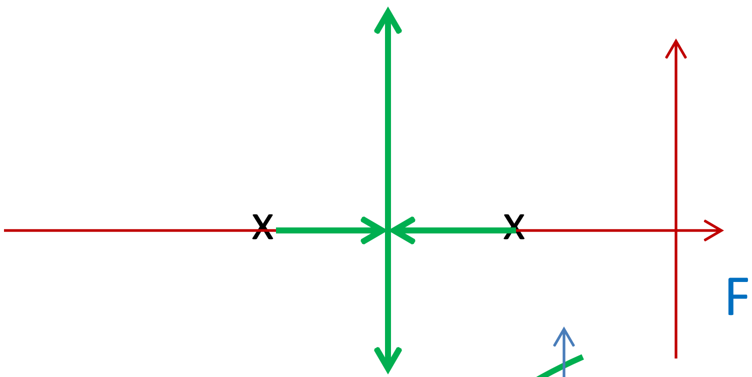


Fig. 1

$G(S) = K/(S+1)(S+2)(S+3)$

- We observe that addition of a pole affects stability of a CL system, as is seen from Fig.1 & 2

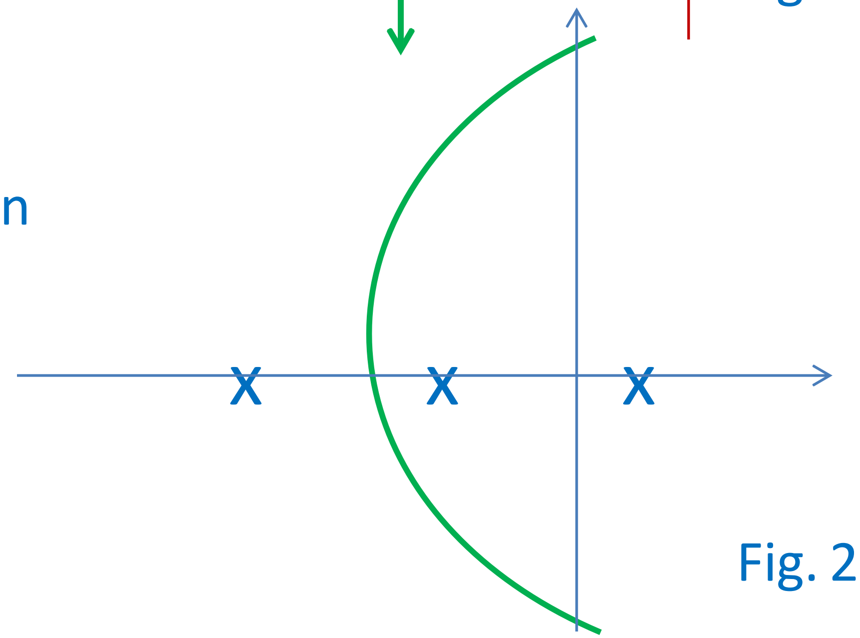


Fig. 2

Root Locus Problems

❑ Problem1:

For $G(S) = K(S + b)/S(S + a)$ & $H(S) = 1$ show that the loci of the complex roots are part of a circle with

center at $(-b,0)$,and

radius $= \sqrt{(b^2 – ab)}$

❑ Solution:

The angle criterion: $\arg\{(S + b)/S(S + a)\} = +/- 180°$

At, $S = \sigma + j\,\omega$ we have : $\arg\{(\sigma + j\,\omega + b)/(\sigma + j\,\omega)(\sigma + j\,\omega + a)\}$

or, $\tan^{-1}(\,\omega/\sigma + b) - \tan^{-1}(\,\omega/\sigma) - \tan^{-1}(\,\omega/\sigma + a) = -\pi$

$\tan^{-1}(\,\omega/\sigma) + \tan^{-1}(\,\omega/\sigma + a) = \pi + \tan^{-1}(\,\omega/\sigma + b)$

Take tan on both sides & simplify, to get

$(\sigma + b)(2\sigma + a) = \sigma\,(\sigma + a) - \omega^2$

$\sigma^{2} + \omega^2 + 2b\sigma + ab = 0$

Root Locus Problems (contd)..

- Add & subtract b^2 term to get

$$(\sigma^2 + 2b\sigma + b^2) - b^2 + \omega^2 + ab = 0$$

$$(\sigma + b)^2 + \omega^2 = b^2 - ab$$ is the equation of the circle with center at (-b,0) & radius = $\sqrt{(b^2 - ab)}$

For b = 1 & a = -1

center = (-1,0) & radius = $\sqrt{2}$

- Problem 2:

$H(S) = 1 \quad G(S) = 1/S(S + \alpha)$

Draw root locus as α varies between $0 \le \alpha \le \infty$

Solution:

'α' appears in the denominator polynomial of G(S). 'K' always appeared in the numerator of G(S). Therefore we manipulate to get 'α' in the numerator.

The Characteristic equation $Q(S) = 1 + G(S)H(S) = 0$

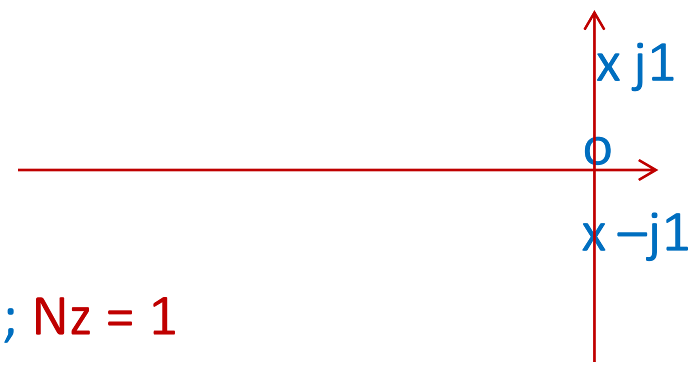
Root Locus Problems (contd)..

- $Q(S) = S^2 + \alpha S + 1 = 0$

From Q(S), we rewrite G(S) in a way that 'α' appears in the numerator

Therefore, we write

$$G(S) = \alpha S / S^2 + 1$$



The root locus for parameter 'α':

1. $\alpha = 0$ points: $S\mathbf{1} = +j1$ & $S\mathbf{2} = -j1$; $N_p = 2$
2. $\alpha = \infty$ points: $S = 0$; (another zero at ∞); $N_z = 1$
3. $N_p - N_z = 1$; No. of loci = 2
4. Locus on the real axis covers entire axis in the LH of S-plane
5. No. of asymptotes = 1
6. No Centroid (because only one asymptote)
7. Angle of asymptote (for $q = 0$) = $180°$

Root Locus Problems (contd)..

- Breakaway point:

  α S/ S² + 1 = -1; α = -(S² + 1)/S; dα/dS = 0 $\Longrightarrow$ S² - 1 =0; S = +/- 1

  The breakaway point is S = -1 because it is a point on the loci

- Angle of Departure: (from pole at S = j1)

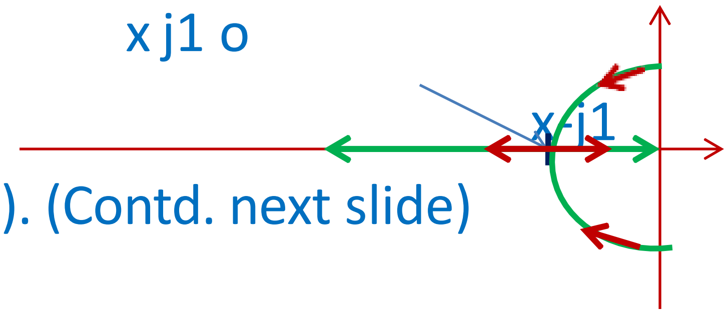
  Angle subtended at S= j1 by zero at S=0 is 90°

  Angle subtended at S = j1 by pole at S= -j1 = 90°

  Total angle subtended, ϕ = 90 – 90 = 0°

  Angle of departure θp = 180° + ϕ = 180°

- The Root Loci:

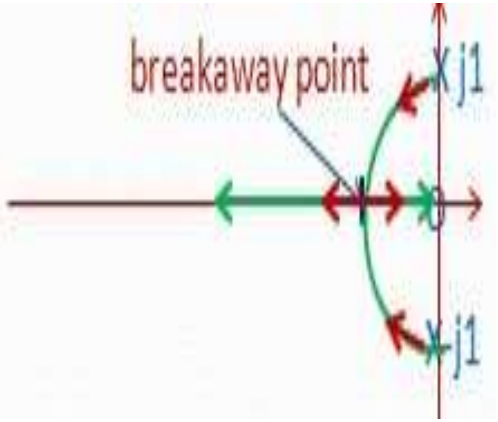


- It is a circle with radius = 1 & center (0,0). (Contd. next slide)

Root Locus Problems (contd)..

- Let us fix the location of closed loop poles for damping ratio $\xi = 0.5$ & determine time domain parameters. We redraw the locus.

  $\xi = \cos(\theta) = 0.5$; $\theta = 60°$. Draw a line at 60° from –ive real axis as shown.

  The intersections A & B on the locus define the location of the closed loop system.

  Since the locus is a circle with unity radius, the vector OA = 1 & therefore $\omega\mathbf{n} = 1$ rads/sec.

  $-\xi\omega n = -0.5$; $\omega\mathbf{d} = \omega\mathbf{n}\sqrt{(1-\xi^2)} = 0.866$ rads/sec

- The CL poles are $-\xi\omega n \pm j\,\omega\mathbf{d} = -0.5 \pm j\,0.866$
- The Characteristic equation is $(S+0.5+j\,0.866)(S+0.5-j\,0.866)= S^2 + S +1=0$

  The derived Ch. Eq. is : $S^2 + \alpha S +1 =0$

  On comparing we get $\alpha = 1$.

Root Locus Problems (contd)..

❑ Problem 3:

Suppose that the Characteristic equation is given as:

$Q(S) = S^3 + K S^2 + 2S + 1 = 0$

You are asked to draw root locus for $0 \le K \le \infty$. How to draw?

Solution:

1. Collect all the terms containing 'K'.
2. Divide terms containing 'K' by the balance terms
3. Write $Q(S) = 1 + N'(S)/D'(S)=0$
4. Write $G(S) = N'(s)/D'(S)$
5. Plot root locus
6. In the present case: $Q(S) = 1 + K S^2/ S^3 + 2S +1 = 0$
7. $G(S) = K S^2/S^3 + 2S + 1$; Factorize denominator polynomial

PROBLEM: Construction of Root Locus

Draw the root locus for the open loop transfer function G(s) and settling time ts=4sec given, find the range of values of k and show that the loci of the complex roots are part of a circle with (-1,0) as centre and radius $=\sqrt{2}$

where
$$G(s) = \frac{k(s+1)}{s(s-1)}$$

Step-1: The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane.

- ❑ The k=0 points:

  s=0, s= 1

  no. of poles (n)= 2

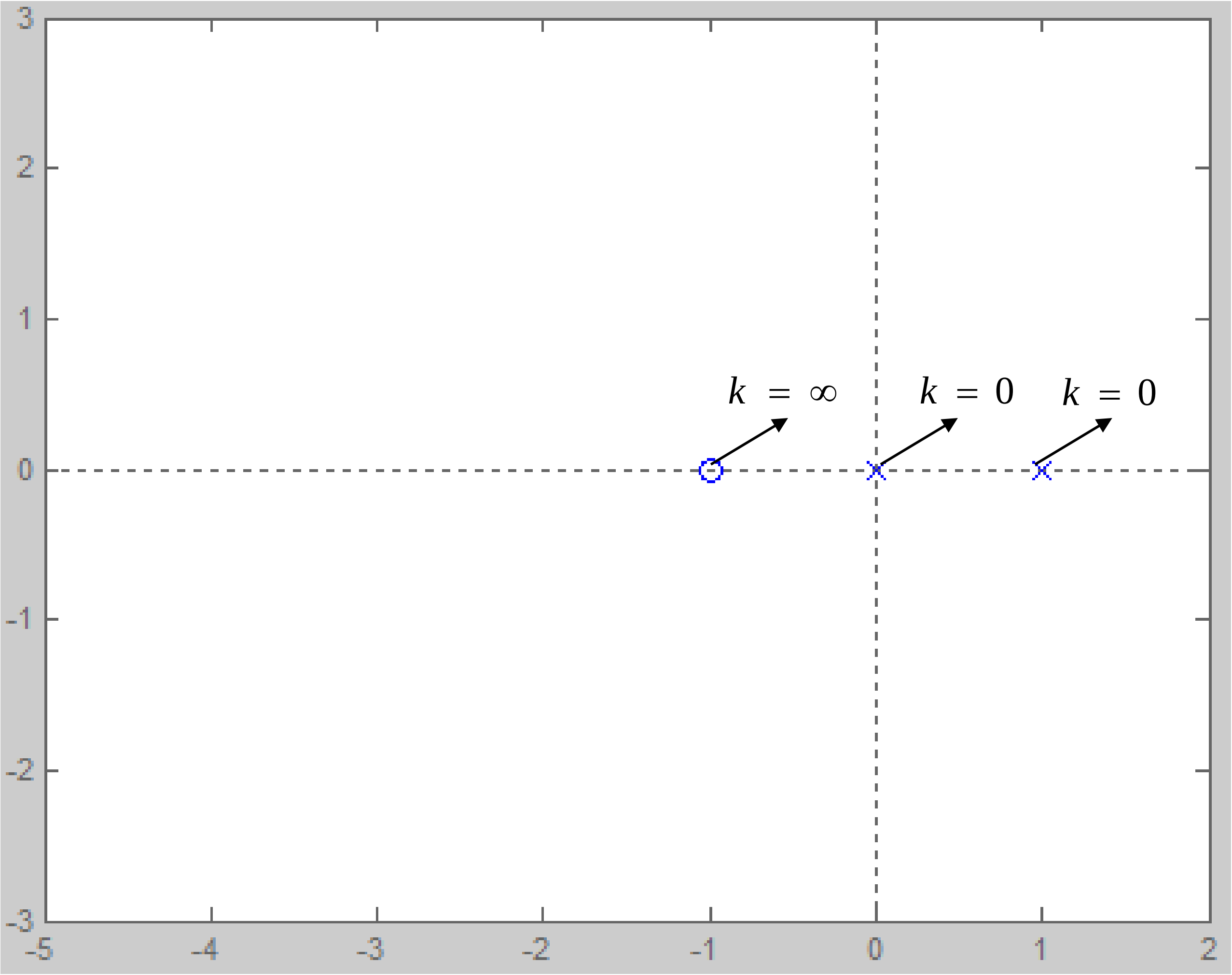
- ❑ The k=∞ points:

  s= -1

  no. of zeros (m)= 1

PROBLEM: Construction of Root Locus (contd)..

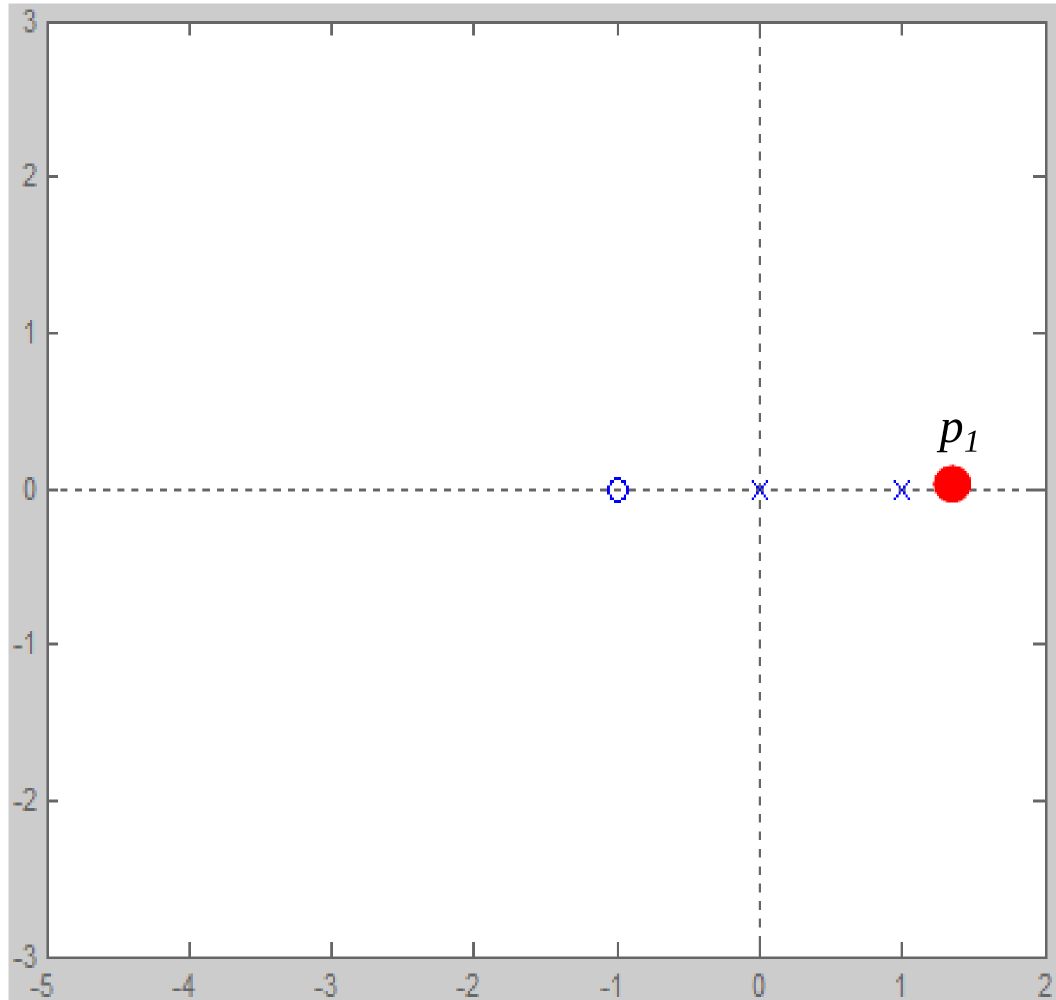
The poles and zeros in s-plane after step-1.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

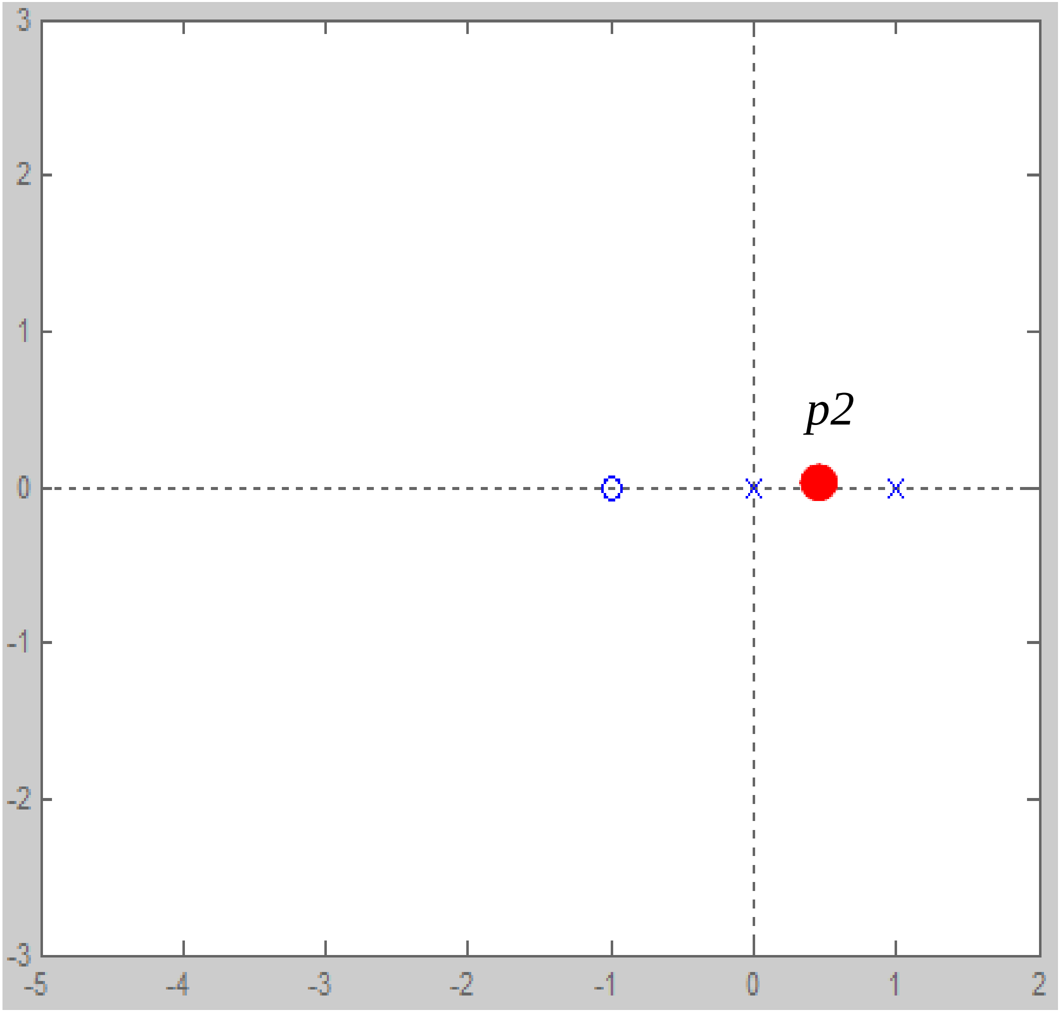
- To determine the root loci on real axis we select some test points.
- e.g: p_1 (on positive real axis).
- No. of real poles and zeros on the right of test point is zero (which is even)
- Hence, there is no root locus on the positive real axis.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

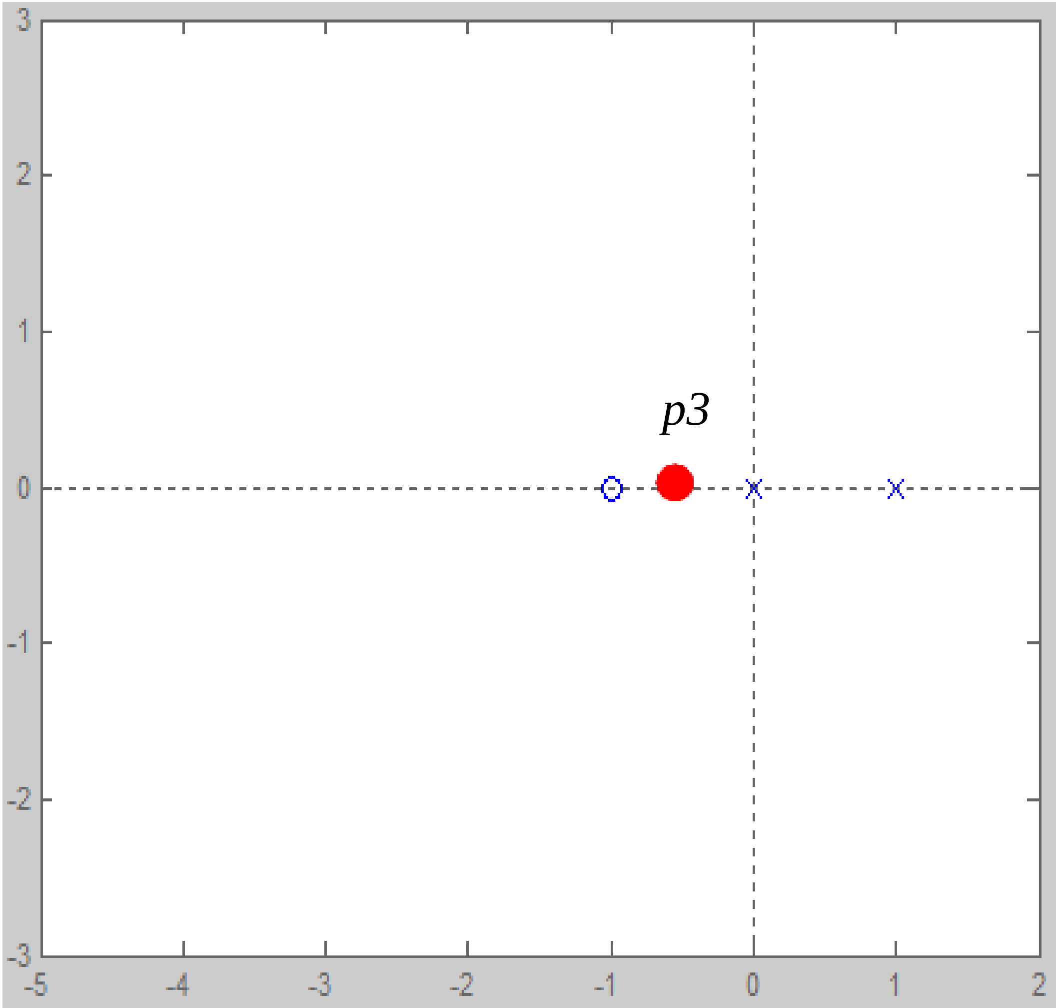
- Next, select a test point on the positive real axis between 1 and 0.
- No. of real poles and zeros on the right of test point is one (which is odd)
- Therefore, from 1 to 0 is part of the root locus.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

- Next, select a test point on the negative real axis between 0 and -1.
- No. of real poles and zeros on the right of test point is two (which is even)
- Therefore, from 0 to -1 is not part of the root locus.



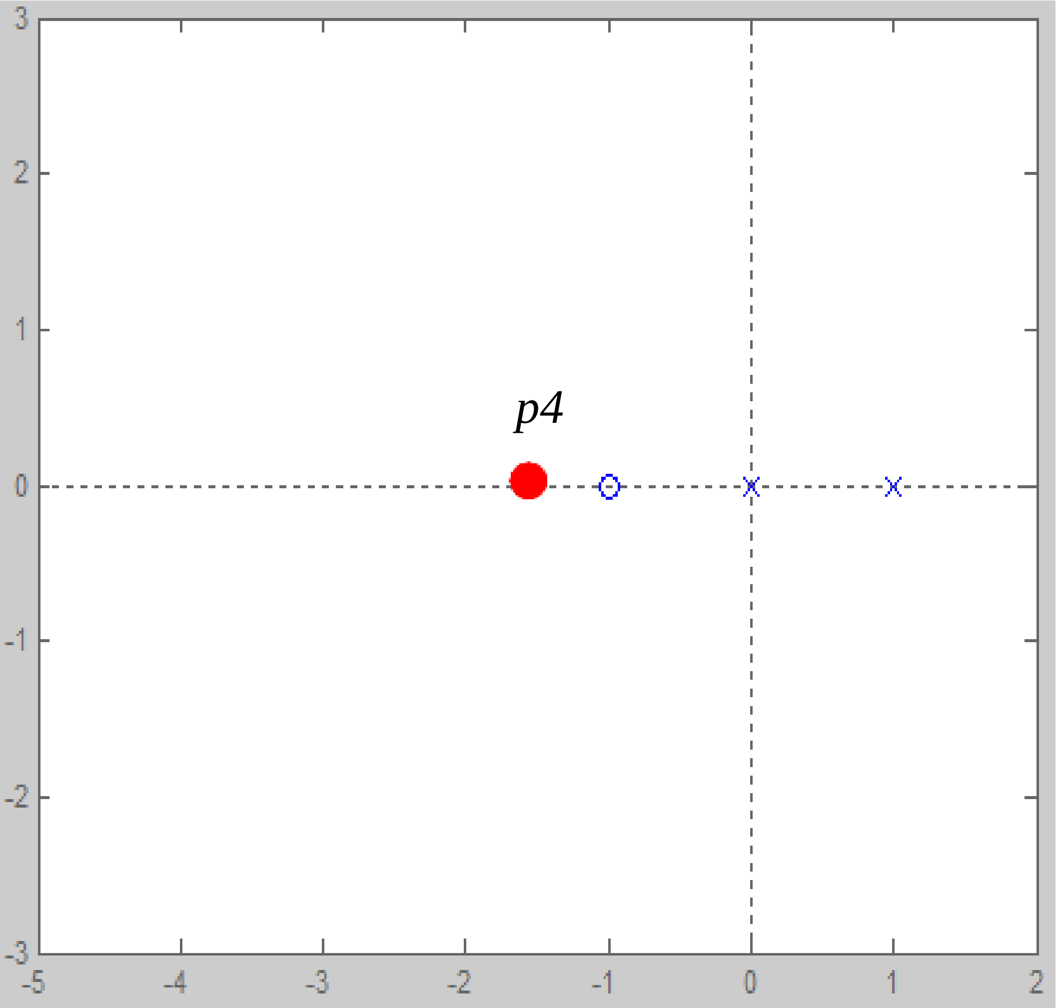
PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.

- Next, select a test point on the negative real axis between -1 and - ∞.

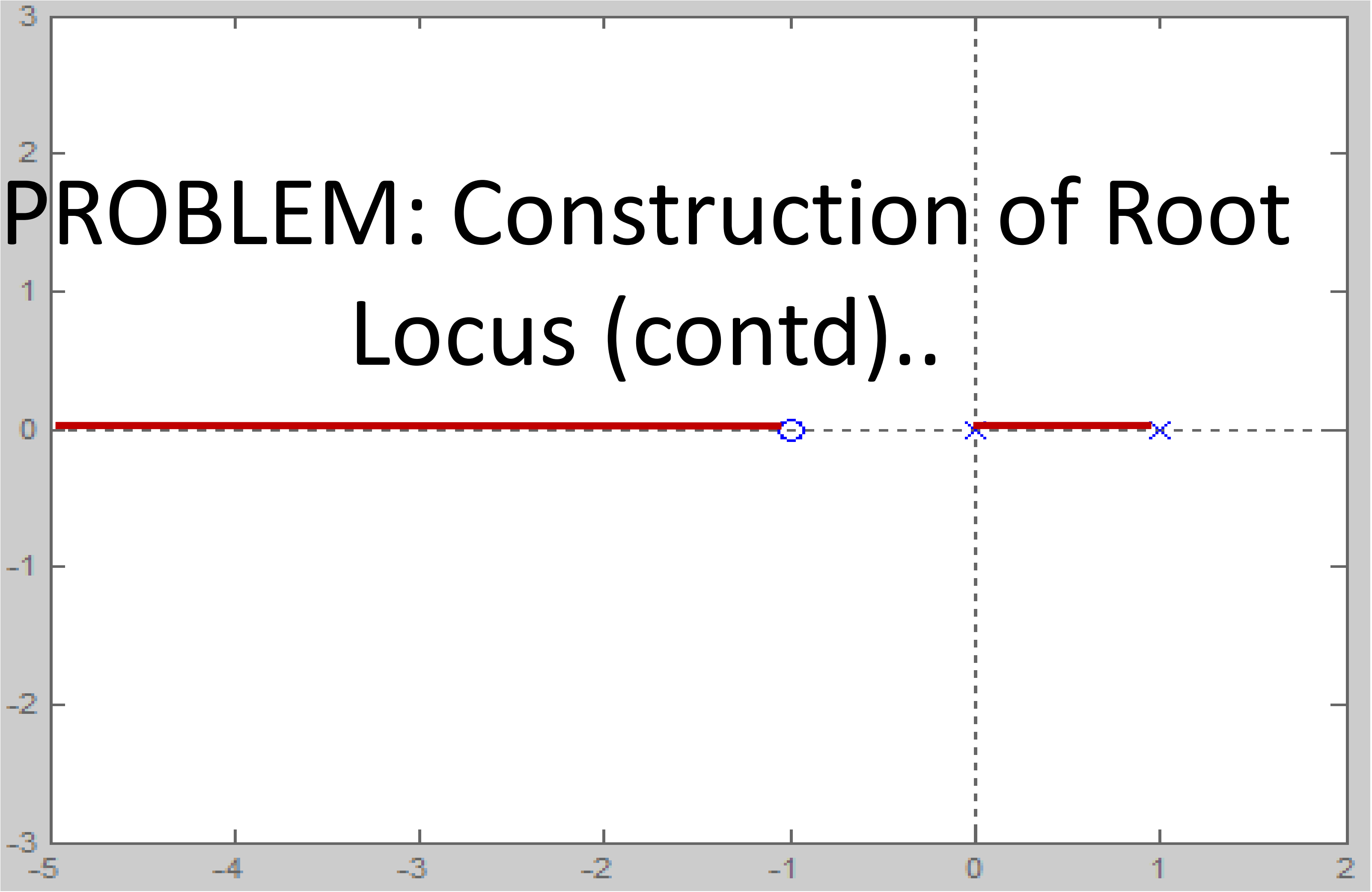
- No. of real poles and zeros on the right of test point is three (which is odd)

- Therefore, from -1 to - ∞ is part of the root locus.



PROBLEM: Construction of Root Locus (contd)..

Step-2: Determine the root loci on the real axis.



PROBLEM: Construction of Root Locus (contd)..

Step-3: Determine the *asymptotes* of the root loci and angles.

$$Angle \quad of \quad asymptotes = \phi = \frac{\pm 180°(2q+1)}{n-m}$$

Where

n-----> number of poles (2)

m-----> number of zeros (1)

$$\phi = \frac{\pm 180°(2q+1)}{2-1}$$

$$\phi = \pm 180° \quad \text{when} \quad q = 0$$

- ❑ No. of asymptotes = n-m = 1
- ❑ The angle of asymptote is 180°.
- ❑ No centroid for this system

PROBLEM: Construction of Root Locus (contd)..

Step-4: Determine the *breakaway/break-in point.*

- The breakaway/break-in point is the point from which the root locus branches leaves/arrives real axis.
- The breakaway or break-in points can be determined from the roots of dK/ds=0
- It should be noted that not all the solutions of dK/ds=0 correspond to actual breakaway points.
- If a point at which dK/ds=0 is on a root locus, it is an actual breakaway or break-in point.
- The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s-1)} = 0$$

$$K = -\frac{s(s-1)}{s+1}$$

PROBLEM: Construction of Root Locus (contd)..

- The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}\left[\frac{s(s-1)}{s+1}\right]$$

$$\frac{dK}{ds} = \frac{(s+1)(2s-1)-(s^2-s)(1)}{(s+1)^2}$$

- Set *dK/ds=0* in order to determine breakaway point.

$$\frac{(s+1)(2s-1)-(s^2-s)(1)}{(s+1)^2} = 0 \qquad s^2 + 2s - 1 = 0$$

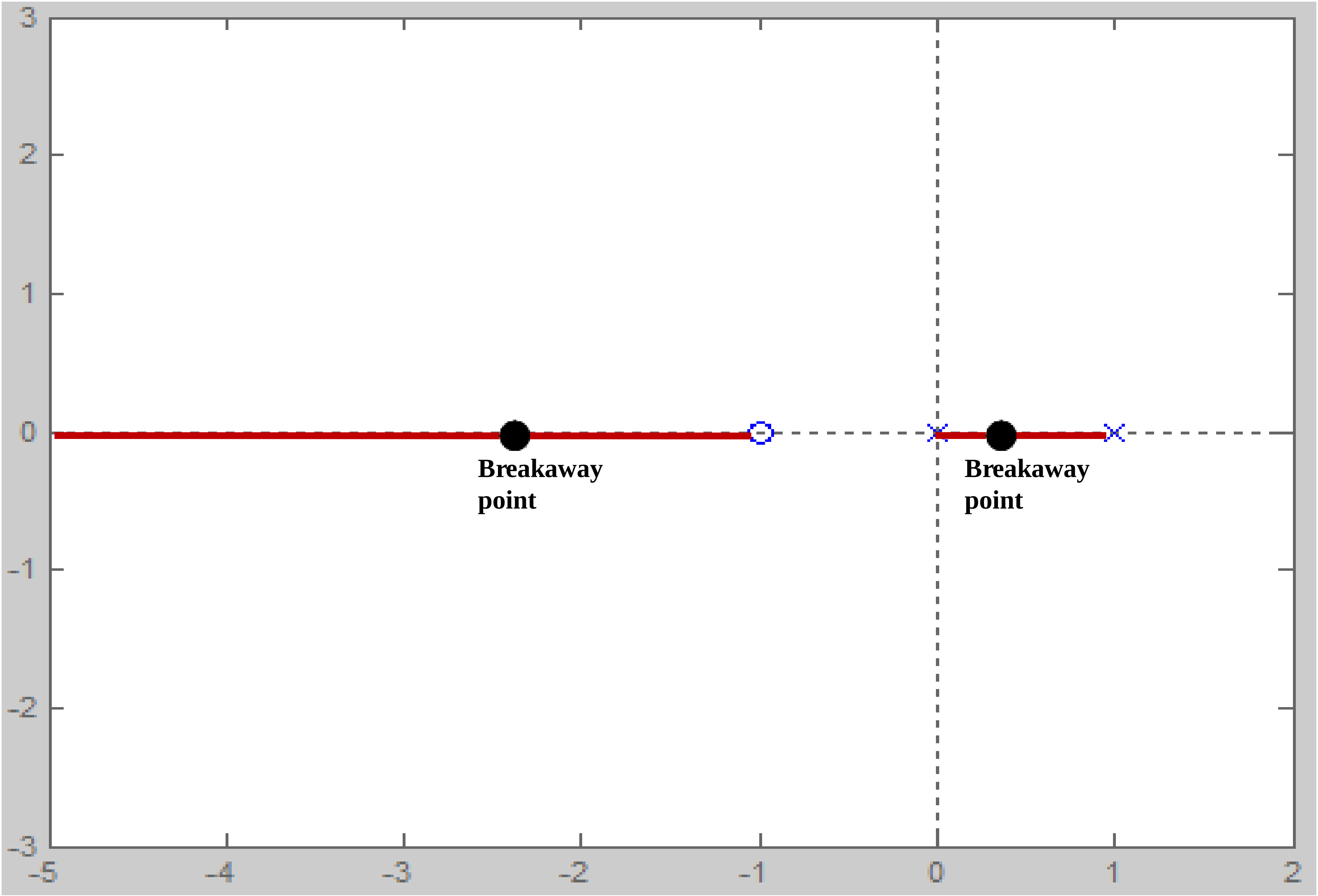
- By solving the equation roots are at

$$s = +0.414$$
$$= -2.414$$

- By substituting these s values in k equation, the value of k is positive real for s=0.414 (k=0.17), s=-2.414 (k=5.828). so these points are actual breakaway points.

PROBLEM: Construction of Root Locus (contd)..

Step-4: Determine the *breakaway/break-in point.*



PROBLEM: Construction of Root Locus (contd)..

Step-5: Determine the points where root loci cross the imaginary axis and range of K for stable operation

- The characteristic equation of closed loop system:

$$s(s-1)+k(s+1)=0$$

$$s^2+(k-1)s+k=0$$

$$\begin{array}{c|cc} s^2 & 1 & k \\ s^1 & k-1 & 0 \\ s^0 & k & \end{array}$$

$$k>0$$

$$k>1$$

$$s^2+1=0$$

$$s=\pm j1$$

- The root loci cuts the imaginary axis at $s=\pm j1$

PROBLEM: Construction of Root Locus (contd)..

Step-5: Determine the points where root loci cross the imaginary axis and range of K for stable operation

- The characteristic equation of closed loop system:

$$s(s-1)+k(s+1)=0$$

$$s^2+(k-1)s+k=0$$

$$s^2+2\xi\omega_n s+\omega_n^2=0$$

$$\omega_n=\sqrt{k}$$

$$\xi\omega_n=(\frac{k-1}{2})$$

$$t_s=4=\frac{4}{\xi\omega_n}$$

$$\xi\omega_n=1=(\frac{k-1}{2})$$

$$k=3$$

- The location of closed loop poles for k=3, ts=4 sec

$$s^2+2s+3=0$$

$$s=-1\pm j\sqrt{2}$$

PROBLEM: Construction of Root Locus (contd)..

- To show that the loci of the complex roots are part of a circle with (-1,0) as centre and radius = $\sqrt{2}$

- Apply the angle criterion:

$$\angle G(s) = \angle k \frac{(s+1)}{s(s-1)} = \pm\pi(2q+1)$$

$$s = \sigma + j\omega$$

$$\angle k + \angle \sigma + j\omega + 1 - \angle \sigma + j\omega - \angle \sigma + j\omega - 1 = -\pi$$

$$\pi + \tan^{-1}\left(\frac{\omega}{\sigma+1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma-1}\right)$$

- Apply the tan on both sides

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\frac{\frac{\omega}{\sigma} + \left(\frac{\omega}{\sigma-1}\right)}{1 - \frac{\omega}{\sigma}\left(\frac{\omega}{\sigma-1}\right)} = \frac{\tan(\pi) + \left(\frac{\omega}{\sigma+1}\right)}{1 - \tan(\pi)\left(\frac{\omega}{\sigma+1}\right)}$$

PROBLEM: Construction of Root Locus (contd)..

- By cross multiply and simplify:

$$\frac{\omega}{\sigma} + \frac{\omega}{\sigma - 1} = \frac{\omega}{\sigma + 1}\left[1 - \frac{\omega^2}{\sigma(\sigma - 1)}\right]$$

$$\sigma^2 + \omega^2 + 2\sigma - 1 = 0$$

- By add and subtract '1' and rearrange

$$(\sigma^2 + 2\sigma + 1) - 1 + \omega^2 - 1 = 0$$

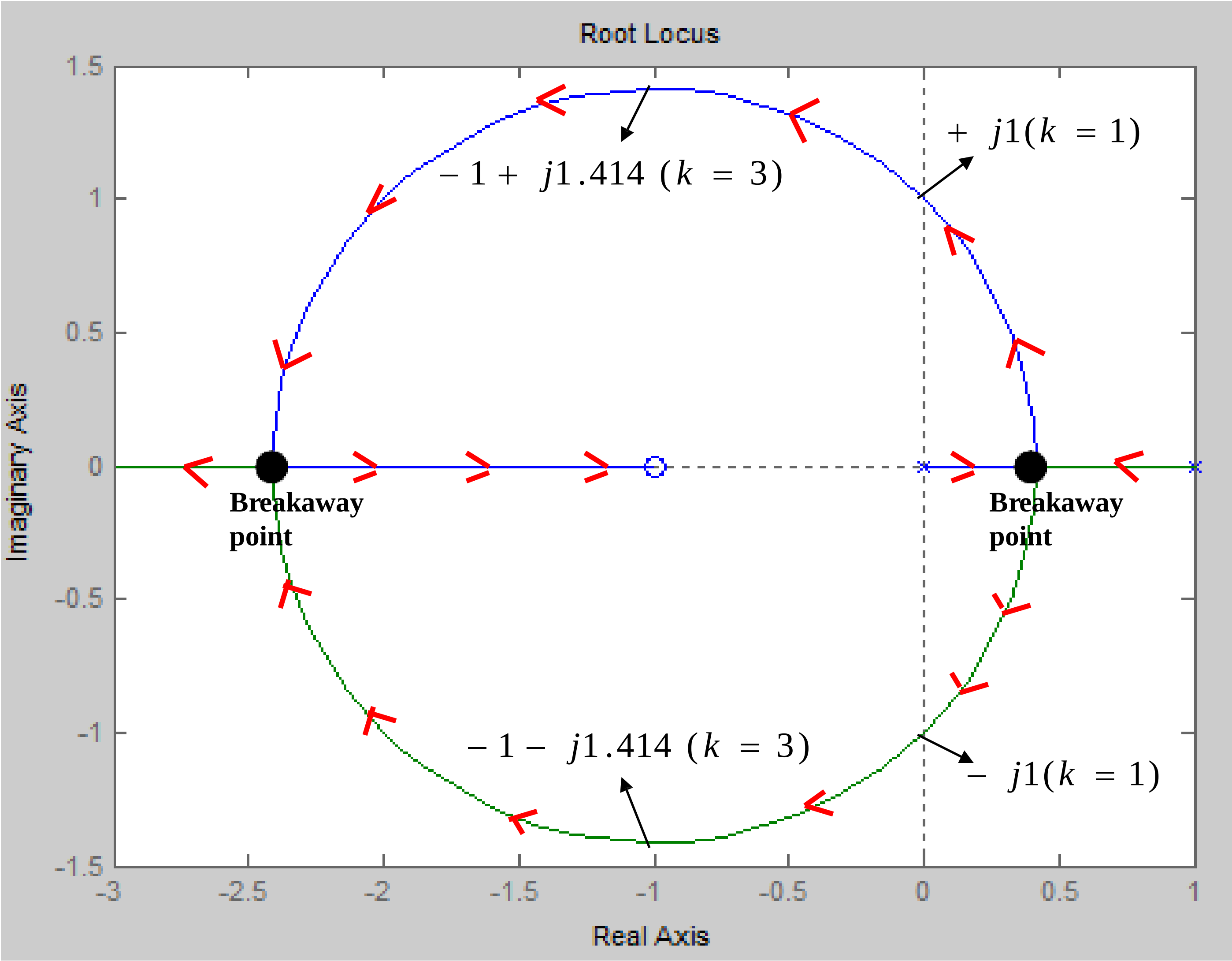
$$(\sigma + 1)^2 + \omega^2 = 2$$

- This is the equation of the circle with center at (-1,0) and radius $\sqrt{2}$

PROBLEM: Construction of Root Locus (contd)..

- Complete root locus for the given system

$$G(s) = \frac{k(s+1)}{s(s-1)}$$



PROBLEM: Construction of Root Locus

The characteristic equation of a feedback control system is

$$s^4 + 3s^3 + 12s^2 + (k - 16)s + k = 0$$

Sketch the root locus plot for 0<k<∞ and show that the system is conditionally stable (stable only for a range of gain k). Determine the range of gain for which the system is stable.

Solution:

To sketch the root locus, we require the open-loop transfer function G(s)H(s)

- $$1 + G(s)H(s) = s^4 + 3s^3 + 12s^2 - 16s + ks + k = 0$$

$$1 + G(s)H(s) = s(s^3 + 3s^2 + 12s - 16) + k(s + 1) = 0$$

$$1 + \frac{k(s+1)}{s(s^3 + 3s^2 + 12s - 16)} = 1 + \frac{k(s+1)}{s(s-1)(s^2 + 4s + 16)} = 0$$

PROBLEM: Construction of Root Locus (contd)..

- $G(s)H(s) = \dfrac{k(s+1)}{s(s^3 + 3s^2 + 12s - 16)} = \dfrac{k(s+1)}{s(s-1)(s+2+j3.42)(s+2-j3.42)}$
- The k=0 points: s=0, s= 1, s=-2+j3.42, s=-2-j3.42

  no. of poles (n)= 4
- The k=∞ points: s=-1

  no. of zeros (m)=1
- No. of root locus branches (n)=4
- Root locus exists on the real axis from s=1 to s=0 and to the left of s=-1
- No asymptotes (n-m)=3
- Angles of asymptotes $\phi = \pm 60^o, \pm 180^o$
- Centroid $\sigma = -0.66$
- The breakaway points are given by dk/ds=0.

  where $k = \dfrac{s(s-1)(s^2 + 4s + 16)}{s+1}$

PROBLEM: Construction of Root Locus (contd)..

- $$\frac{dk}{ds} = (s+1)\frac{d}{ds}(s^4 + 3s^3 + 12s^2 - 16s) - (s^4 + 3s^3 + 12s^2 - 16s)\frac{d}{ds}(s+1) = 0$$

  $$(s+1)(4s^3 + 9s^2 + 24s - 16) - s^4 - 3s^3 - 12s^2 + 16s = 0$$

  $$3s^4 + 10s^3 + 21s^2 + 24s - 16 = 0$$

- By solving the above equation out of four roots only, s=0.45 and s= -2.26 are actual break points.

- Out of these s=0.45 is the breakaway point and s=-2.26 is the break-in point.

- Corresponding to these points k values are 2.64 and 77.66

- The angle departure of the root locus from the complex pole is $\theta_d = \pm 55.27^o$

PROBLEM: Construction of Root Locus (contd)..

Determine the points where root loci cross the imaginary axis and range of K for stable operation

- The characteristic equation of closed loop system:

$$s^4 + 3s^3 + 12s^2 + (k-16)s + k = 0$$

$$\begin{array}{l|ccc}
s^4 & 1 & 12 & k \\
s^3 & 3 & k-16 & \\
s^2 & \dfrac{36-k+16}{3} & k & \\
s^1 & \dfrac{\dfrac{52-k}{3}(k-16)-3k}{\dfrac{52-k}{3}} & & \\
s^0 & k & &
\end{array}$$

$$k > 0$$

$$52 - k > 0$$

$$k < 52$$

$$52k + 16k - k^2 - 832 - 9k > 0$$

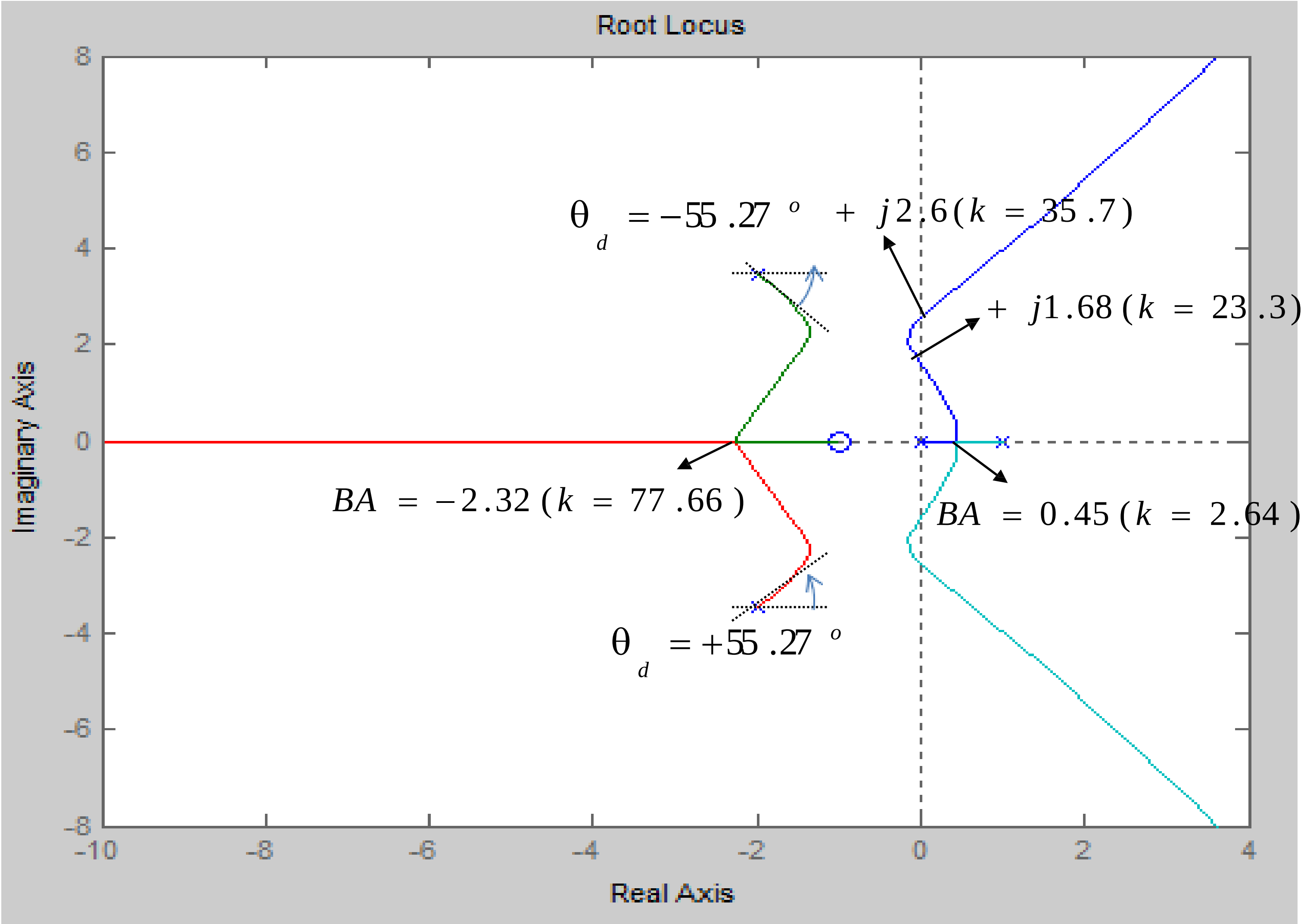
$$k^2 - 59k + 832 < 0$$

$$k > 23.3 \, and k < 35.7$$

- The range of values of k for stability is 23.3<k<35.7. The corresponding oscillation frequencies are 1.68 rad/sec and 2.6 rad/sec

PROBLEM: Construction of Root Locus (contd)..

- Complete root locus of the given system is



Unit-IV
FREQUENCY DOMAIN ANALYSIS

Frequency Domain Specifications

- We have studied about time domain specifications like, rise time ,t**r**; peak time, t**p**; settling time, t**s**; peak overshoot, M**p**.
- Now, we define frequency domain specifications for a given system and determine their correlation with the time domain specifications.
- This correlation between time & frequency domain is necessary as it enables us to derive time domain specifications from frequency domain ones & vice-versa.
- Further, we may like to analyze a given system either in time domain or frequency domain & hence we need to have a set of specification in each domain for evaluating a given system's response.
- Like in time domain, here too we consider a second order system for deriving frequency domain specifications.

Frequency Domain Specifications (contd)..

- Given, a closed loop transfer function, T(S) = C(S)/R(S), as

  $T(S) = C(S)/R(S) = \omega\mathbf{n}^2 / (S^2 + 2\xi\, \omega\mathbf{n}\, S + \omega\mathbf{n}^2)$

- For determining frequency response, we let S = jω in T(S) because we are interested in real frequencies which lie on the Imaginary axis of the S-plane.

$$T(j\omega) = \omega\mathbf{n}^2 / (-\omega^2 + j2\xi\, \omega\mathbf{n}\, \omega + \omega\mathbf{n}^2)$$

$$T(j\omega) = \omega\mathbf{n}^2 / \omega\mathbf{n}^2 \{ (1-(\omega/ \omega\mathbf{n}))^2 + j2\xi\, \omega/ \omega\mathbf{n} \}$$

- Let u = $\omega/ \omega\mathbf{n}$; u: normalized frequency

  $\omega\mathbf{n}$: natural frequency of oscillation of the system

  ω : input signal frequency

- Thus, $T(j\omega) = 1/\{ (1-u^2) + j\, 2\xi\, u$- .... (1)

$|T(j\omega)| = M(u) = 1/\surd, (1-u^2)^2 + 4\xi^2 u^2$- ..... (2)

$\arg\{T(j\omega)\} = \phi = - \tan^{-1} ,2\xi\, u/(1-u^2)$- ....(3)

Frequency Domain Specifications (contd)..

- The magnitude & phase response are part of frequency response. Equations(2) & (3) corresponding to magnitude & phase response tell us that,
- if we feed an input signal $r(t) = A \sin(\omega t)$ to the system, the output signal will have

$$\text{magnitude} = A/ \surd, (1-u^2)^2 + 4\xi^2 u^2\}$$, and the

$$\text{phase introduced} = - \tan^{-1}\{2\xi\, u/(1-u^2)\}$$

- Thus the output signal, under steady state, will be

$$c(t) = A/{*}\surd, (1-u^2)^2 + 4\xi^2 u^2\}] \sin(\omega t - \tan^{-1}\{2\xi\, u/(1-u^2)\})$$

- We observe that the output amplitude is dependent on the input frequency, and so is the phase lag introduced in the output signal.

Frequency Domain Specifications (contd)..

- Reproducing equations (2) & (3), we have

  $M(u) = 1/\sqrt{(1-u^2)^2 + 4\xi^2 u^2}$ ..... (2)

  $\phi = -\tan^{-1}\{2\xi u/(1-u^2)\}$ ....(3)

  Plotting M & ϕ vs. u, $u = \omega/\omega_n$

u	M	ϕ
0.0	1.0	0 ($\omega=0$)
1.0	$1/(2\xi)$	$-\pi/2$ ($\omega = \omega_n$)
∞	0	$-\pi$ ($\omega_n \rightarrow \infty$)

- Observation:

  At $\omega = \omega_n$, the value of 'M' is inversely proportional to ξ.

  The lower the ξ higher the 'M' implies higher peak in the magnitude response.

Frequency Domain Specifications (contd)..

Resonant Frequency:

The frequency where 'M' has a peak value is called resonant frequency. At this frequency, the slope of the magnitude curve, M, is zero. Differentiate 'M' w.r.t 'u' in equation (1)

Therefore, $dM/du \big|_{u = u_r} = 0 \Rightarrow u_r^2 = 1 - 2\xi^2 \Rightarrow u_r = \sqrt{(1-2\xi^2)}$

$\Rightarrow \omega_r = \omega_n \sqrt{(1-2\xi^2)}$

Resonant frequency : $\omega_r = \omega_n \sqrt{(1-2\xi^2)}$ .... (4)

Resonant Peak, Mr:

The maximum value of magnitude is known as 'Resonant peak'

$M(u) = 1/\sqrt{\{(1-u^2)^2 + 4\xi^2 u^2\}}$; at resonant frequency $u=u_r$, we get M_r.

Substitute for $u= u_r$ in $M(u)$, to get $M_r = 1/\{2\xi \sqrt{(1-\xi^2)}\}$ ..... (5)

Frequency Domain Specifications (contd)..

❑ Phase angle, ϕr at Resonant Frequency:

Phase angle: $\phi = - \tan^{-1}\{2\xi\, u/(1-u^2)\}$

Substitute for $u = u\mathbf{r}$ in ϕ, to get

$$\phi \mathbf{r} = - \tan^{-1}\{\sqrt{(1-2\,\xi^2)}/\xi\} \quad \ldots\ldots (6)$$

From equations (4) & (5), as reproduced below

$$\omega r = \omega \mathbf{n} \sqrt{(1-2\,\xi^2)} \quad \ldots. (4)$$

$$M\mathbf{r} = 1/\{2\xi \sqrt{(1-\xi^2)}\} \ldots.. (5)$$

It is seen that as ξ approaches '0'

ωr approaches $\omega \mathbf{n}$, and

$M\mathbf{r}$ approaches ∞

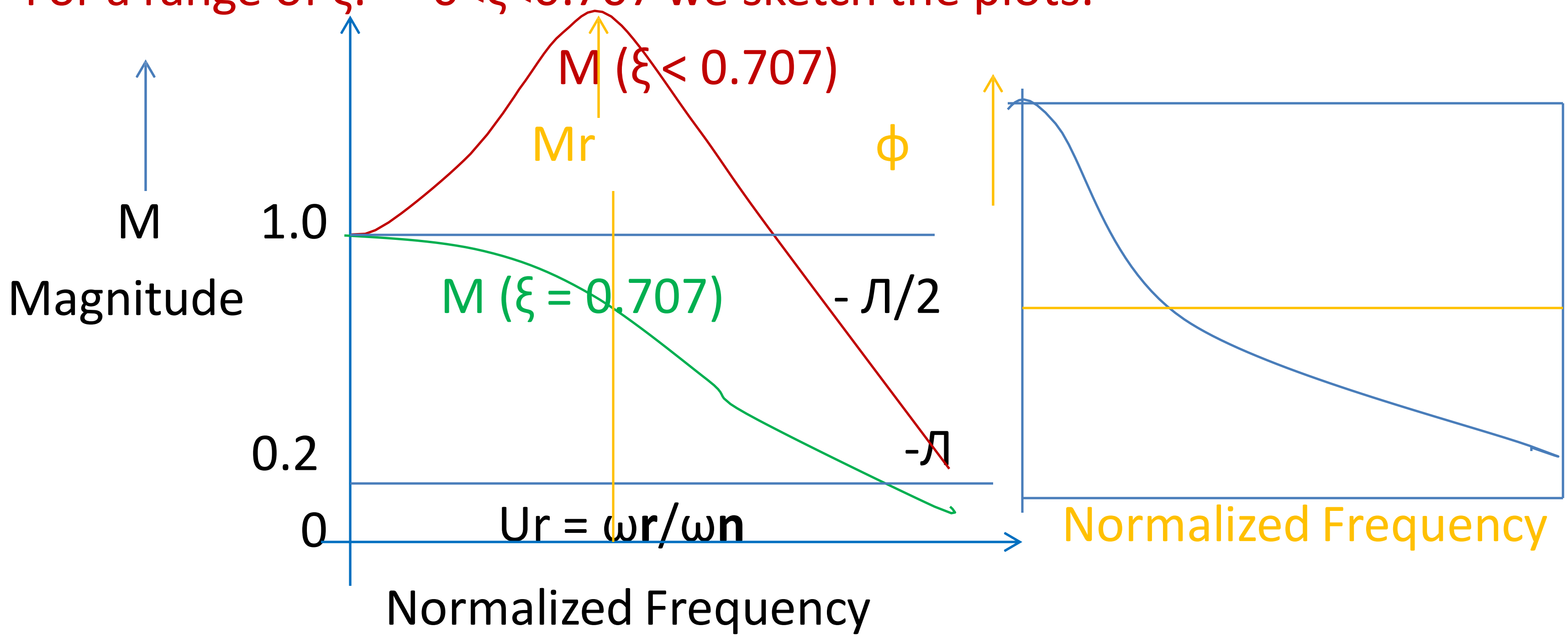
At $\xi = 0.707$; $M\mathbf{r} = 1$ & $\omega r = 0$

Therefore there is no resonant peak & hence no resonant frequency.

Frequency Domain Specifications

❑ The magnitude & phase plot:

For a range of ξ: 0<ξ<0.707 we sketch the plots.



❑ We observe that for ξ≥ 0.707, the magnitude plot decreases monotonically from M=1 at u=0. Thus there is no resonant peak for ξ≥ 0.707 & the greatest value of M = 1.0

Frequency Domain Specifications (contd)..

- **Bandwidth, ωb:**

  The frequency at which $M = 0.707\ (1/\sqrt{2})$ is called cut off frequency, ω**c**.

- The range of frequencies for which $M \geq 1/\sqrt{2}$ is defined as bandwidth, ω**b** of a system. Since control systems are low pass filters, ω**b** = ω**c** .

- At u = u**b** = ω**b** / ω**n**; (the normalized bandwidth), the expression for M is

$$M(u_b) = 1/\sqrt{,}\ (1- u_b\ ^2)^2 + 4\xi^2\ u_b\ ^2 - = 1/\sqrt{2}$$

  Solving the above equation, we get

  $u_b^4 - 2(1-2\xi^2)u_b^2 - 1 = 0$ Let $u_b^2 = x$; solve for x & then for ub. $U_b = \sqrt{x}$

- Solving for ub we get: $u_b = \sqrt{\ {*}1-2\xi^2 + \sqrt{(2-4\xi^2+4\xi^4)}]}$

Frequency Domain Specifications (contd)..

- **Bandwidth:**

The denormalized bandwidth is given by,

$$\omega b = \omega \mathbf{n} \sqrt{*1-2\xi^2 + \sqrt{(2-4\xi^2+4\xi^4)}]}$$

Thus, we observe that bandwidth is a function of damping, ξ only.

ξ	ωb
0.2	1.51 ω**n**
0.5	1.272 ω**n**
0.707	0.999 ω**n**

Thus we observe that as damping increases the bandwidth reduces.

Frequency Domain Specifications (contd)..

- Correlation between time and frequency domain parameters:

Time Domain:

$$M\mathbf{p} = \exp(-\pi\xi/\sqrt{(1-\xi^2)})$$

$$t\mathbf{p} = \pi/\omega\mathbf{n}\sqrt{(1-\xi^2)}; \quad \omega\mathbf{d} = \omega\mathbf{n}\sqrt{(1-\xi^2)}$$

Frequency Domain:

$$M\mathbf{r} = 1/\{2\xi\sqrt{(1-\xi^2)}\}; \quad \omega r = \omega\mathbf{n}\sqrt{(1-2\xi^2)}$$

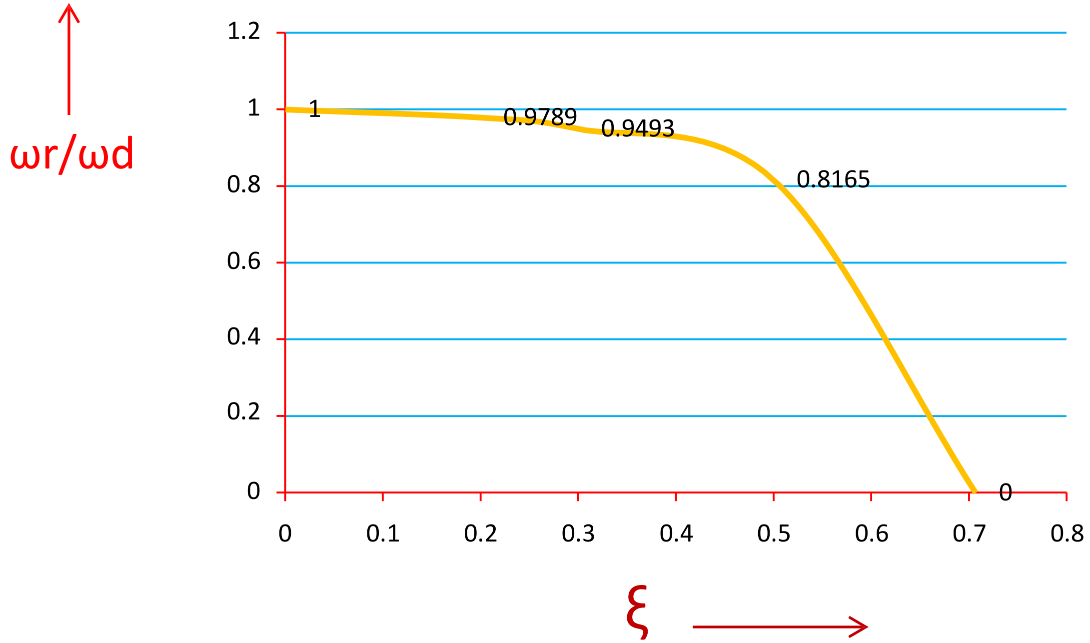
- From the above equations we understand that no matter in which domain (frequency or time) we are analyzing a system performance, the other domain (time or frequency) parameters can be easily estimated using the above set of relationships.
- For example, working in time domain from the root locus we can fix ξ, $\omega\mathbf{n}$, for a desired location of closed loop poles and then we can determine frequency domain parameters using above equations.

Frequency Domain Specifications (contd)..

- Correlation between time & frequency domain parameters:

$$\omega_r / \omega_d = \sqrt{(1-2\xi^2)} / \sqrt{(1-\xi^2)}$$

ξ	ω_r / ω_d
0.707	0.0
0.5	0.8165
0.3	0.9493
0.2	0.9789
0.0	1.0



POLAR PLOT

❑ Polar Plot:

Magnitude and phase of $G(j\omega)$ is plotted in X-Y plane (graph sheet)

$$G(j\omega) = Re[G(j\omega)] + Img\,[G(j\omega)]$$

$$G(j\omega) = |G(j\omega)|\ \arg,G(j\omega)\} = M \exp(-j\phi)$$

As ω is varied from '0' to '∞'; the '$M(\omega = \omega 1)$' value is marked on the graph sheet at an angle of $\phi(\omega = \omega 1)$

❑ Example 1:

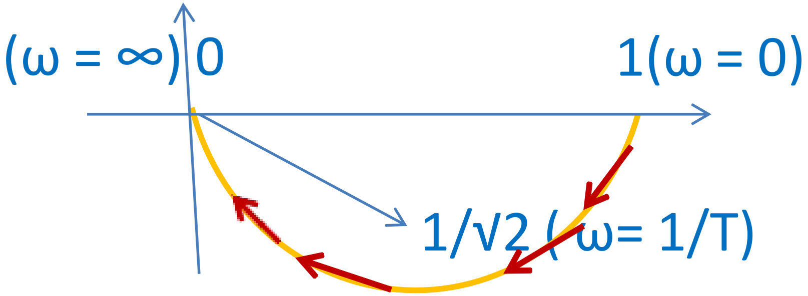
$$G(S) = 1/(1 + TS) \Rightarrow G(j\omega) = 1/(1 + j\,\omega T)$$

$$\Rightarrow M(\omega) = 1/\sqrt{(1 + (\omega T)^2)}; \qquad \phi(\omega) = -\tan^{-1}(\omega T)$$

$\omega \rightarrow 0$; $M = 1$ $\qquad \phi = 0°$

$\omega \rightarrow \infty$; $M = 0$ $\qquad \phi = -\pi/2$

$\omega = 1/T$; $M = 1/\sqrt{2}$ $\qquad \phi = -\pi/4$



POLAR PLOT (Contd..)

❑ Observations:

1. The $\omega = 0$ & $\omega = \infty$ are important points in a polar plot.
2. The angle subtended by $G(j\omega)$ or $G(j\omega)\,H(j\omega)$ at these frequencies indicate the number of quadrants the polar plot is going to traverse in the $G(j\omega)$ or $G(j\omega)\,H(j\omega)$ plane.
3. As we shall see later the intersection of the polar plot with the negative real axis of the $G(j\omega)$ or $G(j\omega)\,H(j\omega)$ plane is a very important information because it allows us to determine the stability of a CL system, as also its relative stability.
4. Polar plot need not be drawn for all the frequencies from 0 to ∞; the necessary points are $\omega = 0$ & $\omega = \infty$ and those values of ω at which the polar plot intersects with the negative real axis of the $G(j\omega)$ or $G(j\omega)$ $H(j\omega)$ plane.

POLAR PLOT (Contd..)

- Example 2:

  $G(S)$ or $G(S)H(S) = 1/S(1+TS)$

  $G(j\omega) = 1/j\omega\,(1 + j\,T\omega)$; $M(\omega) = 1/\omega\,\sqrt{(1 + T^2\omega^2)}$;

  $\phi(\omega) = -\pi/2 - \tan^{-1}(T\omega)$

  $\omega = 0$; $M = \infty$; $\phi = -\pi/2$ Angle measured in CW direction: -

  $\omega = \infty$; $M = 0$; $\phi = -\pi$ Angle measured in CCW direction: +

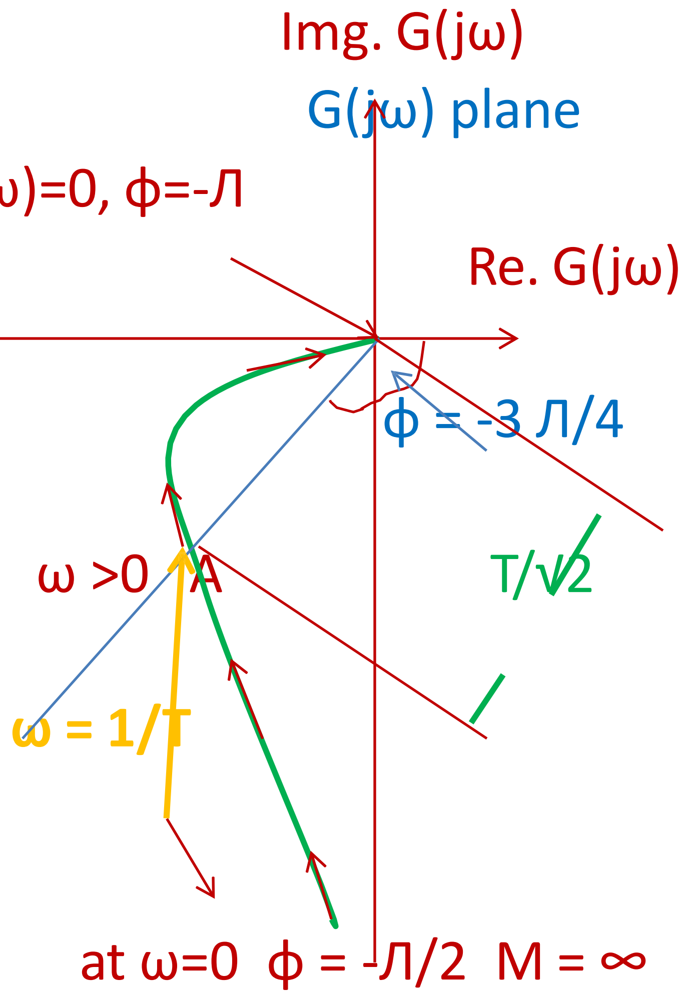
  $\omega = 1/T$; $M = T/\sqrt{2}$ $\phi = -3\pi/4$

- Note: we observe that between $\omega = 0$ & $\omega = \infty$ the angle changes by $\pi/2$; therefore the polar plot will traverse only in one quadrant.

  The polar plot is shown in the next slide

POLAR PLOT (Contd..)

- Polar plot:



- At ω =1/T; M = T/√2 ϕ = -3Л/4

  In order to plot this point, we draw an angle ϕ = -3Л/4 and then mark the point M = T/√2

  At point A,

  M = T/√2, ϕ = -3Л/4, ω = 1/T

POLAR PLOT (Contd..)

- Example 3:

  $G(S) = 1/(1+T_1 S)(1+T_2 S)$; $G(j\omega) = 1/(1 + j\,\omega T_1)\,(1 + j\,\omega T_2)$

$$M(\omega) = 1/\sqrt{(1+\omega^2 T_1^2)}\,\sqrt{(1+\omega^2 T_2^2)}$$

$$\phi(\omega) = -\tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega)$$

  $\omega = 0$; $M = 1$; $\phi = 0$ Angle measured in CW direction: -

  $\omega = \infty$; $M = 0$; $\phi = -\pi$ Angle measured in CCW direction: +

- We observe that ϕ changes from 0 to $-\pi$ as ω changes from 0 to ∞.
- Therefore, the polar plot will traverse two quadrants in the $G(j\omega)$ or $G(j\omega)\,H(j\omega)$ plane.
- Since the polar plot traverses two quadrants, we need to determine point(s) of intersection between polar plot & the Imaginary & negative real axis of the $G(j\omega)$ plane.

POLAR PLOT (Contd..)

- Intersection with real & imaginary axis of the G(jω) plane:
- Procedure:

1. Rationalize G(jω) or G(jω) H(jω)
2. Separate in to real & imaginary parts of G(jω) or G(jω) H(jω)
3. For intersection on real axis; imaginary part = 0. Make imaginary part = 0 by making its numerator = 0. We get value of ω at point of intersection. Calculate the value of real part at this value of ω. Draw a vector of this length from the origin to get intersection on the real axis.
4. For intersection on imaginary axis; real part = 0. Make real part = 0 by making its numerator = 0. We get value of ω at point of intersection. Calculate the value of imaginary part at this value of ω. Draw a vector of this length from the origin to get intersection on the real axis.

POLAR PLOT (Contd..)

- Determination of Intersection point(s):
- $G(j\omega)$ can be written as, $G(j\omega) = 1/[(1-\omega^2T_1T_2) + j\,\omega(T_1 + T_2)+$

  Rationalize: multiply & divide $G(j\omega)$ by $[(1-\omega^2T_1T_2)$ **-** $j\,\omega(T_1 + T_2)+$; that is conjugate of the denominator.

  We get,

  $G(j\omega) = [(1-\omega^2T_1T_2)$ **+** $j\,\omega(T_1 + T_2)+/[(1-\omega^2T_1T_2)^2 + \omega^2(T_1 + T_2)^2+$

  Real part $= (1-\omega^2T_1T_2)/*(1-\omega^2T_1T_2)^2 + \omega^2(T_1 + T_2)^2+$

  Imaginary part $= \omega(T_1 + T_2)/*(1-\omega^2T_1T_2)^2 + \omega^2(T_1 + T_2)^2+$

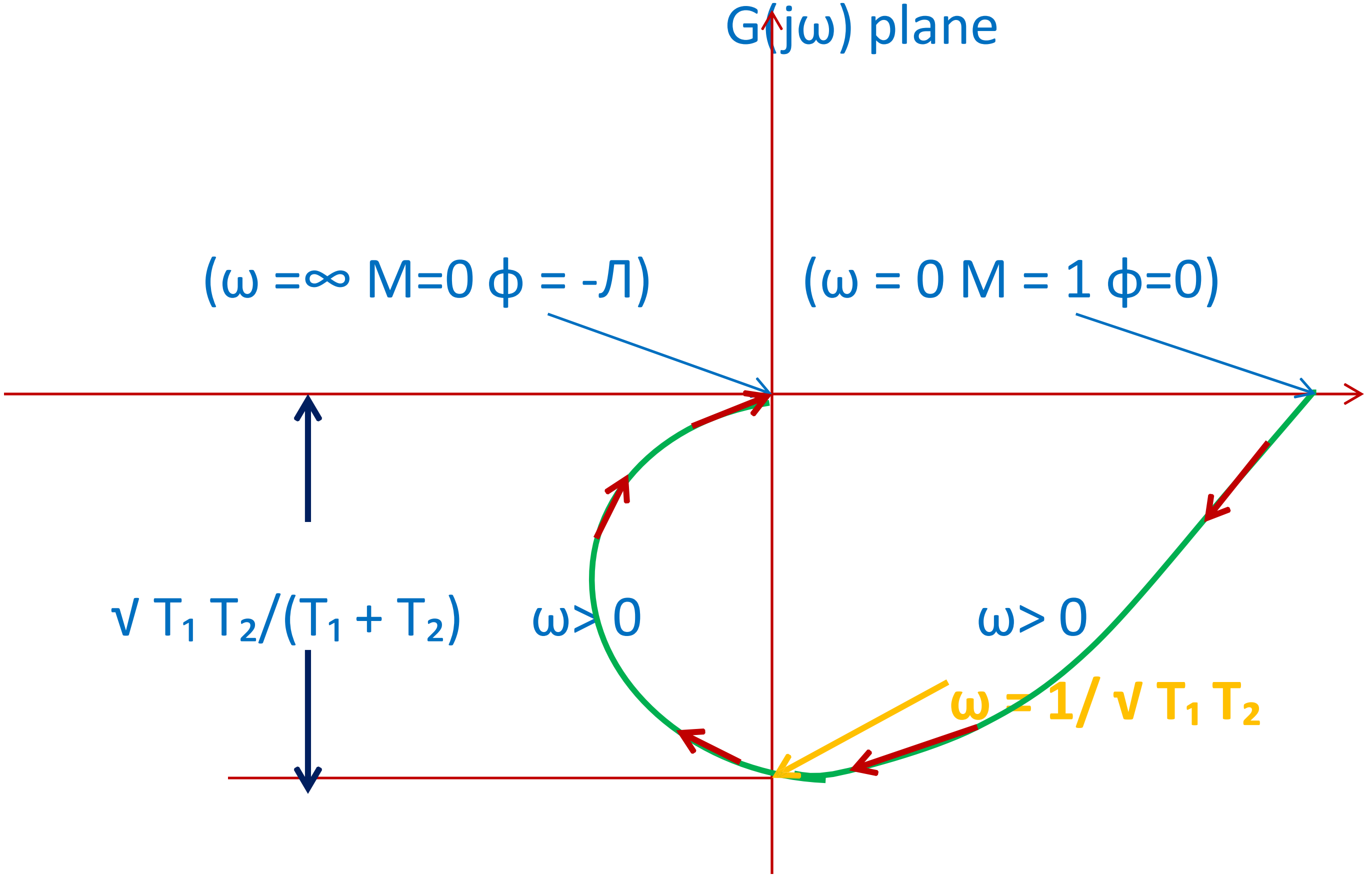
- We see from the above that

  Imag. Part cannot be zero, &

  Real part = 0 for $1-\omega^2T_1T_2 =0$; $\omega^2 = 1/ T_1T_2$

  at intersection on imaginary axis, the frequency $\omega = 1/\surd\, T_1T_2$

POLAR PLOT (Contd..)



POLAR PLOT

- Example 4:

  $G(S) = 1/(1+T_1 S)(1+T_2 S)(1+T_3 S);$

  $G(j\omega) = 1/(1 + j\,\omega T_1)(1 + j\,\omega T_2)(1 + j\,\omega T_3)$

  $M(\omega) = 1/\sqrt{(1+\omega^2 T_1^2)}\sqrt{(1+\omega^2 T_2^2)}\sqrt{(1+\omega^2 T_3^2)}$

  $\phi(\omega) = -\tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega) - \tan^{-1}(T_3\omega)$

  $\omega = 0;\quad M = 1;\quad \phi = 0$ — Angle measured in CW direction: -

  $\omega = \infty;\quad M = 0;\quad \phi = -3Л/2$ — Angle measured in CCW direction: +

- We observe that ϕ changes from 0 to –3Л/2 as ω changes from 0 to ∞.
- Therefore, the polar plot will traverse three quadrants in the G(jω) or G(jω) H(jω) plane.
- Since the polar plot traverses three quadrants, we need to determine point(s) of intersection between polar plot & the Imaginary & negative real axis of the G(jω) plane.

POLAR PLOT (Contd..)

- Intersection on the Real & Imaginary axis of G(j ω) plane:

  Following the procedure as explained earlier, we have:

- For intersection on Imaginary Axis:

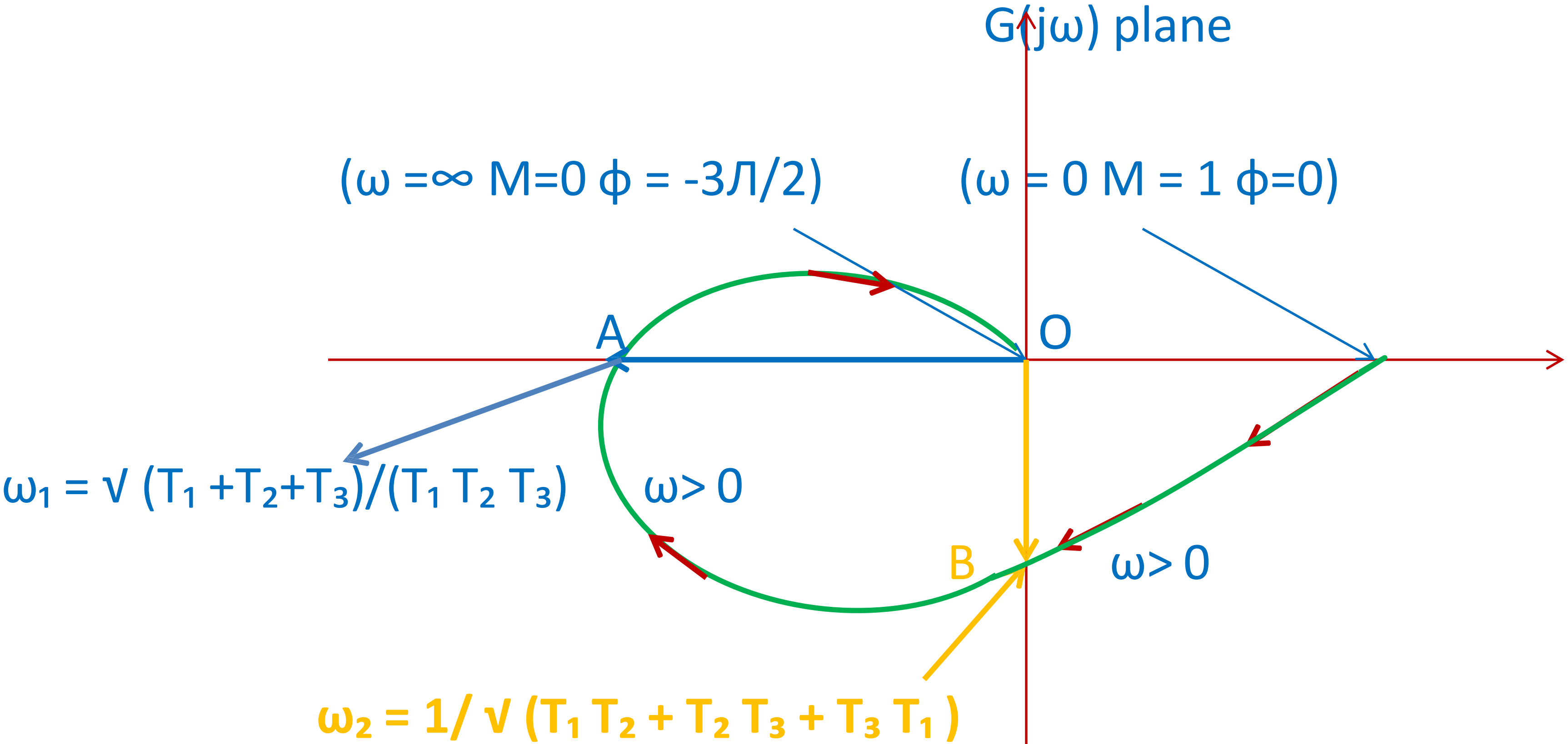
$$\omega = 1/\sqrt{(T_1 T_2 + T_3 T_1 + T_2 T_3)}$$

- For intersection on real Axis:

$$\omega = \sqrt{\ *T_1 + T_2 + T_3/T_1 T_2 T_3\ ]}$$

  For the above values of ω, determine the magnitude of the points with imaginary intersection.

POLAR PLOT (Contd..)



OA: magnitude of $G(j\omega)$ at $\omega = \omega_1$

OB : magnitude of $G(j\omega)$ at $\omega = \omega_2$

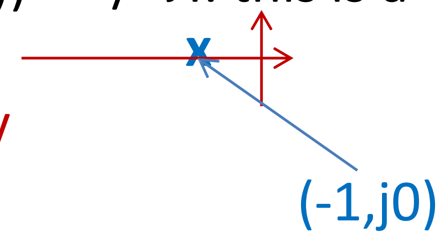
POLAR PLOT (Relative Stability)

Relative Stability:

1. It is defined for systems that are open loop stable.
2. We have the Characteristic equation $Q(S) = 1 + G(S)H(S) = 0$
3. For real frequencies (frequency response) $S = j\omega$
4. Therefore, $Q(j\omega) = 1 + G(j\omega)\, H(j\omega) = 0$
5. Or, $G(j\omega)\, H(j\omega) = -1$
6. therefore, $|G(j\omega)\, H(j\omega)| = 1$ & $\arg(G(j\omega)\, H(j\omega)) = \phi = +/- \pi$
7. When loop gain = $|G(j\omega)\, H(j\omega)| = 1$ & $\arg(G(j\omega)\, H(j\omega)) = +/- \pi$
8. Phase introduced due to error detector = 180°
9. Therefore, total phase in the loop = 360° & $|G(j\omega)\, H(j\omega)| = 1$
10. The CL system response is oscillatory & it is on the verge of instability

POLAR PLOT (Relative Stability Contd..)

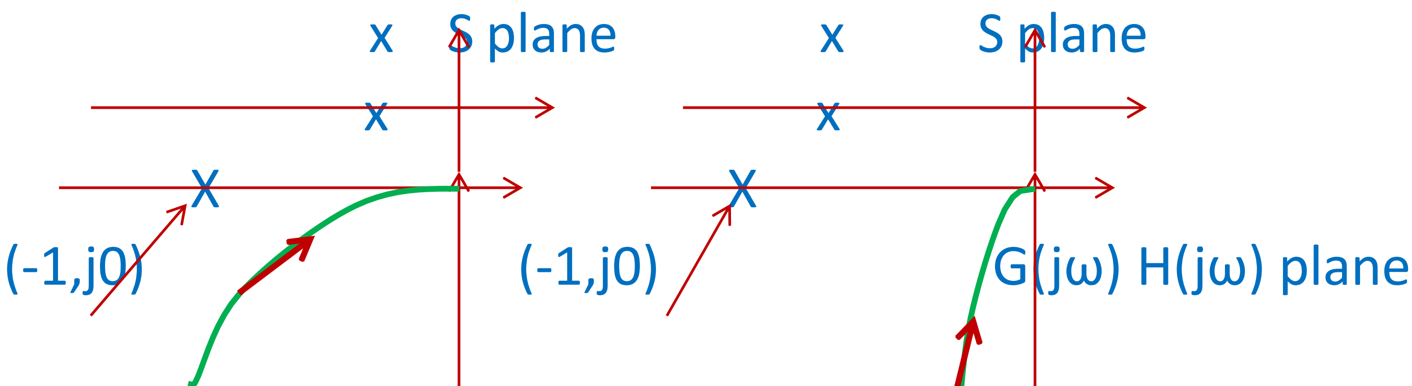
11. loop gain = |G(jω) H(j ω)| = 1 & arg(G(jω) H(j ω)) = +/- Л: this is a point (-1, j0) in the G(jω) H(j ω) plane.



12. Stability of a closed loop system is determined by

non-encirclement of (-1,j0) point. As the polar plot gets closer to (-1,j0) point, the CL system tends towards instability.

Polar plot & Location of closed loop poles:



We observe that polar plot closer to (-1,j0) point implies CL poles are closer to the Imaginary axis of the S-plane

POLAR PLOT (Relative Stability Contd..)

- As the CL poles move closer to the Imaginary axis of the S plane, the system takes more time to settle down (reach steady state) & is therefore relatively less stable than the one which has CL poles far removed from the Imaginary axis of the S plane.
- In frequency domain it implies that as the polar plot moves closer to the (-1,j0) in the $G(j\omega)\ H(j\omega)$ plane, the CL system becomes relatively less & less stable.
- Therefore proximity of the polar plot to the (-1,j0) point determines CL system's relative stability.
- If the polar plot passes through (-1,j0) point then the CL system is on the verge of instability
- If the polar plot encircles the (-1,j0) point then the CL system is unstable.

POLAR PLOT (Relative Stability Contd..)

- Example of Relative stability:

Plot 1:

Intersects negative real axis at ‘b’

Plot 2:

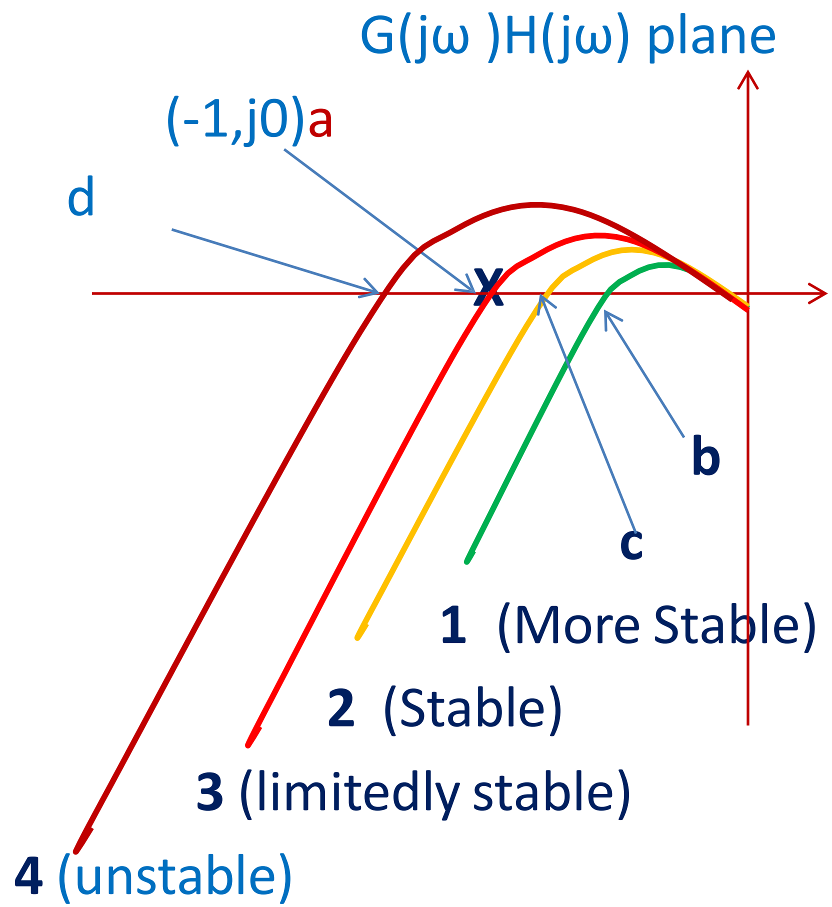
Intersects negative real axis at ‘c’

Plot 3:

Passes through (-1,j0) point

Plot 4:

Encircles (-1,j0) point & intersects negative real axis at ‘d’



|b|<|c|<|a|<|d|

Relative Stability Index: Gain & Phase Margin

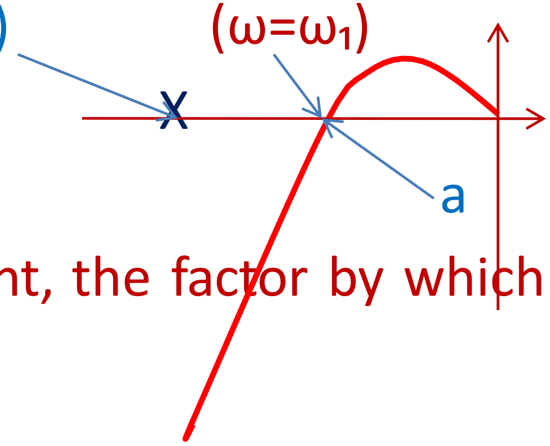
❑ Gain Margin:

1. The margin between actual gain ‘K’ (of the system) and the critical gain causing oscillations (in the system output) is called Gain Margin (GM)
2. Critical gain: the value of ‘K’ at which the Polar plot- { $G(j\omega)H(j\omega)$} plot - passes through (-1,j0) point.
3. Definition of GM: It is the factor by which the system gain can be increased to drive it to the verge of instability.
4. At $\omega = \omega_1$, the magnitude of intersection with the negative real axis is ‘a’; the phase angle = Л
5. For the plot to pass through (-1,j0) point, the factor by which the gain is to be increased = 1/a. GM = 1/a

GH plane

(-1,j0)

(ω=ω_1)



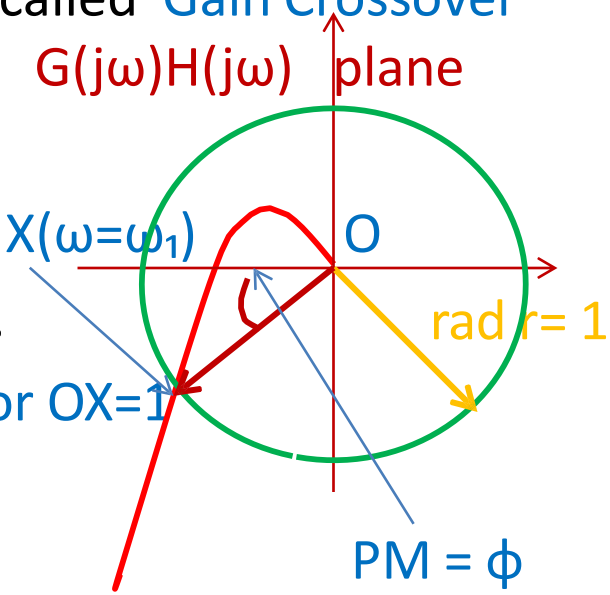
Relative Stability Index: Gain & Phase Margin

1. $|G(j\omega)H(j\omega)| = a$, at $\omega = \omega_1$
2. arg $\{G(j\omega)H(j\omega)\} = \phi = \pi$, at $\omega = \omega_1$
3. $\omega = \omega_1$ is the frequency at which $\phi = 180°$.
4. $\omega = \omega_1$ is called 'Phase Crossover Frequency'
5. Phase crossover frequency: is defined as the frequency at which the phase offered by the system is π
6. Gain Margin is now defined in terms of phase crossover frequency as
7. 'reciprocal of the gain at the frequency at which phase angle becomes 180'
8. Thus GM value is obtained at phase crossover frequency.
9. GM = 1/a; In decibels: GM = 20 Log(1/a) = - 20 Log(a)

Relative Stability Index: Gain & Phase Margin

❑ Phase Margin:

1. It is calculated at 'Gain Crossover Frequency'
2. The frequency at which $|G(j\omega)H(j\omega)| = 1$ is called 'Gain Crossover frequency'
3. Draw a **unit circle** as shown.
4. The point of intersection of unit circle
5. with polar plot is X , say, the frequency is ω_1.
6. The $|G(j\omega)H(j\omega)|$(at $\omega=\omega_1$) = length of vector OX=1



7. Therefore $\omega=\omega_1$ is the gain cross over frequency.
8. The angle made by OX with the negative real axis of the $G(j\omega)H(j\omega)$ plane is Phase Margin (PM), ϕ, of the system.

Relative Stability Index: Gain & Phase Margin

❑ Phase Margin & Stability of CL system:

1. It is defined as the amount of additional phase lag at the gain cross over frequency required to bring the system to the verge of instability.
2. It is measured in the CCW direction from the negative real axis of the $G(j\omega)\,H(j\omega)$ plane.
3. If $\omega = \omega_1$ is the gain cross over frequency, then phase margin (PM) is computed as:
4. $PM = \phi = \arg\{G(j\omega_1)\,H(j\omega_1)\} + 180°$
5. Since systems introduce phase lag , $\arg\{G(j\omega_1)\,H(j\omega_1)\}$ is always negative.
6. If PM is positive, the CL system is stable
7. If PM is negative the CL system is unstable
8. If PM = 0 the CL system is on the verge of instability

Relative Stability Index: Gain & Phase Margin

❑ GM & Stability of CL system:

GM is calculated as the inverse of the |G(jω)H(jω)|= 'a' at the point of its intersection with negative real axis of the GH plane.

GM = 1/a ; or, GM = - 20 Log (a) in dB.

1. If GM is positive, CL system is stable
2. If GM is negative, CL system is unstable
3. If GM = '0', CL system is on the verge of instability

❑ Interpretation of Relative Stability from GM & PM Values:

1. Large GM or large PM imply sluggish CL system
2. GM close to '1' or PM close to '0°' imply highly oscillatory system
3. GM of about 6 dB or PM of 30-35° imply reasonably good degree of relative stability
4. Generally a good GM automatically guarantees a good PM & vice-versa.

Relative Stability Index: Gain & Phase Margin

- Special Cases:

  We have said that generally a good GM yields good PM & vice versa. In certain cases, it may not hold.

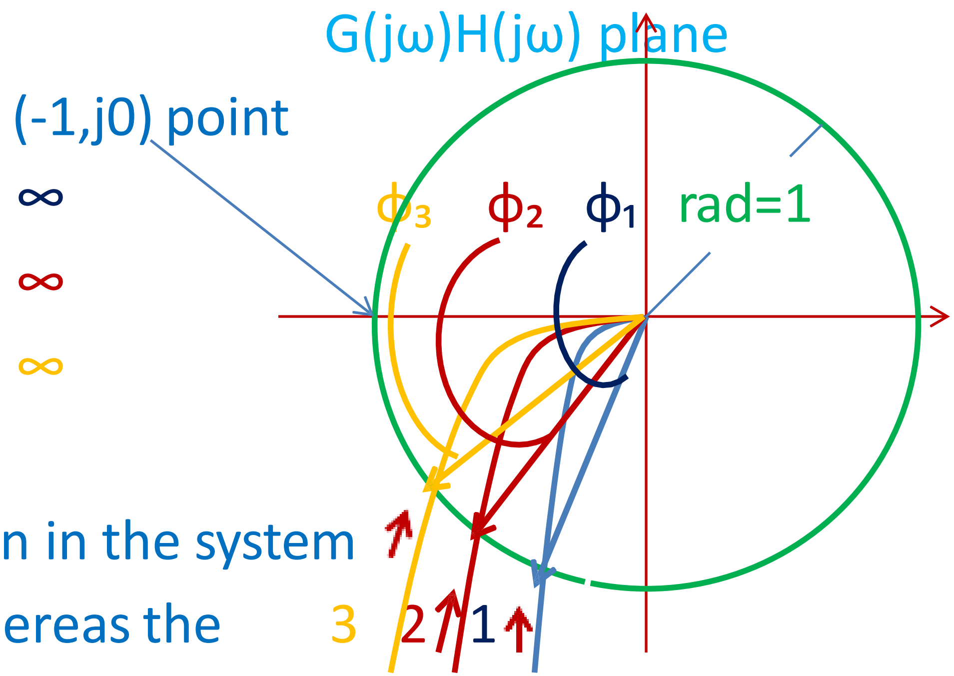
- Case 1:

  Plot 1: gain K_1 ;PM = ϕ_1; GM = ∞

  Plot 2: gain K_2; PM = ϕ_2; GM = ∞

  Plot 3: gain K_3; PM = ϕ_3; GM = ∞

  $K_3 > K_2 > K_1$; $\phi_3 < \phi_2 < \phi_1$



- We see that as we increase gain in the system the Phase Margin reduces whereas the
- Gain Margin does not change. Therefore in such cases we need to focus only on PM because GM is not adjustable.

Relative Stability Index: Gain & Phase Margin

- Case 2:

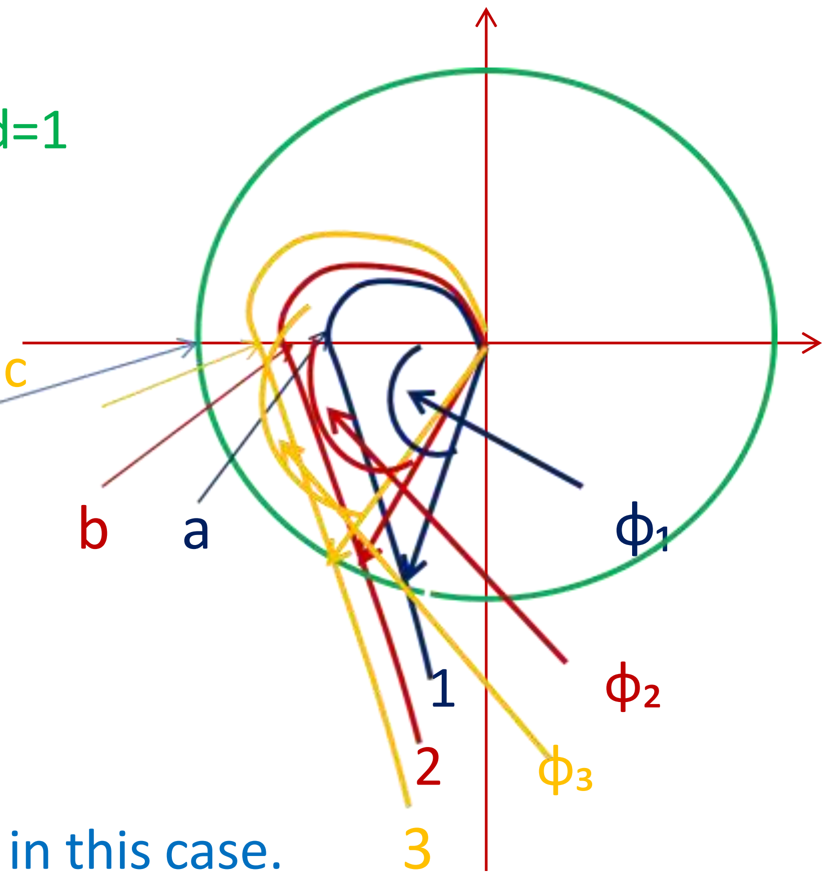
  Plot 1: gain K_1 ;PM = ϕ_1; GM = 1/a rad=1

  Plot 2: gain K_2; PM = ϕ_2; GM = 1/b

  Plot 3: gain K_3; PM = ϕ_3; GM = 1/c

  $K_3 > K_2 > K_1$; $\phi_3 < \phi_2 < \phi_1$

  (-1,j0) point



- We see that as we increase gain the GM reduces appreciably , but the PM does not vary much.
- Therefore, we need to monitor GM in this case.

Polar Plot: Correlation between PM & ξ

- Correlation between Phase Margin & Damping ξ:

  Let $G(S) = \omega n^2/S(S + 2\xi\omega n)$; for a unity feedback system

- At the gain cross over frequency, $\omega = \omega_1$

$$|G(j\,\omega)H(j\omega)| = 1.0$$

$$\text{or,}\quad \omega n^2/\,\omega_1\sqrt{(\omega_1^2 + 4\,\xi^2\,\omega n^2)} = 1.0$$

$$\text{or,}\quad \omega_1^2(\omega_1^2 + 4\,\xi^2\,\omega n^2) = \omega n^4$$

$$\text{or,}\quad (\omega_1/\,\omega n)^4 + 4\,\xi^2\,(\omega_1/\,\omega n\,)^2 - 1 = 0;\ \text{let } (\omega_1/\omega n\,)^2 = x$$

$$\text{or,}\quad x^2 + 4\,\xi^2\,x - 1 = 0$$

$$\text{or,}\quad x = -2\,\xi^2 +/- \sqrt{(1 + 4\,\xi^4)}$$

$$\text{or,}\ (\omega_1/\,\omega n\,)^2 = \sqrt{(1 + 4\,\xi^4)} - 2\,\xi^2$$

$$\text{or,}\quad \omega_1 = \omega n\,\sqrt{(\sqrt{(1 + 4\,\xi^4)} - 2\,\xi^2)}$$

- The above equation relates ξ with gain cross over frequency, ω_1

Polar Plot: Correlation between PM & ξ

$$\arg\{G(j\,\omega)H(j\omega)\} = -\,90° - \tan^{-1}(\omega/2\,\xi\,\omega\mathbf{n})$$

at $\omega = \omega_1$, $\phi_1 = -\,90° - \tan^{-1}(\omega_1/2\,\xi\,\omega\mathbf{n})$

$$PM = \phi = 180° + \phi_1 = 180° - 90° - \tan^{-1}(\omega_1/2\,\xi\,\omega\mathbf{n})$$

$$\phi = 90° - \tan^{-1}(\omega_1/2\,\xi\,\omega\mathbf{n})$$

- Substitute for ω_1 to get,

$$\phi = 90° - \tan^{-1}*\sqrt{(\sqrt{(1+4\,\xi^4)} - 2\,\xi^2)}\,/2\,\xi\,]$$

or, $*\sqrt{(\sqrt{(1+4\,\xi^4)} - 2\,\xi^2)}\,/2\,\xi\,] = \tan(90° - \phi) = \cot\phi$

or, $\tan\phi = 2\,\xi\,/\,*\sqrt{(\sqrt{(1+4\,\xi^4)} - 2\,\xi^2)}]$

or, $\phi = \tan^{-1}\{2\,\xi\,/\,*\sqrt{(\sqrt{(1+4\,\xi^4)} - 2\,\xi^2)}]\}$

- The above equation gives a relationship between ξ & ϕ for an under damped system.
- In the range $\xi \le 0.707$, a reasonably good approximation is given by

$$\xi = 0.01\,\phi$$

Polar plot Examples: Computation of GM & PM

$G(j\omega) = K/j\omega\,(1+j\,0.2\omega)(1+j\,0.05\,\omega)$

- For K = 1:

  PM = ϕ = 76° ;

- Intersection on negative real axis, a = -0.04

  GM = 20 Log |1/a| = 28 dB

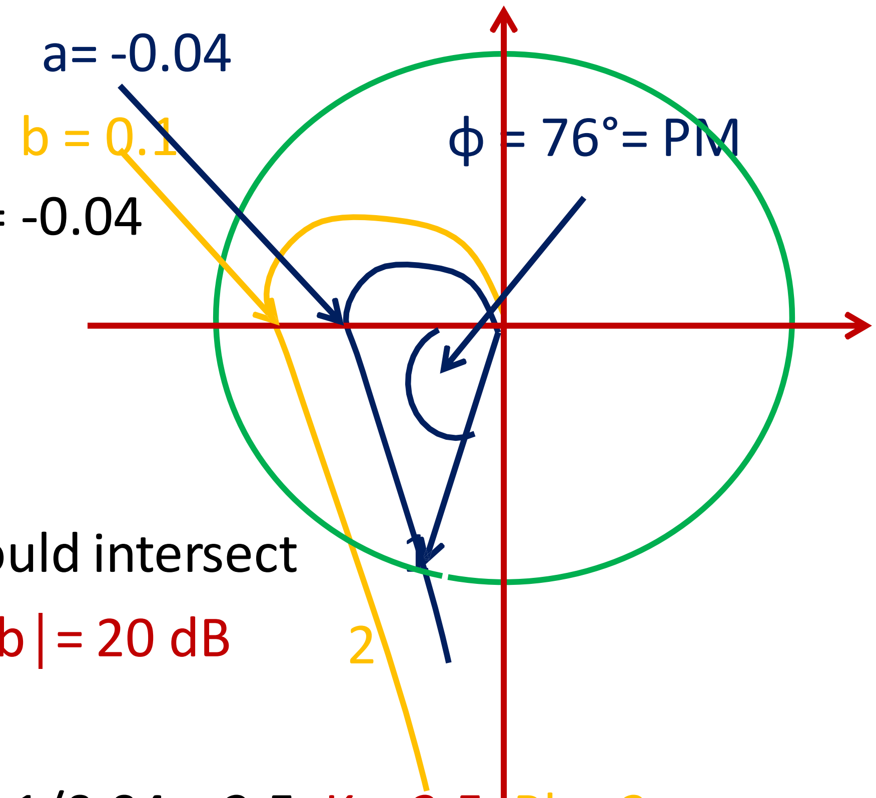
- Suppose we desire a GM = 20 dB, &

  PM = 40°

- For a GM = 20 dB, the polar plot should intersect the negative real axis at : 20 Log |1/b| = 20 dB

  therefore, b = 0.1

- This is achieved if K is increased by 0.1/0.04 = 2.5; K = 2.5. Plot 2

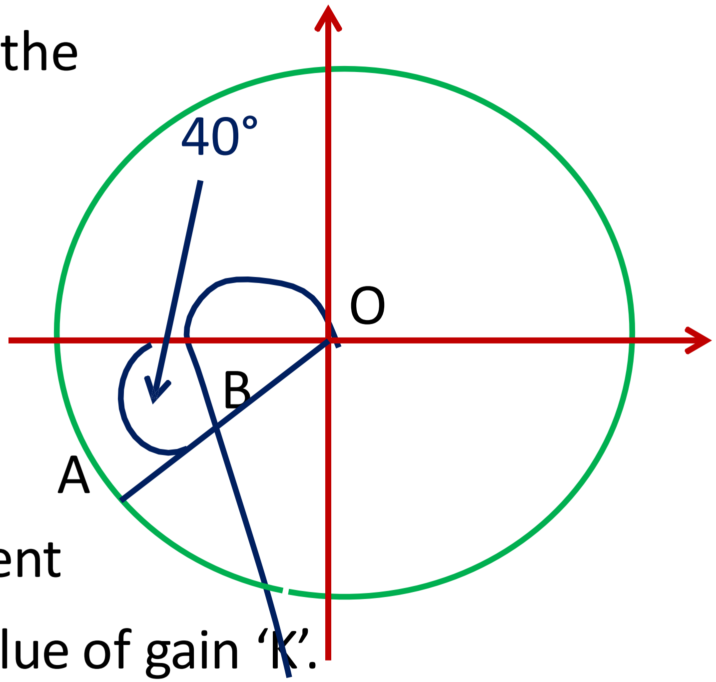


Polar plot Examples: Computation of GM & PM

- To achieve PM = 40°, we have:
- Draw an angle of 40° in CCW direction from the negative real axis of GH plane, as shown
- We see that for PM = 40° , gain 'K' is to be increased by the ratio OA/OB

$$OA/OB = 1/0.191 = 5.24$$

$$K = 5.24$$



- Thus we note that GM & PM are two different
- Specifications not achievable for a single value of gain 'K'.

Analytical Method: Gain & Phase Margin

- **Example:**
- $G(S) = K/S(1+0.2S)(1+.05S) \Rightarrow G(j\omega) = K/j\omega(1+j0.2\omega)(1+j0.05\omega)$
- We know that for determining GM, we need to find intersection on negative real axis (Imaginary part = 0).
- Determine value of ω for which Imaginary part = 0.

Simplify $G(j\omega)$ to get $G(j\omega) = K/[-0.25\,\omega^2 + j\omega\,(1-0.01\,\omega^2)]$

Rationalize $G(j\omega)$ to get,

$$G(j\omega) = -0.25K\,\omega^2/Den \;-\; j\,\omega(1-0.01\,\omega^2)/Den$$

Where, $\quad Den = [(-0.25\,\omega^2)^2 + (\omega(1-0.01\,\omega^2))^2]$

For Imaginary part = 0, $\Rightarrow 1-0.01\,\omega^2 = 0; \Rightarrow \omega = 10 = \omega_1$

ω_1: phase cross over frequency. Magnitude of $G(j\omega)$ at $\omega = \omega_1$

$|G(j\omega)| = K/0.25(\omega_1)^2 = K/25 = a$ (Contd.)

Analytical Method: Gain & Phase Margin

- For a desired GM = 20 dB, we have

$$20 \text{ Log } (1/a) = 20 \text{ , or, } a = 1/10 = 0.1$$

$$K/25 = a; \; K = 2.5$$

- Calculation of PM:

Let $\omega = \omega_2$ be the gain crossover frequency;

PM = 180° + arg{G(jω)}; Desired PM = 40°

$\arg\{G(j\omega)\} = -90° - \tan^{-1}(0.2\,\omega_2) - \tan^{-1}(0.05\,\omega_2)$

$PM = -90° - \tan^{-1}(0.2\,\omega_2) - \tan^{-1}(0.05\,\omega_2) + 180° = 40°$

$\tan^{-1}(0.2\,\omega_2) - \tan^{-1}(0.05\,\omega_2) = 50°$; Apply tan on

$0.25\,\omega_2/[1-0.01\,\omega_2^2] = \tan 50° = 1.2$ rads; $\omega_2 = 4$ rads/sec

$|G(j\omega)|$ at $\omega = \omega_2$ is $= K/[\omega_2 \sqrt{,1+(0.2\,\omega_2)^2} - \sqrt{,1+(0.05\,\omega_2)^2}\} = 1$

For $\omega_2 = 4$, $K = 5.2$

BODE PLOT

- From the frequency response of open loop transfer function G(S) or G(S)H(S), closed loop system stability & relative stability is determined; as in polar plots & root locus methods.

1. We draw two plots for each transfer function
2. Magnitude plot in dB
3. Phase plot
4. Both the plots are drawn on semi log paper
5. Magnitude in dB is given by 20 Log $|G(j\omega)|$

   or 20 Log $|G(j\omega)H(j\omega)|$

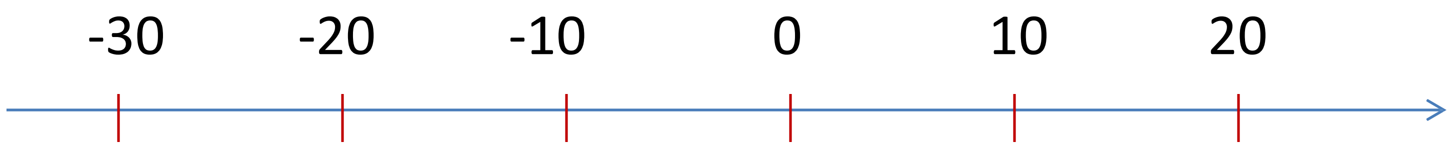
   Angle $\phi(\omega)$ is plotted in degrees

BODE PLOT (Contd..)

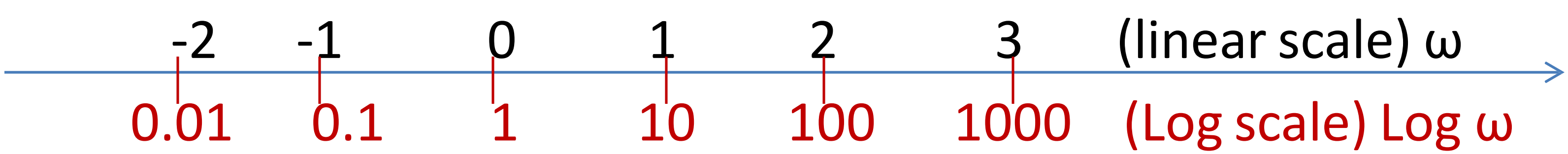
- Note on Log Scale:

  The advantage of Log scale is that we can handle a very large data size

- Linear Scale:



- In linear scale each segment is incremented equally.
- Log Scale:
- In log scale, we decide the multiplication factor ‘x’. Let x = 10



BODE PLOT (Contd..)

- Conversion to Log scale:

  Log **10** ω = 0 (on linear scale) ⇒ ω = 1 (on log scale)

  Log **10** ω = 1 (on linear scale) ⇒ ω = 10 (on log scale)

  Log **10** ω = 2 (on linear scale) ⇒ ω = 100 (on log scale)

  Log **10** ω = -1 (on linear scale) ⇒ ω = 0.1 (on log scale)

  Log **10** ω = -2 (on linear scale) ⇒ ω = 0.01 (on log scale)

- We observe from the above that

  1. on the positive side increment by '1' on linear scale corresponds to multiplication by '10' on the Log scale ,and
  2. on the negative side increment by '-1' on linear scale corresponds to division by '10' on the Log scale
  3. We also observe that on the Log scale we cannot start with a value of ω = 0, but it can assume a very small value

BODE PLOT (Contd..)

- Thus, we observe that increment by ‘1’ on linear scale causes multiplication by ‘10’ on Log scale and hence enabling data compression and thus facilitating usage of large chunks of data.
- Further observations on Log scale:
  1. Between $\omega = 1$ & $\omega = 10$ on the log scale, if we want to mark $\omega = 2$ then we write: Log **10** ^{2} = 0.301 (which is 30.1% of the segment length between ‘1’ & ‘10’ on the Log scale
  2. Between $\omega = 1$ & $\omega = 10$ on the log scale, if we want to mark $\omega = 3$ then we write: Log **10** ^{3} = 0.477 (which is 47.7% of the segment length between ‘1’ & ‘10’ on the Log scale
  3. Between $\omega = 1$ & $\omega = 10$ on the log scale, if we want to mark $\omega = 5$ then we write: Log **10** ^{5} = 0.699 (which is 69.9% of the segment length between ‘1’ & ‘10’ on the Log scale

  Thus we see that the marking is not linear.

BODE PLOT (Contd..)

- Representation of Transfer Functions:
- We have two ways of representing a transfer function:
- Pole-Zero Form:

$$G(S) = K * \prod_{j=1}^{m} (S + Zj)] / * \prod_{i=1}^{n} (S + Pi)] \ ; m \leq n$$

- Time – Constant Form:

$$G(S) = \{K \prod Zj / \prod Pi\} \ \{* \prod_{j=1}^{m} (1+S/ Zj)] / * \prod_{i=1}^{n} (1+ S/Pi)]\}$$

- Let $K_1 = K \prod Zj/\prod Pi$; $Tzj = 1/Zj$; $Tpi = 1/Pi$; Tzj & Tpi are time constants

$$G(S) = K_1 \ * \prod_{j=1}^{m} (1+ Tzj\ S)] / * \prod_{i=1}^{n} (1+ Tpi\ S)]$$

BODE PLOT (Contd..)

- Example:

  Given, $G(S) = 10\ (S + 2)\ (S+4)/(S + 5)\ (S + 10)$ in pole- zero form

  Convert in to time constant form

- Solution:

  $G(S) = (10)(2)(4)(1 + S/2)(1+ S/4)\ /\ (5)(10)(1 + S/5)(1 + S/10)$

  $K_1 = (10)(2)(4)/(5)(10) = 8/5$

  $G(S) = (8/5)\ (1+0.5\ S)(1+0.25S)/(1+0.2S)(1+0.1S)$

- Where, Tz1 = 0.5; Tz2 = 0.25; Tp1 = 0.2; Tp2 = 0.1 are time constants
- Convert Time constant form in to Pole-Zero form:

  $G(S) = (8/5)(.5)(.25)(S + 1/.5)(S + 1/.25)/[(.2)(.1)(S+1/.2)\ (S+1/.1)]$

  $G(S) = K\ (S + 2)(S + 4)/(S + 5)(S + 10)$

  $K = (8/5)(.5)(.25)/(.2)(.1) = 10$

- In Bode & Polar plots we use Time Constant form

BODE PLOT (Method for Drawing)

- Example:

$G(S) = 1/(1+TS) \Rightarrow G(j\omega) = 1/(1 + j\,T\omega)$

$\Rightarrow |G(j\omega)| = 1/\sqrt{(1 + (T\omega)^2)}$; $\arg[G(j\omega)] = -\tan^{-1}(\omega T)$

- The Log – magnitude in dB is given by:

$20 \text{ Log } \mathbf{10}\ |G(j\omega)| = M(\omega) = 20 \text{ Log } \mathbf{10}\ [1/\sqrt{(1 + (T\omega)^2)}]$

$M(\omega) = -10 \text{ Log } \mathbf{10}\ (1 + (T\omega)^2)$ ---------- 1

- Two cases are considered:

1. For $T\omega \lll 1$ (low frequency asymptote); $M(\omega) = 0.0$ because $(T\omega)^2$ can be neglected as compared to '1'
2. For $T\omega \ggg 1$ (high frequency asymptote); $M(\omega) = -20 \text{ Log } \mathbf{10}\ (T\omega)$....... 2; '1' can be neglected

ωT (rads)	M(ω) in dB
1	0
10	-20

ωT (rads)	M(ω) in dB
100	-40
1000	-60 (cont)

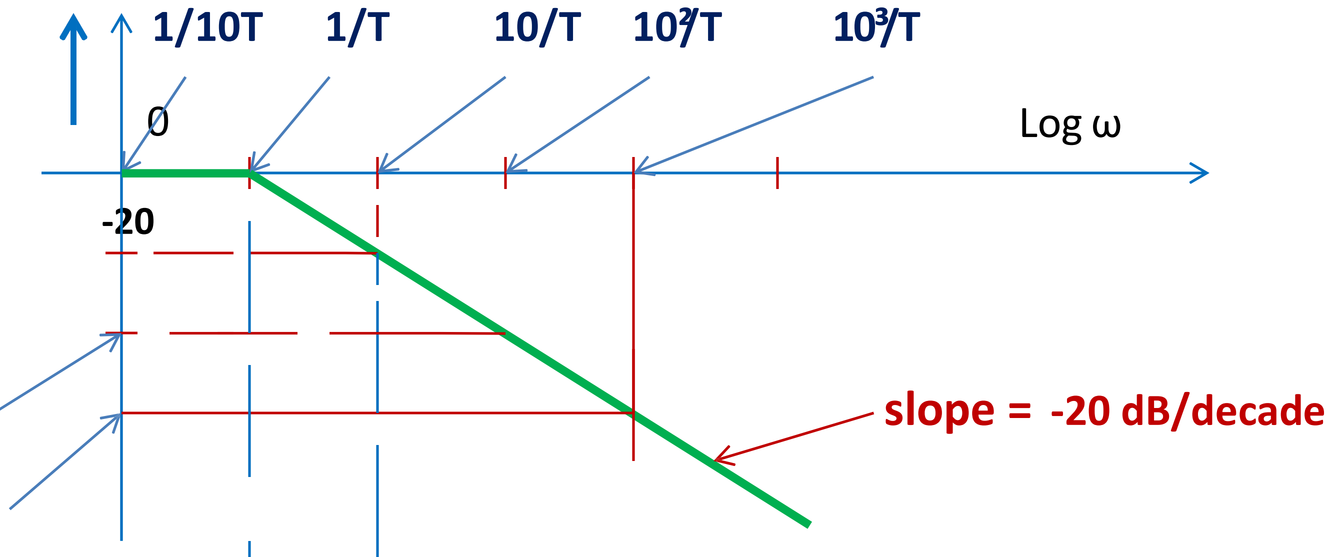
BODE PLOT (Method for Drawing) Contd..

- We observe from the table in the previous slide that,
  1. For a decade change in frequency (1 to 10, 10 to 100, & so on) the magnitude changes by -20 dB.
  2. Therefore the slope of the magnitude plot is -20 dB/decade change in frequency.
- We have two plots: for $\omega T <<< 1$ & $\omega T >>> 1$
- For $\omega T <<< 1$; $M(\omega) = 0$ & for $\omega T >>> 1$; $M(\omega)$ has slope of -20 dB/decade
- At $\omega T = 1$; $M(\omega)$ in equation (2) = 0 dB & $M(\omega)$ in equation (1) =0 therefore the two meet at $\omega T = 1$, if we extend the low frequency asymptote; (as they are both = 0)
- This meeting point is called 'Corner Frequency' & is derived from $\omega T = 1$; or, $\omega = 1/T$ is the corner frequency.

BODE PLOT (Method for Drawing) Contd..

- The Log-magnitude in dB is plotted as:

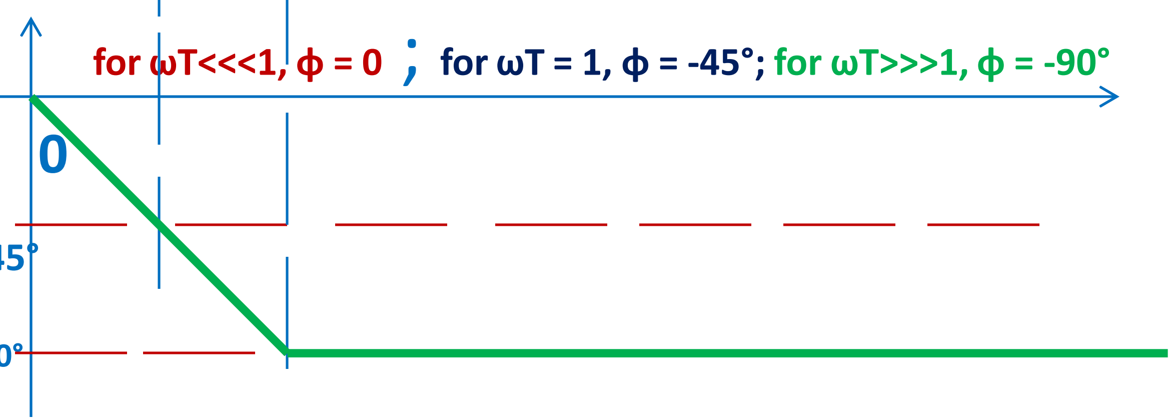
mag. in dB



(Log mag. Plot – semi log graph paper)

- The Angle Plot:

φ(ω)



Bode Plots: Different types of Transfer Functions

- Example: First order 'zero'

$$G(S) = (1+TS) \qquad G(j\omega) = (1 + j\,T\omega)$$

$$|G(j\omega)| = \sqrt{(1 + (T\omega)^2)}\ ; \qquad \arg[G(j\omega)] = \tan^{-1}(\omega T)$$

- The Log - magnitude in dB is given by:

$$20 \text{ Log } \mathbf{10}\ |G(j\omega)| = M(\omega) = 20 \text{ Log } \mathbf{10}\ [\sqrt{(1 + (T\omega)^2)}]$$

$$M(\omega) = 10 \text{ Log } \mathbf{10}\ (1 + (T\omega)^2) \text{ ---------- } 1$$

- Two cases are considered:

1. For $T\omega <<< 1$ (low frequency asymptote); $M(\omega) = 0.0$ because $(T\omega)^2$ can be neglected as compared to '1'

2. For $T\omega >>> 1$ (high frequency asymptote); $M(\omega) = 20$ Log **10** $(T\omega)$… 2; '1' can be neglected

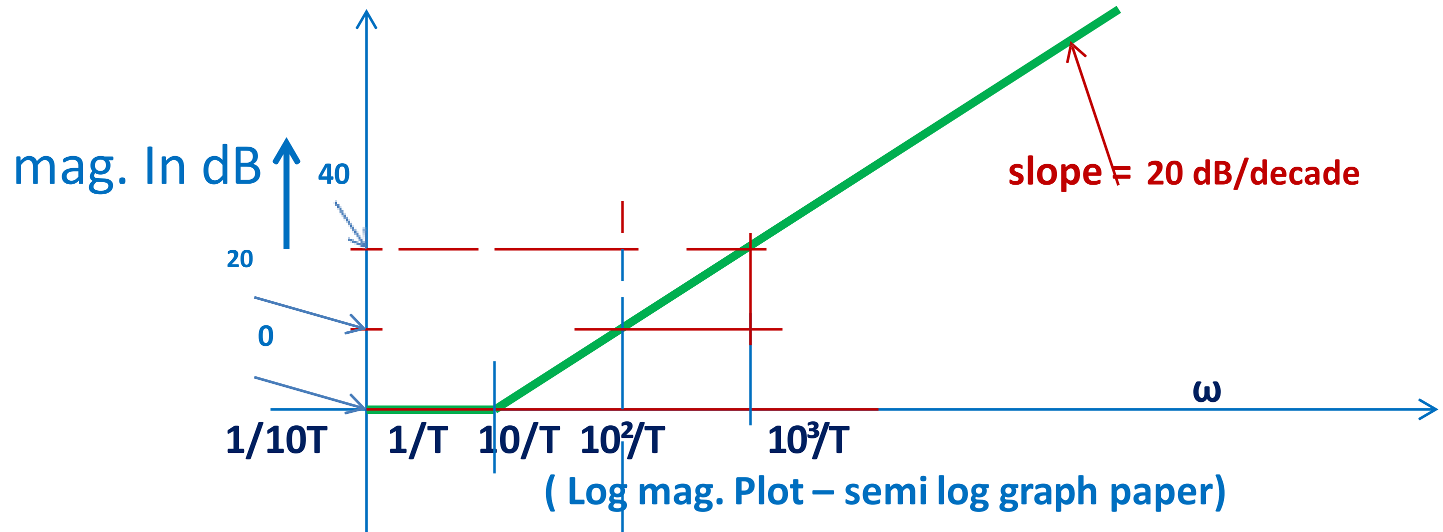
ωT (rads)	M(ω) in dB	ωT (rads)	M(ω) in dB
1	0	100	40

Bode Plots: Different types of Transfer Functions

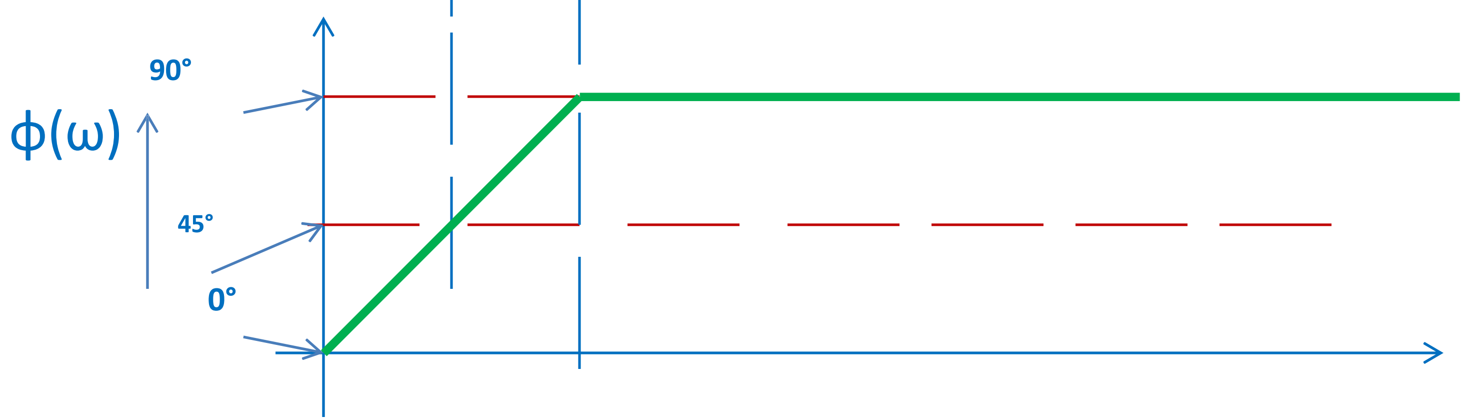
- We observe from the table in the previous slide that,
  - For a decade change in frequency (1 to 10, 10 to 100, & so on) the magnitude changes by 20 dB.
  - Therefore the slope of the magnitude plot is 20 dB/decade change in frequency.
  - We have two plots: for $\omega T <<< 1$ & $\omega T >>> 1$
  - For $\omega T <<< 1$; $M(\omega) = 0$ & for $\omega T >>> 1$; $M(\omega)$ has slope of 20 dB/decade
  - At $\omega T = 1$; $M(\omega)$ in equation (2) = 0 dB & $M(\omega)$ in equation (1) =0 therefore the two meet at $\omega T = 1$, if we extend the low frequency asymptote; (as they are both = 0)
  - This meeting point is called 'Corner Frequency' & is derived from $\omega T = 1$; or, $\omega = 1/T$ is the corner frequency.

Bode Plots: Different types of Transfer Functions

- The Log-magnitude in dB is plotted as:



- The Angle Plot: **for ωT<<<1, ϕ = 0 ; for ωT = 1, ϕ = 45°; for ωT>>>1, ϕ = 90°**



Bode Plots: Different types of Transfer Functions

- Example:

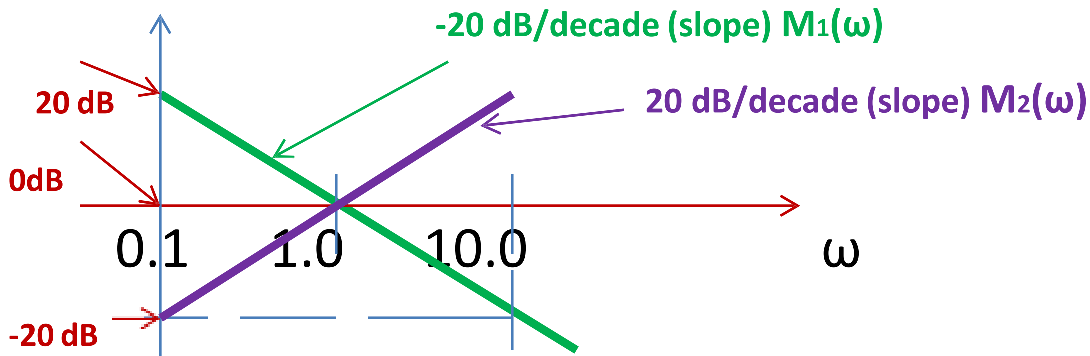
Consider 1) $G1(S) = 1/S$ & 2) $G2(S) = S$

1) $G1(j\omega) = 1/j\omega$; $|G1(j\omega)| = 1/\omega$ & $G2(j\omega) = j\omega$; $|G2(j\omega)| = \omega$
2) The Log – magnitude in dB is given by:

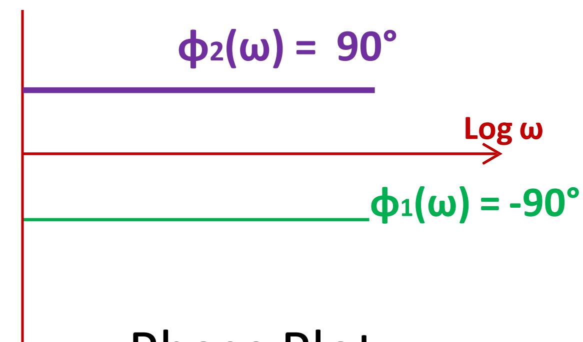
$20 \text{ Log } \mathbf{10} \, |G\mathbf{1}(j\omega)| = M\mathbf{1}(\omega) = 20 \text{ Log } \mathbf{10} \, [1/\omega] = -20 \text{ Log } \mathbf{10} \, (\omega)$

$20 \text{ Log } \mathbf{10} \, |G\mathbf{2}(j\omega)| = M\mathbf{2}(\omega) = 20 \text{ Log } \mathbf{10} \, [\omega] = 20 \text{ Log } \mathbf{10} \, (\omega)$

Angle : $\phi\mathbf{1}(\omega) = -90°$ Angle : $\phi\mathbf{2}(\omega) = 90°$



Log-magnitude plot



Phase Plot

Bode Plots: Different types of Transfer Functions

- We have drawn Bode plots for first order transfer functions having a simple (order 1) pole or a simple (order 1)zero. We now generalize it to multiple order poles & zeros which may be present in a given transfer function.

$G\mathbf{1}(S) = 1/(1 + TS)^m$ (pole of order ‘m’), &

$G\mathbf{2}(S) = (1 + TS)^m$ (zero of order ‘m’)

$G\mathbf{1}(j\omega) = 1/(1 + j\,T\omega)^m$; $|G\mathbf{1}(j\omega)| = 1/*\surd(1+(\omega T)^2+$

Log-magnitude (in dB) = $20\ \mathrm{Log}\mathbf{10}\ \{1/*\surd(1+(\omega T)^2+^m-$

$= -10\ m\ \mathrm{Log}\mathbf{10}\ \{(1+(\omega T)^2+$ 1

Angle = $-\ m \tan^{-1}(\omega T)$

$G\mathbf{2}(j\omega) = (1 + j\,T\omega)^m$; $|G\mathbf{2}(j\omega)| = *\surd(1+(\omega T)^2+$

Log-magnitude (in dB) = $20\ \mathrm{Log}\mathbf{10}\ *\surd(1+(\omega T)^2+$

$= 10\ m\ \mathrm{Log}\mathbf{10}\ \{(1+(\omega T)^2+$ 2

Angle = $m \tan^{-1}(\omega T)$

Bode Plots: Different types of Transfer Functions

- For G**1**(S) :

  Log-magnitude (in dB) = -10 m Log**10** {(1+(ωT)^2]

- For G**2**(S) :

  Log-magnitude (in dB) = 10 m Log**10** {(1+(ωT)^2]

- Thus we observe that, for ωT>>>1, the

  slope of log-mag. plot for pole of order 'm' = -20 m dB/decade

  slope of log-mag. plot for zero of order 'm' = 20 m dB/decade

- While the respective angles are given by

  $$-/+\ m \tan^{-1}(\omega T)$$

  where m = 1,2,3 … is the order of the pole & zero. So as 'm' increases the slopes and the angle increase.

Bode Plots: Different types of Transfer Functions

- Multiple Poles & Zeros at the Origin of the S plane:

  Consider 1) $G\mathbf{1}(S) = 1/S^m$ & 2) $G\mathbf{2}(S) = S$

  1) $G\mathbf{1}(j\omega) = 1/(j\omega)^m$; $|G\mathbf{1}(j\omega)| = 1/\omega^m$ & $G\mathbf{2}(j\omega) = (j\omega)^m$; $|G\mathbf{2}(j\omega)| = \omega$
  2) The Log – magnitude in dB is given by:

  $20 \text{ Log } \mathbf{10}\ |G\mathbf{1}(j\omega)| = M\mathbf{1}(\omega) = 20 \text{ Log } \mathbf{10}\ [1/\omega^m] = -20 \text{ m Log } \mathbf{10}\ (\omega)$

  $20 \text{ Log } \mathbf{10}\ |G\mathbf{2}(j\omega)| = M\mathbf{2}(\omega) = 20 \text{ Log } \mathbf{10}\ [\omega^m] = 20 \text{ m Log } \mathbf{10}\ (\omega)$

  Angle : $\phi\mathbf{1}(\omega) = -\ m\ 90°$ Angle : $\phi\mathbf{2}(\omega) = m\ 90°$

- Here again we observe that the slope for log-magnitude plot of

  G**1**(S) is -20m dB/decade & angle is –m 90°, &

  G**2**(S) is 20m dB/decade & angle is m 90°

- where, m = 1,2,3 …. Is the order of the pole and zero

  As ‘m’ increases, slopes & angle increase

Bode Plots: Different types of Transfer Functions

- $G(S) = K(1+T_1 S)(1+T_2 S)/S^m(1 + T_3 S)(1 + T_4 S)$

  We have a combination of poles & zeros. There can be any number of poles & zeros in a transfer function. We need to plot Log-magnitude plot in dB & Angle plot in degrees

- Log-magnitude plot:

  $G(j\omega) = K(1 + j T_1\omega)(1 + j T_2\omega)/(j\omega)^m(1 + j T_3\omega)(1 + j T_4\omega)$

  $20 \log |G(j\omega)| = 20 \log |K(1+j T_1\omega)(1+j T_2\omega)/(j\omega)^m(1+j T_3\omega)(1+j T_4\omega)|$

  $20 \log K + 20 \log \sqrt{(1+ (T_1\omega)^2} + 20 \log \sqrt{(1 + (T_2\omega)^2}$

  $-20\, m \log \omega - 20 \log \sqrt{(1 + (T_3\omega)^2} - 20 \log \sqrt{(1+(T_4\omega)^2}$ ….. 1

- From equation (1) we make out that log-magnitude plot in dB, for a given G(S), is obtained by algebraically adding asymptotic plot of each pole & zero including the constant gain term 'K'

Bode Plots: Different types of Transfer Functions

- Example:

$$G(S) = 10\,(1+S)(1+10S)/S(1+5S)(1+20S)$$

- Bode Plot:

$$G(j\omega) = 10(1+j\,1\omega)(1+j\,10\omega)/j\omega(1+j\,5\omega)(1+j\,20\omega)$$

1. $K = 10$; magnitude in dB = 20 log 10 = 20 dB
2. $(1+j1\omega)$; corner frequency $\omega T = 1$; $\omega = 1/T$; $\omega = 1$; up to $\omega = 1$, magnitude = 0; for $\omega \geq 1$, magnitude plot has a slope of 20 dB/decade
3. $(1+j\,10\,\omega)$; corner frequency $\omega T = 1$; $\omega = 1/10$; $\omega = 0.1$; up to $\omega = 0.1$, magnitude = 0; for $\omega \geq 0.1$, magnitude plot has a slope of 20 dB/decade
4. ω; corresponds to pole at origin; magnitude plot has a slope of -20 dB /decade

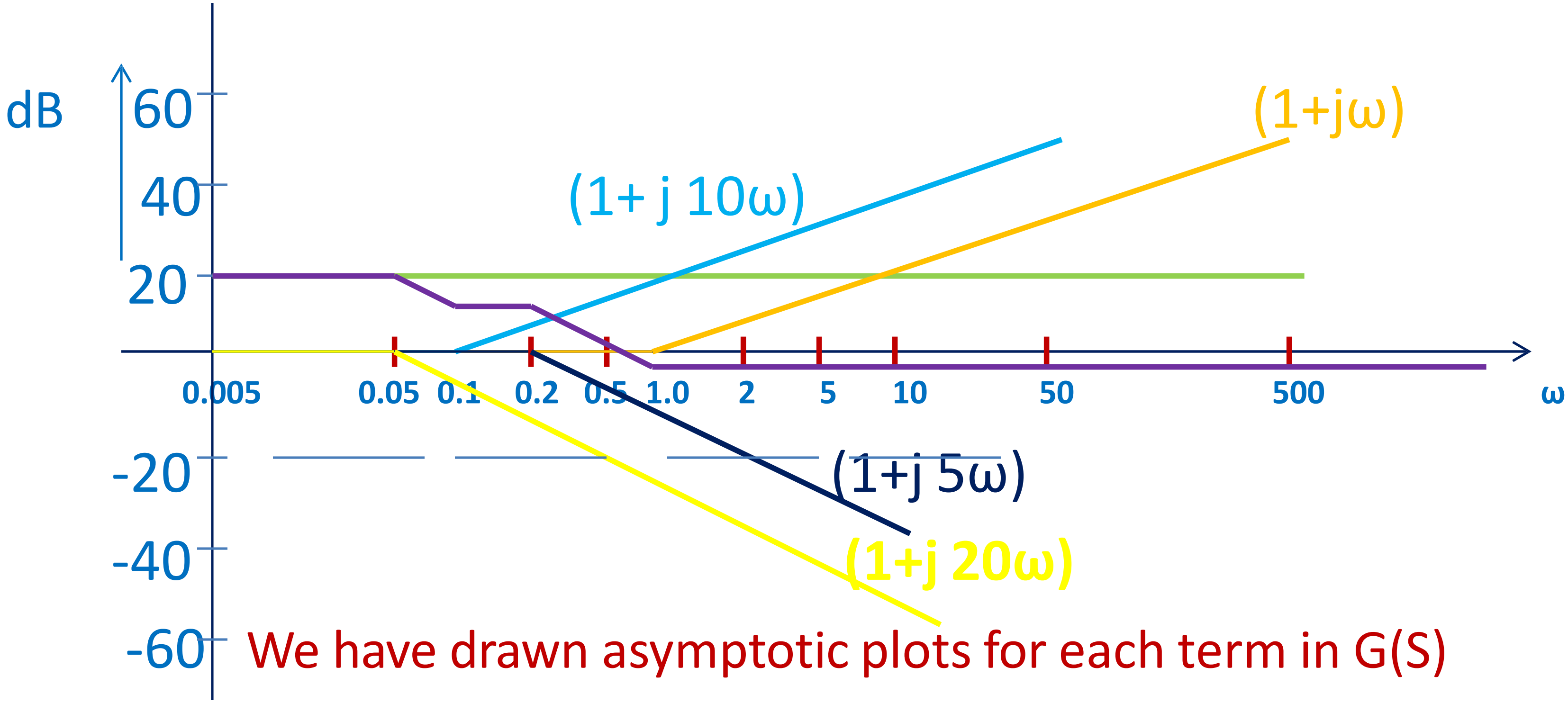
Bode Plots: Different types of Transfer Functions

- (1+j5ω); corner frequency ωT = 1; ω= 1/5; ω =0.2; up to ω= 0.2, magnitude = 0; for ω≥0.2, magnitude plot has a slope of -20 dB/decade
- (1+j 20 ω); corner frequency ωT = 1; ω= 1/20; ω =0.05; up to ω= 0.05, magnitude = 0; for ω≥0.05, magnitude plot has a slope of -20 dB/decade.
- The lowest corner frequency is 0.05; therefore we take lowest frequency in log ω scale as 0.005

The complete log- magnitude plot is shown in the next slide

Bode Plots: Different types of Transfer Functions

- Complete log-magnitude plot: —— **complete log-magnitude plot**

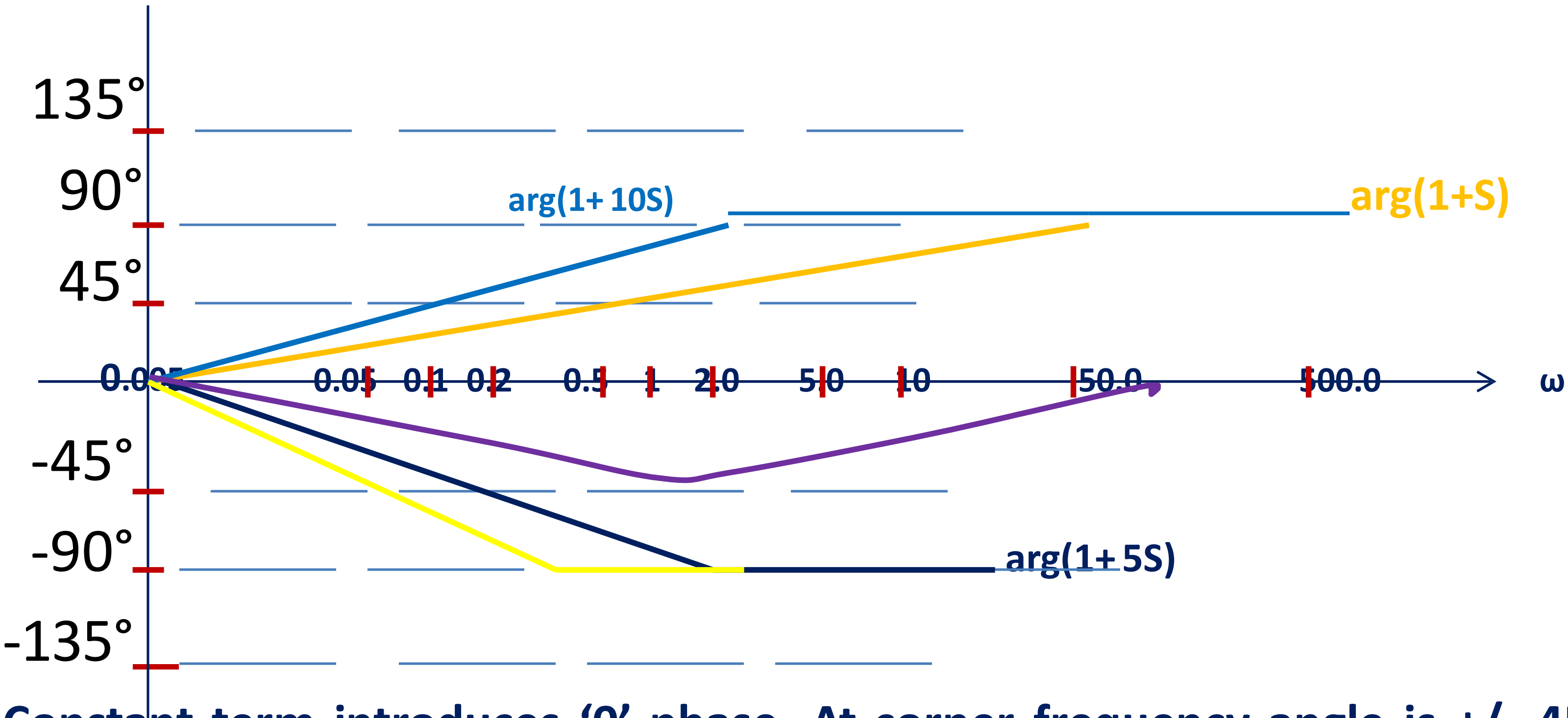


We have drawn asymptotic plots for each term in G(S)

- Now, we algebraically add all the plots keeping in mind that slope change occurs at corner frequency only; corner frequencies are 0.05, 0.1; slope change begins at these frequencies.

Bode Plots: Different types of Transfer Functions

- Complete Angle plot:____ complete Angle plot



- Constant term introduces ‘0’ phase. At corner frequency angle is +/- 45°. At ten times the corner frequency angle can be taken as +/- 90°. These are asymptotic plots for angle of each term in G(S).
- Complete Angle plot is obtained by algebraically adding all the individual plots.

BODE PLOT: For 2nd order Under damped Transfer Functions

- Under damped systems have complex conjugate poles. Let us consider normalized form of a second order system, given by

$$G(j\,u) = 1/(1 + j2\xi u - u^2);$$

$$|G(j\,u)| = 1/\sqrt{*}(1-u^2)^2 + (2\xi u)^2]$$

- The log-magnitude plot is given by

$$20 \log |G(j\,u)| = M(u) = -10 \log[(1-u^2)^2 + (2\xi u)^2]$$

For $u \lll 1$; higher order terms in u are neglected to obtain

$$M(u) = 0 \text{ dB}$$

For $u \ggg 1$; $M(u) = -10 \log u^4 = -40 \log u$; $(2\xi u)^2 \ll u^4$ because $\xi < 1$

- Therefore, log magnitude plot consists of 2 straight line asymptotes

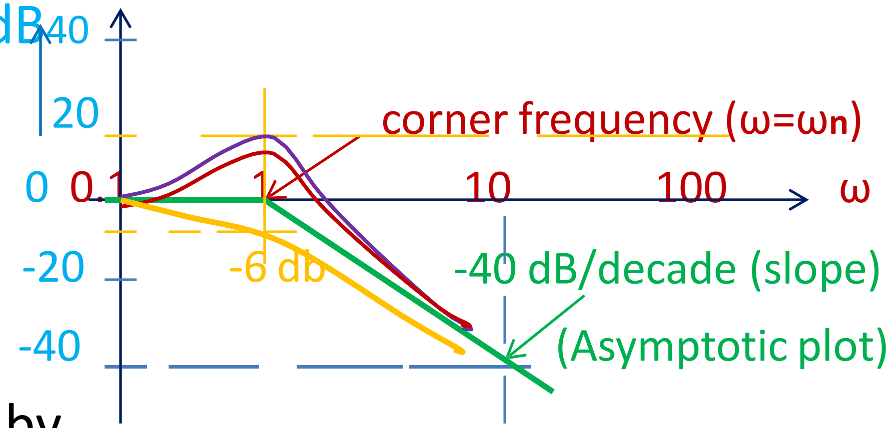
  - one horizontal line at '0' dB for $u \lll 1$

  - the other, a line with a slope of -40 dB/decade for $u \ggg 1$

- These 2 asymptotes meet on '0'dB line at u = 1; i.e. at $\omega = \omega$**n**.

BODE PLOT: For 2nd order Under damped Transfer Functions

- The asymptotic plot for 2nd order system is:
- Asymptotic plots are approx. dB plots; error at u = 1.



- Exact Plot:

  The log-magnitude plot is given by

  $M(u) = -10 \log[(1-u^2)^2 + (2\xi u)^2]$; Actual plots are drawn around Asymptotic plot.

  We directly substitute for u = 1 & determine M(u) for different ξ values. M(u), u=1, is function of ξ.

u=1	M(u)
$\xi = 0.05$	20 dB
$\xi = 0.1$	14 dB
$\xi = 1.0$	-6 dB

BODE PLOT: For 2nd order Under damped Transfer Functions

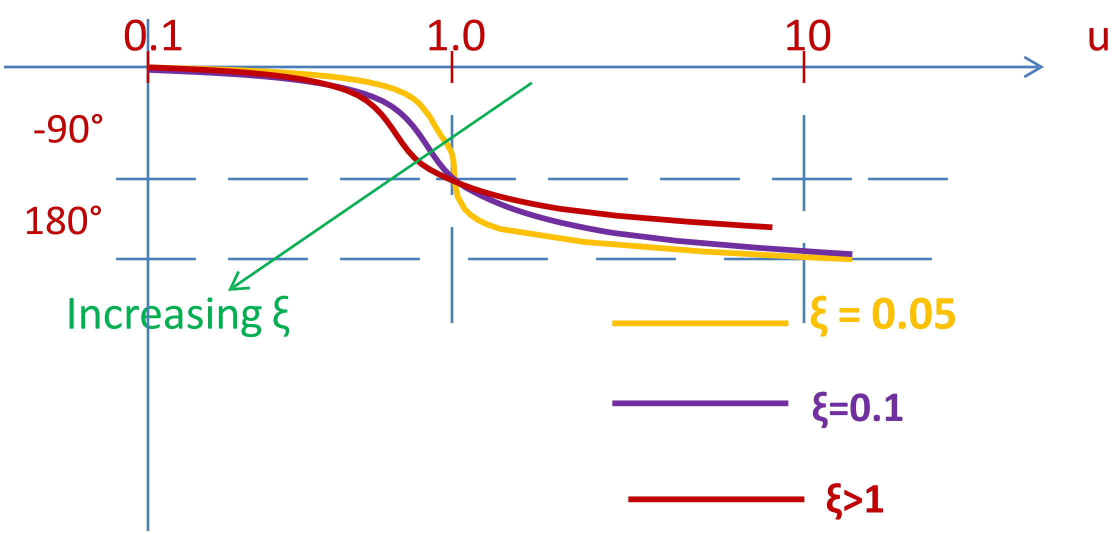
- The Phase Plot:

  The phase angle is given by: $\phi(u) = -\tan^{-1}(2\xi u/1-u^2)$;

  We observe that $\phi(u)$ is a function of u & ξ. However, at u=1, for any value of ξ, $\phi(u) = -90°$.

  for $u = 0$; $\phi(u) = 0$ & for $u = \infty$, $\phi(u) = -180°$

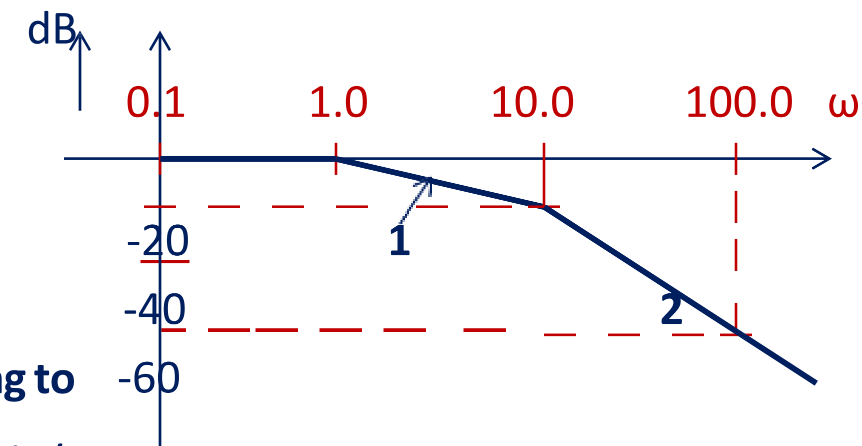
- For $0<u<1$ & $1<u<\infty$, $\phi(u)$ is dependent on ξ value.



Determination of Transfer Function from Bode Plot

- The problem of Synthesis:
- Given a transfer function, we know how to draw Bode plot.
- Now we will have the reverse problem:

Given the Bode (log-magnitude) plot how to determine the transfer function. This is the process of system identification from a given frequency response.



Solution:

Slope of plot '1' = -20 dB/decade

Slope of plot '2' = -40 dB/decade

Corner frequency (ωT = 1) corresponding to

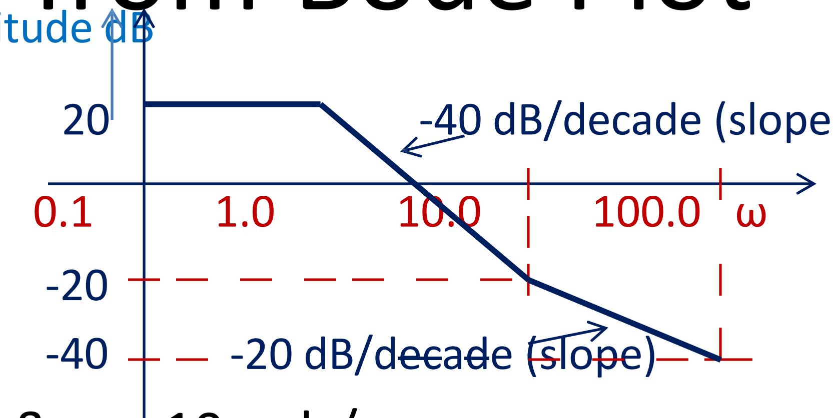
plot '1' = 1 rad/sec & plot '2' = 0.1 rads/sec

- The gain up to 1st corner frequency (= 1 rad/sec) = 0 dB; therefore K = 1

The transfer function, G(S) = 1/(1 + S)(1 + 0.1S)

Determination of Transfer Function from Bode Plot

- Determine G(S)



- Corner frequencies are at ω = 1 & ω = 10 rads/sec

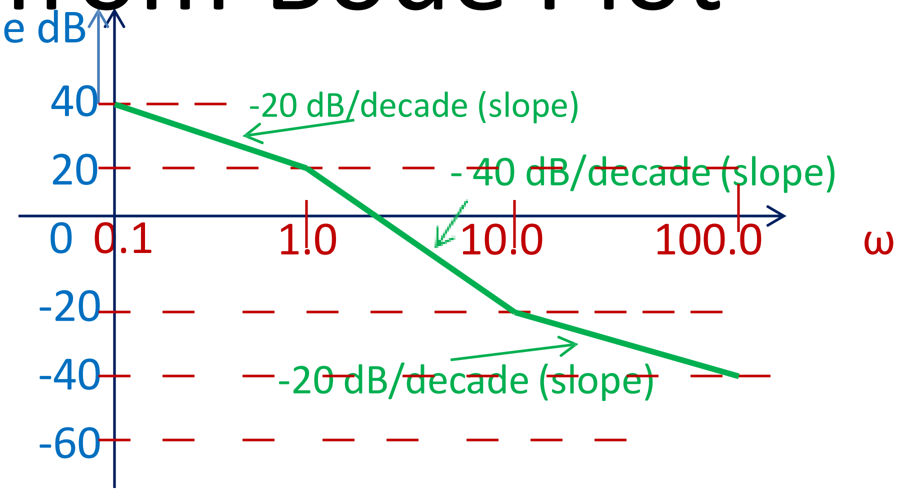
  Up to ω = 1 rads/sec, the gain(magnitude) = 20 dB. We determine 'K' from it. 20 Log **10** K = 20 dB; therefore K = 10.

- At ω = 1 rads/sec, magnitude plot falls with a slope of -40 dB/decade. This corresponds to a double pole term like,$1/(1+S)^2$ in G(S). From ω = 10 rads/sec, the slope changes to -20 dB/decade, therefore there is a zero term like (1 + 0.1S) in G(S).

- Therefore $G(S) = K\,(1 + 0.1S)/(1 + S)^2$

Determination of Transfer Function from Bode Plot

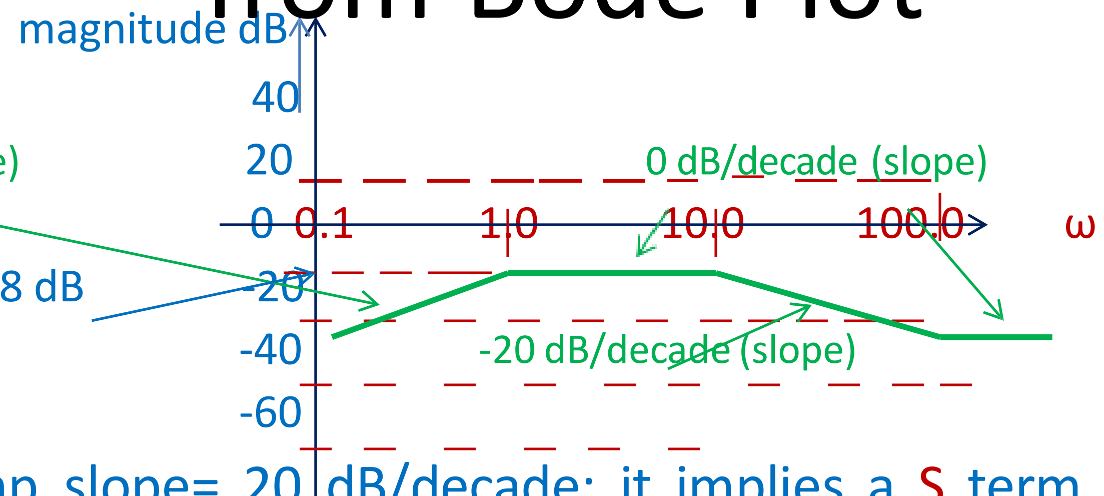
- Determine G(S):



- There is a ramp with a slope: -20 dB/decade, starting at ω = 0.1 r/s. It implies a term 1/S in G(S). At ω = 1 r/s; its magnitude should be '0' dB, but it is 20 dB. It implies 'K' = 10 in G(S). From ω = 1 r/s to ω= 10 r/s, the slope is -40 dB/decade. It implies a term 1/(1 + S) in G(S). From ω = 10 r/s to ω= 100 r/s, the slope is -20 dB/decade. It implies a term (1 + 0.1 S) in G(S).
- Therefore, the transfer function is: G(S) = 10 (1+ 0.1 S)/S(S + 1)

Determination of Transfer Function from Bode Plot

- Determine G(S):



- Starting, there is a ramp slope= 20 dB/decade; it implies a S term in G(S); its magnitude should = 0 at ω = 1 r/s, but it is not so. It implies a gain term 'K' in G(S). To determine 'K' we write
- 20 Log K + 20 log ω = -8 at ω = 1 r/s; or, 20 log K = -8; K = 0.3981
- From ω = 1 to 10 r/s ; slope is '0'; implies a term 1/(S +1) in G(S). From ω = 10 to 100 r/s ; slope is -20 dB/decade; implies a term 1/(1+ 0.1S) in G(S). From ω=1000 r/s onwards, the slope is '0'; implies a term (1 + 0.01 S) in G(S).
- Therefore, G(S) = 0.3981 (1 + 0.01 S)/(S + 1)(1 + 0.01 S)

Nyquist Method for finding Stability of CL System

- Stability study is carried out graphically from the open loop frequency response.
- Nyquist Stability Criterion:
- The characteristic equation: $Q(S) = 1 + G(S)H(S) = 0$

  $G(S)H(S) = K\ (S+Z_1)(S+Z_2)\ \ldots\ldots(S + Z\mathbf{m})/(S+P_1)(S+P_2)\ldots\ (S + P\mathbf{n});\ m \leq n$

  $Q(S) = 1+ K\ (S+Z_1)(S+Z_2)\ \ldots\ldots(S + Z\mathbf{m})/(S+P_1)(S+P_2)\ldots\ (S + P\mathbf{n})$

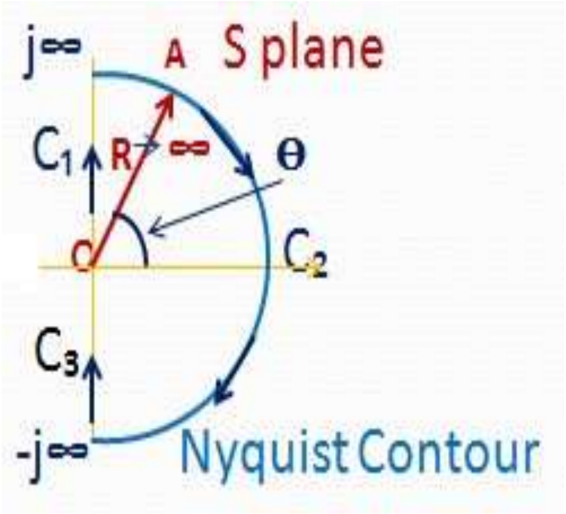
  On simplification, we write:

  $Q(S) = (S+Z_1')(S+Z_2')\ \ldots\ldots(S + Z\mathbf{n}')/(S+P_1)(S+P_2)\ldots\ (S + P\mathbf{n})$
- We observe that
  - Zeros of Q(S) at $S = -Z_1'$, $S = -Z_2'$, ……$S = -Z\mathbf{n}'$ are the roots of the characteristic equation
  - Poles of Q(S) at $S = -P_1$, $S = -P_2$, … $S = -P\mathbf{n}$ are the same as open loop poles of the system
  - For stable system, zeros of Q(S), roots of characteristic equation, must be in the LH of the S-plane.

Nyquist Method for finding Stability of CL System

- Even if some open loop poles lie in the RH of the S plane, all the zeros of Q(S), poles of CL system, must lie in the LH of the S plane. It means that an unstable open loop system can be made stable with an appropriate design of CL system.
- The Nyquist Contour:

  Since we interested in finding out whether there are any zeros of Q(S) in the RH of the S plane, we choose a contour that completely encloses RH of the S plane. This is called Nyquist Contour.



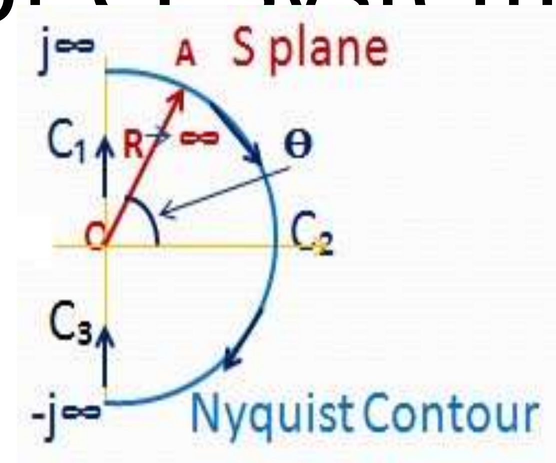
- In CW direction, starting from the origin of the S plane, we traverse Nyquist Contour. along the paths C_1 C_2 and C_3.

  Since $R \rightarrow \infty$, entire RH is enclosed

Nyquist Method for finding Stability of CL System

- From the Nyquist Contour we observe that for $S = j\omega$, along path C_1 frequency, ω, varies from '0' to ∞

  along path C_3 frequency, ω, varies from $-\infty$ to '0'.



- The path C_2 is a circle of infinite radius ($R \rightarrow \infty$). Any point on C_2 can be represented in polar form as: $S = R \exp(+/- j\theta)$. Along C_2, while traversing in the direction of arrows, the angle Θ varies from 90° to -90°.
- The Nyquist Contour as defined in the aforesaid lines, encloses all the right half S plane zeros & poles of $1 + G(S)H(S)$.

Nyquist Method for finding Stability of CL System

- The Stability Criterion & Nyquist Theorem:

  Let,

  Z: be the number of zeros of Q(S) in RH of the S plane

  P: be the number of poles of Q(S) in RH of the S plane

- Nyquist Theorem:

  As point $S = S_0$ moves along the Nyquist contour in the S plane, in the Q(S) plane a closed contour Γ**q** is traversed which encloses the origin 'N' times in CCW direction; where N = P-Z.

- For every point $S = S_0$ on the Nyquist contour, Q(S) has a value. If we plot the values of Q(S) in the plane called 'Q(S) plane', then, according to Nyquist theorem, we will obtain a closed path, Γ**q,** which will enclose the origin of 'Q(S) plane' 'N' times.

- Stability Criterion:

  We know that zeros of Q(S), Z, are the closed loop system poles &

  therefore should lie in the LH of the S plane for system stability.

Nyquist Method for finding Stability of CL System

- Stability Criterion (contd.):

  Therefore, Z = 0 (for stable CL system).

- So for a stable CL system, we have two situations:

  for P ≠ 0:

  $\Rightarrow$ N = P-Z = P

  that the CCW encirclements of the origin of 'Q(S) plane' should be equal to the number of poles, P, of Q(S) (open loop poles of G(S)H(S)) in the RH of the S plane.

- The above assertion implies that even if the open loop system is unstable, the CL system can be stable.

- For P = 0: (no poles of G(S)H(S) in RH of the S plane) the number of encirclements N = 0 for a stable CL system

Nyquist Method for finding Stability of CL System

- Modified Stability Criterion:

  We know that, $Q(S) = 1 + G(S)H(S)$

  $\Rightarrow G(S)H(S) = Q(S) - 1$

- Therefore, we say that while,

  Γq encircles the origin in Q(S) plane

  Γ**GH** will encircle (-1,j0) point in the GH plane

- In G(S)H(S) plane, we state the Nyquist Stability Criterion as:

  For $P \neq 0$:

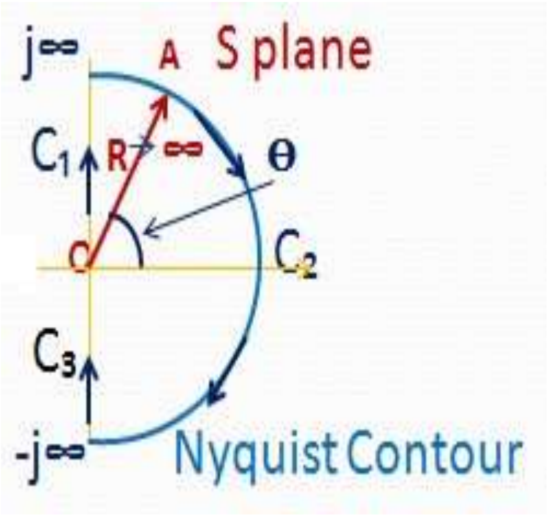
  If the contour Γ**GH** of the open loop transfer function G(S)H(S), corresponding to the Nyquist contour in the S plane, encircles the point (-1,j0) in the CCW direction as many times as the number of right of S-plane poles of G(S)H(S), the CL system is stable.

  For $P = 0$: The CL system is stable if no encirclements of (-1,j0) point.

Nyquist Method for finding Stability of

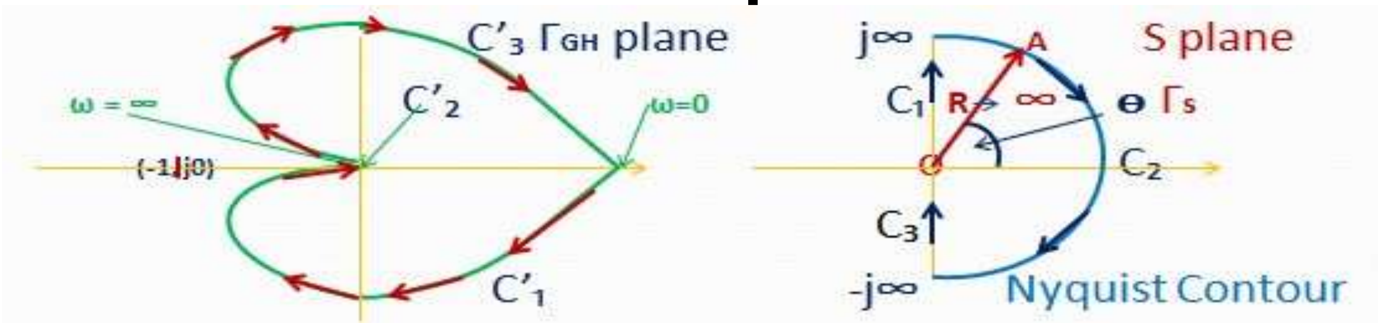
- **Mapping of Nyquist contour in to GH contour:**

  Following steps are followed:



1. Convert G(S)H(S) in to G(jω) H(jω)
2. For S = jω; $0 \le \omega \le \infty$ (segment C_1) draw polar (Nyquist) plot in GH plane
3. For contour C_2: $S = R\exp(j\Theta)$; $R \longrightarrow \infty$. Substitute $S = R\exp(j\Theta)$ in G(S)H(S) and let $R \longrightarrow \infty$ for $\infty \le S \le -\infty$. The entire segment maps to '0' in the GH plane
4. For $-\infty \le \omega \le 0$ (segment C_3) draw polar plot for negative frequencies; which is mirror image of plot for C_1.

Nyquist Method: Examples



- G(S)H(S) = $K/(1+T_1 S)(1+T_2 S)$; C'$_3$ is mirror image of C'$_1$

1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'$_1$.
2. Corresponding to C_2 in Γ**s** plane we have; $S = R \exp(j\theta)$ in G(S)H(S); $R \longrightarrow \infty$

   $G(S)H(S) = K/(T_1 R e^{j\theta} + 1)(T_2 R e^{j\theta} + 1)$ as $R \longrightarrow \infty$ therefore

   $G(S)H(S) = 0\, e^{-j2\theta}$; $|G(S)H(S)| = 0$; $\arg\{G(S)H(S)\} = -2\theta$

   On C_2 ; θ varies from +90° to -90° as we move from +j∞ to -j∞

   arg {G(S)H(S)} varies from -180° to + 180° . This is C'$_2$ in Γ**GH** plane.
3. C_3 in Γ**S** plane is mapped as C'$_3$ (Nyquist plot) in Γ**GH** plane. (Contd.)

Nyquist Method: Examples (Contd..)

- For the example in previous slide:

  We have drawn the Nyquist plot for a given G(S)H(S). Now we need to determine the stability of its closed loop system.

- The number of encirclements, N, of (-1,j0) point is given by:

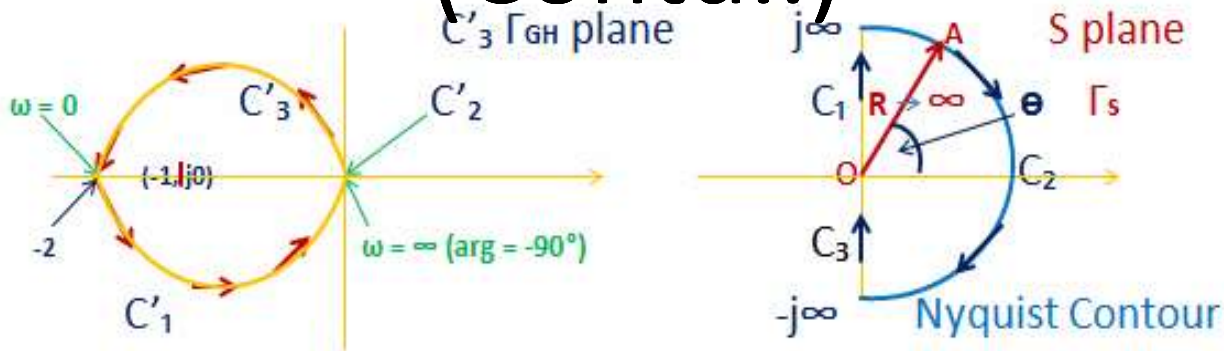
$$N = P - Z$$

- For closed loop system to be stable, Z = 0

  In this example, P = 0 because no poles of G(S)H(S) are in the RH of S plane.

- Therefore N should be equal to '0', i.e. that there should be no encirclement of (-1,j0) point. We see from the Nyquist diagram that it does not encircle (-1,j0) point & hence the closed loop system is stable.

Nyquist Method: Examples (Contd..)



- $G(S)H(S) = (S+2)/(S+1))(S-1)$; C'_3 is mirror image of C'_1

1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'_1.
2. Corresponding to C_2 in Γ**s** plane we have; $S = R\exp(j\Theta)$ in G(S)H(S);

   $R \longrightarrow \infty$

   $G(S)H(S) = (2+ R\,e^{j\theta})/(1+ R\,e^{j\theta})(R\,e^{j\theta} -1)$ as, $R \longrightarrow \infty$ therefore

   $G(S)H(S) = 0\,e^{-j\theta}$; $|G(S)H(S)| = 0$; $\arg\{G(S)H(S)\} = -\Theta$

   on C_2 ; ϴ varies from +90° to -90° as we move from +j∞ to -j∞

   arg{G(S)H(S)} varies from -90° to +90° . This is C'_2 in Γ**GH** plane.
3. C_3 in Γ**S** plane is mapped as C'_3 (Nyquist plot) in Γ**GH** plane. (Contd. .)

Nyquist Method: Examples (Contd..)

- Having drawn the Nyquist diagram, we need to determine the stability of related CL system.
- Observation:

G(S) H(S) has a pole in the RH of the S plane; therefore P = 1

$$N = P - Z$$

Z = 0 for stable CL system

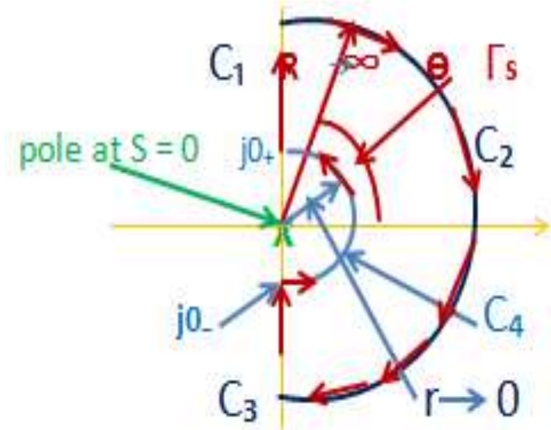
Therefore, $N = P = 1$

⇒ that the Nyquist plot should encircle (-1,j0) point once in the CCW direction for the CL system to be stable.

- From the Nyquist diagram we that it is encircling (-1,j0) point once in CCW direction. Therefore, the CL system is stable

Nyquist Method: Examples (Contd..)

- Case: G(S)H(S) has a pole at the origin of the S plane:



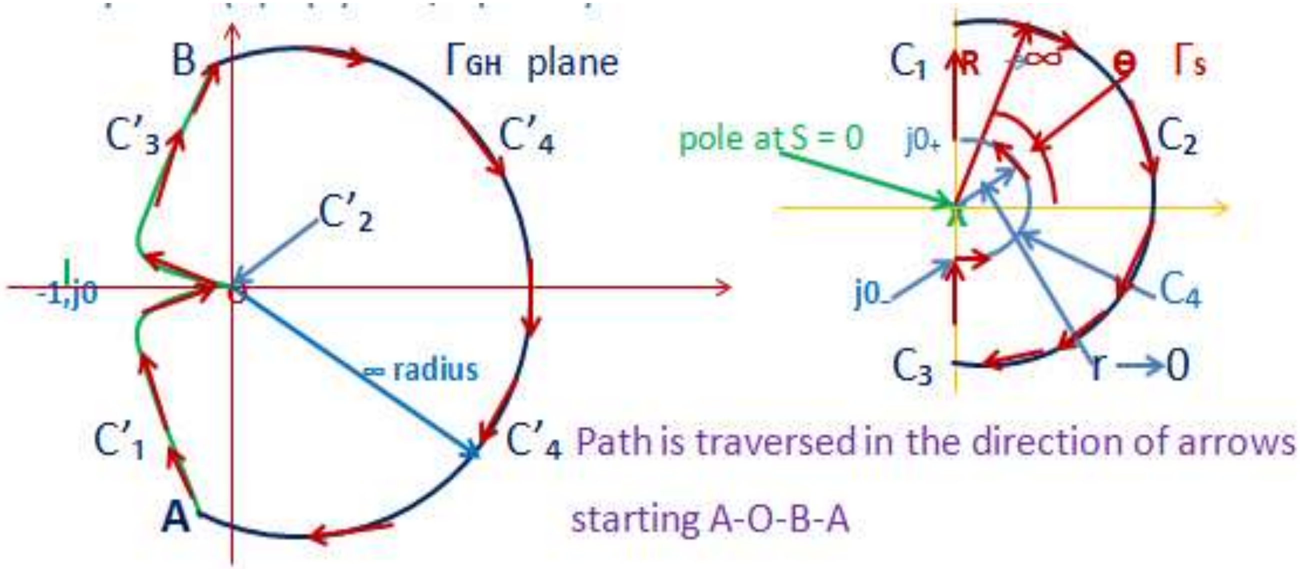
- Since there is a pole at the origin in the S plane, while drawing the Nyquist contour we bypass the origin because pole is a singularity. Bypassing is done by drawing a circle of very small radius 'r'; as $r \rightarrow 0$. A point on the semi circle, C_4, is represented by

$$S = r\,e^{j\phi}$$

- The Nyquist contour is traversed starting 1) $s = j0_+$ to $j\infty$ (C_1) 2) $S = j\infty$ to $-j\infty$ (C_2), 3) $S = -j\infty$ to $j0_-$ (C_3) and 4) $S = j0_-$ to $j0_+$ (C_4)

Nyquist Method: Examples (Contd..)

- Example: $G(S)H(S) = K/S(1+TS)$



- A : $\omega = j0_+$; $|G(j\omega)\,H(j\omega)| = \infty$; arg = -90° B: $\omega = j0_+$; $|G(j\omega)\,H(j\omega)| = \infty$; arg = -90°

  o: $\omega = j\infty$ to $-j\infty$; $|G(j\omega)\,H(j\omega)| = 0$; arg = -180° to 180°

  C_1 is mapped in to C'_1 & C_3 is mapped in to C'_3 (Nyquist/polar plot)

  C_2 is mapped in to C'_2(origin); C_4 is mapped in to C'_4. (Contd.)

Nyquist Method: Examples (Contd..)

- $G(j\omega)H(j\omega) = K/j\omega(1+j\omega T)$
  1. C_1: mapping in to Γ_{GH} plane: polar plot, C'_1
  2. C_2: mapping in to Γ_{GH} plane: point C'_2 for $S = R\,e^{j\theta}$
  3. $G(S)H(S) = K/\,R\,e^{j\theta}\,(1+T\,R\,e^{j\theta})$ as $R \rightarrow \infty$
  4. $G(S)H(S) = |G(S)H(S)|\,e^{j\theta}$; $0\,e^{-j2\theta}$; $\arg(G(S)H(S)) = -2\theta$
  5. Since θ changes from +90 to -90 ; $\arg(G(S)H(S))$ changes from -180° to + 180°. So we get point 'O' in Γ_{GH} plane.
  6. C_4 mapping in to C'_4 in Γ_{GH} plane for $S = r\,e^{j\phi}$ as $r \rightarrow 0$
  7. $G(S)H(S) = K/\,r\,e^{j\phi}\,(1+T\,r\,e^{j\phi})$ as $r \rightarrow 0$
  8. $G(S)H(S) = |G(S)H(S)|\,e^{j\phi}$; $\infty\,e^{-j\phi}$; $\arg(G(S)H(S)) = -\phi$
  9. Since ϕ changes from -90 to +90 ; $\arg(G(S)H(S))$ changes from 90° to -90°. So we get C'_4 Γ_{GH} plane.

Nyquist Method: Examples

- For the example in previous Lecture:

  We have drawn the Nyquist plot for a given G(S)H(S). Now we need to determine the stability of its closed loop system.

- The number of encirclements, N, of (-1,j0) point is given by:

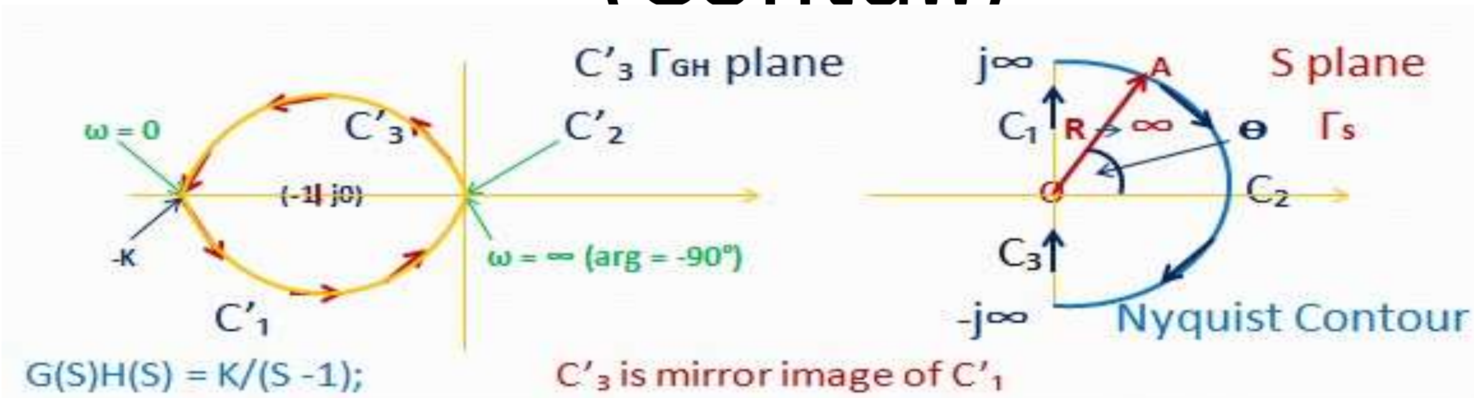
$$N = P-Z$$

- For closed loop system to be stable, Z = 0

  In this example, P = 0 because no poles of G(S)H(S) are in the RH of S plane.

- Therefore N should be equal to '0', i.e. that there should be no encirclement of (-1,j0) point. We see from the Nyquist diagram that it does not encircle (-1,j0) point & hence the closed loop system is stable.

Nyquist Method: Examples (Contd..)



1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'_1.
2. Corresponding to C_2 in Γ**s** plane we have; $S = R\exp(j\Theta)$ in G(S)H(S); $R \longrightarrow \infty$

   $G(S)H(S) = K/(R\,e^{j\Theta} - 1)$ as, $R \longrightarrow \infty$ therefore

   $G(S)H(S) = 0\,e^{-j\Theta}$; $|G(S)H(S)| = 0$; arg, G(S)H(S)- = $-\Theta$

   On C_2 ; Θ varies from +90° to -90° as we move from +j∞ to -j∞

   arg{G(S)H(S)} varies from -90° to +90° . This is C'_2 in Γ**GH** plane.
3. C_3 in Γ**S** plane is mapped as C'_3 (Nyquist plot) in Γ**GH** plane. (Contd. .)

Nyquist Method: Examples (Contd..)

- Having drawn the Nyquist diagram, we need to determine the stability of related CL system.
- Observation:

G(S) H(S) has a pole in the RH of the S plane; therefore P = 1

$$N = P - Z$$

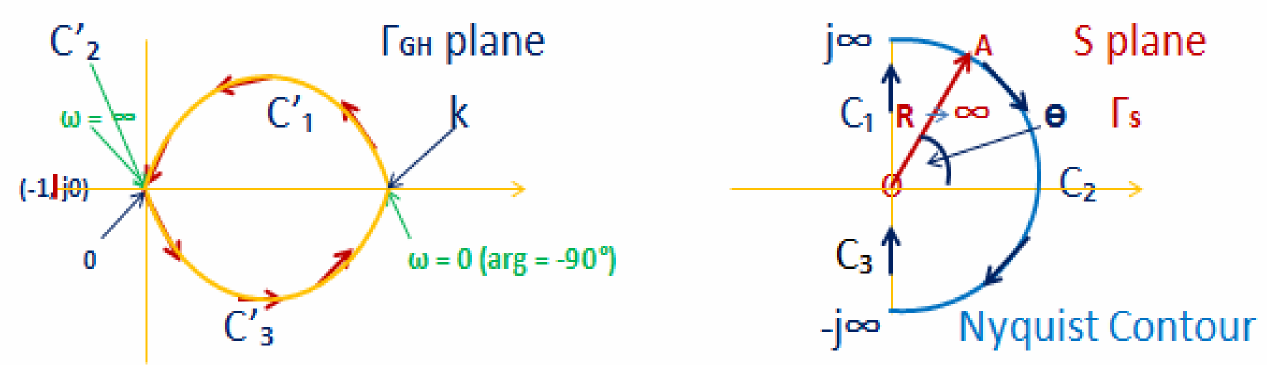
Z = 0 for stable CL system

Therefore, $N = P = 1$

⇨ that the Nyquist plot should encircle (-1,j0) point once in the CCW direction for the CL system to be stable.

- From the Nyquist diagram we that it is encircling (-1,j0) point once in CCW direction. Therefore, the CL system is stable

Nyquist Method: Examples (Contd..)



G(S)H(S) = K/(1-S);   C'$_3$ is mirror image of C'$_1$

1. Corresponding to C_1 in Γ**s** plane we have the Nyquist plot in Γ**GH** plane as C'_1.
2. Corresponding to C_2 in Γ**s** plane we have; $S = R \exp(j\theta)$ in $G(S)H(S)$; $R \to \infty$

   $G(S)H(S) = K/\,(R\,e^{j\theta} - 1)$ as, $R \to \infty$ therefore

   $G(S)H(S) = 0\,e^{-j\theta}$; $|G(S)H(S)| = 0$; arg, $G(S)H(S)$- = $-\theta$

   On C_2 ; θ varies from +90° to -90° as we move from $+j\infty$ to $-j\infty$

   arg{G(S)H(S)} varies from -90° to +90° . This is C'_2 in Γ**GH** plane.
3. C_3 in Γ**S** plane is mapped as C'_3 (Nyquist plot) in Γ**GH** plane. (Contd.)

STATE SPACE ANALYSIS OF CONTINUOUS SYSTEMS

Basically, there are 2 approaches to the analysis & Design of control systems. 1. The Transfer function approach. 2. State variable Approach.

Transfer function Approach	State variable Approach
1. Transfer function Approach is also called conventional Approach (or) Classical Approach	1. State variable Approach is called modern Approach
2. It is applicable to linear-Time Invariant systems, It is generally limited to single input single o/p systems.	2. It is applicable to linear as well as non-linear, Time variant as well as Time invariant, single i/p single o/p as well as multi i/p multi o/p systems.
3. In this initial conditions are neglected.	3. They are considered.
4. It is basically a frequency domain Approach.	4. It is a time domain Approach.
5. It is based on Trial & Error procedure	5. It is not based on Trial & Error procedure.
6. Only i/p, o/p & Error signals are considered important. The i/p & o/p variables must be measurable.	6. ~~The i/p & o/p variables~~ State variable need not represent physical variables. They need not be measurable & observable.

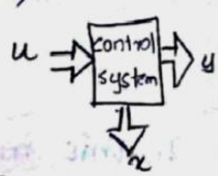
7. Internal variables cannot be fed back	7. State variables can be fed back.
8. Transfer function of a system is unique	8. State model of a system is not unique

* Concepts of state, state variable and state Model :

State : The state of a dynamic system is a minimal set of variables such that the knowledge of these variables at $t=t_0$ together with the knowledge of inputs for $t \geq t_0$ completely determines the behaviour of the system for $t \geq t_0$

State variable :

Set of variables which describe the system at any time instant are called state variables. In the state variable formulation of a system, in general, a system consists of 'm' inputs, 'p' outputs, 'n' state variables.



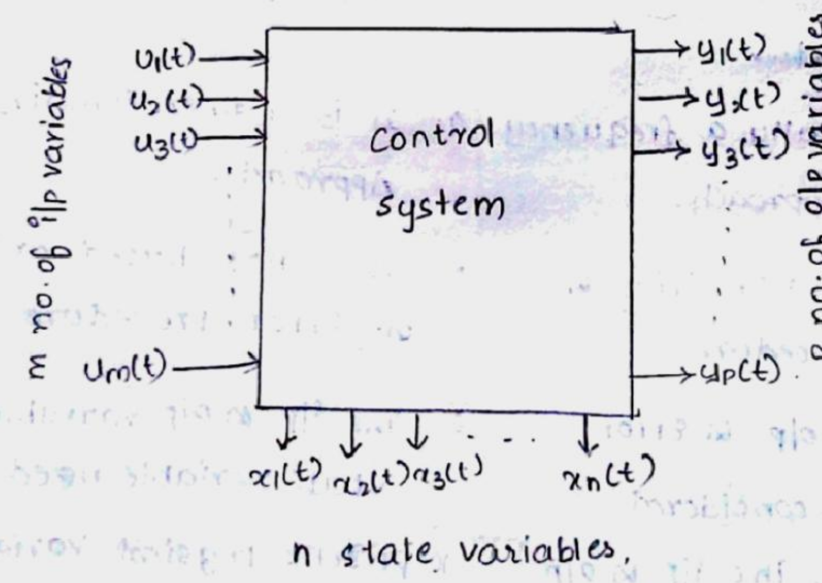


Fig : State Space Representation of a System.

Different variables may be represented by the vectors.

Input vector $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$; Output vector $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

State Equations :

State variable Representation can be arranged in the form of 'n' no. of 1^{st} order differential eqns as shown below.

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, \ldots x_n ; u_1, u_2, \ldots u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2, \ldots x_n ; u_1, u_2, \ldots u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2, \ldots x_n ; u_1, u_2, \ldots u_m)$$

'n' no. of differential equations may be written in vector notations as

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$

- The set of all possible values which the i/p vector $u(t)$ can have at a time t forms the i/p space of the system
- The set of all possible values which the o/p vector $y(t)$ can have at a time t forms the o/p space of the system
- The set of all possible values which the state vector $x(t)$ can have at a time t forms the state space of system

State model of Linear system :

- The state model of a system consists of state eqn & o/p equation.
- The state eqn of a system is a function of state variables & i/p's.
- For linear time invariant systems, the first derivatives

of state variables can be expressed as linear combination of state variables and inputs.

$$\dot{x}_1 = (a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n) + (b_{11}u_1 + b_{12}u_2 + \dots b_{1m}u_m)$$

$$\dot{x}_2 = (a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n) + (b_{21}u_1 + b_{22}u_2 + \dots b_{2m}u_m)$$

$$\vdots$$

$$\dot{x}_n = (a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n) + (b_{n1}u_1 + b_{n2}u_2 + \dots b_{nm}u_m)$$

where coefficients a_{ij} & b_{ij} are constants

In the matrix form, above eqns can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\dot{X}(t) = AX(t) + BU(t) \longrightarrow ①.$$

where $X(t)$ is state vector of order $n \times 1$

$U(t)$ is input vector of order $m \times 1$.

A = system matrix with order $n \times n$

B = Input matrix with order $n \times m$.

Eqn - ① is called STATE EQUATION of LTI system.

* The o/p at any time are functions of state variables and input.

$\therefore$ output vector $y(t) = f(x(t), u(t))$

The o/p variables can be expressed as linear combination of state variables and inputs.

$$y_1 = (c_{11}x_1 + c_{12}x_2 + \dots c_{1n}x_n) + (d_{11}u_1 + d_{12}u_2 + \dots d_{1m}u_m)$$

$$y_2 = (c_{21}x_1 + c_{22}x_2 + \dots c_{2n}x_n) + (d_{21}u_1 + d_{22}u_2 + \dots d_{2m}u_m)$$

$$\vdots$$

$$y_p = (c_{p1}x_1 + c_{p2}x_2 + \dots c_{pn}x_n) + (d_{p1}u_1 + d_{p2}u_2 + \dots d_{pm}u_m)$$

In matrix form, above Eqns can be written as.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$Y(t) = C\,X(t) + D\,U(t) \longrightarrow (2)$$

where $X(t)$ = State vector of order $n \times 1$

$U(t)$ = Input vector of order $m \times 1$

$Y(t)$ = Output vector of order $p \times 1$

C = output matrix of order $p \times n$

D = Transmission matrix of order $p \times m$.

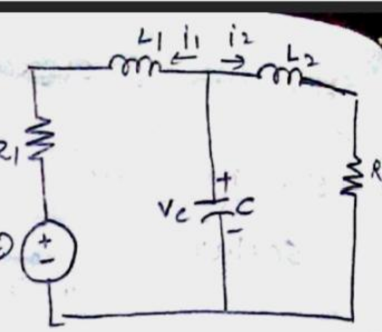
Eqn-(2) is called OUTPUT EQUATION of LTI system. The state Eqn & o/p Eqn together called as state model of the system. Hence the state model of the LTI system is given by.

$$\dot{X}(t) = A\,X(t) + B\,U(t) \quad \text{- state Eqn}$$

$$Y(t) = C\,X(t) + D\,U(t) \quad \text{- o/p Eqn.}$$

1. Obtain the state model of Electrical network shown in figure by choosing minimum no. of state variables.

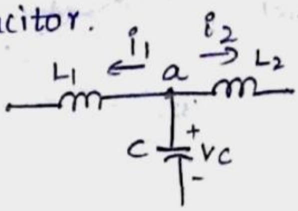
Sol:- Let the state variables x_1, x_2, x_3 are related to physical quantities.



$x_1 = i_1$ = current through L_1.

$x_2 = i_2$ = current through L_2

$x_3 = V_C$ = voltage across capacitor.



consider the node

at node - a.

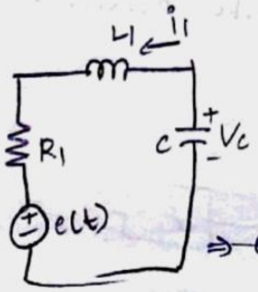
$$i_1 + i_2 + C\frac{dV_C}{dt} = 0$$

$$\Rightarrow x_1 + x_2 + \frac{C\,dx_3}{dt} = 0$$

$$\Rightarrow x_1 + x_2 + C\dot{x}_3 = 0 \Rightarrow \dot{x}_3 = \frac{-(x_1+x_2)}{C}$$

$$\dot{x}_3 = -\frac{x_1}{C} - \frac{x_2}{C} \longrightarrow ①$$

Now consider



$\cancel{\Rightarrow e(t) + i_1R_1 + \frac{1}{L_1}\int i_1\,dt + \cdot}$

$$e(t) = -i_1R_1 - L_1\frac{di_1}{dt} + V_C.$$

Let $e(t) = u(t)$

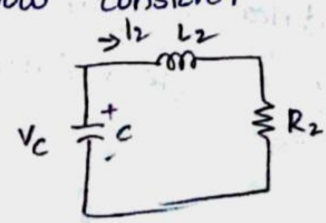
$$\Rightarrow u(t) = -x_1R_1 - L_1\dot{x}_1 + x_3$$

$$\Rightarrow -\dot{x}_1 = \frac{u(t) + x_1R_1 - x_3}{L_1}$$

$\Rightarrow \dot{x}_1 = -\frac{x_1 R_1}{L} + \frac{x_3}{L_1} - \frac{u(t)}{L}$

$$\Rightarrow \dot{x}_1 = -\frac{u(t)}{L_1} - x_1\frac{R_1}{L_1} + \frac{x_3}{L_1} \longrightarrow ②$$

Now consider



$$V_C = \frac{di_2}{dt}L_2 + i_2R_2$$

$$\Rightarrow x_3 = \dot{x}_2L_2 + x_2R_2.$$

$$\Rightarrow \dot{x}_2 = -\frac{x_2R_2}{L_2} + \frac{x_3}{L_2} \rightarrow ③$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & \frac{1}{L_1} \\ 0 & -R_2/L_2 & 1/L_2 \\ -\frac{1}{C} & -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1/L_1 \\ 0 \\ 0 \end{bmatrix} [u].$$

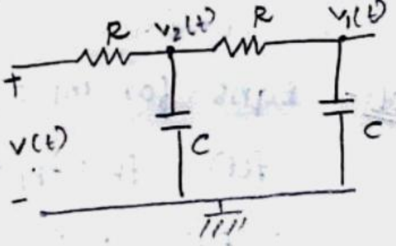
↳ state Equation

$y_1 = i_1 R_1 = x_1 R_1$ → Assume voltage across & current through R_2 are o/p variables y_1 & y_2 resp ∴ The o/p eqns are given by $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & R_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ → Output Equation.

$y_2 = i_2 R_2 = x_2 R_2$

$$\Rightarrow \cancel{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

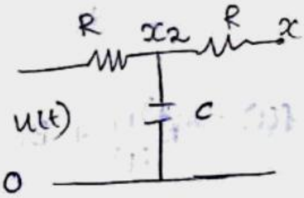
2. Obtain state model for the Electrical network by choosing $v_1(t)$ & $v_2(t)$ as state variables & $v(t)$ as $u(t)$



Sol:- Apply KCL at node-1

Let $v_1(t) = \dot{x}_1$, $v_2(t) = \dot{x}_2$.

$\Rightarrow$ ~~$v_2(t) - v(t)$~~

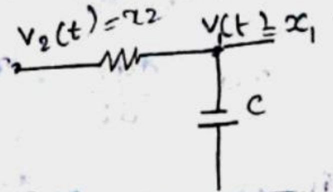


$$\Rightarrow \frac{x_2 - u(t)}{R} + \frac{x_2 - x_1}{R} + C\frac{dx_2}{dt} = 0$$

$$\Rightarrow x_2\left[\frac{2}{R}\right] - \frac{x_1}{R} + C\dot{x}_2 - \frac{u(t)}{R} = 0$$

$$\Rightarrow \dot{x}_2 = \frac{x_1}{RC} - x_2\left[\frac{2}{RC}\right] + \frac{u(t)}{RC} \longrightarrow (1)$$

Now consider



Apply KCL

$$C\frac{dx_1}{dt} + \frac{x_1 - x_2}{R} = 0$$

$$\frac{x_1}{R} + C\dot{x}_1 - \frac{x_2}{R} = 0.$$

$$\dot{x}_1 = -\frac{x_1}{RC} + \frac{x_2}{RC} \longrightarrow (2).$$

matrix form

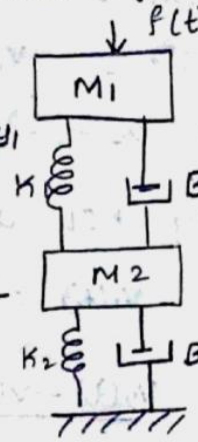
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/RC & +1/RC \\ +1/RC & -2/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ +1/RC \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

→ State Equation

$y_1(t) = V_1(t) = x_1.$

Matrix form $y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ → output Equation.

3. construct the state model of mechanical system shown below.



Sol:- Eqns for M_1:

$f(t) = f_{K_1} + f_{b_1} + f_{M_1}$

$$\Rightarrow f(t) = M\frac{d^2y_1}{dt^2} + B_1\frac{d(y_1-y_2)}{dt} + K_1(y_1-y_2)$$

$$\Rightarrow f(t) = M\frac{d^2y_1}{dt^2} + B_1\frac{dy_1}{dt} - B_1\frac{dy_2}{dt} + K_1y_1 - K_1y_2 \rightarrow (1)$$

Let Eqns for M_2:

$$0 = f_{K_2} + f_{b_2} + f_{K_1} + f_{b_1} + f_{M_2}$$

$$0 = K_2y_2 + B_2\frac{dy_2}{dt} + K_1(y_2-y_1) + B_1\frac{d}{dt}(y_2-y_1) + M_2\frac{d^2y_2}{dt^2} \quad (2)$$

Let $f(t) = u$, $x_1 = y_1$, $x_2 = y_2$, $x_3 = \frac{dy_1}{dt}$,

$x_4 = \frac{dy_2}{dt}$, $\dot{x}_3 = \frac{d^2y_1}{dt^2}$, $\dot{x}_4 = \frac{d^2y_2}{dt^2}$

(1) $\Rightarrow$ $U = M_1\dot{x}_3 + B_1x_3 - B_1x_4 + K_1x_1 - K_1x_2.$

$U = M_1\dot{x}_3 + B_1(x_3-x_4) + K_1(x_1-x_2).$

(2) $\Rightarrow$ $M_2\dot{x}_4 + B_2 x_4 + B_1(x_4 - x_3) + K_1(x_2 - x_1) + K_2 x_2 = 0$

(1) $\Rightarrow$ $\dot{x}_3 = \dfrac{u - B_1(x_3 - x_4) + K_1(x_1 - x_2)}{M_1} \rightarrow$ -

(2) $\Rightarrow$ $\dot{x}_4 = \dfrac{-[B_2 x_4 + B_1(x_4 - x_3) + K_1(x_2 - x_1) + K_2 x_2]}{M_2}$

$$\Rightarrow \dot{x}_3 = -\frac{K_1}{M_1}x_1 + \frac{K_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{u}{M_1} \rightarrow (3).$$

$$\dot{x}_4 = -\left[\frac{-K_1}{M_2}x_1 + \frac{(K_1+K_2)}{M_2}x_2 - \frac{B_1}{M_2}x_3 + \frac{(B_1+B_2)}{M_2}x_4.\right]$$

$$\Rightarrow \dot{x}_4 = \frac{K_1}{M_2}x_1 - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4 \rightarrow (4)$$

we have

$$x_1 = y_1 \Rightarrow \dot{x}_1 = \frac{dy_1}{dt} = x_3. \rightarrow (5)$$

$$x_2 = y_2 \Rightarrow \dot{x}_2 = \frac{dy_2}{dt} = x_4. \rightarrow (6)$$

Matrix form of Eqns (5), (6), (3) & (4).

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_1/M_1 & K_1/M_1 & \frac{-B_1}{M_1} & \frac{B_1}{M_1} \\ K_1/M_2 & \frac{-(K_1+K_2)}{M_2} & \frac{B_1}{M_2} & \frac{-(B_1+B_2)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix} [u]$$

$y_1 = x_1$, $y_2 = x_2$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

14.4 CONVERSION OF STATE VARIABLE MODELS TO TRANSFER FUNCTIONS

We shall derive the transfer function of a SISO system from the Laplace-transformed version of the state and output equations. Refer to Section 14.2 for the matrix operations used in the derivation.

Consider the state variable model (Eqns (14.13)):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t); \mathbf{x}(t_0) \triangleq \mathbf{x}^0$$
$$y(t) = \mathbf{c}\mathbf{x}(t) + du(t) \tag{14.26}$$

Taking the Laplace transform of Eqns (14.26), we obtain

$$s\mathbf{X}(s) - \mathbf{x}^0 = \mathbf{A}\mathbf{X}(s) + \mathbf{b}U(s)$$
$$Y(s) = \mathbf{c}\mathbf{X}(s) + dU(s)$$

where

$$\mathbf{X}(s) \triangleq \mathscr{L}[\mathbf{x}(t)]; U(s) \triangleq \mathscr{L}[u(t)]; Y(s) \triangleq \mathscr{L}[y(t)]$$

Manipulation of these equations gives

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}^0 + \mathbf{b}U(s); \ \mathbf{I} \text{ is } n \times n \text{ identity matrix}$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}^0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s) \tag{14.27a}$$

$$Y(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}^0 + [\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]U(s) \tag{14.27b}$$

Equations (14.27) are algebraic equations. If $\mathbf{x}^0$ and $U(s)$ are known, $\mathbf{X}(s)$ and $Y(s)$ can be computed from these equations.

In the case of a zero initial state (i.e., $\mathbf{x}^0 = \mathbf{0}$), the input–output behavior of the system (14.26) is determined entirely by the transfer function,

$$\frac{Y(s)}{U(s)} = G(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d \tag{14.28}$$

We can express the inverse of the matrix $(s\mathbf{I} - \mathbf{A})$ as

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{(s\mathbf{I} - \mathbf{A})^{+}}{|s\mathbf{I} - \mathbf{A}|} \tag{14.29}$$

where

$$|s\mathbf{I} - \mathbf{A}| = \text{determinant of the matrix } (s\mathbf{I} - \mathbf{A})$$
$$(s\mathbf{I} - \mathbf{A})^{+} = \text{adjoint of the matrix } (s\mathbf{I} - \mathbf{A})$$

Using Eqn. (14.29), the transfer function $G(s)$ given by Eqn. (14.28) can be written as

$$G(s) = \frac{\mathbf{c}(s\mathbf{I} - \mathbf{A})^{+}\mathbf{b}}{|s\mathbf{I} - \mathbf{A}|} + d \tag{14.30}$$

14.4.1 Eigenvalues of a Matrix

For a general nth-order matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

the matrix $(s\mathbf{I} - \mathbf{A})$ has the following appearance:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{bmatrix}$$

If we imagine calculating *det* $(s\mathbf{I} - \mathbf{A})$, we see that one of the terms will be the product of diagonal elements of $(s\mathbf{I} - \mathbf{A})$:

$$(s - a_{11})(s - a_{22})\cdots(s - a_{nn}) = s^n + \alpha_1' s^{n-1} + \cdots + \alpha_n',$$

a polynomial of degree n with the leading coefficient of unity. There will be other terms coming from the off-diagonal elements of $(s\mathbf{I} - \mathbf{A})$, but none will have a degree as high as n. Thus, $|s\mathbf{I} - \mathbf{A}|$ will be of the following form:

$$|s\mathbf{I} - \mathbf{A}| = \Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \tag{14.31}$$

where α_i are constant scalars.

This is known as the *characteristic polynomial* of the matrix **A**. It plays a vital role in the dynamic behavior of the system. The roots of this polynomial are called the *characteristic roots* or *eigenvalues* of matrix **A**. These roots, as we shall see in Section 14.6, determine the essential features of the unforced dynamic behavior of the system (14.26).

State Transition matrix & Solution to state equation:

Here we will investigate ~~state equation~~ the solution of the state equation

$$\frac{d x(t)}{dt} = \dot{x}(t) = A\, x(t) + B u(t) \leftarrow \text{state equation}$$

$$\& \quad x(t_0) = X(0)$$

Take laplace transform

$$S X(S) - X(0) = A\, X(S) + B\, U(S)$$

$$X(S)\, [SI - A] = X(0) + B\, U(S)$$

$$\boxed{X(S) = [SI - A]^{-1} X(0) + (SI - A)^{-1} B\, U(S)}$$

Solution to $x(t)$ is Inverse laplace transform of $X(S)$

$$L^{-1}[X(S)] = x(t)$$

Apply I.L.T to $X(S)$

$$x(t) = L^{-1}\left[[SI - A]^{-1} X(0)\right] + L^{-1}\left[[SI - A]^{-1} B\, U(S)\right]$$

$$\boxed{x(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} B\, U(\tau)\, d\tau}$$

$$x(t) = \underbrace{e^{At}\,x(0)}_{\text{Zero input response}} + \underbrace{\left[\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau\right]}$$

If the initial time $(t_0) = 0$

the initial state $x(0)$ is known at $t = 0$

From above equation

$$\boxed{x(t) = e^{At}\,x(0)}$$

From this equation it is observed that the initial state $x(0)$ at $t = 0$ is driven to a state $x(t)$ at time 't'.

This transition in state is carried out by the matrix exponential (e^{At})

Due to this property, (e^{At}) is known as state transition matrix and is denoted by $\phi(t)$

$$\boxed{\phi(t) = e^{At}} = \text{State Transition matrix}$$

$$\phi(t) = e^{At} = L^{-1}\left[(SI - A)^{-1}\right]$$

proof of $\boxed{L^{-1}\left([SI-A]^{-1}\right) = e^{At}}$

$$[SI-A]^{-1} = \left[SI\left[1 - \frac{A}{SI}\right]\right]^{-1}$$

$$= \frac{1}{SI}\left[1 - \frac{A}{S}\right]^{-1}$$

from the expansion

$$(1-x)^{-1} = \left[1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right]$$

$$\Rightarrow \frac{1}{SI}\left[1 + \frac{A}{S} + \frac{A^2}{S^2} + \frac{A^3}{S^3} + \frac{A^4}{S^4} + \cdots\right]$$

$$[SI-A]^{-1} = \left[\frac{I}{S} + \frac{A}{S^2} + \frac{A^2}{S^3} + \frac{A^3}{S^4} + \frac{A^4}{S^5} + \cdots\right]$$

Apply Inverse ~~to~~ laplace Transform

$$= \left[I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \cdots\right]$$

$$\boxed{[SI-A]^{-1} = e^{At}}$$

Properties of $\phi(t) = e^{At}$

* $\phi(0) = I$
* $\phi^{-1}(t) = \phi(-t)$
* $\phi^n(t) = \phi n(t)$
* $\phi(t_1)\,\phi(t_2) = \phi(t_1+t_2)$

$$\because (e^{At})^{-1} = e^{-At} = e^{A(-t)} = \phi(-t)$$

$$(e^{At})^n = e^{nAt} = e^{A(nt)} = \phi(nt)$$

Controlability & observability

We study the controllability and observability of linear time-invariant systems described by state variable model of the following form

$$
\textcircled{1}\quad \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \rightarrow \text{state equation} \\ y(t) = C\,x(t) + D\,u(t) \rightarrow \text{output equation} \end{cases}
$$

where A, B, C, D are respectively — $(n \times n)$ $(n \times 1)$ $(1 \times n)$ (1×1) matrices

$x(t)$ is $(n \times 1)$ state vector

$y(t)$ and $u(t)$ are output and input variables.

Controlability:

for the linear system given by eqn (1), if there exists an input $u_{[0, t_1]}$ which transfers the initial state $x(0) = x^0$ to the state x' in a finite time (t_1), the state x^0 is said to be controllable.

If all initial states are controllable the system is said to be completely controllable, or simply controllable. Otherwise, the system is said to be uncontrollable.

* controllability of a system depends on A & B matrix and ~~not on~~ & is independent of output matric 'C'

so controllability of the system is frequently referred to as the controllability of the pair $\{A, B\}$

Observability

for the linear system given by eqn (1)

If the knowledge of the output "y" and the input "u" over a finite interval of time $[0, t_1]$ suffices (means be enough or adequate) to determine the state $x(0) = x^0$, the state x^0 is said to be observable. If all initial states are observable, the system is said to be completely observable or simply Observable. Otherwise the system is said to be unobservable.

(*) Observability of system ~~is~~ depends on matrix 'A' and 'C' and is independent of 'B'.

(*) Therefore observability of the system is frequently referred to as the observability of the pair {A, C}

Controllability Test:

The necessary and sufficient condition for the system given by eqn (1) to be controllable is that the $n \times n$ controllability matrix $(U)_{n\times n}$ has rank equal to 'n'

$$[U]_{n\times n} = [B \quad AB \quad A^2B \quad A^3B \ldots A^{n-1}B]$$

rank of matrix$(U) = n$

i.e., $\boxed{\rho[U] = n}$

$$\Rightarrow |U| \neq 0 \quad \text{(Determinant} \neq 0)$$

Observability Test:

The necessary and sufficient condition for the system (given by eqn ①) to be completely observable is that the $n \times n$ observability matrix 'V' has rank equal to n i.e, $\rho(V) = n$

$$\text{Rank}[V] = n$$

where observability matrix 'V'

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Example 14.10 Consider the electrical network shown in Fig. 14.20. Differential equations governing the dynamics of this network can be obtained by various standard methods. By use of nodal analysis, for example, we get

$$C_1 \frac{de_1}{dt} + \frac{e_1 - e_2}{R_3} + \frac{e_1 - e_0}{R_1} = 0$$

$$C_2 \frac{de_2}{dt} + \frac{e_2 - e_1}{R_3} + \frac{e_2 - e_0}{R_2} = 0$$

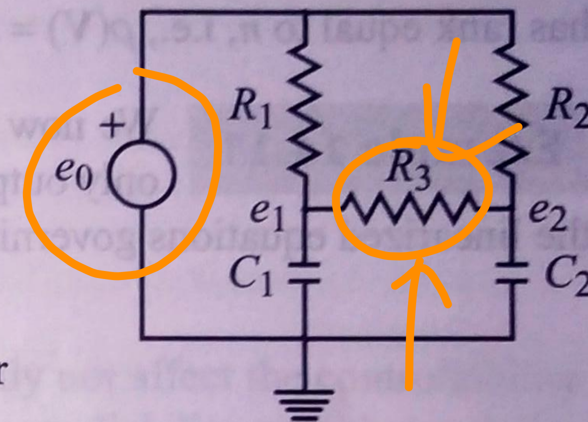


Fig. 14.20

The appropriate state variables for the network are the capacitor voltages e_1 and e_2. Thus, the state equations of the network are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}e_0$$

where

$$\mathbf{x} = [e_1 \; e_2]^T$$

$$\mathbf{A} = \begin{bmatrix} -\left(\dfrac{1}{R_1} + \dfrac{1}{R_3}\right)\dfrac{1}{C_1} & \dfrac{1}{R_3 C_1} \\ \dfrac{1}{R_3 C_2} & -\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)\dfrac{1}{C_2} \end{bmatrix}; \mathbf{b} = \begin{bmatrix} \dfrac{1}{R_1 C_1} \\ \dfrac{1}{R_2 C_2} \end{bmatrix}$$

The controllability matrix of the system is

$$\mathbf{U} = [\mathbf{b} \;\; \mathbf{A}\mathbf{b}] = \begin{bmatrix} \dfrac{1}{R_1 C_1} & -\dfrac{1}{(R_1 C_1)^2} + \dfrac{1}{R_3 C_1}\left(\dfrac{1}{R_2 C_2} - \dfrac{1}{R_1 C_1}\right) \\ \dfrac{1}{R_2 C_2} & -\dfrac{1}{(R_2 C_2)^2} + \dfrac{1}{R_3 C_2}\left(\dfrac{1}{R_1 C_1} - \dfrac{1}{R_2 C_2}\right) \end{bmatrix}$$

We see that under the condition

$$R_1C_1 = R_2C_2,$$

$\rho(\mathbf{U}) = 1$ and the system becomes 'uncontrollable'. This condition is the one required to balance the bridge, and in this case, the voltage across the terminals of R_3 cannot be influenced by the input e_0.

Stability of Linear systems:

Controlability and observability, introduced in previous section are two important qualitative properties of linear systems. Another important qualitative property is stability.

Consider an n-dimensional linear time-invariant system discribed by a state variable model of form

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,r(t)\ ;\quad x(t=0) = x^0 \\ y(t) &= c\,x(t) + D\,r(t) \end{aligned}} \quad \text{(A)}$$

where x is $n\times1$ state vector

r is the scalar input

y is the scalar output

A is $n\times n$ real constant matrix

"B" & "C" are $n\times1$ & $1\times n$ real constant vectors and 'D' is a constant scalar.

The system represented by a state model of form (A) has two sources of excitation

(i) the initial state x^0 representing initial energy storage

(ii) external input - $r(t)$; $t \geq 0$

The response of linear system of form (A) to these two sources of excitation can always be decomposed as the

→ zero-state response [i.e, response with $(x^0 = 0)$]

→ zero-input response [i.e, response with $(r(t) = 0;\ t \geq 0)$]

It is customary to study the stability of these two responses separately.

In the stability study of linear systems, we are generally concerned with the following questions

1. # If the system is relaxed $(x^0 = 0)$, will the bounded input $r(t)$; $t \geq 0$, produce a bounded output $y(t)$ for all t ?

2. for the system with $r(t) = 0$; $t \geq 0$

$$\dot{x}(t) = A\,x(t) + 0 \quad ; \quad x(t=0) = x^0$$

$$\Rightarrow \boxed{\dot{x}(t) = A\,x(t) \ ; \ x(t=0) = x^0} \Rightarrow \text{(B)}$$

at $x(t) = 0$ → $\dot{x}(t) = 0$ → system is at steady state

Therefore, a relaxed system $\boxed{x^0 = 0}$ with $\boxed{r(t) = 0;\ t \geq 0}$, will continue to be in the zero state for all time.

This condition can be viewed as the equilibrium state of the system:

The system is said to be in equilibrium state $x^e = 0$ when both internal energy storage and the external input are zero.

Taking changes in the initial state as the stimulus and the system state $x(t)$ as the response, we pose the following question.

If the system with zero input $(r(t) = 0;\ t \geq 0)$ is perturbed from its equilibrium state $x^e = 0$ at $t = 0$, will the system state $x(t)$ return to x^e, remain close to x^e, or diverge from x^e) ?

The first notion of stability is concerned with the Boundedness of the output, and is called Bounded-input, Bounded Output (BIBO) Stability. (The conditions of BIBO Stability is given in terms of pole locations of the input-output transfer functions).

The second notion of stability is concerned with the 'Boundedness' of the state of an unforced system in response to arbitrary initial state and is called zero-input stability.

The goal of this section is to study the concept of stability of dynamical systems described in terms of state variables.

State variable model

$$\dot{x}(t) = A\,x(t) \; ; \; x(t=0) = x^0$$

is most appropriate model for the study of dynamic evolution of the state $x(t)$ in response to the initial state x^0.

We classify stability as follows:

1. **Unstable:** There is at least one finite initial state x^0 such that $x(t)$ grows thereafter without bound at $t \to \infty$
2. **Asymptotically Stable:** Every initial state x^0 excites a bounded response, which, in addition, approaches '0' as $t \to \infty$
3. **Marginally Stable:** For all initial states x^0, $x(t)$ remains thereafter within finite bounds for $t > 0$.

Solution of State model

$$\dot{x}(t) = A\,x(t) \quad ; \quad (x(t=0) = x^0)$$

$\downarrow$

Apply Laplace Transform

$$S\,X(S) - X(0) = A\,X(S)$$

$$X(S)\,[SI - A] = x^0$$

$$\Rightarrow \boxed{X(S) = [SI - A]^{-1}\,x^0}$$

where
$$[SI - A]^{-1} = \frac{[SI - A]^{+}}{|SI - A|} = \frac{\operatorname{adj}(SI - A)}{\det(SI - A)}$$

C

$\because$ each element of $(SI - A)^{-1}$ is a strictly proper rational function, and can be expanded in partial fraction expansion.

→ The nature of response $x(t) = L^{-1}[X(S)]$ depends on the roots of the characteristic polynomial $|SI - A|$. The roots of $|SI - A|$ are the eigen values of matrix A.

The conditions for zero-input stability given in terms of eigen values of matrix 'A', are as follows.

1. If all the eigen-values of matrix 'A' have negative real parts, the system is Asymptotically Stable.

2. If any eigenvalue of matrix 'A' has positive real part, or if there is a repeated eigen value with zero real part, the system is unstable.

3. If condition (1) is satisfied except for the presense of one or more non repeated eigenvalues with zero real part, the system is marginally stable.

These conditions can ~~be~~ easily be established from inverse Laplace transform of eqn (C)

consider the system

$$\dot{X} = AX + BU$$

$$Y = CX$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Test the controllability of the system.

controllability matrix $U = [B \quad AB]$

AB

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1}$$

$$AB = \begin{bmatrix} -2 & +1 \\ 1 & -2 \end{bmatrix}_{2\times 1}$$

$$AB = \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{2\times 1}$$

$$U = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$|U| = -1 - (-1)$$
$$= 0$$

$$\Rightarrow \rho(U) \neq 2 \\ = 1$$

$\Rightarrow$ Given The second order system is not completely controllable.

The eigen values of matrix 'A' are the roots of characteristic equation $|SI-A|=0$

$$|SI-A| = \left| S\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix}\right| = 0$$

$$= \begin{vmatrix}S+2 & -1\\-1 & S+2\end{vmatrix} = 0$$

$$= (S+2)^2 - 1 = 0$$

Characteristic eqn $\boxed{S^2+4S+3=0}$

roots $S = -1, -3$

The transient response are therefore e^{-t} & e^{-3t}

The transfer function of the given system is calculated as

$$G(S) = C\,(SI-A)^{-1}\,B$$

$$= \begin{bmatrix}0 & 1\end{bmatrix} \begin{bmatrix}S+2 & -1\\-1 & S+2\end{bmatrix}^{-1} \begin{bmatrix}1\\1\end{bmatrix}$$

$$= \underset{1\times 2}{\begin{bmatrix}0 & 1\end{bmatrix}} \underset{2\times 2}{\begin{bmatrix}\dfrac{S+2}{(S+1)(S+3)} & \dfrac{1}{(S+1)(S+3)}\\[2ex] \dfrac{1}{(S+1)(S+3)} & \dfrac{S+2}{(S+1)(S+3)}\end{bmatrix}} \underset{2\times 1}{\begin{bmatrix}1\\1\end{bmatrix}}$$

$$= \underset{1\times 2}{\begin{bmatrix}0 & 1\end{bmatrix}} \underset{2\times 1}{\begin{bmatrix}\dfrac{S+2}{(S+1)(S+3)} + \dfrac{1}{(S+1)(S+3)}\\[2ex] \dfrac{1}{(S+1)(S+3)} + \dfrac{S+2}{(S+1)(S+3)}\end{bmatrix}}$$

$$G(S) = 0 + \left(\frac{1}{(S+1)(S+3)} + \frac{S+2}{(S+1)(S+3)}\right)$$

$$= \frac{\cancel{(S+3)}}{(S+1)\cancel{(S+3)}}$$

$$\boxed{G(S) = \frac{1}{S+1}}$$

we ~~foun~~ find that because of pole-zero cancelation, both the eigenvalues of matrix 'A' do not appear as poles in G(S). i.e., The dynamic mode e^{-3t} of system does not show up in input-output characterization given by the transfer function G(S).

consider the system

$$\dot{X} = AX + BU$$

$$Y = CX$$

with $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & -1 \end{bmatrix}$

Test the observability of the system.

Observability matrix

$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$$

<u>CA</u>

$$\begin{bmatrix} 1 & -1 \end{bmatrix}_{1\times2} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}_{2\times2}$$

$$CA = \begin{bmatrix} (-2-1) & (1+2) \end{bmatrix}_{1\times2}$$

$$CA = \begin{bmatrix} -3 & 3 \end{bmatrix}_{1\times2}$$

$$|V| = 3-3 = 0$$

$$\Rightarrow \rho(V) \neq 2$$
$$= 1$$

given second order system is not competely observable.

The eigenvalues of matrix 'A' are $-1, -3$

- The transfer function of the given system is calculated as

$$G(s) = C[sI - A]^{-1} B$$

$$= \underset{1\times 2}{\begin{bmatrix} 1 & -1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} \frac{s+2}{(s+1)(s+3)} & \frac{1}{(s+1)(s+3)} \\ \frac{1}{(s+1)(s+3)} & \frac{s+2}{(s+1)(s+3)} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$= \underset{1\times 2}{\begin{bmatrix} 1 & -1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} \left(\frac{s+2}{(s+1)(s+3)} + 0\right) \\ \left(\frac{1}{(s+1)(s+3)} + 0\right) \end{bmatrix}}$$

$$= \frac{s+2}{(s+1)(s+3)} - \frac{1}{(s+1)(s+3)}$$

$$= \frac{\cancel{(s+1)}}{\cancel{(s+1)}(s+3)}$$

$$\boxed{G(s) = \frac{1}{s+3}}$$

The dynamic mode e^{-t} of the system does not show up in the $G(s)$.

<u>Note</u> : The system under consideration is not a completely observable system.

State-space models of linear discrete time systems:

So far we treated the analysis of linear continuous-time system using state variable methods. In this section, we give a concise review of the same methods for linear, discrete-time systems. Since the theory of linear discrete-time systems - very closely - parallels the theory of linear continuous-time systems, many of the results are similar.

We will be mostly concerned with single-input, single output (SISO) system configuration of the type shown in the Block diagram of below fig.

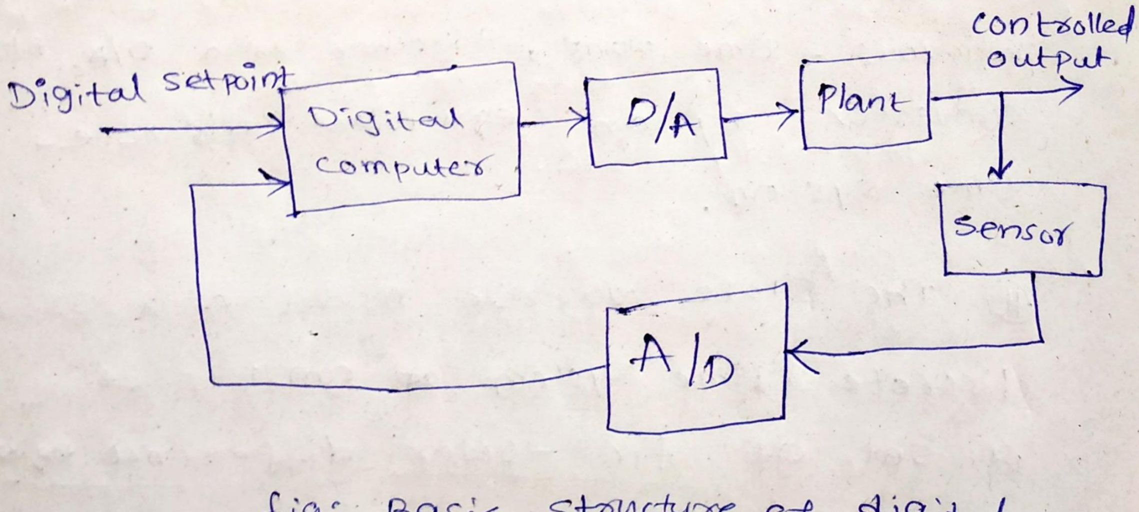


fig: Basic structure of digital control systems.

The plant in the figure, is a physical process characterised by continuous - time input and output variables.

A digital computer is used to control the continuous- time plant.

The interface system that takes care of the communication between the digital computer and the continuous-time plant consists of analog-to-Digital (A/D) converter and Digital-to-Analog (D/A) converter.

In order to analyse such a system, it is often convenient to represent the continuous - time plant, together with D/A, A/D converter by an equivalent discrete-time system.

The State variable model of a SISO discrete - time system ~~of~~ consists of a set of first-order difference equations relating state variables $x_1(k), x_2(k), \ldots x_n(k)$ of discrete-time system to the input $u(k)$; the ouput $y(k)$ is algebraically related to the

state variables and the input.

Assume that the ~~system~~ input is switched on to the system at $k=0$

i.e., $[u(k) = 0]$ for $k < 0$

then the initial state is given by

$X(0) = X^0$; a specified ($n \times 1$ vector)

The dynamics of a linear time invariant system are discribed by equations of the form.

eqn (1)

state equation + output equation

$$\boxed{\begin{aligned} X(K+1) &= F\,X(K) + G\,u(K) \; ; \; (X(0) = X^0) \\ Y(K) &= C\,X(K) + D\,u(K) \end{aligned}}$$

⟹ two eqn's together give state variable model of the system.

$$X(K) = \begin{bmatrix} x_1(K) \\ x_2(K) \\ \vdots \\ x_n(K) \end{bmatrix} = n \times 1 \text{ state vector of } n^{th} \text{ order system}$$

$u(K)$ = system input

$Y(K)$ = defined output.

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & f_{n3} & & f_{nn} \end{bmatrix} = n \times n \text{ constant matrix}$$

$$G = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix} = n \times 1 \text{ constant } \cancel{\text{column}} \text{ column matrix}$$

$$C = [c_1 \quad c_2 \quad c_3 \; \cdots \; c_n] = 1 \times n \text{ constant row matrix.}$$

D = scalar, representing direct coupling between input and output.

Conversion of state variable models to transfer functions.

In the study of linear time-invariant discrete time equations, we may also apply the Z-transform techniques.

Taking the Z-transform of eqns in (D)

$$X(K+1) = F\,X(K) + G\,U(K) \quad ; \quad Y(K) = C\,X(K) + D\,U(K)$$

⇓ Z-transform ⇓

$$Z\,X(Z) - Z\,X^0 = F\,X(Z) + G\,U(Z) \quad ; \quad Y(Z) = C\,X(Z) + D\,U(Z)$$

$$[ZI - F]\,X(Z) = Z\,X^0 + G\,U(Z) \quad ; \quad \left(\begin{array}{l}\text{where} \\ I = n \times n \text{ Identity matrix}\end{array}\right)$$

$$X(Z) = (ZI - F)^{-1} Z X^0 + [ZI - F]^{-1} G\,U(Z)$$

$$\Rightarrow Y(Z) = C\,[ZI - F]^{-1} Z X^0 + [C\,(ZI - F]^{-1} G + D]\,U(Z)$$

~~From above equation~~

* If X^0 and $U(Z)$ are known, $X(Z)$ can be computed using above eqn

* In case of zero input state (i.e., $X^0 = 0$) the input-output behaviour of the system is determined entirely by the transfer function

$$G(Z) = \frac{Y(Z)}{U(Z)} = C\,(ZI - F]^{-1} G + D$$

$$= C\,\frac{(ZI - F)^{+} G}{|ZI - F|} + D$$

where $(ZI-F)^{+}$ = adjoint of the matrix $(ZI-F)$;

$|ZI-F|$ = determinant of the matrix $(ZI-F)$

* $|\lambda I - F|$ is characteristic polynomial of matrix F. The roots of this polynomial are the characteristic roots or eigenvalues of matrix F.

Stability: (Stability of linear discrete-time system)

The conditions for zero-input stability given interms of eigenvalues of matrix F, are as follows:

1. If all the eigenvalues of matrix 'F' have magnitudes less than unity, the system is asymptotically stable.

2. If any eigenvalue of matrix 'F' has magnitude greater than unity, (or) if there is a repeated eigenvalue with unity magnitude. the system is unstable.

3. If condition (1) is satisfied except for the presence of one or more non-repeated eigenvalues with unity magnitude, the system is marginally stable.