ELECTRIC FLUX
&
ELECTRIC FLUX DENSITY

ELECTRIC FLUX

- In electromagnetism, **electric flux** is the rate of flow of the electric field through a given area.
- The total number of lines of force passing through the unit area of a surface.
- It is scalar quantity.
- Its unit is $N.m^2/C$ or $V.m$

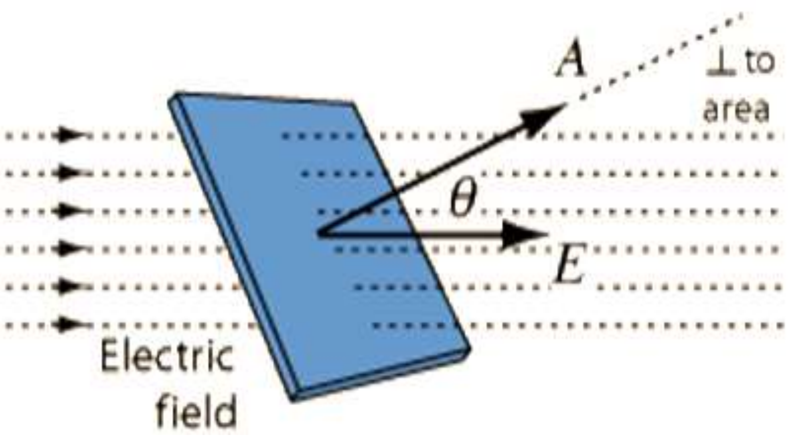
MATHEMATICALLY

- Mathematically the electric flux is defined as:
- **"The dot product of electric field intensity** (E) **and the vector area** (∆A) **is called electric flux."**

$$\Delta\Phi_e = E.\Delta A$$

$$\Delta\Phi_e = E.\Delta A \cos\theta$$

- **Where θ is angle between E and ∆A**



PROPERTIES

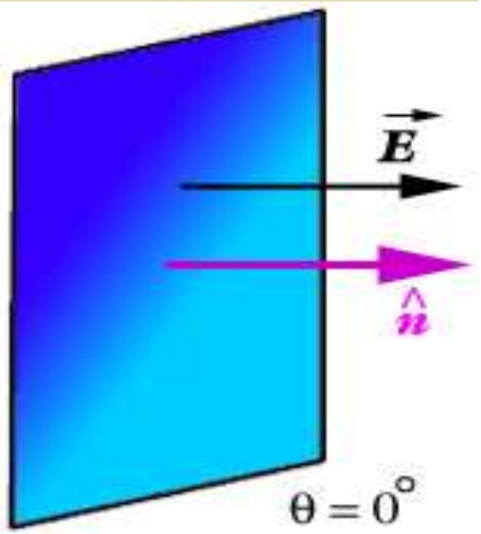
- **Maximum Flux**
- If the surface is placed perpendicular to the electric field then maximum electric lines of force will pass through the surface. Consequently maximum electric flux will pass through the surface
- line are perpendicular than θ=0

$$\phi_e = \vec{E} \cdot \vec{\Delta A}$$

$$\phi_e = E \; \Delta A \cos 0^\circ$$

$$\phi_e = E \; \Delta A \; (1)$$

$$\boxed{\phi_e = E \; \Delta A}$$



CONT…

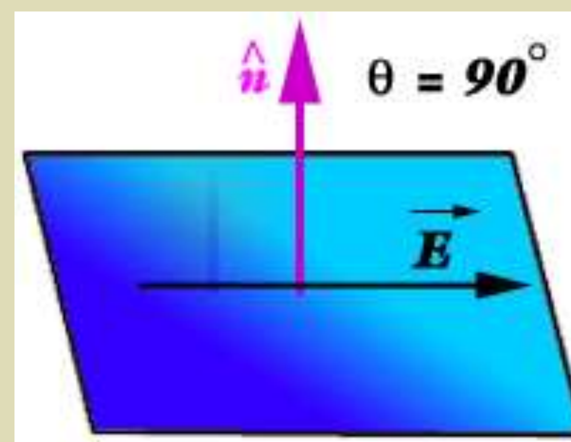
- **Zero Flux**
- If the surface is placed parallel to the electric field then no electric lines of force will pass through the surface. Consequently no electric flux will pass through the surface.

$$\phi_e = \vec{E} \cdot \vec{\Delta A}$$

$$\phi_e = E \quad \Delta A \cos 90^\circ$$

$$\phi_e = E \quad \Delta A \ (0)$$

$$\boxed{\phi_e = 0}$$



ELECTRIC FLUX THROUGH A SPHERE

- Consider a small positive point charge **+q** placed at the Centre of a closed sphere of radius "**r**".
- The relation is not applicable in this situation because the direction of electric intensity varies point to point over the surface of sphere.
- In order to overcome this problem the sphere is divided into a number of small and equal pieces each of area **ΔA.**
- The direction of electric field in each segment of sphere is the same i.e. outward normal.

CONT…

- Now we will determine the flux through each segment.
  Electric flux through the first segment:

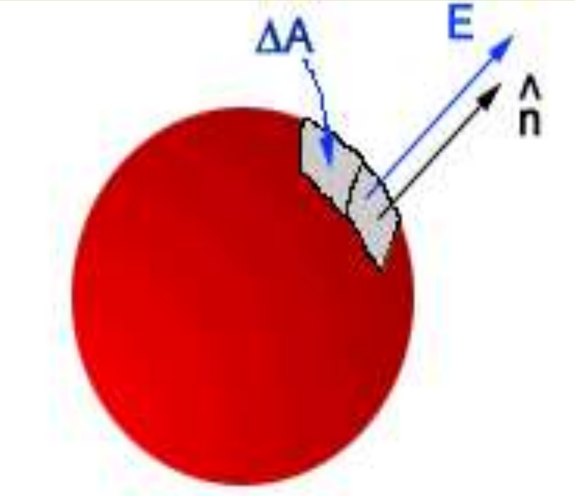
$$\Delta\Phi_1 = E.\Delta A\cos\theta$$

- **But θ=0**

$$\Delta\Phi_1 = E.\Delta A\cos 0$$

$$\Delta\Phi_1 = E.\Delta A\cos(1)$$

$$\Delta\Phi_1 = E.\Delta A$$



CONT…

- Electric flux through the second segment:

$$\Delta\Phi_2 = E.\Delta A$$

- Similarly, Electric flux through other segments:

$$\Delta\Phi_3 = E.\Delta A$$

$$.$$

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$$\Delta\Phi_n = E.\Delta A$$

CONT…

- Being a scalar quantity, the total flux through the sphere will be equal to the algebraic sum of all these flux i.e.

$$\Phi = \sum_{i=1}^{i=n} \Phi i$$

$$\Phi = \sum_{i=1}^{i=n} (E\Delta A)_i$$

$$\Phi = E\sum_{i=1}^{i=n} (\Delta A)_i$$

- But

$$E = \frac{1}{4\pi \in_\circ} \cdot \frac{q}{r^2}$$

$$\Phi = \frac{1}{4\pi\varepsilon_o} \frac{q}{r^2} \sum_{i=1}^{i=n} (\Delta A)_i$$

CONT…

- As $\sum_{i=1}^{i=n}(\Delta A)_i = 4\pi r^2$ (Area of sphere)
- Then equation is $$\Phi = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}4\pi r^2$$

$$\Phi = \frac{q}{\varepsilon_o}$$

- This expression shows that the total flux through the sphere is $1/e_o$ times the charge enclosed (q) in the sphere.

ELECTRIC FLUX DENSITY

- The Electric Flux Density is called Electric Displacement denoted by D, is a vector field that appears in Maxwell's equations.
- It is equal to the electric field strength multiplied by the permittivity of the material through which the electric field extends.
- It is measured in coulombs per square meter.
- The Electric Flux Density (D) is related to the Electric Field (E) by:

$$\mathbf{D} = \varepsilon \mathbf{E}$$ ------Equation (1)

CONT…

- In Equation [1], ε is the permittivity of the medium (material) where we are measuring the fields.
- If you recall that the Electric Field is equal to the force per unit charge (at a distance *R* from a charge of value *q1* [C])

$$\mathbf{E} = \frac{q_1}{4\pi\varepsilon R^2}$$ Equation------ (2)

- Then the Electric Flux Density is:

$$\mathbf{D} = \varepsilon\mathbf{E} = \frac{q_1}{4\pi R^2}$$ Equation------ (3)

CONT…

- From Equation [3], the Electric Flux Density is very similar to the Electric Field, but does not depend on the material in which we are measuring (that is, it does not depend on the permittivity .
- Note that the **D** field is a vector field, which means that at every point in space it has a magnitude and direction.
- The Electric Flux Density has units of Coulombs per meter squared [C/m^2].

Definitions

- Flux—The rate of flow through an area or volume. It can also be viewed as the product of an area and the vector field across the area

- Electric Flux—The rate of flow of an electric field through an area or volume—represented by the number of E field lines penetrating a surface

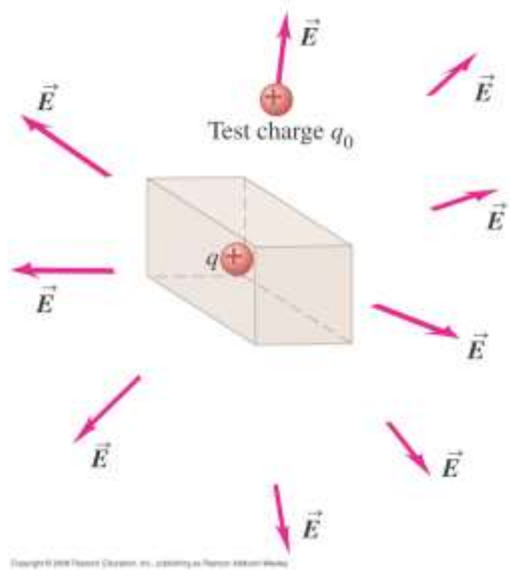
Charge and Electric Flux

Previously, we answered the question – how do we find E-field at any point in space if we know charge distribution?

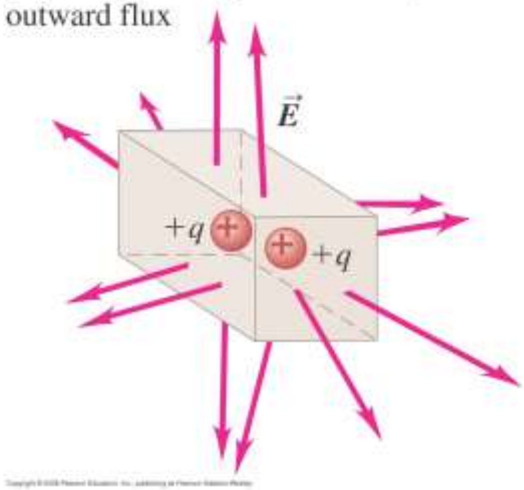
Now we will answer the opposite question – if we know E-field distribution in space, what can we say about charge distribution?

(a) A box containing an unknown amount of charge

(b) Using a test charge outside the box to probe the amount of charge inside the box

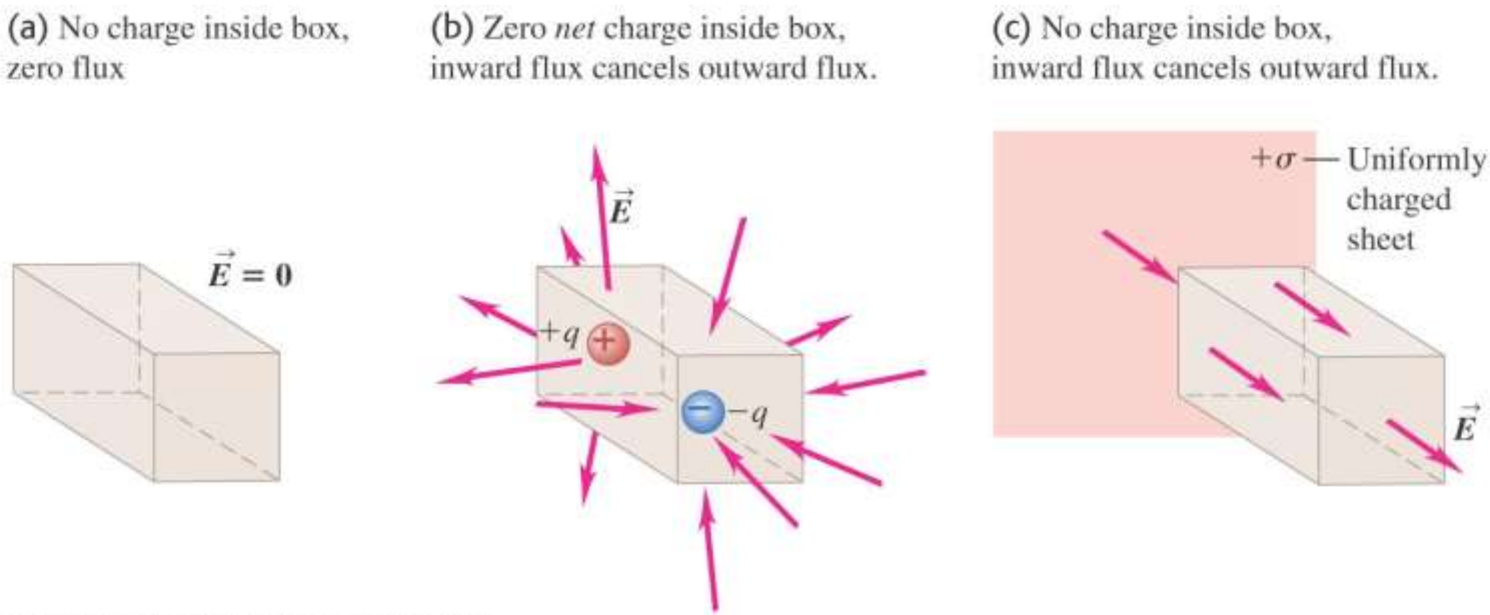


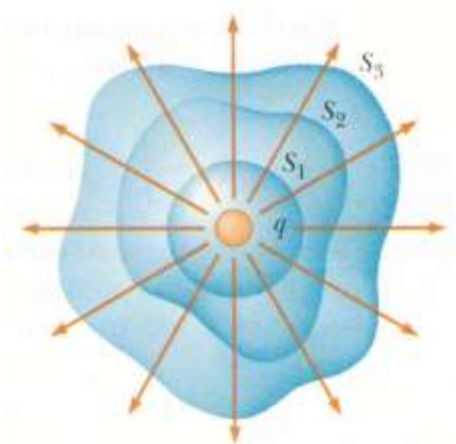
(b) Positive charges inside box, outward flux



Electric flux

Electric flux is associated with the flow of electric field through a surface





$$E \sim \frac{1}{r^2}$$

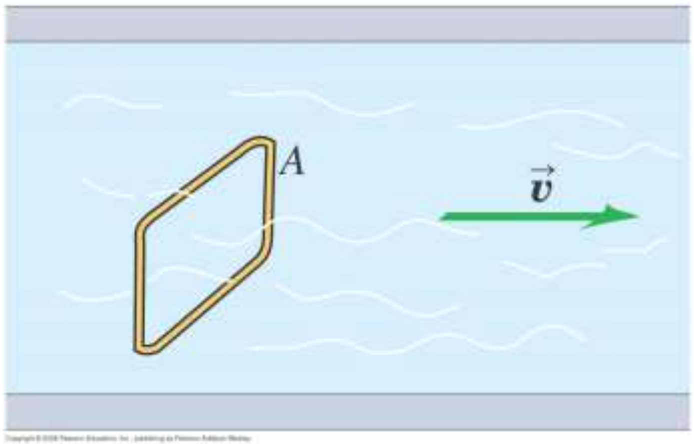
$$S \sim r^2$$

$$E \cdot S = const$$

For an enclosed charge, there is a connection between the amount of charge and electric field flux.

Calculating Electric Flux

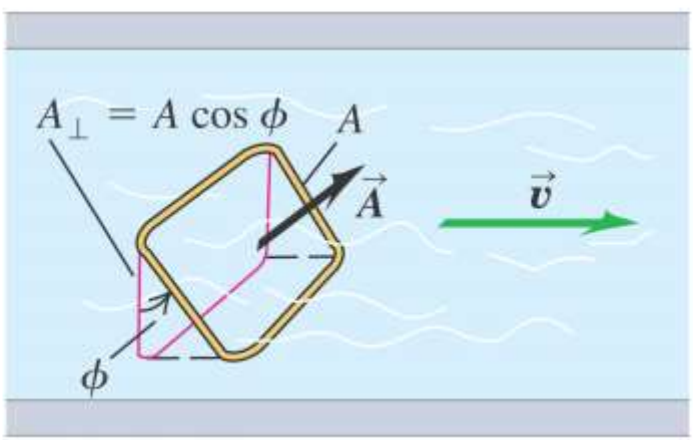
(a) A wire rectangle in a fluid



Amount of fluid passing through the rectangle of area ***A***

$$\frac{dV}{dt} = \upsilon A$$

(b) The wire rectangle tilted by an angle ϕ



$$\frac{dV}{dt} = \upsilon A \cos\phi$$

$$\frac{dV}{dt} = \vec{\upsilon} \cdot \vec{A}$$

Calculating Electric Flux

- The flux for an electric field is

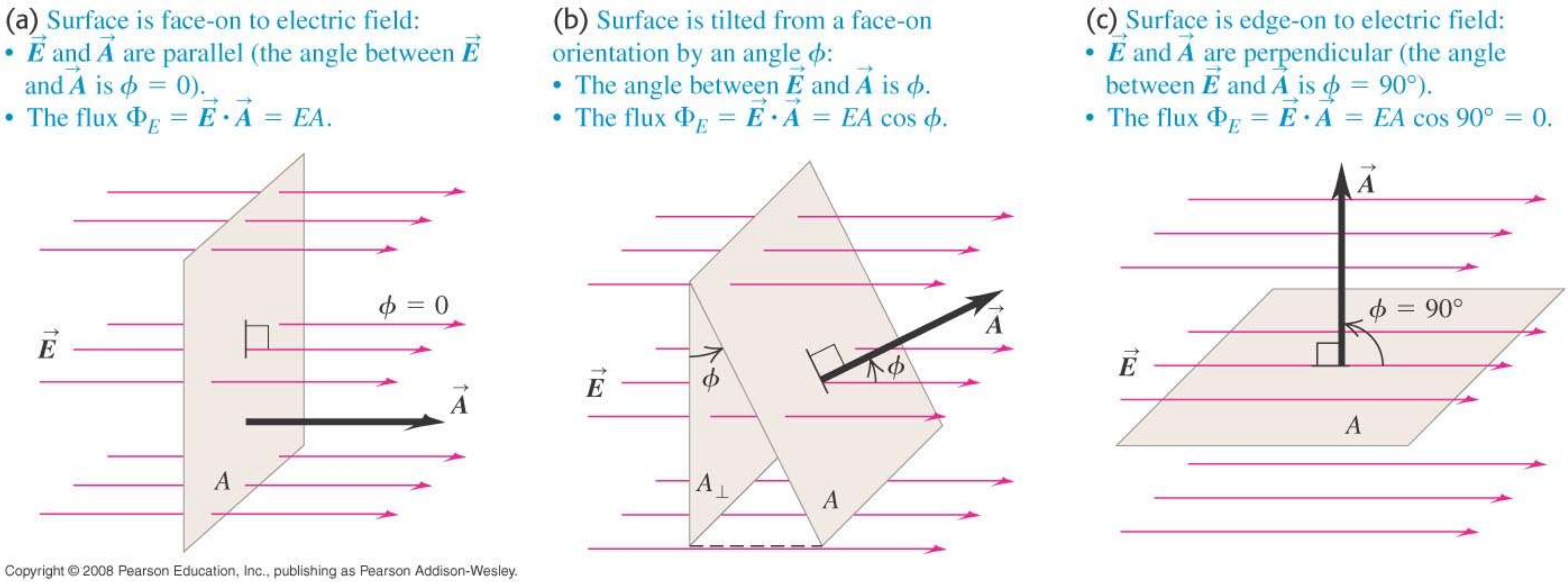
$$\Phi = \vec{E} \bullet \vec{A}$$

- For an arbitrary surface and nonuniform E field

$$\Phi = \oint \vec{E} \bullet d\vec{A}$$

- Where the area vector is a vector with magnitude of the area A and direction normal to the plane of A

Flux of a Uniform Electric Field



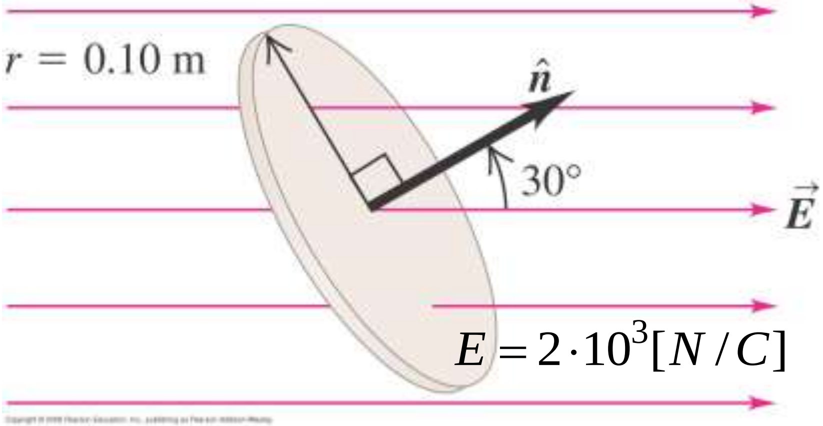
$$\Phi_E = \vec{E}\cdot\vec{A} = EA\cos\phi \qquad \vec{A} = A\cdot\vec{n} \qquad \vec{n}$$ - unit vector in the direction of normal to the surface

Flux of a Non-Uniform Electric Field

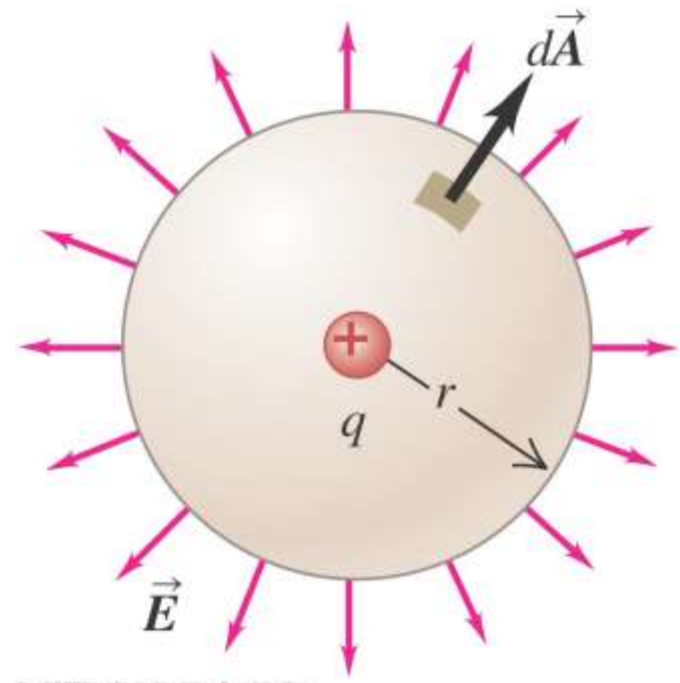
$$\Phi_E = \int_S \vec{E}\cdot d\vec{A}$$

E – non-uniform and
A- not flat

Few examples on calculating the electric flux



Find electric flux



Definitions

- Symmetry—The balanced structure of an object, the halves of which are alike
- Closed surface—A surface that divides space into an inside and outside region, so one can't move from one region to another without crossing the surface
- Gaussian surface—A hypothetical closed surface that has the same symmetry as the problem we are working on—note this is not a real surface it is just an mathematical one

Gauss' Law

- *Gauss' Law* depends on the enclosed charge only

$$\Phi = \oint \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

1. If there is a positive net flux there is a net positive charge enclosed
2. If there is a negative net flux there is a net negative charge enclosed
3. If there is a zero net flux there is no net charge enclosed

- Gauss' Law works in cases of symmetry

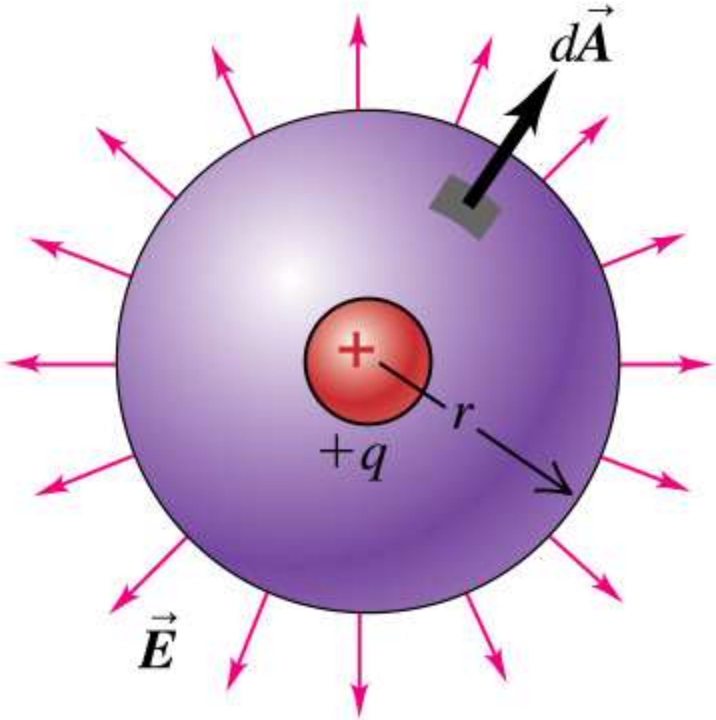
Types of Symmetry

- Cylindrical symmetry—example a can
- Spherical symmetry—example a ball
- Rectangular symmetry—example a box—rarely used

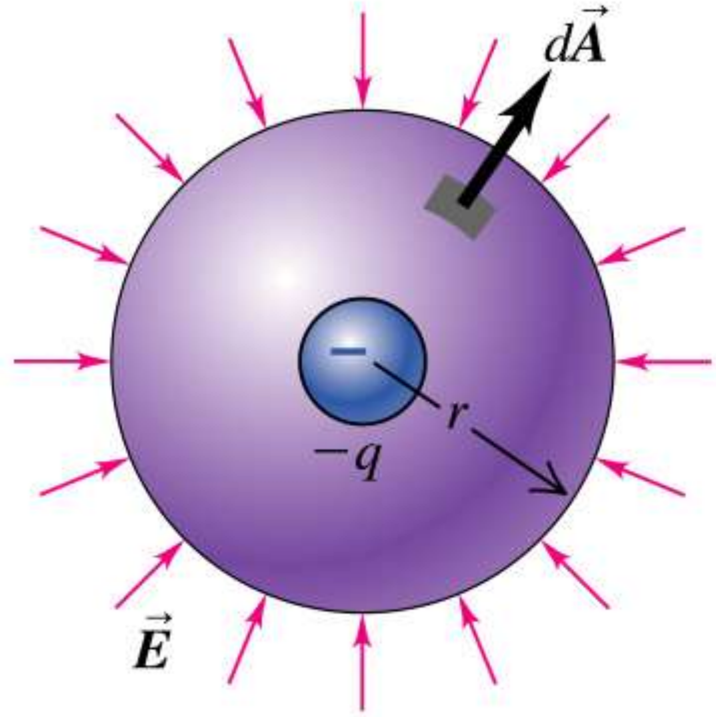
Steps to Applying Gauss' Law

To find the E field produced by a charge distribution at a point of distance r from the center

1. Decide which type of symmetry best complements the problem
2. **Draw a Gaussian surface (mathematical not real) reflecting the symmetry you chose around the charge distribution at a distance of r from the center**
3. Using Gauss's law obtain the magnitude of E

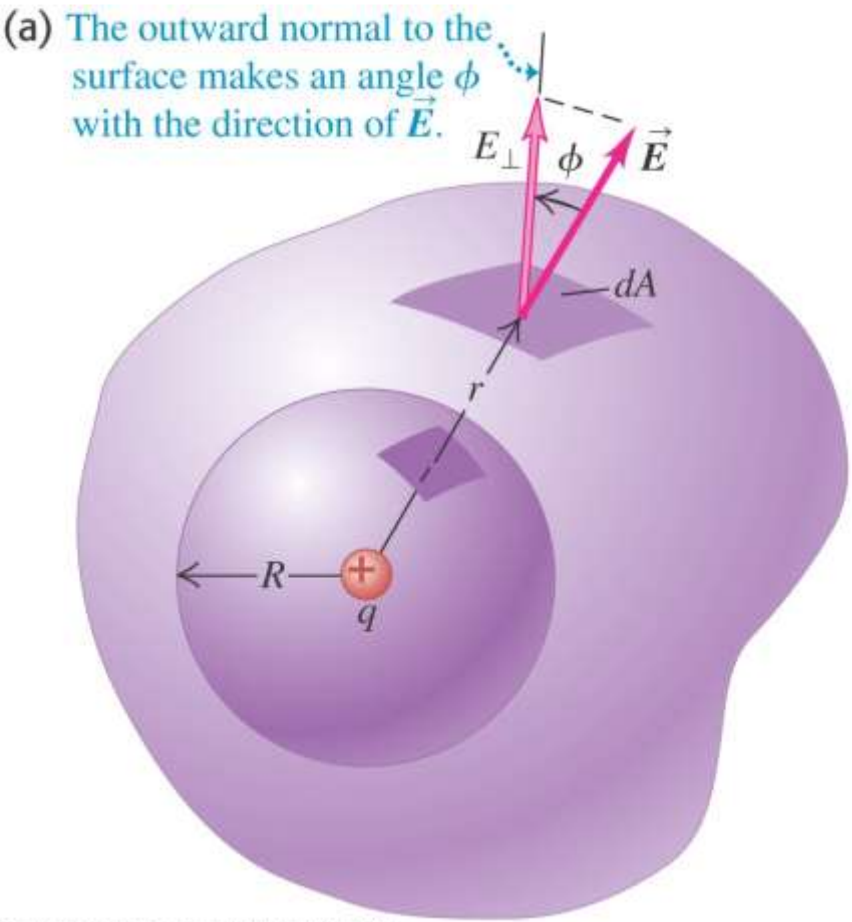


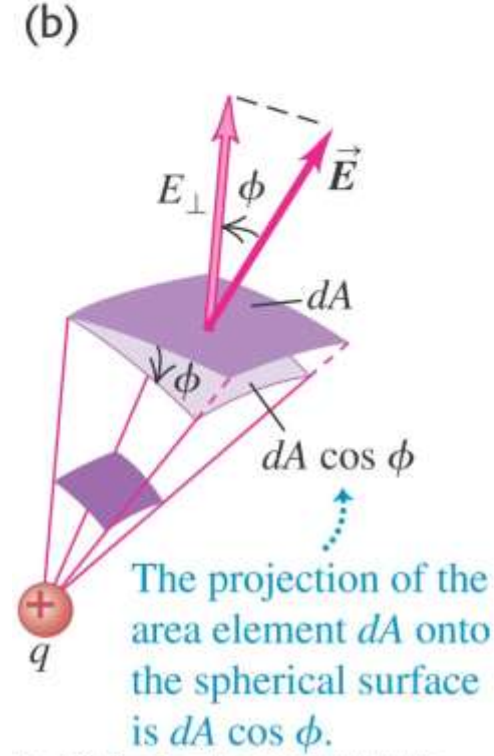
(a) Gaussian surface around positive charge: positive (outward) flux



(b) Gaussian surface around negative charge: negative (inward) flux

Gauss's Law





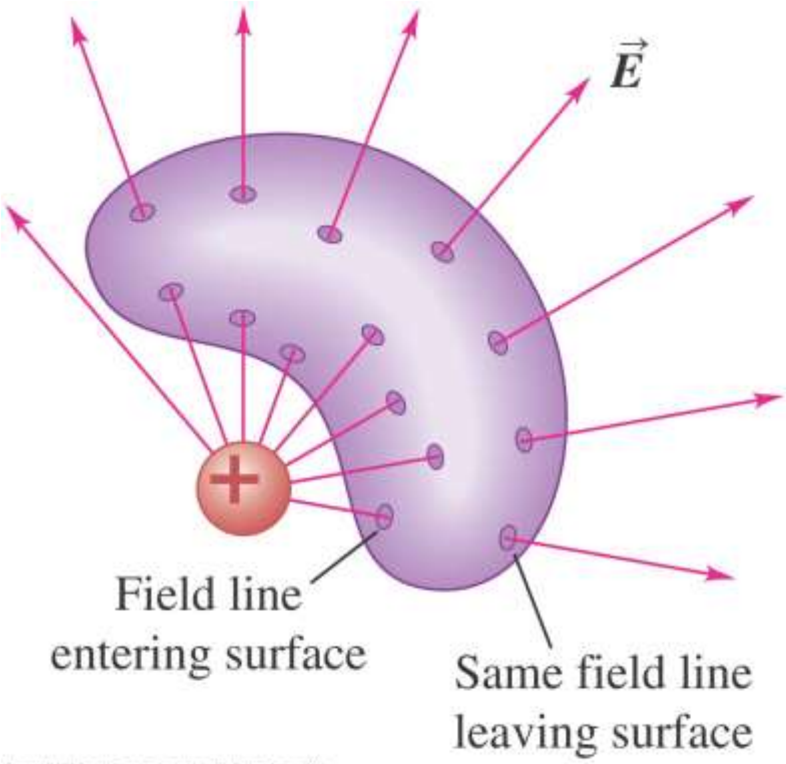
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{\sum q_i}{\varepsilon_0}$$

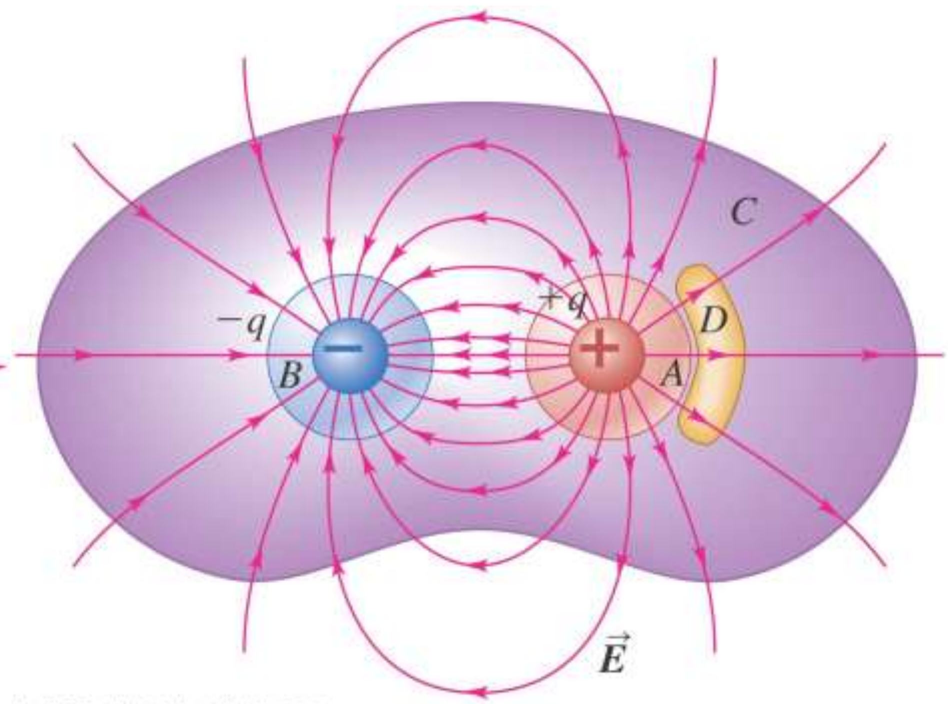
Applications of the Gauss's Law

Remember – electric field lines must start and must end on charges!

If no charge is enclosed within Gaussian surface – flux is zero!

Electric flux is proportional to the algebraic number of lines leaving the surface, outgoing lines have positive sign, incoming - negative





Examples of certain field configurations

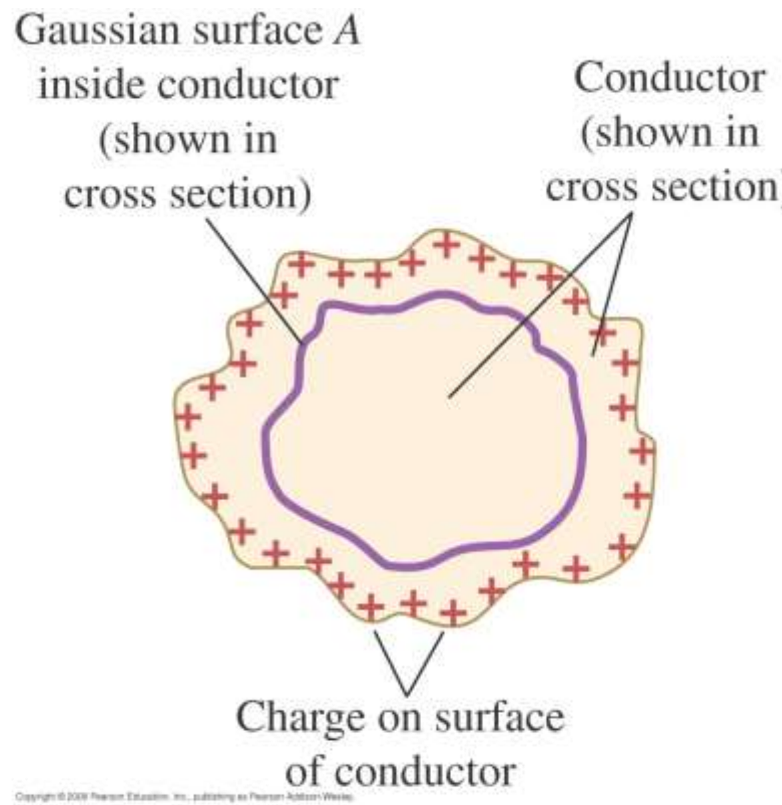
Remember, Gauss's law is equivalent to Coulomb's law

However, you can employ it for certain symmetries to solve the reverse problem – find charge configuration from known E-field distribution.

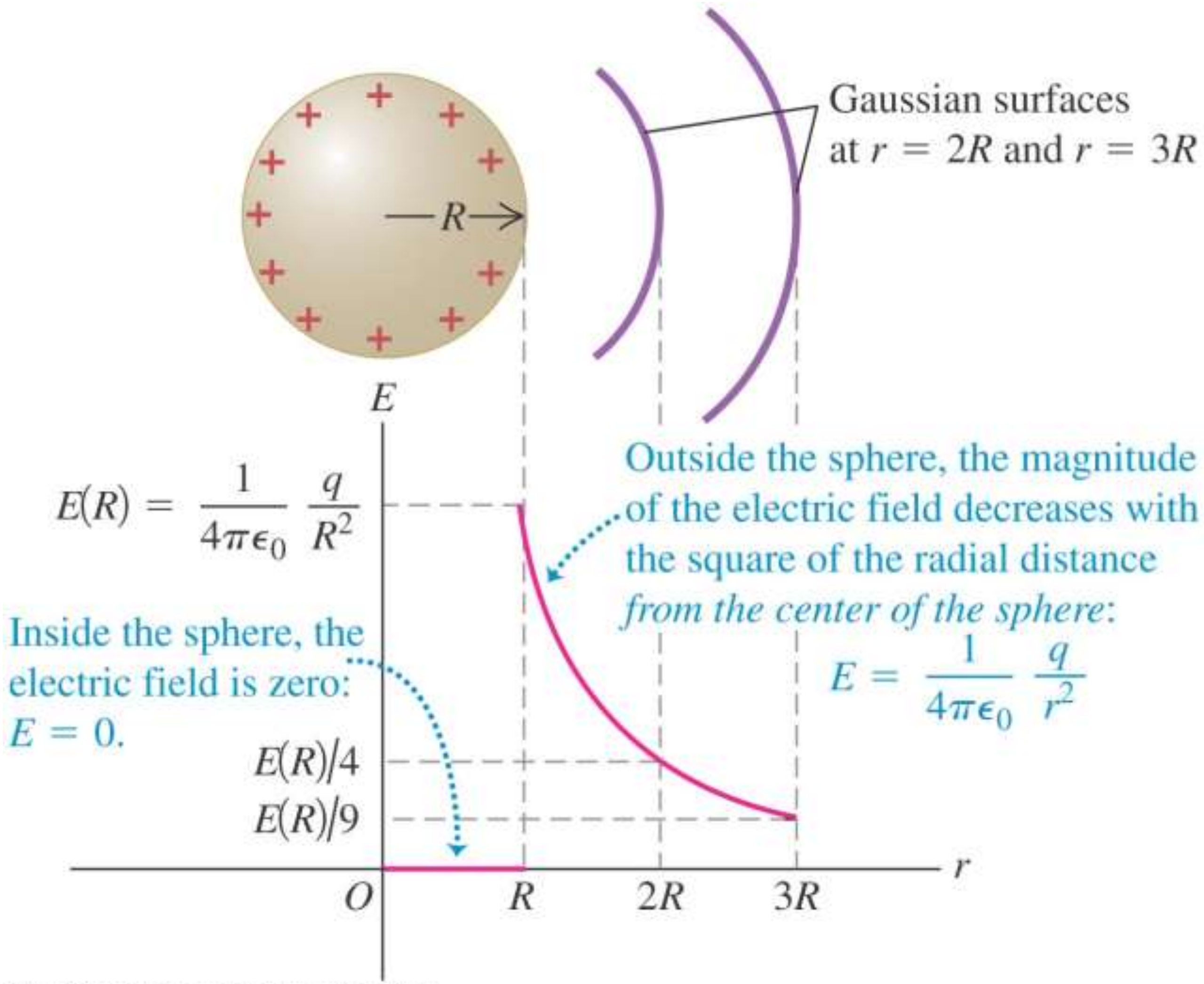
Field within the conductor – zero
(free charges screen the external field)

Any excess charge resides on the surface

$$\int_S \vec{E} \cdot d\vec{A} = 0$$



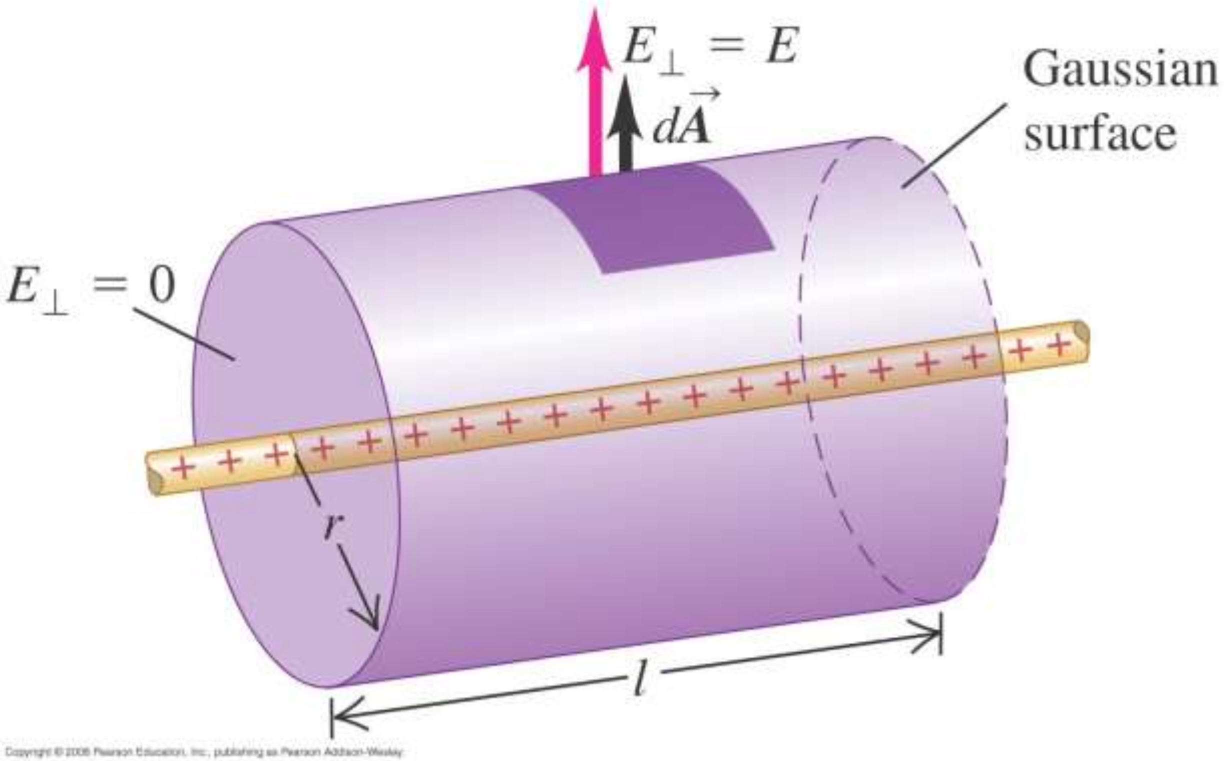
Field of a charged conducting sphere



Field of a thin, uniformly charged conducting wire

Field outside the wire can only point radially outward, and, therefore, may only depend on the distance from the wire

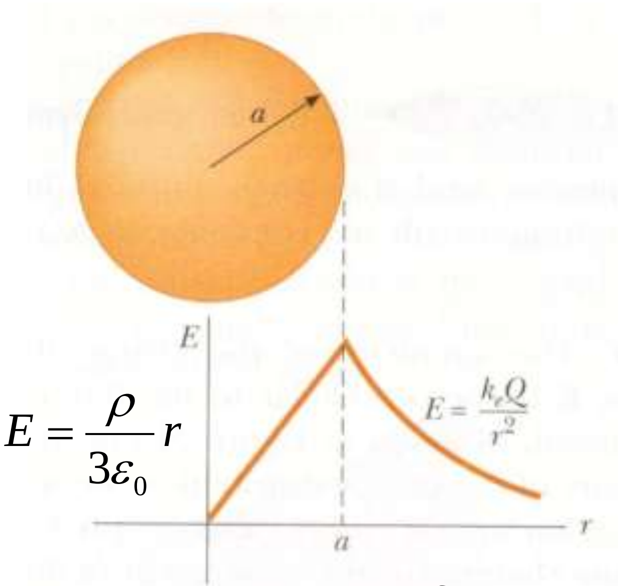
$$\oint \vec{E}\, d\vec{A} = \frac{Q}{\varepsilon_0}$$



$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

λ- linear density of charge

Field of the uniformly charged sphere



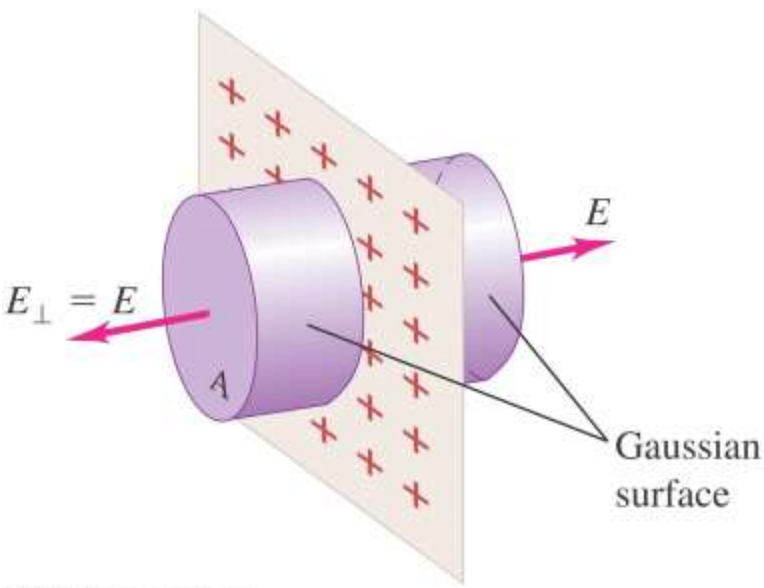
$$E = \frac{\rho}{3\varepsilon_0} r$$

Uniform charge within a sphere of radius ***r***

$$q^{'} = Q\left(\frac{r}{a}\right)^3$$ Q - total charge

$$\rho = \frac{Q}{V}$$ - volume density of charge

Field of the infinitely large conducting plate



$$\sigma = \frac{Q}{A}$$ σ- uniform surface charge density

$$E = \frac{\sigma}{2\varepsilon_0}$$

Charged Isolated Conductors

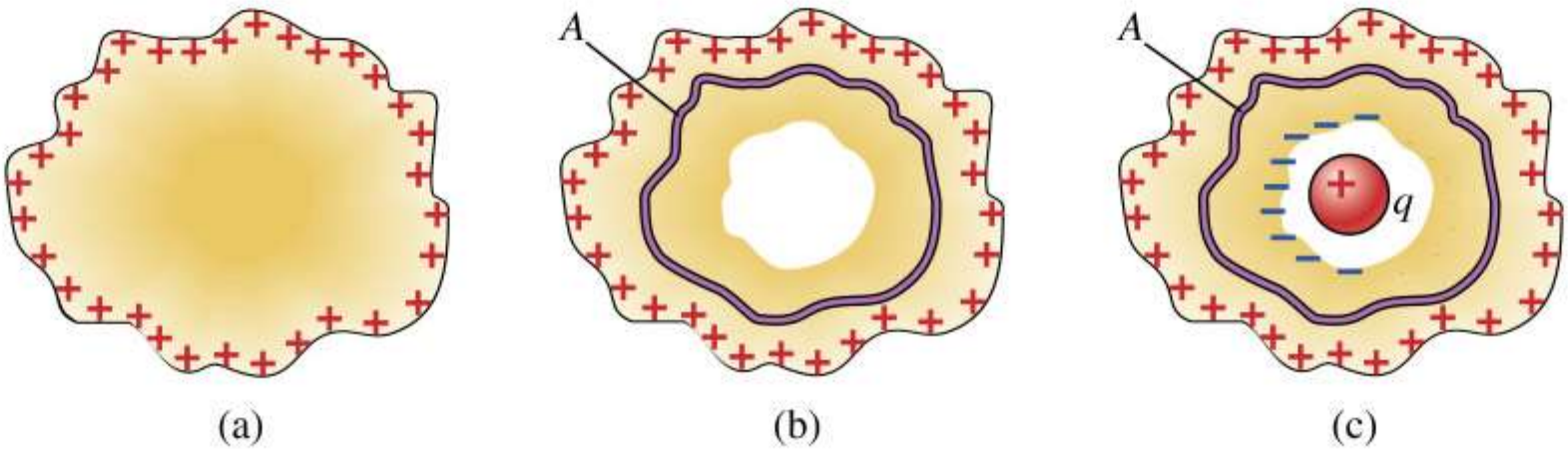
- In a charged isolated conductor all the charge moves to the surface
- The E field inside a conductor must be 0 otherwise a current would be set up
- The charges do not necessarily distribute themselves uniformly, they distribute themselves so the net force on each other is 0.
- This means the surface charge density varies over a nonspherical conductor

Charged Isolated Conductors cont

- On a conducting surface

$$E = \frac{\sigma}{\varepsilon_o}$$

- If there were a cavity in the isolated conductor, no charges would be on the surface of the cavity, they would stay on the surface of the conductor



Charge on solid conductor resides on surface.

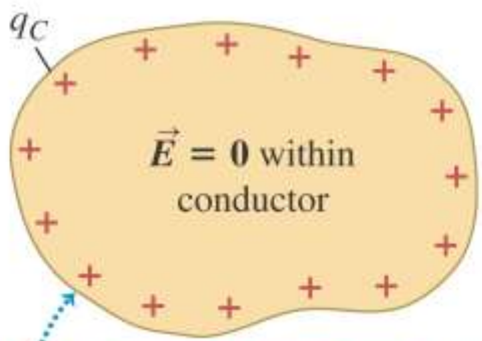
Charge in cavity makes a equal but opposite charge reside on inner surface of conductor.

Properties of a Conductor in Electrostatic Equilibrium

1. The E field is zero everywhere inside the conductor
2. If an isolated conductor carries a charge, the charge resides on its surface
3. The electric field just outside a charged conductor is perpendicular to the surface and has the magnitude given above
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest

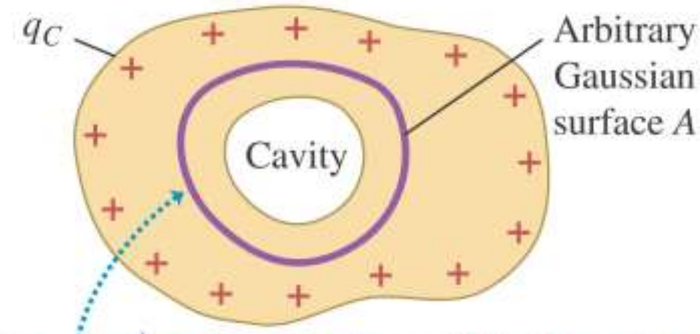
Charges on Conductors

(a) Solid conductor with charge q_C



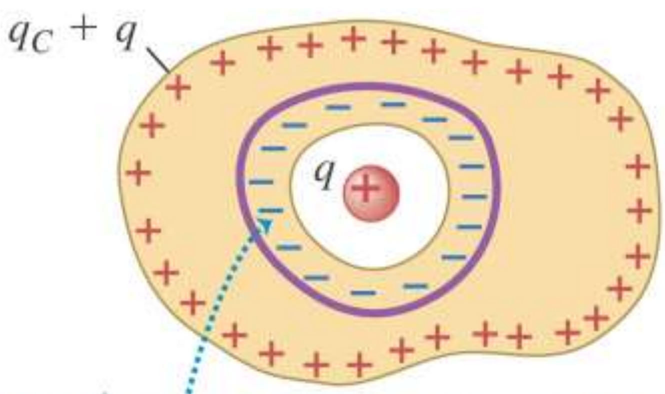
The charge q_C resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = \mathbf{0}$ within the conductor.

(b) The same conductor with an internal cavity



Because $\vec{E} = \mathbf{0}$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

(c) An isolated charge q placed in the cavity



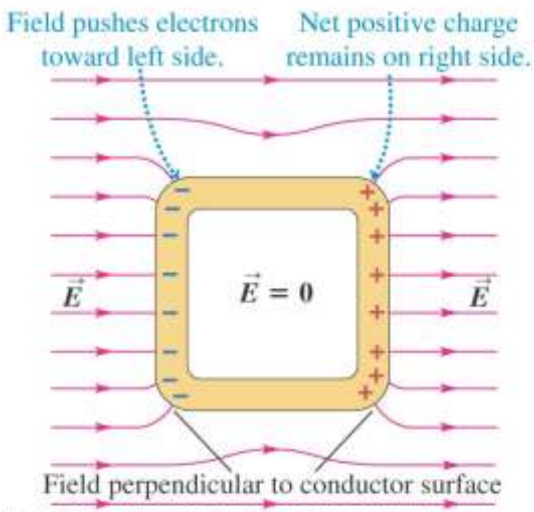
For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.

Field within conductor

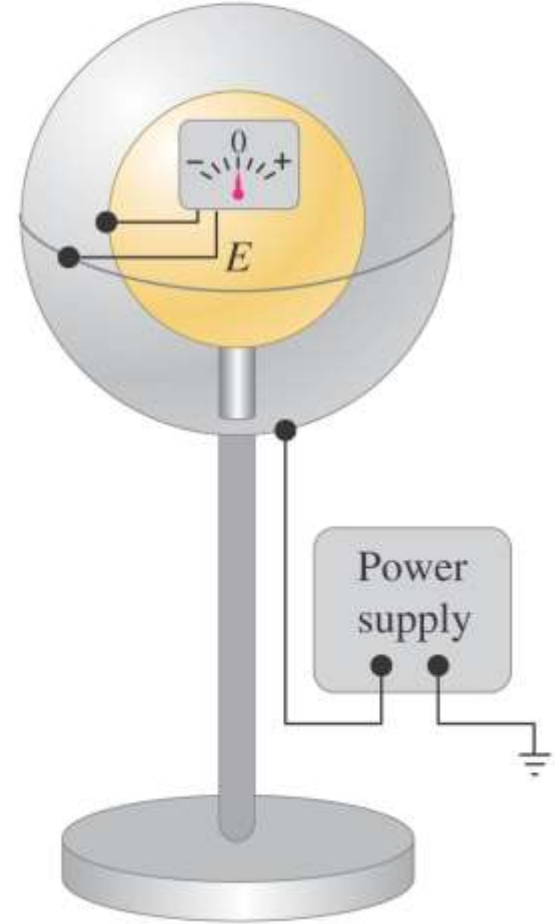
E=0

Experimental Testing of the Gauss's Law

(a)



(b)



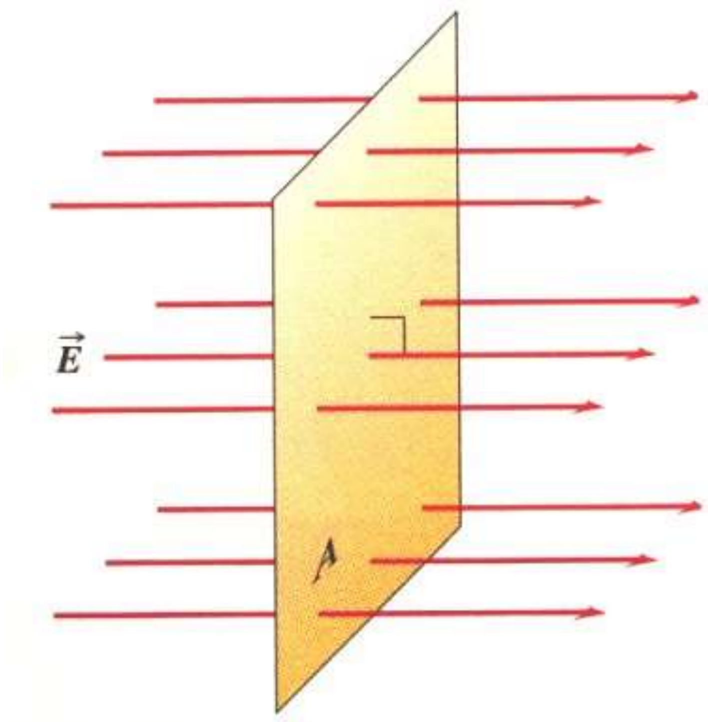
Gauss's Law

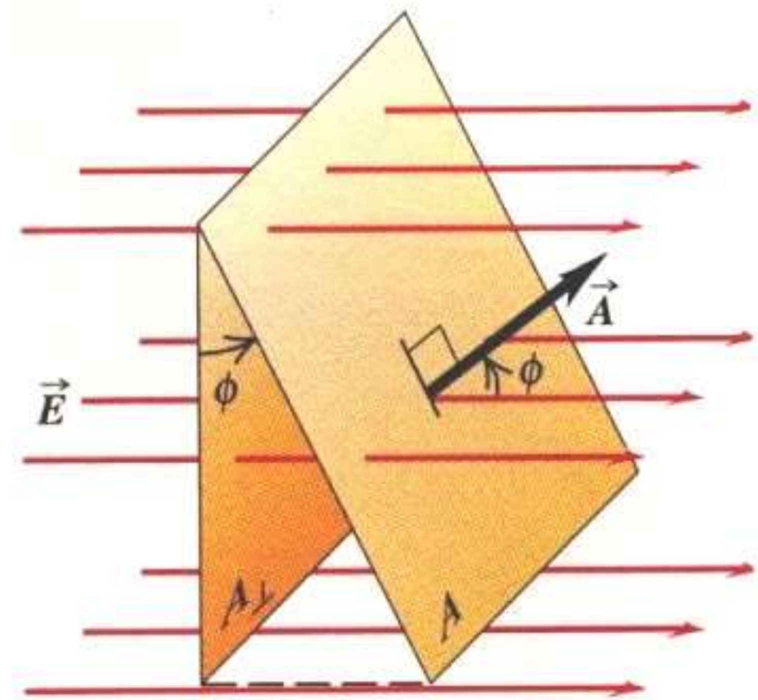
Basic Concepts

- Electric Flux
- Gauss's Law
- Applications of Gauss's Law
- Conductors in Equilibrium

Electric Flux

The electric flux, Φ_E, through a surface is defined as the scalar product of **E** and **A**, $\Phi_E = \mathbf{E} \cdot \mathbf{A}$. **A** is a vector perpendicular to the surface with a magnitude equal to the surface area. This is true for a uniform electric field.





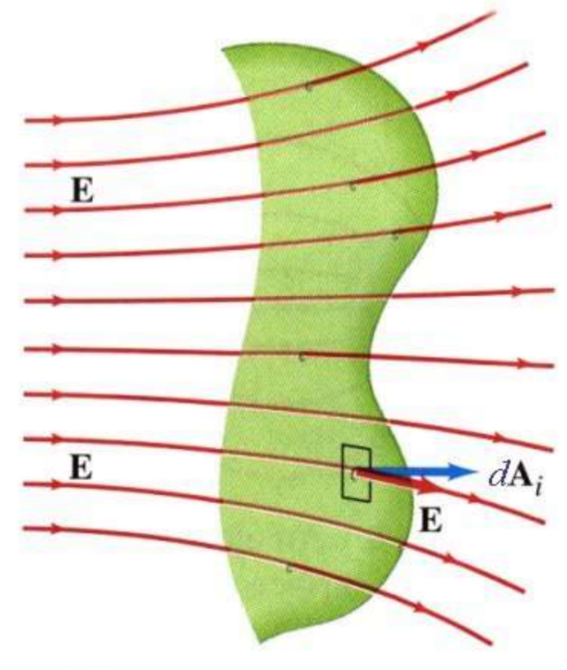
$\Phi_E = EA$

$A_\perp = A\cos\phi$ so $\Phi_E = EA_\perp = EA\cos\phi$

Electric Flux Continued

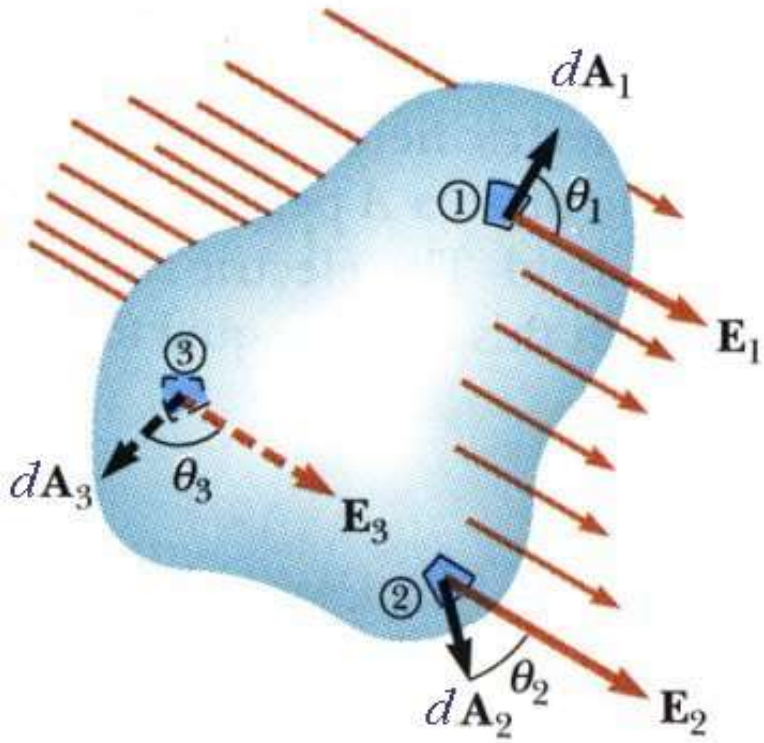
What about the case when the electric field is not uniform and the surface is not flat?

Then we divide the surface into small elements and add the flux through each.



$$\Phi_E = \sum_i \vec{E}_i \cdot d\vec{A}_i \rightarrow \int \vec{E} \cdot d\vec{A}$$

Electric Flux Continued



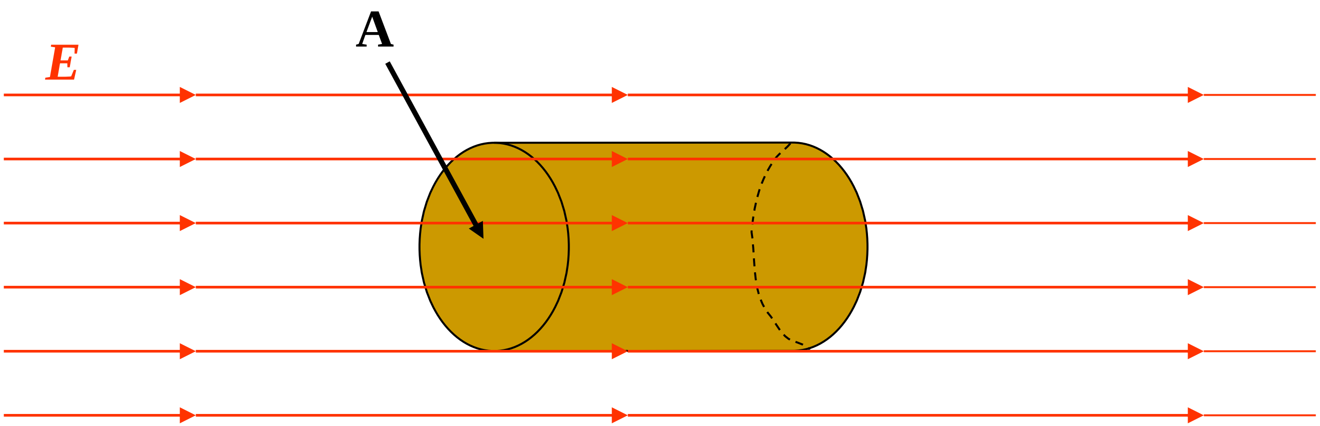
Finally, what about a closed surface?

A closed surface is one that completely encloses a volume. This is handled as before, but we need to resolve ambiguity about direction of **A**. It is defined to point *outward* so flux exiting the enclosed volume is *positive* and flux entering is *negative.*

$$\boxed{\Phi_E = \oint \vec{E} \cdot d\vec{A}}$$

Worked Example 1

Compute the electric flux through a cylinder with an axis parallel to the electric field direction.



The flux through the curved surface is zero since **E** is perpendicular to *d***A** there. For the ends, the surfaces are perpendicular to **E**, and **E** and **A** are parallel. Thus the flux through the left end (*into* the cylinder) is $-EA$, while the flux through right end (*out* of the cylinder) is $+EA$. Hence the net flux through the cylinder is zero.

Gauss's Law

Gauss's Law relates the electric flux through a *closed* surface with the charge Q_{in} inside that surface.

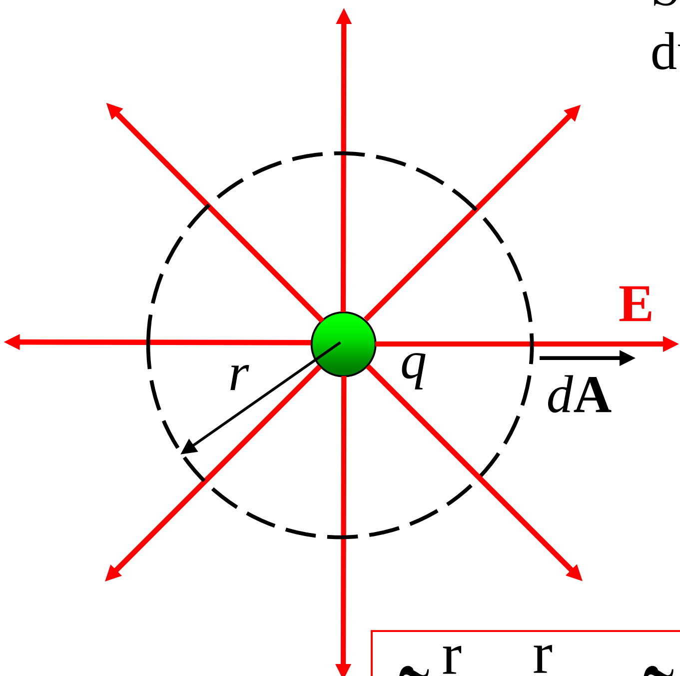
$$\boxed{\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}}$$

This is a useful tool for simply determining the electric field, but only for certain situations where the charge distribution is either rather simple or possesses a high degree of symmetry.

Problem Solving Strategies for Gauss's Law

- Select a Gaussian surface with symmetry that matches the charge distribution
- Draw the Gaussian surface so that the electric field is either constant or zero at all points on the Gaussian surface
- Use symmetry to determine the direction of **E** on the Gaussian surface
- Evaluate the surface integral (electric flux)
- Determine the charge inside the Gaussian surface
- Solve for **E**

Worked Example 2



Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

We choose a Gaussian surface that is a sphere of radius r centered on the point charge. I have chosen the charge to be positive so the field is radial outward by symmetry and therefore everywhere perpendicular to the Gaussian surface.

$$\boxed{\vec{E}\cdot d\vec{A} = E\,dA}$$

Gauss's law then gives:

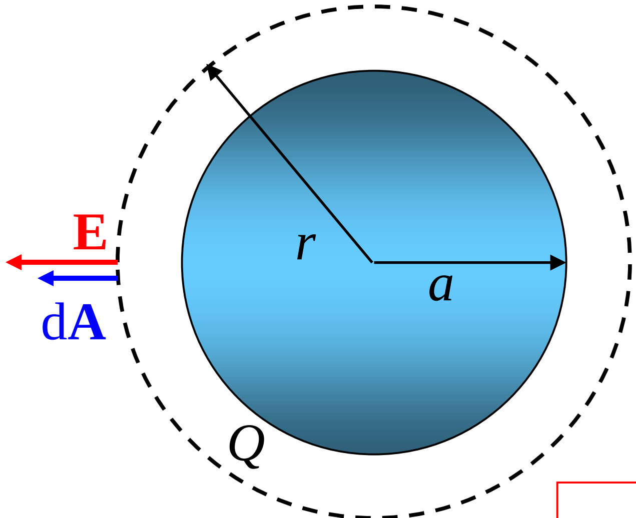
$$\boxed{\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = \frac{Q_{in}}{\varepsilon_0} = \frac{q}{\varepsilon_0}}$$

Symmetry tells us that the field is constant on the Gaussian surface.

$$\boxed{\oint E\,dA = E\oint dA = E\left(4\pi r^2\right) = \frac{q}{\varepsilon_0} \quad \text{so} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = k_e\frac{q}{r^2}}$$

Worked Example 3

An insulating sphere of radius *a* has a uniform charge density ρ and a total positive charge *Q*. Calculate the electric field outside the sphere.



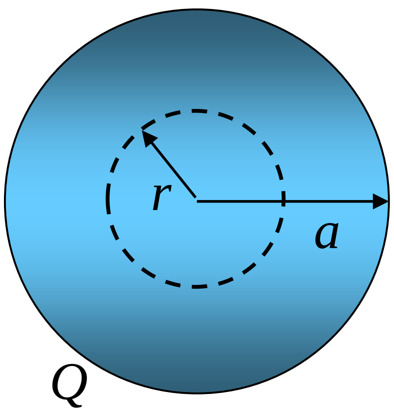
Since the charge distribution is spherically symmetric we select a spherical Gaussian surface of radius $r > a$ centered on the charged sphere. Since the charged sphere has a positive charge, the field will be directed radially outward. On the Gaussian sphere **E** is always parallel to d**A**, and is constant.

❶ Left side: $\oint \vec{E} \cdot d\vec{A} = \oint E\, dA = E \oint dA = E\left(4\pi r^2\right)$

❷ Right side: $\dfrac{Q_{in}}{\varepsilon_0} = \dfrac{Q}{\varepsilon_0}$

❸ $E\left(4\pi r^2\right) = \dfrac{Q}{\varepsilon_0}$ or $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2} = k_e \dfrac{Q}{r^2}$

Worked Example 3 cont'd



Find the electric field at a point inside the sphere.

Now we select a spherical Gaussian surface with radius $r < a$. Again the symmetry of the charge distribution allows us to simply evaluate the left side of Gauss's law just as before.

❶ Left side: $\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = E\oint dA = E\left(4\pi r^2\right)$

The charge inside the Gaussian sphere is no longer Q. If we call the Gaussian sphere volume V' then

❷ Right side: $Q_{in} = \rho V' = \rho \frac{4}{3}\pi r^3$

❸ $E\left(4\pi r^2\right) = \frac{Q_{in}}{\varepsilon_0} = \frac{4\rho\pi r^3}{3\varepsilon_0}$

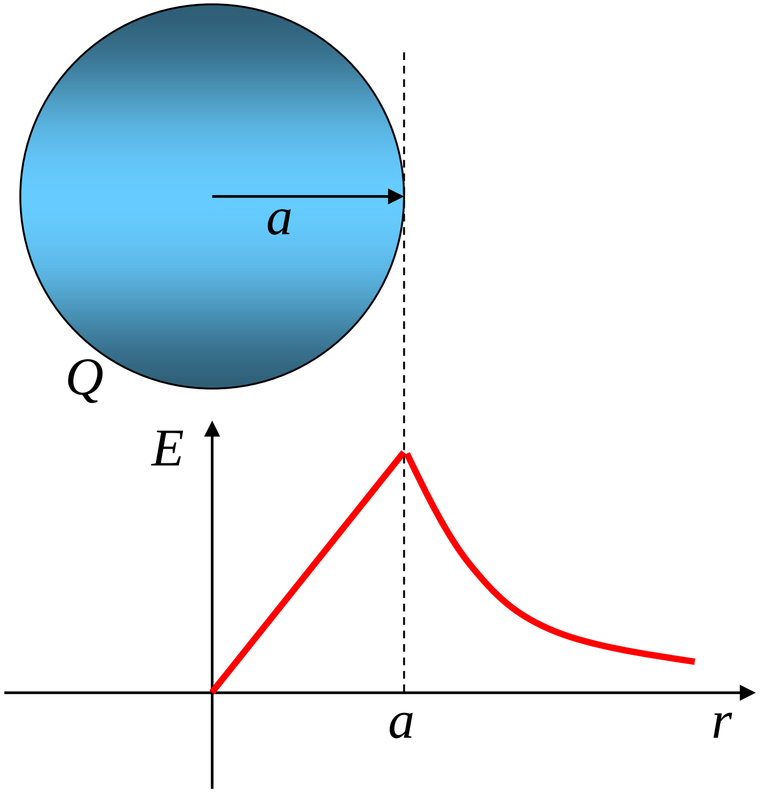
❹ $E = \frac{4\rho\pi r^3}{3\varepsilon_0\left(4\pi r^2\right)} = \frac{\rho}{3\varepsilon_0} r$ but $\rho = \frac{Q}{\frac{4}{3}\pi a^3}$ so $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3} r = k_e \frac{Q}{a^3} r$

Worked Example 3 cont'd

We found for $r > a$, $E = k_e \dfrac{Q}{r^2}$

and for $r < a$, $E = \dfrac{k_e Q}{a^3} r$

Let's plot this:



Conductors in Electrostatic Equilibrium

By electrostatic equilibrium we mean a situation where there is no *net* motion of charge within the conductor

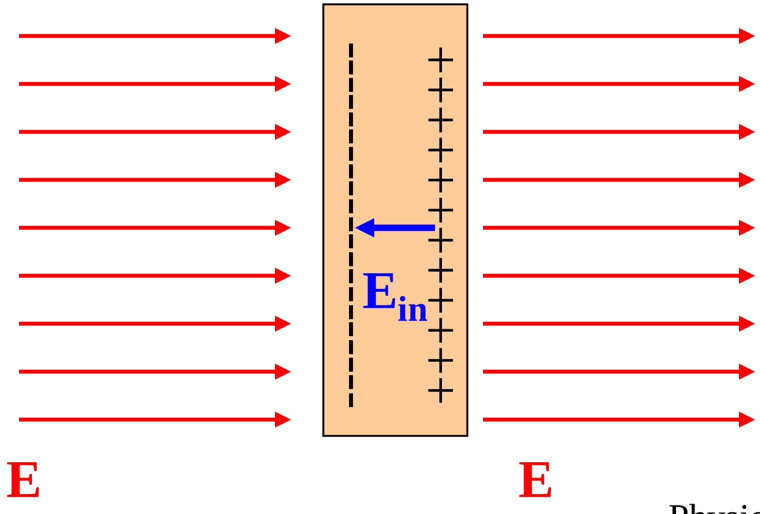
- The electric field is zero everywhere inside the conductor
- Any net charge resides on the conductor's surface
- The electric field just outside a charged conductor is perpendicular to the conductor's surface

Conductors in Electrostatic Equilibrium

- The electric field is zero everywhere inside the conductor

Why is this so?

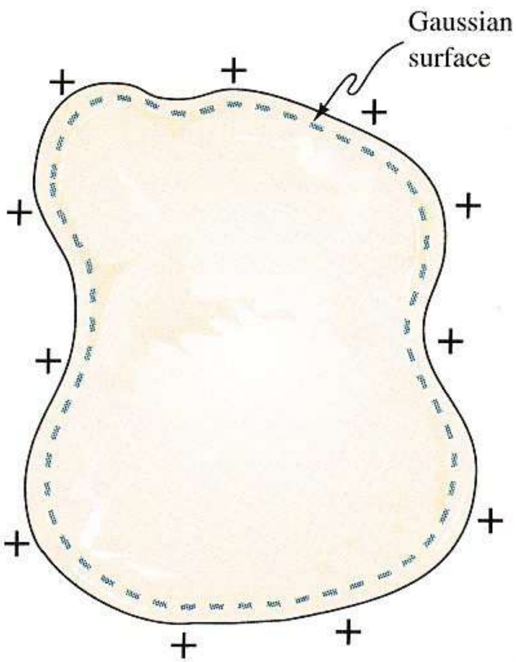
If there was a field in the conductor the charges would accelerate under the action of the field.



The charges in the conductor move creating an internal electric field that cancels the applied field on the inside of the conductor

Worked Example 4

Any net charge on an isolated conductor must reside on its surface and the electric field just outside a charged conductor is perpendicular to its surface (and has magnitude σ/ε_0). Use Gauss's law to show this.



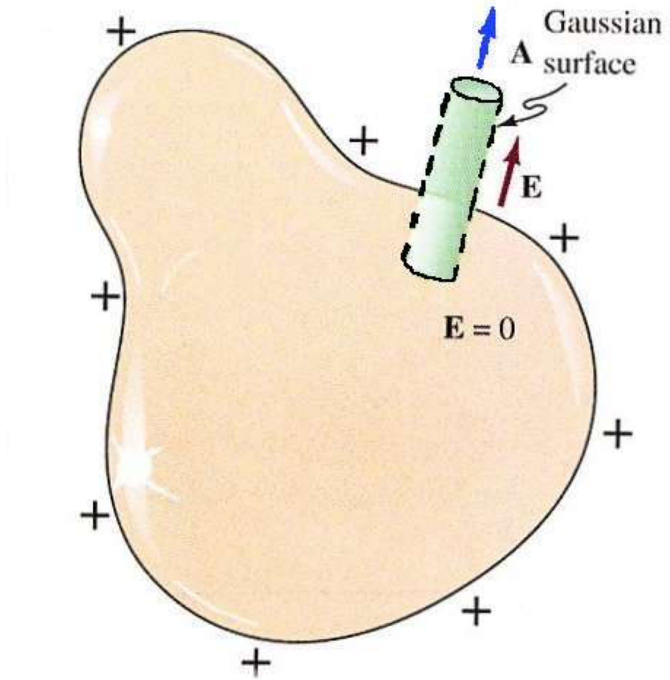
For an arbitrarily shaped conductor we can draw a Gaussian surface inside the conductor. Since we have shown that the electric field inside an isolated conductor is zero, the field at every point on the Gaussian surface must be zero.

$$\boxed{\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}}$$

From Gauss's law we then conclude that the net charge inside the Gaussian surface is zero. Since the surface can be made arbitrarily close to the surface of the conductor, *any net charge must reside on the conductor's surface*.

Worked Example 4 cont'd

We can also use Gauss's law to determine the electric field just outside the surface of a charged conductor. Assume the surface charge density is σ.

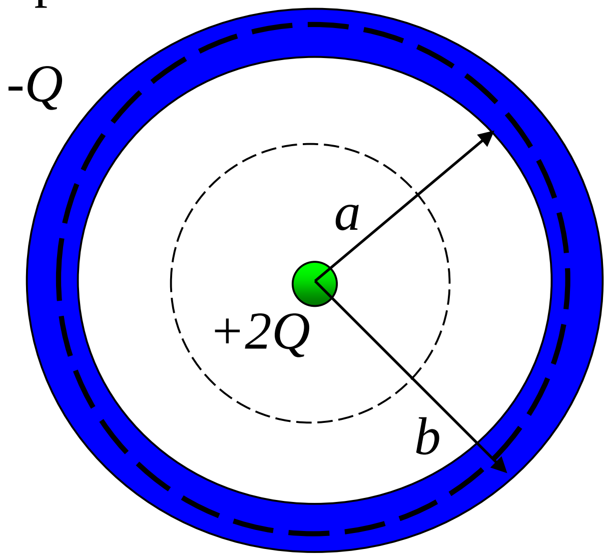


Since the field inside the conductor is zero there is no flux through the face of the cylinder inside the conductor. If **E** had a component along the surface of the conductor then the free charges would move under the action of the field creating surface currents. Thus **E** is perpendicular to the conductor's surface, and the flux through the cylindrical surface must be zero. Consequently the net flux through the cylinder is *EA* and Gauss's law gives:

$$\boxed{\Phi_E = EA = \frac{Q_{in}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \quad \text{or} \quad E = \frac{\sigma}{\varepsilon_0}}$$

Worked Example 5

A conducting spherical shell of inner radius a and outer radius b with a net charge $-Q$ is centered on point charge $+2Q$. Use Gauss's law to find the electric field everywhere, and to determine the charge distribution on the spherical shell.



First find the field for $0 < r < a$

This is the same as Ex. 2 and is the field due to a point charge with charge $+2Q$.

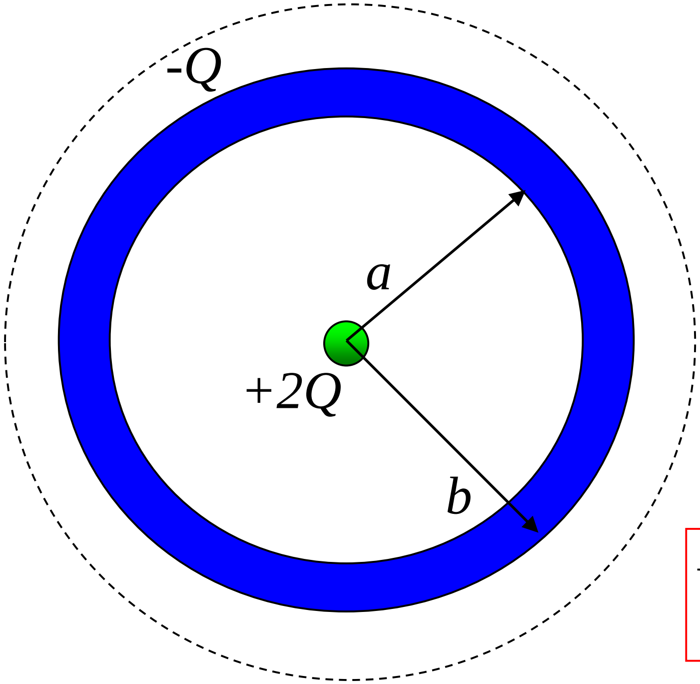
$$\boxed{E = k_e \frac{2Q}{r^2}}$$

Now find the field for $a < r < b$

The field must be zero inside a conductor in equilibrium. Thus from Gauss's law Q_{in} is zero. There is a + 2Q from the point charge so we must have $Q_a = -2Q$ on the inner surface of the spherical shell. Since the net charge on the shell is $-Q$ we can get the charge on the outer surface from $Q_{net} = Q_a + Q_b$.

$$Q_b = Q_{net} - Q_a = -Q - (-2Q) = +Q.$$

Worked Example 5 cont'd



Find the field for $r > b$

From the symmetry of the problem, the field in this region is radial and everywhere perpendicular to the spherical Gaussian surface. Furthermore, the field has the same value at every point on the Gaussian surface so the solution then proceeds exactly as in Ex. 2, but $Q_{in}=2Q-Q$.

$$\boxed{\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = E\oint dA = E\left(4\pi r^2\right)}$$

Gauss's law now gives:

$$\boxed{E\left(4\pi r^2\right) = \frac{Q_{in}}{\varepsilon_0} = \frac{2Q-Q}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = k_e\frac{Q}{r^2}}$$

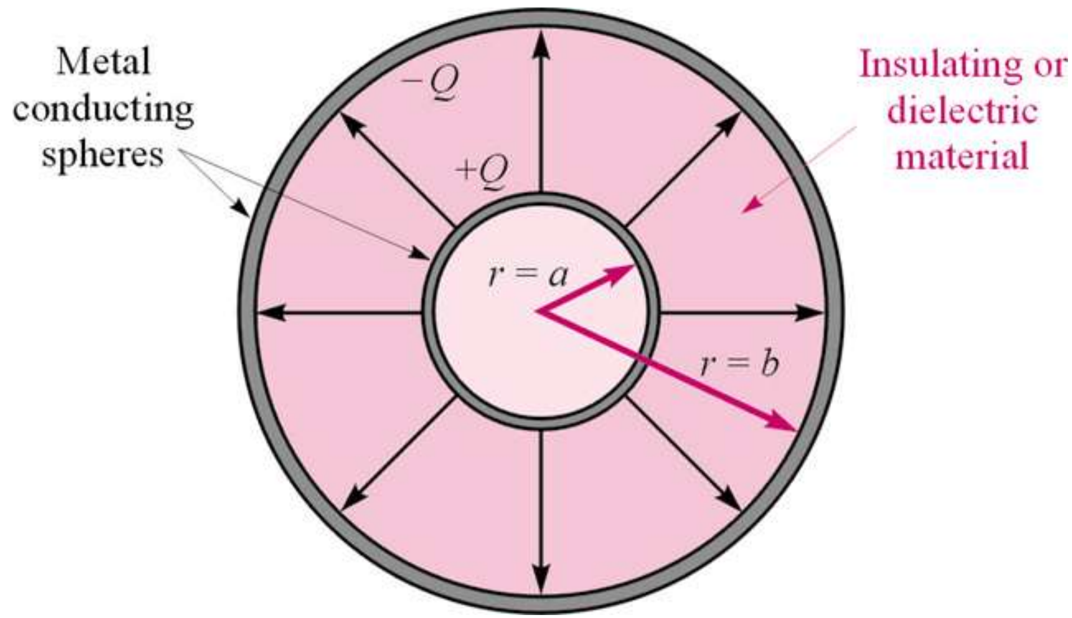
Summary

- Two methods for calculating electric field
  - Coulomb's Law
  - Gauss's Law
- Gauss's Law: Easy, elegant method for symmetric charge distributions
- Coulomb's Law: Other cases
- Gauss's Law and Coulomb's Law are equivalent for electric fields produced by static charges

Electric Flux Density, Gauss's Law, and Divergence

Electric Flux Density

- Faraday's Experiment



$\text{Flux} = \Psi$, same units as Q

Ψ is responsible for creating $-Q$ on outer sphere

Electric Flux Density, **D**

- Units: C/m^2
- Magnitude: Number of flux lines (coulombs) crossing a surface normal to the lines divided by the surface area.
- Direction: Direction of flux lines (same direction as **E**).
- For a point charge: $$\mathbf{D} = \frac{Q}{4\pi r^2}\,\mathbf{a}_r$$
- For a general charge distribution,

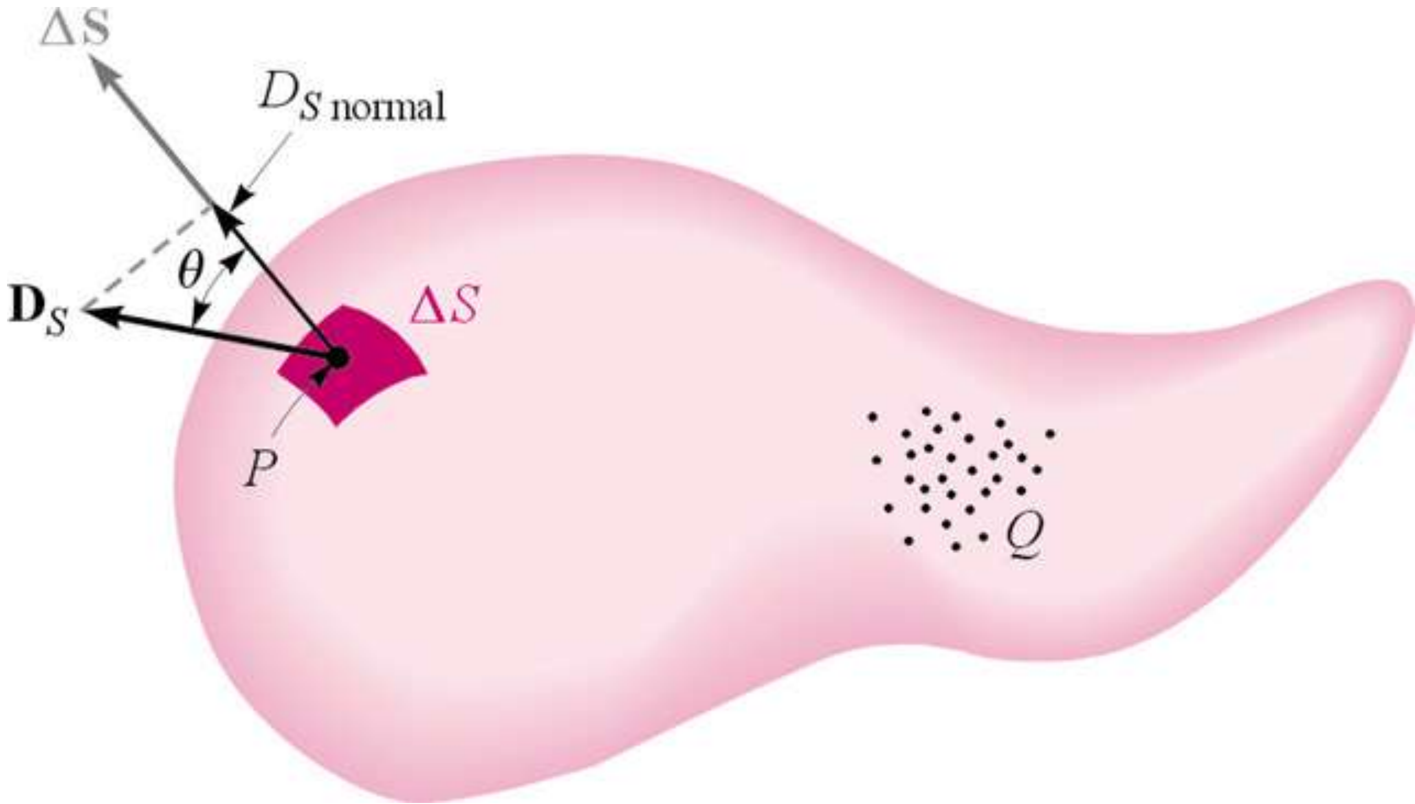
$$\mathbf{D} = \epsilon_0\,\mathbf{E} = \int_{\text{vol}} \frac{\rho_v\,dv}{4\pi R^2}\,\mathbf{a}_r$$

Gauss's Law

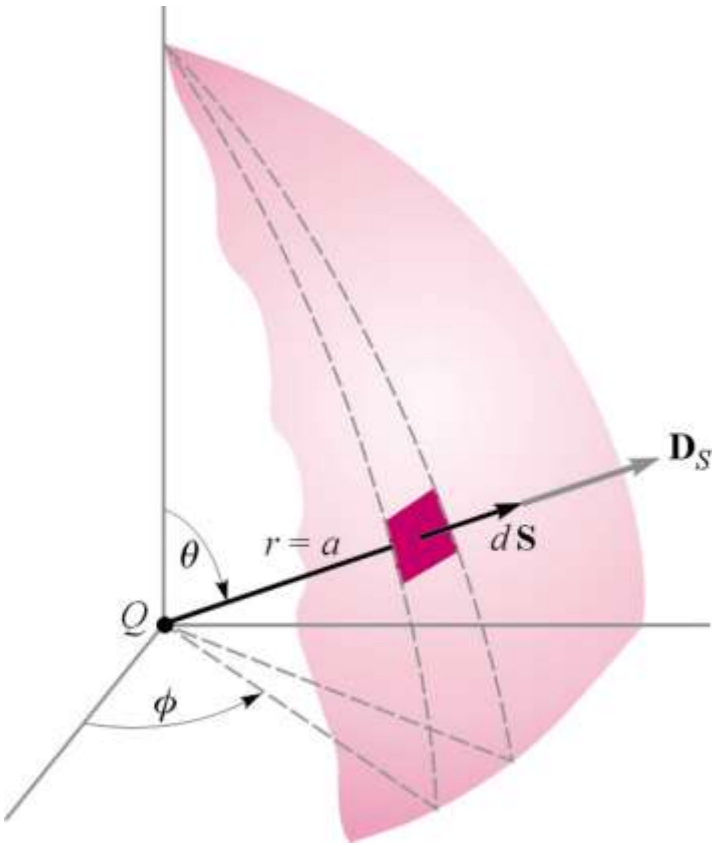
- "The electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

- The integration is performed over a *closed* surface, i.e. *gaussian surface*.



- We can check Gauss's law with a point charge example.



$$\int_0^{2\pi}\int_0^{\pi}\frac{q}{4\pi a^2}\,a^2\,\mathrm{Sin}[\theta]\,d\theta\,d\phi$$

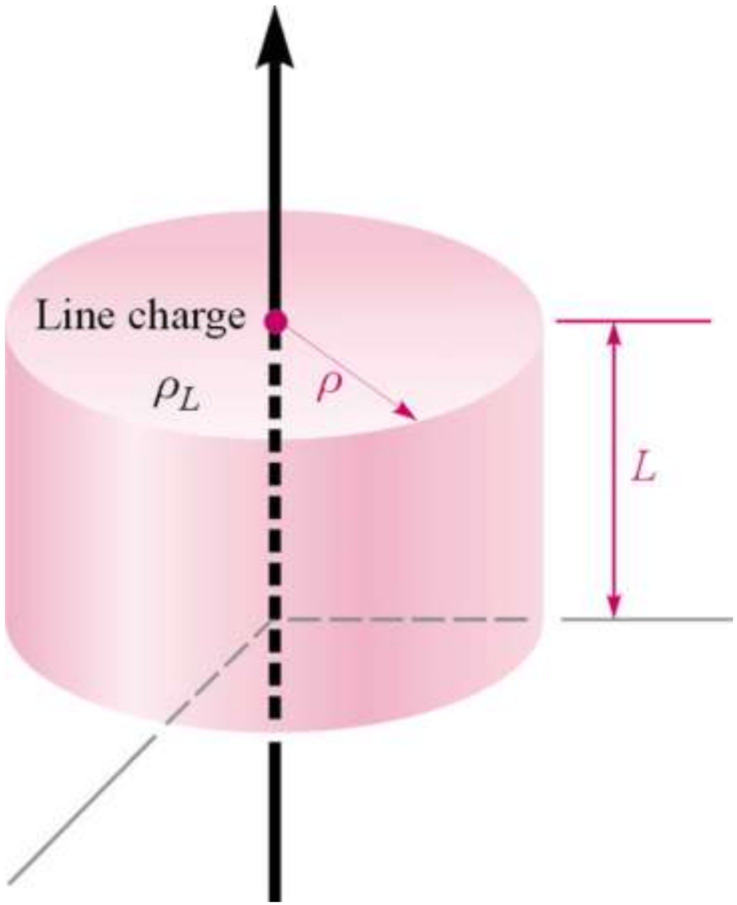
q

Symmetrical Charge Distributions

- Gauss's law is useful under two conditions.

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S\, dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot dS$ is not zero, D_S = constant.

Gauss's law simplifies the task of finding **D** near an infinite line charge.

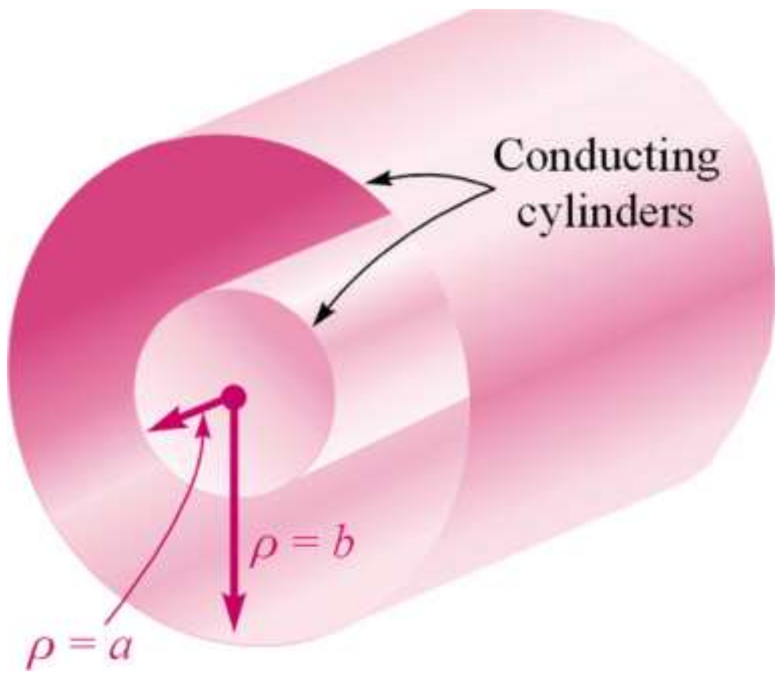


$$Q = \oint_{\text{cyl}} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{\text{sides}} dS + 0 \int_{\text{top}} dS + 0 \int_{\text{bottom}} dS$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S \, 2\pi\rho L$$

$$D_S = D_\rho = \frac{Q}{2\pi\rho L} = \frac{\rho_L}{2\pi\rho}$$

Infinite coaxial cable:

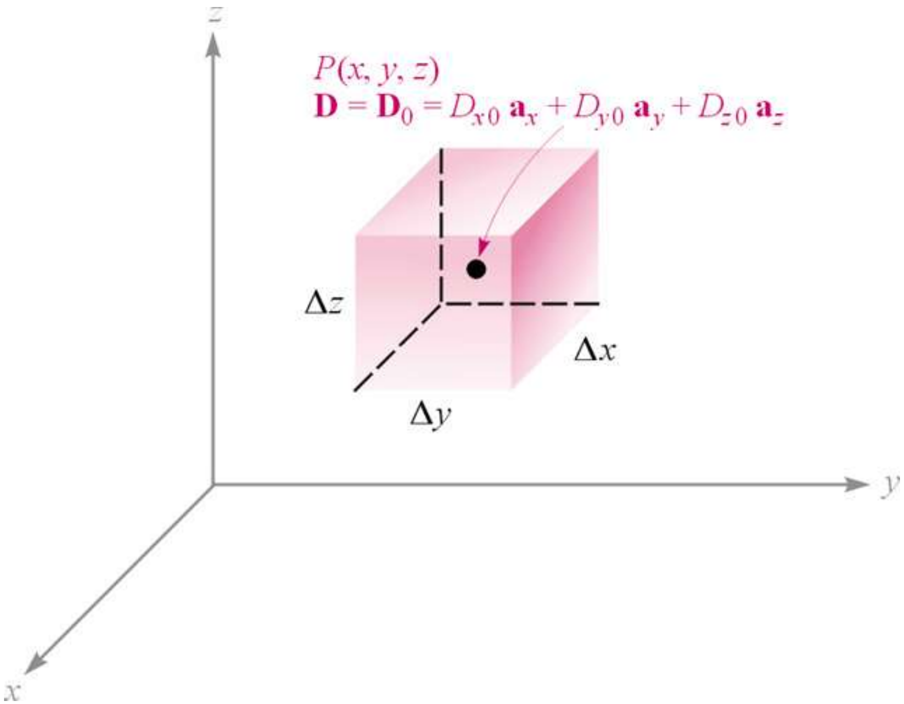


$$D = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho \quad (a < \rho < b)$$

$$D = 0 \quad (\rho > b)$$

Differential Volume Element

- If we take a small enough closed surface, then **D** is almost constant over the surface.



$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

$$\int_{\text{front}} \doteq \left(D_{\text{x0}} + \frac{\Delta x}{2}\,\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{\text{back}} \doteq \left(-D_{\text{x0}} + \frac{\Delta x}{2}\,\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x}\,\Delta x \Delta y \Delta z$$

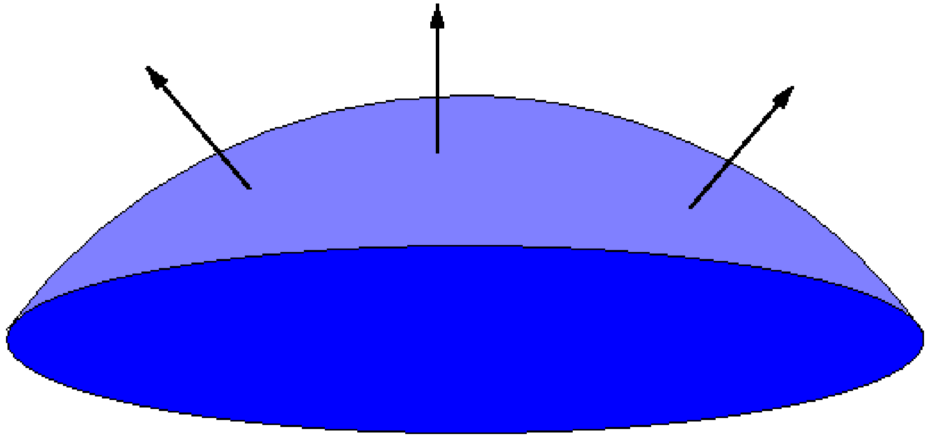
$$\vdots$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

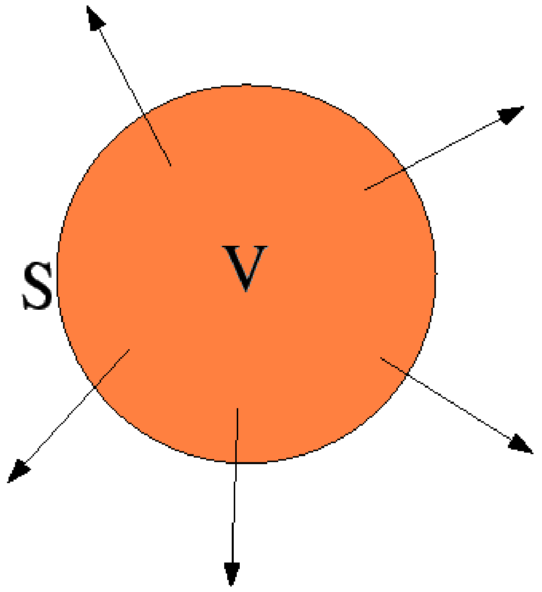
Charge enclosed in volume $\Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times$ volume Δv

Divergence

Divergence is the outflow of flux from a small closed surface area (per unit volume) as volume shrinks to zero.



open surface.



closed surface

-Water leaving a bathtub

- -Closed surface (water itself) is essentially incompressible
- -Net outflow is zero

-Air leaving a punctured tire

- -Divergence is positive, as closed surface (tire) exhibits net outflow

Mathematical definition of divergence

$$\mathrm{div}(\mathbf{D}) = \lim_{\Delta v \to 0} \oint \frac{\mathbf{D}}{\Delta v}\, d\mathbf{S}$$

Surface integral as the volume element (Δv) approaches zero

D is the vector flux density

$$\boxed{\mathrm{div}(\mathbf{D}) = \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} \right)}$$

- *Cartesian*

Divergence in Other Coordinate Systems

Cylindrical

$$\mathrm{div}(\mathbf{D}) = \frac{1}{\rho}\cdot\frac{\delta}{\delta\rho}\left(\rho\cdot D_{\rho}\right) + \frac{1}{\rho}\cdot\frac{\delta D_{\phi}}{\delta\phi} + \frac{\delta D_{z}}{\delta z}$$

Spherical

$$\mathrm{div}(\mathbf{D}) = \frac{1}{r^{2}}\cdot\frac{\delta\left(D_{r}\cdot r^{2}\right)}{\delta r} + \frac{1}{r\cdot\sin(\theta)}\cdot\frac{\delta\left(D_{\theta}\cdot\sin(\theta)\right)}{\delta\theta} + \frac{1}{r\cdot\sin(\theta)}\cdot\frac{\delta D_{\phi}}{\delta\phi}$$

Maxwell's First Equation

$$\int_S \mathbf{A} \cdot d\mathbf{S} = Q$$

Gauss' Law…

$$\frac{\int_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

…per unit volume

Volume shrinks to zero

$$\lim_{\Delta v \to 0} \frac{\int_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

Electric flux per unit volume is equal to the volume charge density

Maxwell's First Equation

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

$$\boxed{\mathrm{div}(\mathbf{D}) = \rho_v}$$

Sometimes called the point form of Gauss' Law

Enclosed surface is reduced to a single point

∇ and the Divergence Theorem

∇ ➔ *del* operator

What is del?

$$\nabla \quad = \frac{\delta(\mathbf{a_x})}{\delta x} + \frac{\delta(\mathbf{a_y})}{\delta y} + \frac{\delta(\mathbf{a_z})}{\delta z}$$

∇'s Relationship to Divergence

$$\mathrm{div}(\mathbf{D}) = \nabla \cdot \mathbf{D}$$

True for all coordinate systems

$$\nabla \bullet \mathbf{A} = [\partial/\partial\mathbf{x}\, \boldsymbol{x} + \partial/\partial\mathbf{y}\, \boldsymbol{y} + \partial/\partial\mathbf{z}\, \boldsymbol{z}] \bullet [\mathrm{A}_{\mathbf{x}}\, \boldsymbol{x} + \mathrm{A}_{\mathbf{y}}\, \boldsymbol{y} + \mathrm{A}_{\mathbf{z}}\, \boldsymbol{z}]$$

$$= (\partial \mathrm{A}_{\mathbf{x}}/\partial\mathbf{x}) + (\partial \mathrm{A}_{\mathbf{y}}/\partial\mathbf{y}) + (\partial \mathrm{A}_{\mathbf{z}}/\partial\mathbf{z})\ ;$$

Other ∇ Relationships

Gradient – results from ∇ operating on a function

$$\nabla \mathbf{f} = (\partial \mathbf{f}/\partial \mathbf{x})\, \boldsymbol{x} + (\partial \mathbf{f}/\partial \mathbf{y})\, \boldsymbol{y} + (\partial \mathbf{f}/\partial \mathbf{z})\, \boldsymbol{z}\ .$$

Represents direction of greatest change

Curl – cross product of ∇ and

$$\nabla \times \mathbf{A} = [\partial/\partial x\, \boldsymbol{x} + \partial/\partial y\, \boldsymbol{y} + \partial/\partial z\, \boldsymbol{z}] \times [A_x \boldsymbol{x} + A_y \boldsymbol{y} + A_z \boldsymbol{z}] =$$

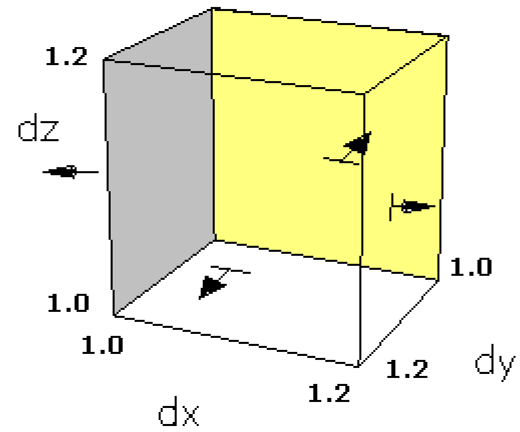
$$(\partial A_y/\partial x)(\boldsymbol{z}) + (\partial A_z/\partial x)(-\boldsymbol{y}) + (\partial A_x/\partial y)(-\boldsymbol{z}) + (\partial A_z/\partial y)(\boldsymbol{x}) + (\partial A_x/\partial z)(\mathbf{y}) + (\partial A_y/\partial z)(-\boldsymbol{x})$$

$$= (\partial A_z/\partial y - \partial A_y/\partial z)\, \boldsymbol{x} + (\partial A_x/\partial z - \partial A_z/\partial x)\, \boldsymbol{y} + (\partial A_y/\partial x - \partial A_x/\partial y)\, \boldsymbol{z}$$

Relates to work in a field

If curl is zero, so is work

Examination of ∇ and flux



Cube defined by $1 < x,y,z < 1.2$

$$\mathbf{D} = 2 \cdot x^2 \cdot y \cdot \mathbf{a}_{\mathbf{x}} + 3 \cdot x^2 \cdot y^2 \cdot \mathbf{a}_{\mathbf{y}}$$

Calculation of total flux

$$Q = \int_S \mathbf{D}\, d\mathbf{S} = \int_{vol} \rho_v \, dv = \Phi$$

$x_1 := 1 \qquad x_2 := 1.2$

$y_1 := 1 \qquad y_2 := 1.2$

$z_1 := 1 \qquad z_2 := 1.2$

$$\Phi_{total} = \Phi_{left} + \Phi_{right} + \Phi_{front} + \Phi_{back}$$

$$\Phi_{x1} := \int_{z_1}^{z_2} \int_{y_1}^{y_2} -2 \cdot x_1^{\,2} \cdot y \, dy \, dz \qquad \Phi_{y1} := \int_{z_1}^{z_2} \int_{x_1}^{x_2} -3 \cdot x^2 \cdot y_1^{\,2} \, dx \, dz$$

$$\Phi_{x2} := \int_{z_1}^{z_2} \int_{y_1}^{y_2} 2 \cdot x_2^{\,2} \cdot y \, dy \, dz \qquad \Phi_{y2} := \int_{z_1}^{z_2} \int_{x_1}^{x_2} 3 \cdot x^2 \cdot y_2^{\,2} \, dx \, dz$$

$$\Phi_{total} := \Phi_{x1} + \Phi_{x2} + \Phi_{y1} + \Phi_{y2}$$

$$\Phi_{total} = 0.103$$

Evaluation of $\nabla \cdot \mathbf{D}$ at center of cube

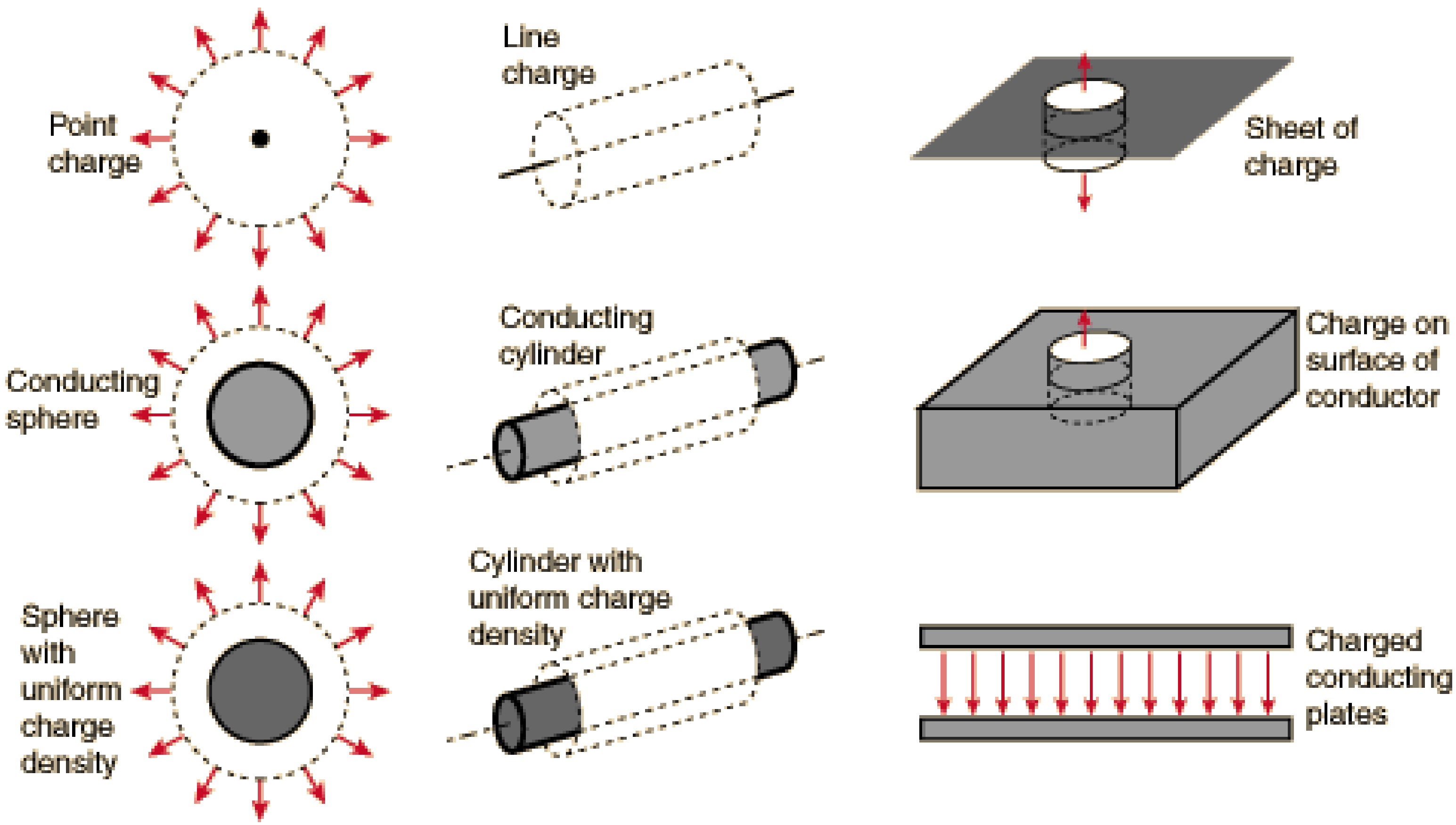
$$\mathrm{div}(\mathbf{D}) = \frac{d}{dx}\left(2 \cdot x^2 \cdot y\right) + \frac{d}{dy}\left(3 \cdot x^2 \cdot y^2\right)$$

$$\mathrm{div}(\mathbf{D}) = 4 \cdot x \cdot y + 6 \cdot x^2 \cdot y$$

$$\mathrm{divD} := 4 \cdot (1.1) \cdot (1.1) + 6 \cdot (1.1)^2 \cdot (1.1)$$

$$\mathrm{divD} = 12.826$$

Applications of Gauss's Law



Chapter 4
Energy and Potential

4.1 Energy to move a point charge through a Field

- Force on Q due to an electric field

$$F_E = QE$$

- Differential work done by an external source moving Q

$$dW = -QE \cdot dL$$

- Work required to move a charge a finite distance

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

4.2 Line Integral

- Work expression without using vectors

$$W = -Q \cdot \int_{\text{initial}}^{\text{final}} E_L \, dL$$

EL is the component of E in the dL direction

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(cartesian)}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)}$$

- Uniform electric field density

$$W = -QE \cdot L_{BA}$$

Example

$$E(x,y) := \begin{pmatrix} y \\ x \\ 2 \end{pmatrix} \qquad Q := 2 \qquad A := \begin{pmatrix} .8 \\ .6 \\ 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Path: $x^2 + y^2 = 1 \qquad z = 1$

Calculate the work to cary the charge from point B to point A.

$$W = -Q\cdot\int_{B_0}^{A_0} E(x,y)_0\,dx - Q\cdot\int_{B_1}^{A_1} E(x,y)_1\,dy - Q\cdot\int_{B_2}^{A_2} E(x,y)_2\,dz$$

Plug path in for x and y in E(x,y)

$$W := -Q\cdot\int_{B_0}^{A_0} E\left(0,\sqrt{1-x^2}\right)_0 dx - Q\cdot\int_{B_1}^{A_1} E\left(\sqrt{1-y^2},0\right)_1 dy - Q\cdot\int_{B_2}^{A_2} E(0,0)_2\,dz \qquad W = -0.96$$

Example

$$E(x,y) := \begin{pmatrix} y \\ x \\ 2 \end{pmatrix} \qquad Q := 2 \qquad A := \begin{pmatrix} .8 \\ .6 \\ 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Path: $y = -3(x-1) \quad z = 1$

(straight line)

Calculate the work to cary the charge from point B to point A.

$$W = -Q \cdot \int_{B_0}^{A_0} E(x,y)_0 \, dx - Q \cdot \int_{B_1}^{A_1} E(x,y)_1 \, dy - Q \cdot \int_{B_2}^{A_2} E(x,y)_2 \, dz$$

Plug path in for x and y in E(x,y)

$$W := -Q \cdot \int_{B_0}^{A_0} E[0, -3(x-1)]_0 \, dx - Q \cdot \int_{B_1}^{A_1} E\left(\frac{-y}{3} + 1, 0\right)_1 dy - Q \cdot \int_{B_2}^{A_2} E(0,0)_2 \, dz \quad W = -0.96$$

- Same amount of work with a different path
- Line integrals are path independent

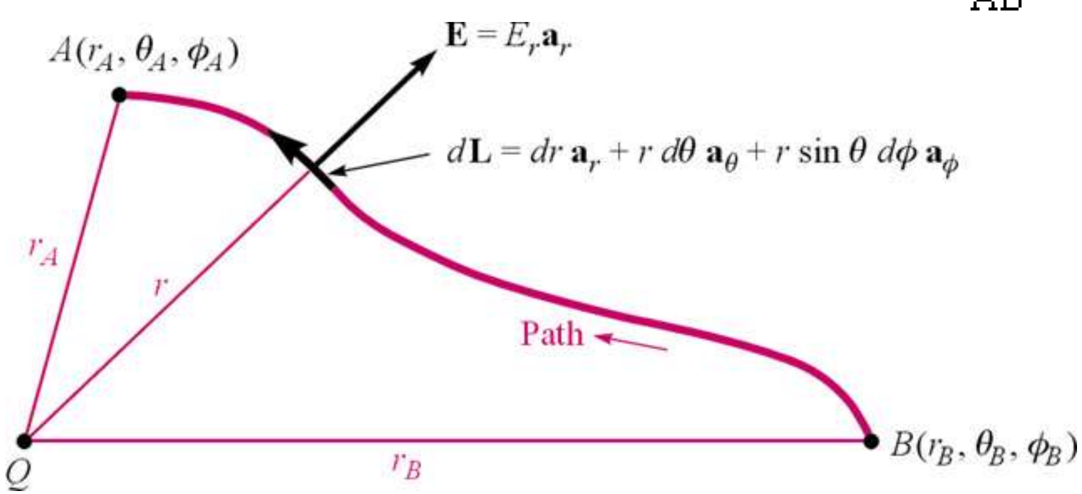
4.3 Potential Difference

- Potential Difference

$$V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \qquad V_{AB} = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

- Using radial distances from the point charge

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 \cdot r^2}\, dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$



4.3 Potential

- Measure potential difference between a point and something which has zero potential "ground"

$$V_{AB} = V_A - V_B$$

Example – D4.4

$$E(x, y, z) := \begin{pmatrix} 6x^2 \\ 6y \\ 4 \end{pmatrix}$$

a) Find Vmn

$$M := \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} \qquad N := \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix} \qquad V_{MN} := -\int_{N_0}^{M_0} 6x^2\,dx - \int_{N_1}^{M_1} 6y\,dy - \int_{N_2}^{M_2} 4\,dz \qquad V_{MN} = -139$$

b) Find Vm if V=0 at Q(4,-2,-35)

$$Q := \begin{pmatrix} 4 \\ -2 \\ -35 \end{pmatrix} \qquad V_M := -\int_{Q_0}^{M_0} 6x^2\,dx - \int_{Q_1}^{M_1} 6y\,dy - \int_{Q_2}^{M_2} 4\,dz \qquad V_M = -120$$

c) Find Vn if V=2 at P(1, 2, -4)

$$P := \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad V_N := -\int_{P_0}^{N_0} 6x^2\,dx - \int_{P_1}^{N_1} 6y\,dy - \int_{P_2}^{N_2} 4\,dz + 2 \qquad V_N = 19$$

4.4 Potential Field of a Point Charge

- Let V=0 at infinity

$$V = \frac{Q}{4\pi\varepsilon_0 \cdot r}$$

- Equipotential surface:
  - A surface composed of all points having the same potential

Example – D4.5

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Q is located at the origin

a) Find V1 if V=0 at (6,5,4)

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$$V_1 := \frac{Q}{4\pi\,\varepsilon_0}\,\frac{1}{|P_1|} \qquad V_1 = 36.047$$

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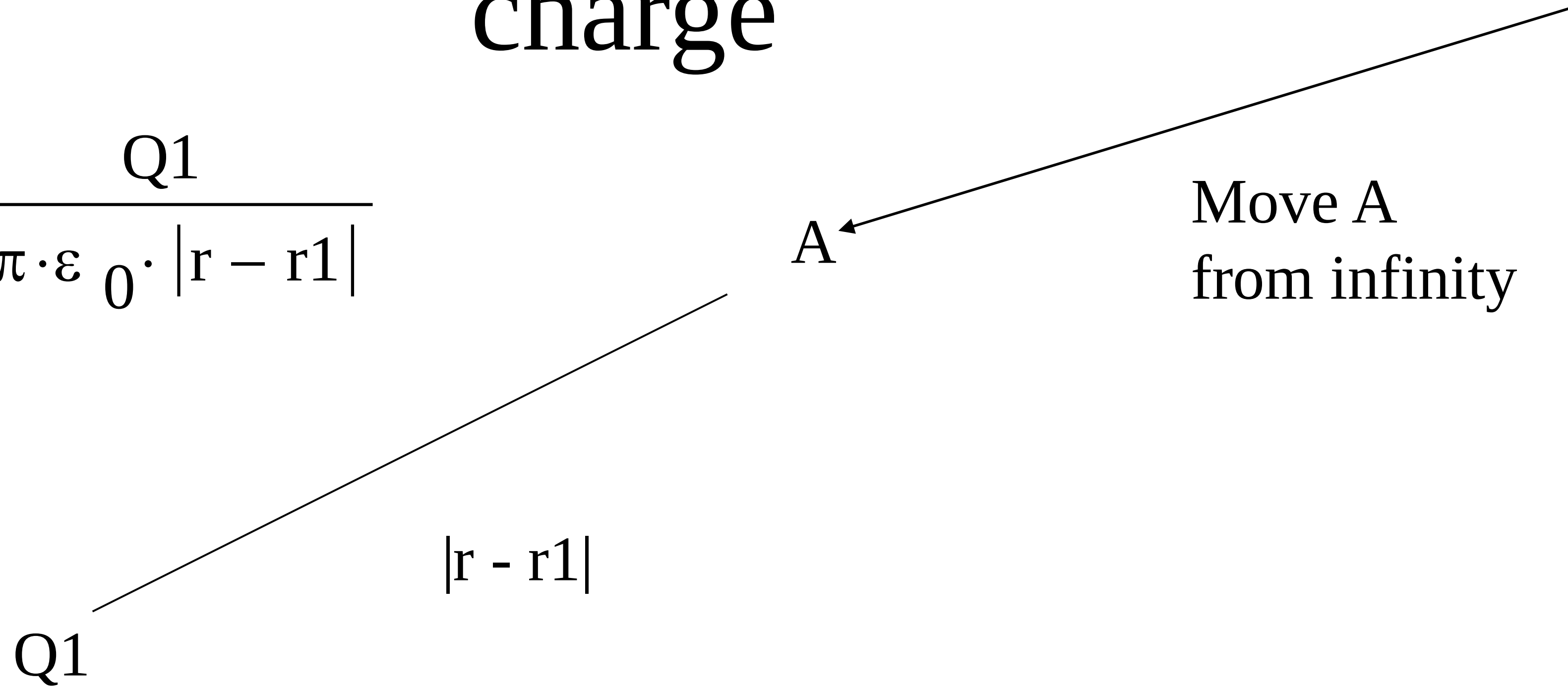
Potential field of single point charge

$$V(r) = \frac{Q1}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r1|}$$



Potential due to two charges

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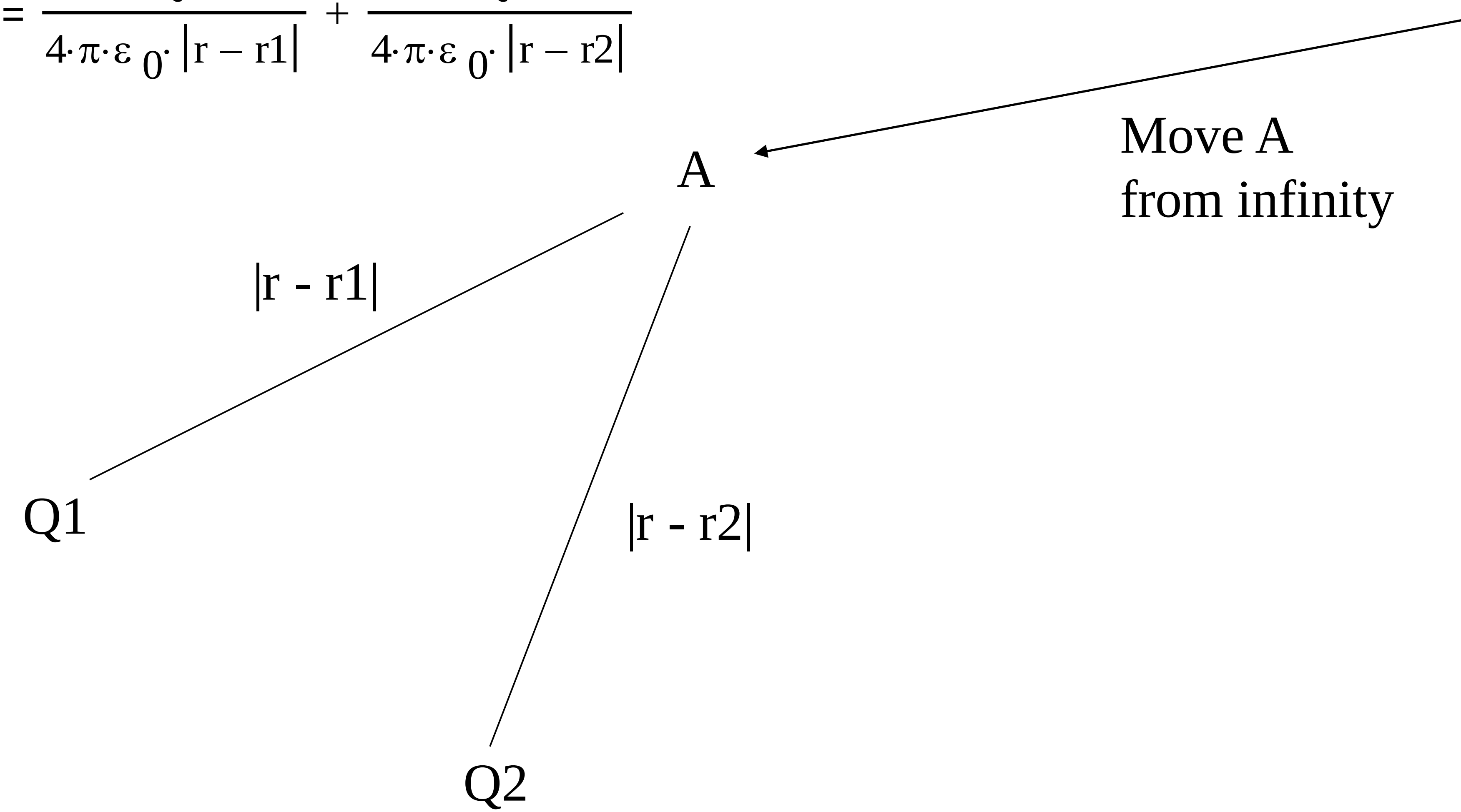

Potential due to *n* point charges

Continue adding charges

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$$V(r) = \sum_{m=1}^{n} \frac{Qm}{4\cdot\pi\cdot\varepsilon_0\cdot\left|r - r_m\right|}$$

Potential as point charges become infinite

Volume of charge

$$V(r) = \int \frac{\rho_v(r_{prime})}{4\cdot\pi\cdot\varepsilon_0\cdot\left|r - r_{prime}\right|}\,dv_{prime}$$

Line of charge

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Example

Find V on the z axis for a uniform line charge ρL in the form of a ring

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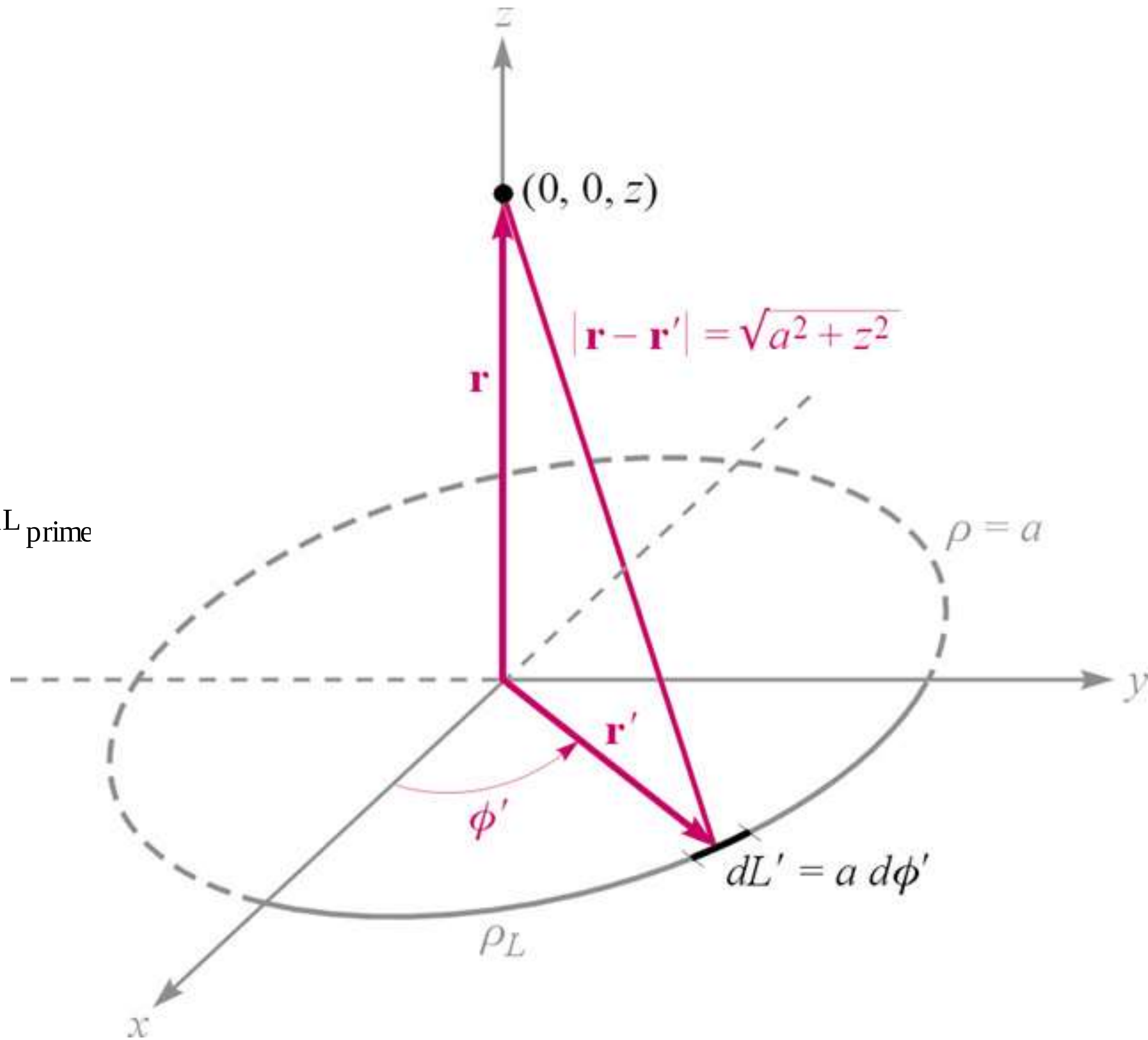

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No work is done (energy is conserved) around a closed path

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4.6
Potential gradient Relationship between potential and electric field intensity

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Two characteristics of relationship:

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Gradient

- The gradient of a scalar is a vector

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The following equations are found on page 104 and inside the back cover of the text:

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Cylindrical

$$\boxed{\mathrm{grad}V = \frac{\delta V}{\delta r}\cdot a_r + \frac{1}{r}\cdot\frac{\delta V}{\delta \theta}\cdot a_\theta + \frac{1}{r\cdot\sin(\theta)}\cdot\frac{\delta V}{\delta \phi}\cdot a_\phi}$$

Spherical

Example 4.3

Given the potential field, $V = 2x^2y - 5z$, and a point P(-4, 3, 6), find the following: potential V, electric field intensity **E**

potential $V_P = 2(-4)^2(3) - 5(6) = 66$ V

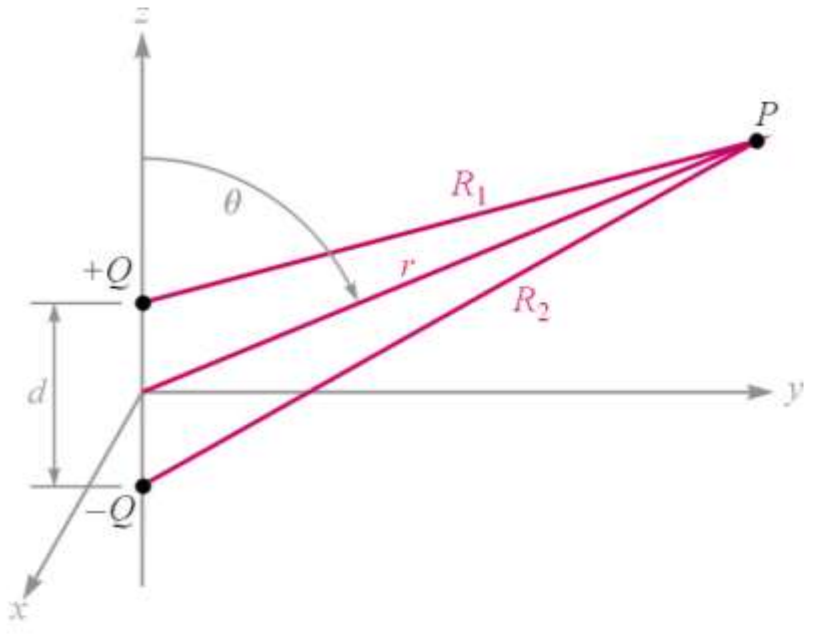
electric field intensity - use gradient operation

$$\mathbf{E} = -4xy\mathbf{a_x} - 2x^2\mathbf{a_y} + 5\mathbf{a_z}$$

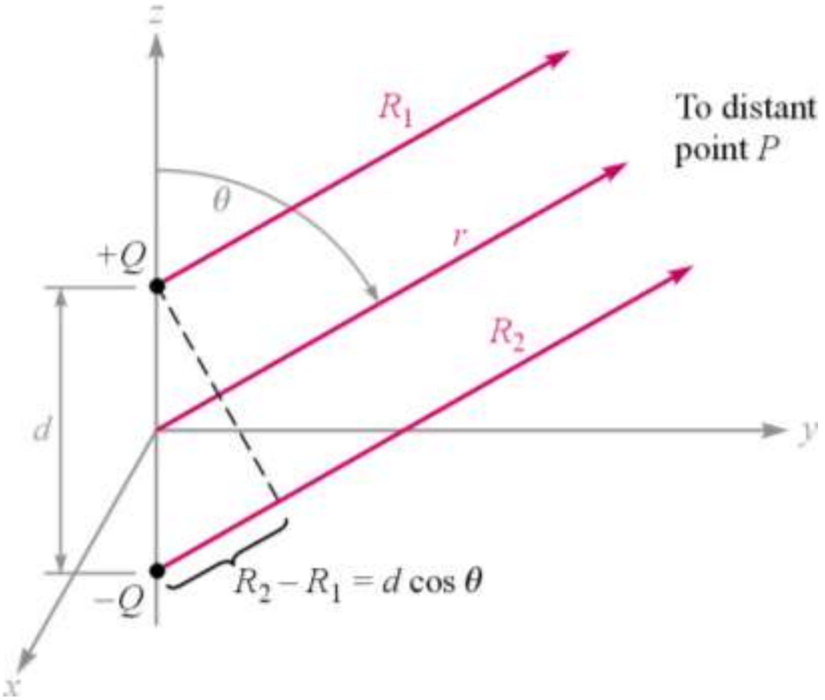
$$\mathbf{E}_P = 48\mathbf{a_x} - 32\mathbf{a_y} + 5\mathbf{a_z}$$

Dipole

The name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P, at which we want to know the electric and potential fields



Potential



To approximate the potential of a dipole, assume R1 and R2 are parallel since the point P is very distant

$$\boxed{V = \frac{Q}{4\cdot\pi\cdot\varepsilon_0}\cdot\left(\frac{1}{R1} - \frac{1}{R2}\right)}$$

$$\boxed{V = \frac{Q\cdot d\cdot\cos(\theta)}{4\cdot\pi\cdot\varepsilon_0\cdot r^2}}$$

Dipole moment

The dipole moment is assigned the symbol p and is equal to the product of charge and separation

$$\boxed{\mathbf{p} = Q*\mathbf{d}}$$

The dipole moment expression simplifies the potential field equation

Example

An electric dipole located at the origin in free space has a moment **p** = 3*ax - 2*ay + az nC*m. Find V at the points (2, 3, 4) and (2.5, 30°, 40°).

$$p := \begin{pmatrix} 3\cdot 10^{-9} \\ -2\cdot 10^{-9} \\ 1\cdot 10^{-9} \end{pmatrix} \qquad \varepsilon_0 := 8.854\cdot 10^{-12}$$

$$P := \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \qquad V := \frac{p}{4\cdot\pi\cdot\varepsilon_0\cdot(|P|)^2}\cdot\frac{P}{|P|} \qquad V = 0.23$$

$$P_{spherical} := \begin{pmatrix} 2.5 \\ 30\cdot\frac{\pi}{180} \\ 40\cdot\frac{\pi}{180} \end{pmatrix}$$

Transform this into rectangular coordinates

$$P_{rectangular} := \begin{pmatrix} 2.5\cdot\sin\left(30\cdot\frac{\pi}{180}\right)\cdot\cos\left(40\cdot\frac{\pi}{180}\right) \\ 2.5\cdot\sin\left(30\cdot\frac{\pi}{180}\right)\cdot\sin\left(40\cdot\frac{\pi}{180}\right) \\ 2.5\cdot\cos\left(30\cdot\frac{\pi}{180}\right) \end{pmatrix} \qquad P_{rectangular} = \begin{pmatrix} 0.958 \\ 0.803 \\ 2.165 \end{pmatrix}$$

$$V := \frac{p}{4\cdot\pi\cdot\varepsilon_0\cdot\left(\left|P_{rectangular}\right|\right)^2}\cdot\frac{P_{rectangular}}{\left|P_{rectangular}\right|} \qquad V = 1.973$$

Potential energy

Bringing a positive charge from infinity into the field of another positive charge requires work. The work is done by the external source that moves the charge into position. If the source released its hold on the charge, the charge would accelerate, turning its potential energy into kinetic energy.

The potential energy of a system is found by finding the work done by an external source in positioning the charge.

Empty universe

Positioning the first charge, Q1, requires no work (no field present)

Positioning more charges does take work

Total positioning work = potential energy of field = W_E =
$Q_2V_{2,1} + Q_3V_{3,1} + Q_3V_{3,2} + Q_4V_{4,1} + Q_4V_{4,2} + Q_4V_{4,3} + ...$

Manipulate this expression to get
$W_E = 0.5(Q_1V_1 + Q_2V_2 + Q_3V_3 + \ldots)$

Where is energy stored?

The location of potential energy cannot be precisely pinned down in terms of physical location - in the molecules of the pencil, the gravitational field, etc?

So where is the energy in a capacitor stored?

Electromagnetic theory makes it easy to believe that the energy is stored in the field itself

Energy and Potential

Energy to move a point charge through a Field

- Force on Q due to an electric field

$$F_E = QE$$

- Differential work done by an external source moving Q

$$dW = -QE \cdot dL$$

- Work required to move a charge a finite distance

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

Line Integral

- Work expression without using vectors

$$W = -Q \cdot \int_{\text{initial}}^{\text{final}} E_L \, dL$$

EL is the component of E in the dL direction

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(cartesian)}$$

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- Uniform electric field density

$$W = -QE \cdot L_{BA}$$

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- Potential Difference

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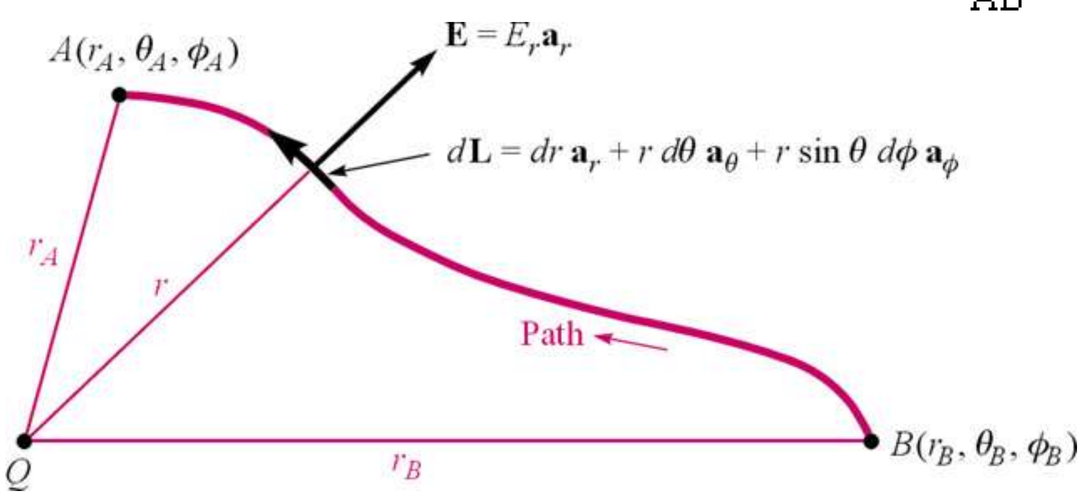
$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 \cdot r^2}\, dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$



Potential

- Measure potential difference between a point and something which has zero potential “ground”

$$V_{AB} = V_A - V_B$$

Potential Field of a Point Charge

- Let V=0 at infinity

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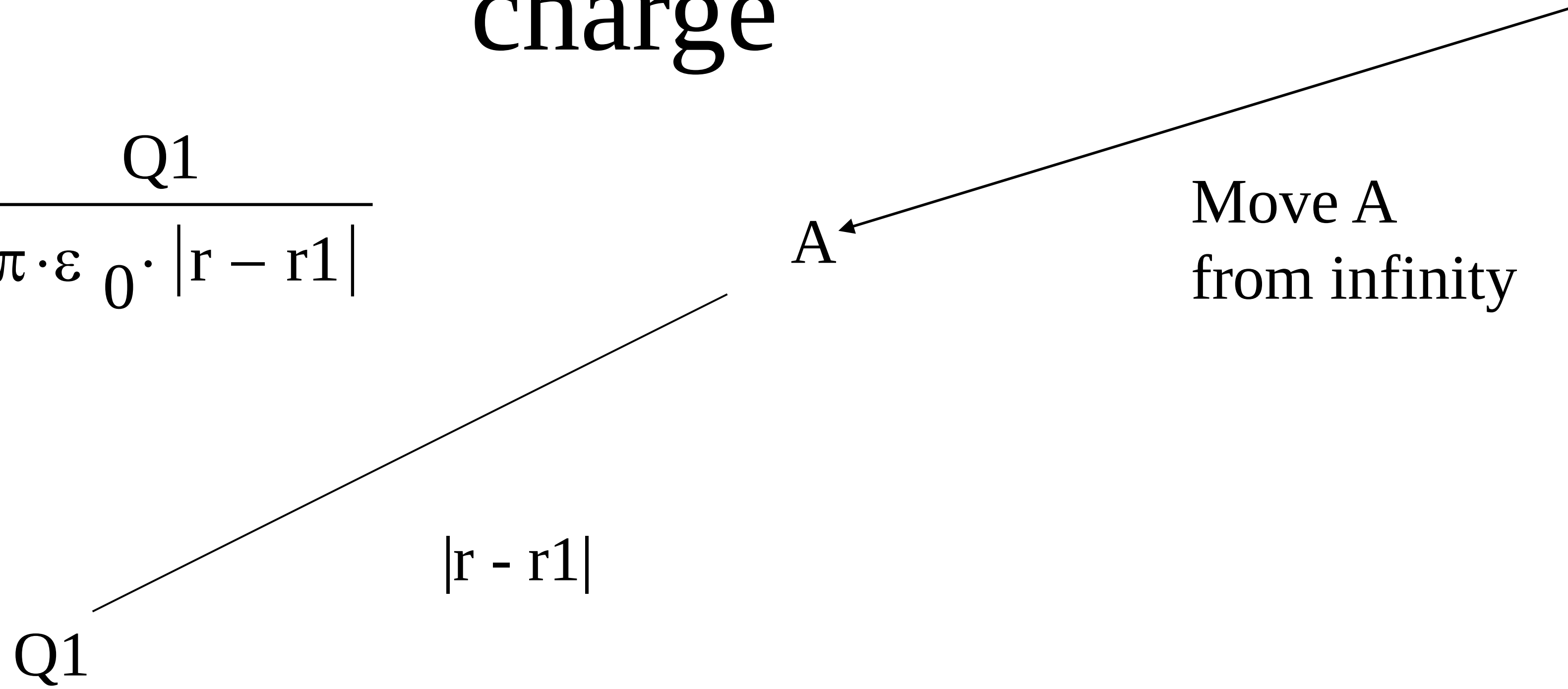

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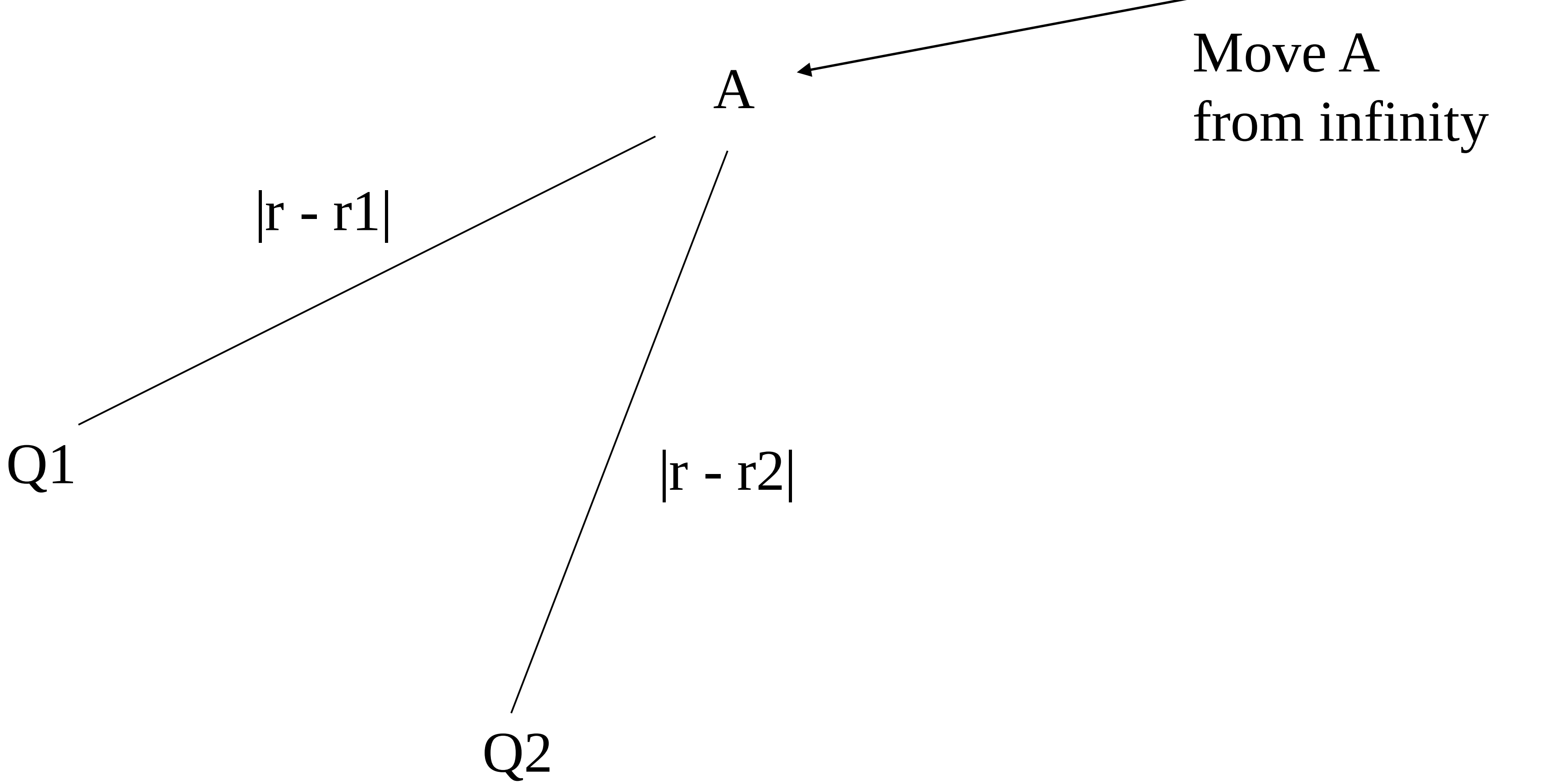

Potential due to *n* point charges

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Spherical

Chapter 4
Energy and Potential

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- Uniform electric field density

$$W = -QE \cdot L_{BA}$$

Example

$$E(x,y) := \begin{pmatrix} y \\ x \\ 2 \end{pmatrix} \qquad Q := 2 \qquad A := \begin{pmatrix} .8 \\ .6 \\ 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Path: $x^2 + y^2 = 1 \qquad z = 1$

Calculate the work to cary the charge from point B to point A.

$$W = -Q\cdot\int_{B_0}^{A_0} E(x,y)_0\,dx - Q\cdot\int_{B_1}^{A_1} E(x,y)_1\,dy - Q\cdot\int_{B_2}^{A_2} E(x,y)_2\,dz$$

Plug path in for x and y in E(x,y)

$$W := -Q\cdot\int_{B_0}^{A_0} E\left(0,\sqrt{1-x^2}\right)_0 dx - Q\cdot\int_{B_1}^{A_1} E\left(\sqrt{1-y^2},0\right)_1 dy - Q\cdot\int_{B_2}^{A_2} E(0,0)_2\,dz \qquad W = -0.96$$

Example

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Path: $y = -3(x-1) \quad z = 1$

(straight line)

Calculate the work to cary the charge from point B to point A.

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- Same amount of work with a different path
- Line integrals are path independent

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- Potential Difference

$$V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \qquad V_{AB} = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

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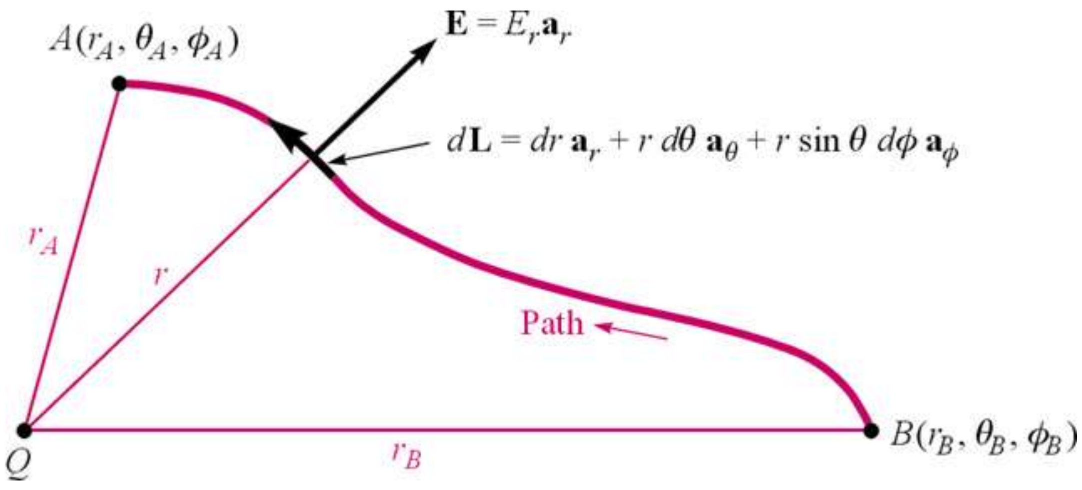

4.3 Potential

- Measure potential difference between a point and something which has zero potential "ground"

$$V_{AB} = V_A - V_B$$

Example – D4.4

$$E(x,y,z) := \begin{pmatrix} 6x^2 \\ 6y \\ 4 \end{pmatrix}$$

a) Find Vmn

$$M := \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} \qquad N := \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix} \qquad V_{MN} := -\int_{N_0}^{M_0} 6x^2\,dx - \int_{N_1}^{M_1} 6y\,dy - \int_{N_2}^{M_2} 4\,dz \qquad V_{MN} = -139$$

b) Find Vm if V=0 at Q(4,-2,-35)

$$Q := \begin{pmatrix} 4 \\ -2 \\ -35 \end{pmatrix} \qquad V_M := -\int_{Q_0}^{M_0} 6x^2\,dx - \int_{Q_1}^{M_1} 6y\,dy - \int_{Q_2}^{M_2} 4\,dz \qquad V_M = -120$$

c) Find Vn if V=2 at P(1, 2, -4)

$$P := \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad V_N := -\int_{P_0}^{N_0} 6x^2\,dx - \int_{P_1}^{N_1} 6y\,dy - \int_{P_2}^{N_2} 4\,dz + 2 \qquad V_N = 19$$

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- Let V=0 at infinity

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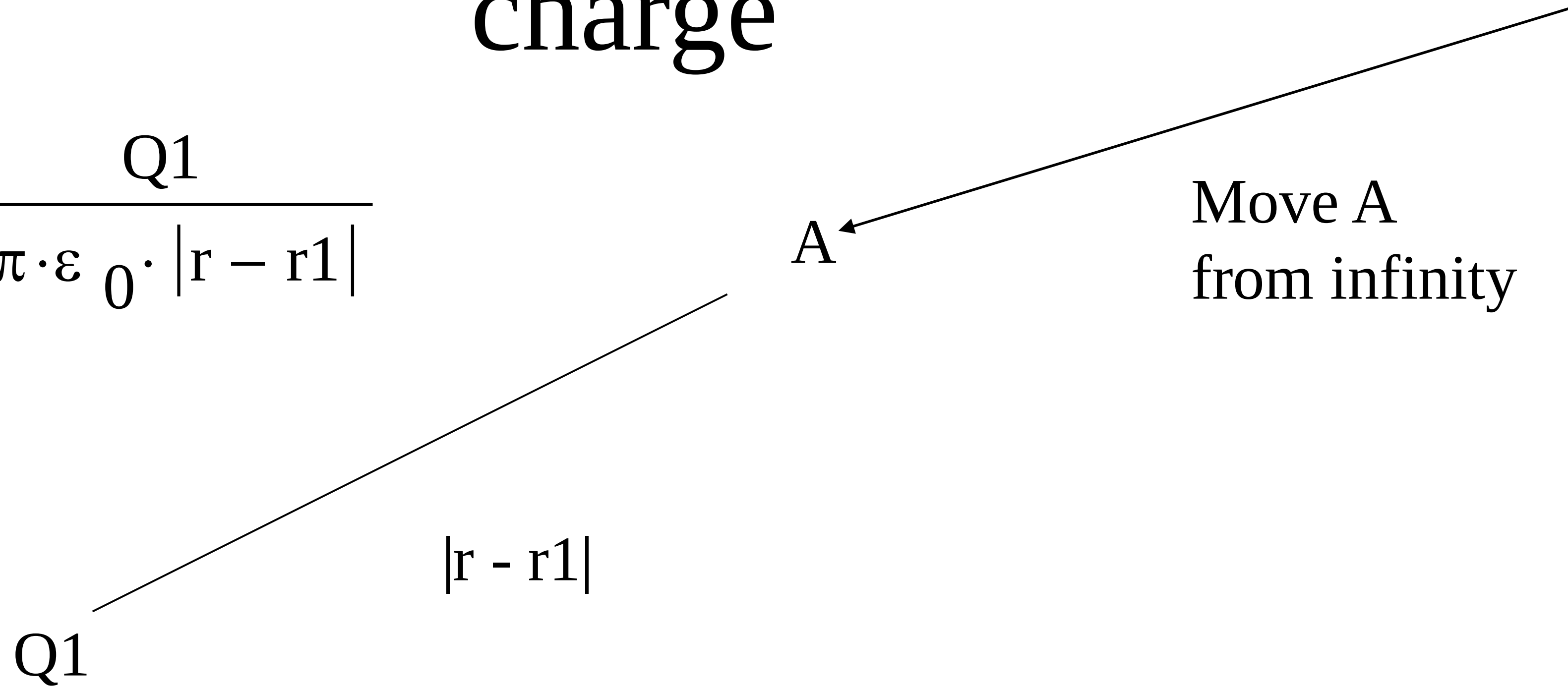

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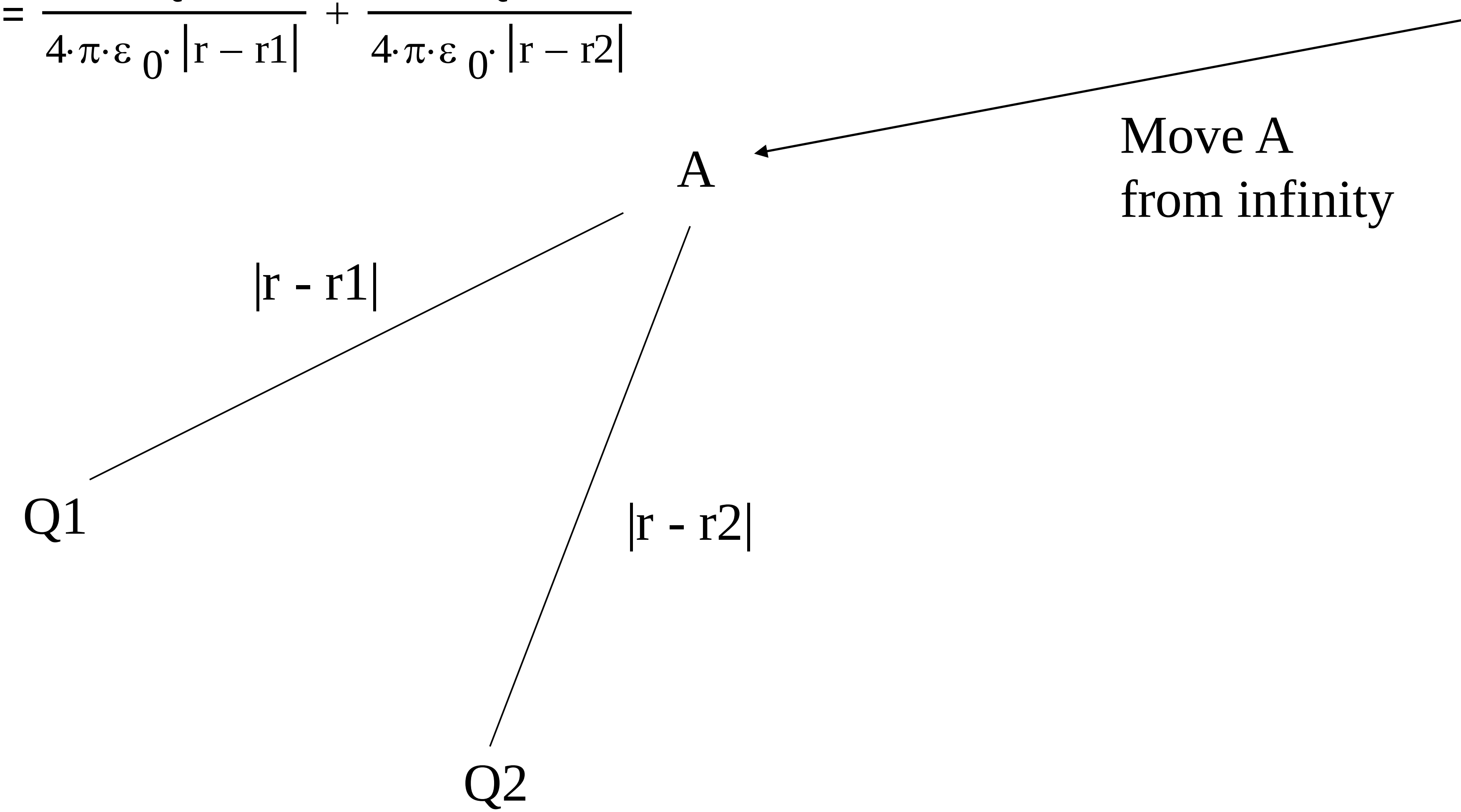

Potential due to *n* point charges

Continue adding charges

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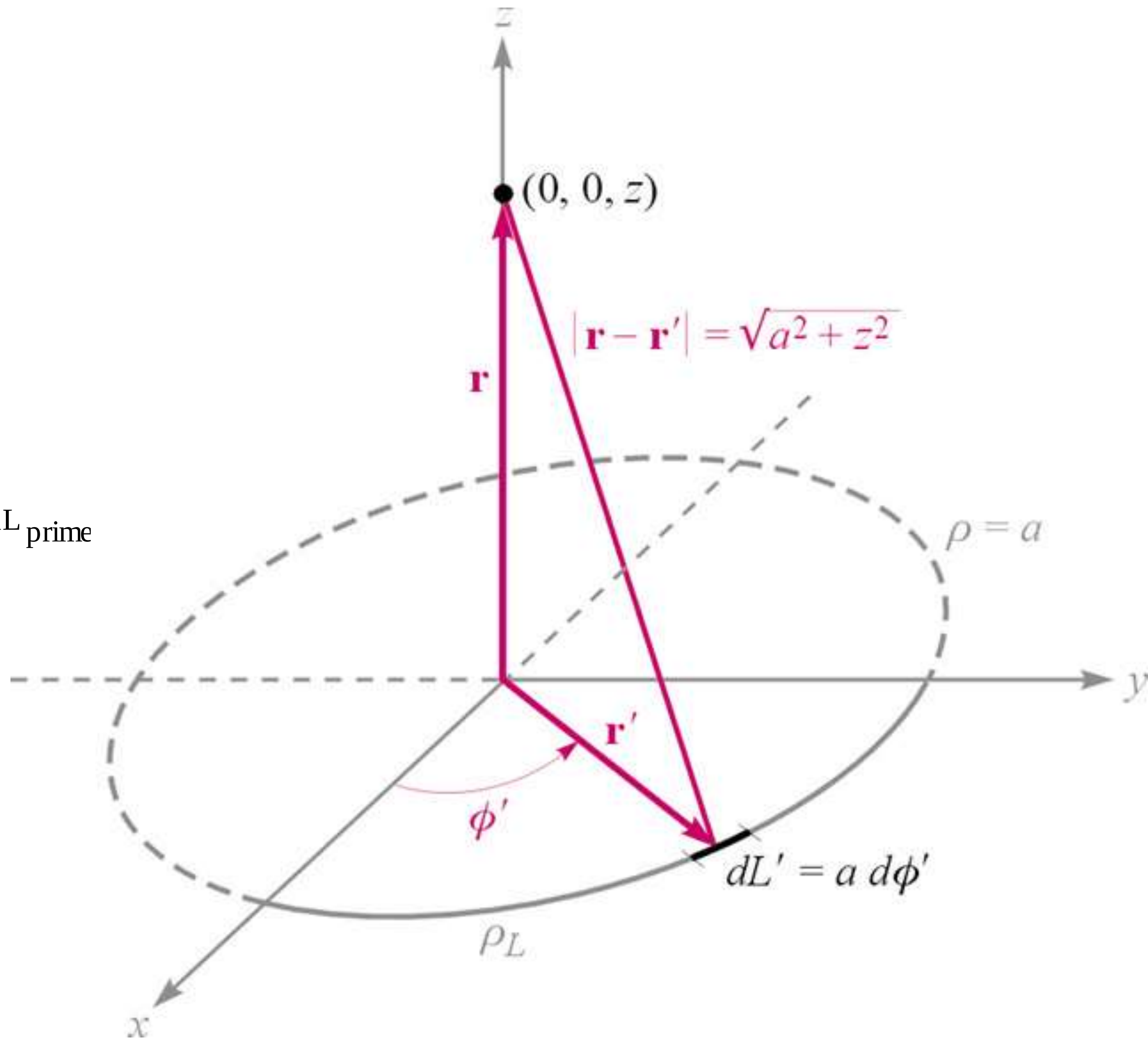

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potential $V_P = 2(-4)^2(3) - 5(6) = 66$ V

electric field intensity - use gradient operation

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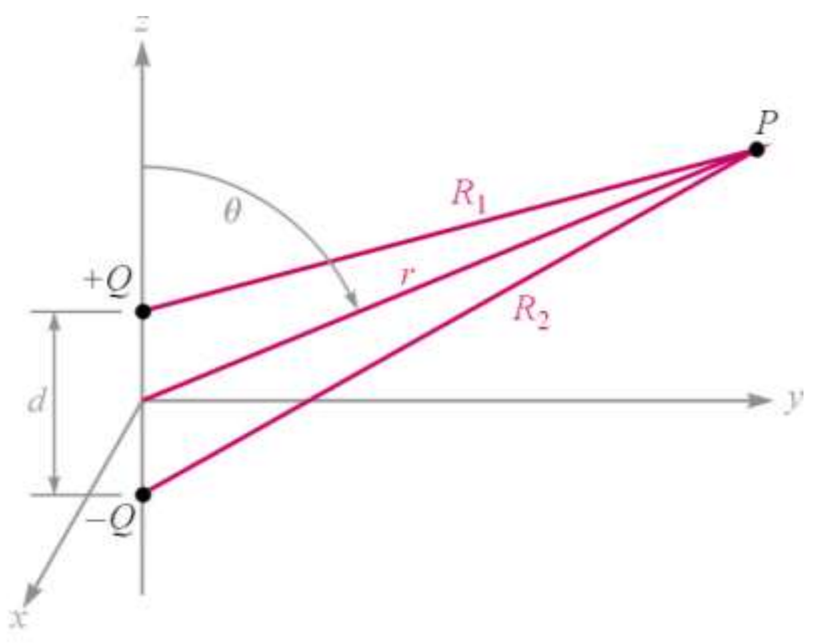
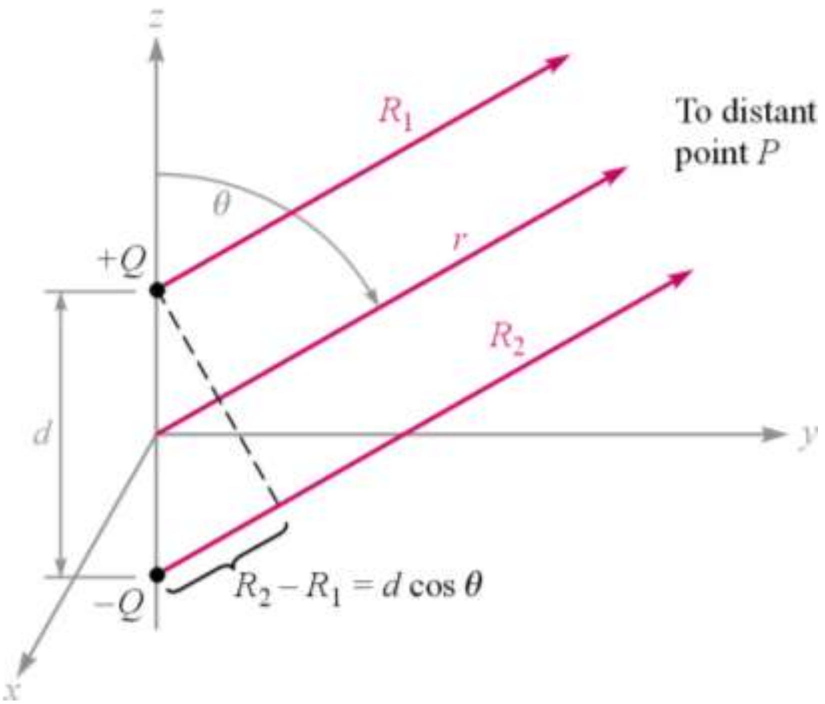
Dipole

The name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P, at which we want to know the electric and potential fields



Potential



To approximate the potential of a dipole, assume R1 and R2 are parallel since the point P is very distant

$$\boxed{V = \frac{Q}{4\cdot\pi\cdot\varepsilon_0}\cdot\left(\frac{1}{R1} - \frac{1}{R2}\right)}$$

$$\boxed{V = \frac{Q\cdot d\cdot\cos(\theta)}{4\cdot\pi\cdot\varepsilon_0\cdot r^2}}$$

Dipole moment

The dipole moment is assigned the symbol p and is equal to the product of charge and separation

$$\boxed{\mathbf{p} = Q*\mathbf{d}}$$

The dipole moment expression simplifies the potential field equation

Example

An electric dipole located at the origin in free space has a moment **p** = 3*ax - 2*ay + az nC*m. Find V at the points (2, 3, 4) and (2.5, 30°, 40°).

$$p := \begin{pmatrix} 3\cdot 10^{-9} \\ -2\cdot 10^{-9} \\ 1\cdot 10^{-9} \end{pmatrix} \qquad \varepsilon_0 := 8.854\cdot 10^{-12}$$

$$P := \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \qquad V := \frac{p}{4\cdot\pi\cdot\varepsilon_0\cdot(|P|)^2}\cdot\frac{P}{|P|} \qquad V = 0.23$$

$$P_{spherical} := \begin{pmatrix} 2.5 \\ 30\cdot\frac{\pi}{180} \\ 40\cdot\frac{\pi}{180} \end{pmatrix}$$

Transform this into rectangular coordinates

$$P_{rectangular} := \begin{pmatrix} 2.5\cdot\sin\left(30\cdot\frac{\pi}{180}\right)\cdot\cos\left(40\cdot\frac{\pi}{180}\right) \\ 2.5\cdot\sin\left(30\cdot\frac{\pi}{180}\right)\cdot\sin\left(40\cdot\frac{\pi}{180}\right) \\ 2.5\cdot\cos\left(30\cdot\frac{\pi}{180}\right) \end{pmatrix} \qquad P_{rectangular} = \begin{pmatrix} 0.958 \\ 0.803 \\ 2.165 \end{pmatrix}$$

$$V := \frac{p}{4\cdot\pi\cdot\varepsilon_0\cdot\left(\left|P_{rectangular}\right|\right)^2}\cdot\frac{P_{rectangular}}{\left|P_{rectangular}\right|} \qquad V = 1.973$$

Potential energy

Bringing a positive charge from infinity into the field of another positive charge requires work. The work is done by the external source that moves the charge into position. If the source released its hold on the charge, the charge would accelerate, turning its potential energy into kinetic energy.

The potential energy of a system is found by finding the work done by an external source in positioning the charge.

Empty universe

Positioning the first charge, Q1, requires no work (no field present)

Positioning more charges does take work

Total positioning work = potential energy of field = W_E =
$Q_2V_{2,1} + Q_3V_{3,1} + Q_3V_{3,2} + Q_4V_{4,1} + Q_4V_{4,2} + Q_4V_{4,3} + ...$

Manipulate this expression to get
$W_E = 0.5(Q_1V_1 + Q_2V_2 + Q_3V_3 + \ldots)$

Where is energy stored?

The location of potential energy cannot be precisely pinned down in terms of physical location - in the molecules of the pencil, the gravitational field, etc?

So where is the energy in a capacitor stored?

Electromagnetic theory makes it easy to believe that the energy is stored in the field itself