Mean of a Random Variable:

Definition *:

Let X be a random variable with a probability distribution f(x). The mean (or expected value) of X is denoted by μ_X (or E(X)) and is defined by:

$$E(X)=\mu_X=\begin{cases} \sum_{all\ x} x\, f(x)\ ; & if\ \ X\ is\ discrete \\ \int_{-\infty}^{\infty} x\, f(x)\, dx\ ; & if\ \ X\ is\ continuous \end{cases}$$

Example *: (Reading Assignment)

Example: (Example A)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the expected number of defective computers purchased

Solution:

Let X = the number of defective computers purchased.

In Example A, we found that the probability distribution of X is:

x	0	1	2
F(x)=p(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

or:

$$f(x) = P(X = x) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}; & x = 0, 1, 2 \\ 0; \; otherwise \end{cases}$$

The expected value of the number of defective computers purchased is the mean (or the expected value) of X, which is:

$$E(X) = \mu_X = \sum_{x=0}^{2} x\, f(x)$$

$$= (0)\, f(0) + (1)\, f(1) + (2)\, f(2)$$

$$= (0)\frac{10}{28} + (1)\frac{15}{28} + (2)\frac{3}{28}$$

$$= \frac{15}{28} + \frac{6}{28} = \frac{21}{28} = 0.75 \qquad \text{(computers)}$$

Example B:

Let X be a continuous random variable that represents the life (in hours) of a certain electronic device. The pdf of X is given by:

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3} \; ; \; x > 100 \\ 0 \; ; \; elsewhere \end{cases}$$

Find the expected Number of this type of devices.

Solution:

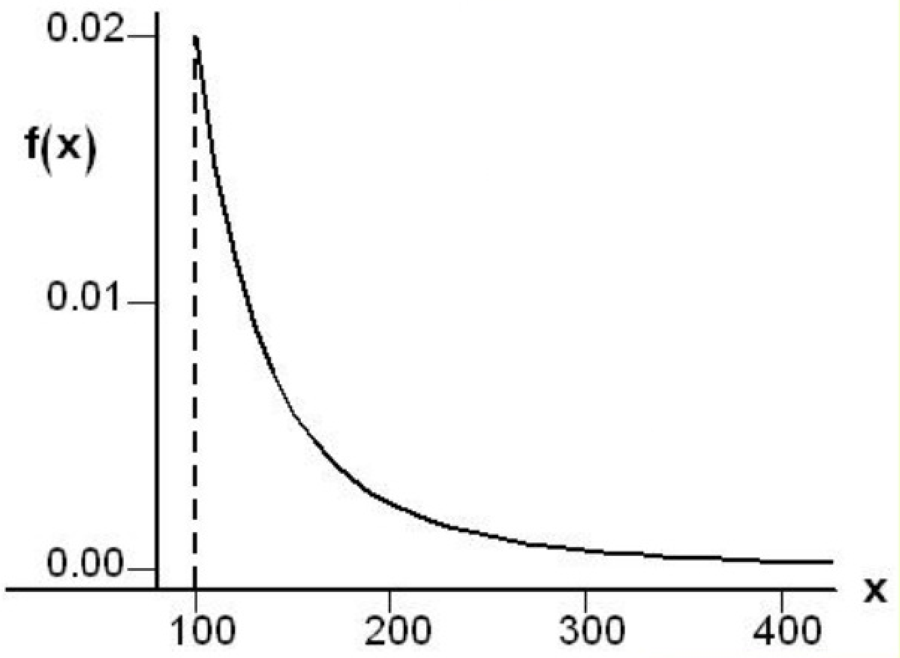
$$E(X)=\mu_X=\int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$=\int_{100}^{\infty} x\,\frac{20000}{x^3}\,dx$$

$$=20000\int_{100}^{\infty}\frac{1}{x^2}\,dx$$

$$=20000\left[-\frac{1}{x}\Bigg|_{x=100}^{x=\infty}\right]$$

$$=-20000\left[0-\frac{1}{100}\right]=200 \quad \text{(hours)}$$



Therefore, we expect that this type of electronic devices to last, on Average of 200 hours

*Theorem *:*

Let X be a random variable with a probability distribution f(x), and let g(X) be a function of the random variable X. The mean (or expected value) of the random variable g(X) is denoted by $\mu_{g(X)}$ (or E[g(X)]) and is defined by:

$$E[g(X)] = \mu_{g(X)} = \begin{cases} \sum\limits_{all\ x} g(x)\, f(x)\ ; & if\ \ X\ is\ discrete \\ \int\limits_{-\infty}^{\infty} g(x)\, f(x)\, dx\ ; & if\ \ X\ is\ continuous \end{cases}$$

Example:

Let X be a discrete random variable with the following probability distribution

x	0	1	2
F(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Find MEAN WHERE $g(X)=(X-1)^2$.

Solution:

$g(X)=(X-1)^2$

$$E[g(X)]=\mu_{g(X)}=\sum_{x=0}^{2} g(x)\,f(x)=\sum_{x=0}^{2}(x-1)^2\,f(x)$$

$$=(0-1)^2 f(0)+(1-1)^2 f(1)+(2-1)^2 f(2)$$

$$=(-1)^2\frac{10}{28}+(0)^2\frac{15}{28}+(1)^2\frac{3}{28}$$

$$=\frac{10}{28}+0+\frac{3}{28}=\frac{10}{28}$$

Example:

In Example B, find $E\left(\frac{1}{X}\right)$

Solution:

$$f(x)=\begin{cases}\dfrac{20{,}000}{x^3}\ ;\ x>100\\ 0\ ;\ elsewhere\end{cases}$$

$$g(X)=\frac{1}{X}$$

$$E\left(\frac{1}{X}\right)=E[g(X)]=\mu_{g(X)}=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx=\int_{-\infty}^{\infty}\frac{1}{x}f(x)\,dx$$

$$=\int_{100}^{\infty}\frac{1}{x}\frac{20000}{x^3}dx=20000\int_{100}^{\infty}\frac{1}{x^4}dx=\frac{20000}{-3}\left[\frac{1}{x^3}\Big|_{x=100}^{x=\infty}\right]$$

$$=\frac{-20000}{3}\left[0-\frac{1}{1000000}\right]=0.0067$$

Variance (of a Random Variable):

The most important measure of variability of a random variable X is called the variance of X and is denoted by Var(X) or σ_X^2

Definition B:

Let X be a random variable with a probability distribution f(x) and mean μ. The variance of X is defined by:

$$\mathrm{Var}(X) = \sigma_X^2 = E[(X-\mu)^2] = \begin{cases} \sum_{all\ x} (x-\mu)^2 f(x); & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx; & if\ X\ is\ continuous \end{cases}$$

Definition:

The positive square root of the variance of X, $\sigma_X = \sqrt{\sigma_X^2}$, is called the standard deviation of X.

Note:

Var(X)=E[g(X)], where g(X)=$(X-\mu)^2$

Theorem :

The variance of the random variable X is given by:

$$\mathrm{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$

where
$$E(X^2) = \begin{cases} \sum\limits_{all\ x} x^2 f(x)\,; & if\ X\ is\ discrete \\ \int\limits_{-\infty}^{\infty} x^2 f(x)\,dx\,; & if\ X\ is\ continuous \end{cases}$$

Example 4.9:

Let X be a discrete random variable with the following probability distribution

x	0	1	2	3
F(x)	0.15	0.38	0.10	0.01

Find Var(X)= σ_X^2 .

Solution:

$$\mu = \sum_{x=0}^{3} x\, f(x) = (0)\, f(0) + (1)\, f(1) + (2)\, f(2) + (3)\, f(3)$$

$$= (0)\,(0.51) + (1)\,(0.38) + (2)\,(0.10) + (3)\,(0.01) = 0.61$$

1. First method:

$$\mathrm{Var}(X) = \sigma_X^2 = \sum_{x=0}^{3} (x-\mu)^2 f(x)$$

$$= \sum_{x=0}^{3} (x - 0.61)^2 f(x)$$

$$=(0-0.61)^2 f(0)+(1-0.61)^2 f(1)+(2-0.61)^2 f(2)+ (3-0.61)^2 f(3)$$

$$=(-0.61)^2 (0.51)+(0.39)^2 (0.38)+(1.39)^2 (0.10)+ (2.39)^2 (0.01)$$

$$= 0.4979$$

2. Second method:

$$\mathrm{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = (0^2)\, f(0) + (1^2)\, f(1) + (2^2)\, f(2) + (3^2)\, f(3)$$

$$= (0)\,(0.51) + (1)\,(0.38) + (4)\,(0.10) + (9)\,(0.01) = 0.87$$

$$\mathrm{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2 = 0.87 - (0.61)^2 = 0.4979$$

*Example *0:*

Let X be a continuous random variable with the following pdf:

$$f(x) = \begin{cases} 2(x-1)\ ;\ 1 < x < 2 \\ 0\ ;\ elsewhere \end{cases}$$

Find the mean and the variance of X.

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{1}^{2} x[\,2(x-1)]\ dx = 2\int_{1}^{2} x\,(x-1)\ dx = 5/3$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_{1}^{2} x^2[\,2(x-1)]\ dx = 2\int_{1}^{2} x^2\,(x-1)\ dx = 17/6$$

$$Var(X) = \sigma_X^2 = E(X^2) - \mu^2 = 17/6 - (5/3)^2 = 1/8$$

B Means and Variances of Linear Combinations of Random Variables:

If $X_1, X_2, \ldots, X_n$ are n random variables and $a_1, a_2, \ldots, a_n$ are constants, then the random variable :

$$Y = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + a_2 X_2 + \quad + a_n X_n$$

is called a linear combination of the random variables $X_1, X_2, \ldots, X_n$.

Theorem 4.5:

If X is a random variable with mean $\mu = E(X)$, and if a and b are constants, then:

$$E(aX \pm b) = a\,E(X) \pm b$$

$$\Leftrightarrow$$

$$\mu_{aX\pm b} = a\,\mu_X \pm b$$

Corollary 1: $E(b) = b$ (a=0 in Theorem 4.5)

$E(aX) = a\,E(X)$ (b=0 in Theorem 4.5)

*Example *6:*

Let X be a random variable with the following probability density function:

$$f(x)=\begin{cases}\frac{1}{3}x^2 \;;-1<x<2\\ 0\;;\; elsewhere\end{cases}$$

Find E(4X+3).

Solution:

$$\mu=E(X)=\int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{-1}^{2} x[\frac{1}{3}x^2]\,dx = \frac{1}{3}\int_{-1}^{2} x^3\,dx = \frac{1}{3}\left[\frac{1}{4}x^4 \Big|_{x=-1}^{x=2}\right] = 5/4$$

E(4X+3) = 4 E(X)+3 = 4(5/4) + 3=8

Another solution:

$$E[g(X)]=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx \quad ;\ g(X) = 4X+3$$

$$E(4X+3) = \int_{-\infty}^{\infty}(4x+3)\,f(x)\,dx = \int_{-1}^{2}(4x+3)\,[\frac{1}{3}x^2]\,dx = \quad =8$$

Theorem:

If $X_1, X_2, \ldots, X_n$ are n random variables and $a_1, a_2, \ldots, a_n$ are constants, then:

$$E(a_1X_1+a_2X_2+ \ldots +a_nX_n) = a_1E(X_1)+ a_2E(X_2)+ \ldots+a_nE(X_n)$$

$$\Leftrightarrow$$

$$E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$$

Corollary:

If X, and Y are random variables, then:

$$E(X \pm Y) = E(X) \pm E(Y)$$

Theorem 4.9:

If X is a random variable with variance $Var(X) = \sigma_X^2$ and if a and b are constants, then:

$$Var(aX \pm b) = a^2 \, Var(X)$$

$$\Leftrightarrow$$

$$\sigma_{aX+b}^2 = a^2 \sigma_X^2$$

Theorem:

If X_1, X_2, …, X_n are n independent random variables and a_1, a_2, …, a_n are constants, then:

$Var(a_1X_1+a_2X_2+\ldots+a_nX_n)$

$$= a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$$

$$\Leftrightarrow$$

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

$$\Leftrightarrow$$

$$\sigma^2_{a_1X_1+a_2X_2+\ldots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \ldots + a_n^2\sigma^2_{X_n}$$

Corollary:

If X, and Y are independent random variables, then:

- $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$
- $Var(aX-bY) = a^2 Var(X) + b^2 Var(Y)$
- $Var(X \pm Y) = Var(X) + Var(Y)$

Example:

Let X, and Y be two independent random variables such that $E(X)=2$, $Var(X)=4$, $E(Y)=7$, and $Var(Y)=1$. Find:

1. $E(3X+7)$ and $Var(3X+7)$
2. $E(5X+2Y-2)$ and $Var(5X+2Y-2)$.

Solution:

1. $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$

   $Var(3X+7)= (3)^2 Var(X)=(3)^2 (4) =36$
2. $E(5X+2Y-2)= 5E(X) + 2E(Y) -2= (5)(2) + (2)(7) - 2= 22$

   $Var(5X+2Y-2)= Var(5X+2Y)= 5^2 Var(X) + 2^2 Var(Y)$

   $= (25)(4)+(4)(1) = 104$

The Binomial, Poisson, and Normal Distributions

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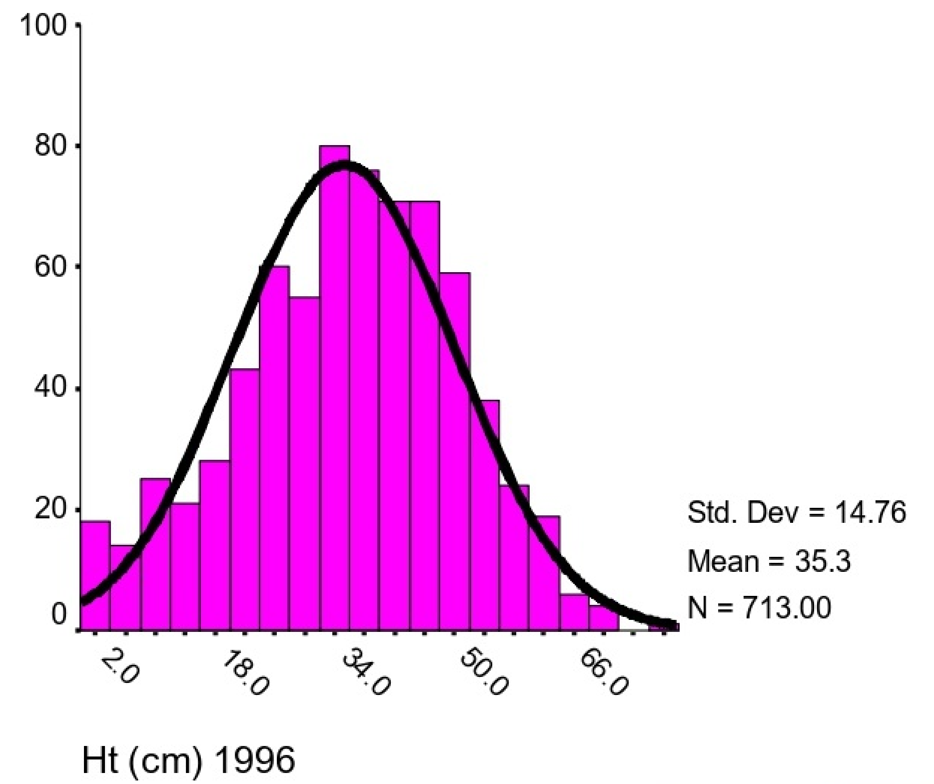
Probability distributions

- We use probability distributions because they work –they fit lots of data in real



Height (cm) of *Hypericum cumulicola* at Archbold Biological Station

Random variable

- The mathematical rule (or function) that assigns a given numerical value to each possible outcome of an experiment in the sample space of interest.

Random variables

- Discrete random variables
- Continuous random variables

The Binomial Distribution
Bernoulli Random Variables

- Imagine a simple trial with only two possible outcomes
  - Success (*S*)
  - Failure (*F*)

Jacob Bernoulli (1654-1705)

- Examples
  - Toss of a coin (heads or tails)
  - Sex of a newborn (male or female)
  - Survival of an organism in a region (live or die)

The Binomial Distribution
Overview

- Suppose that the probability of success is p

- What is the probability of failure?
  - $q = 1 - p$

- Examples
  - Toss of a coin (S = head): $p = 0.5 \Rightarrow q = 0.5$
  - Roll of a die (S = 1): $p = 0.1667 \Rightarrow q = 0.8333$
  - Fertility of a chicken egg (S = fertile): $p = 0.8 \Rightarrow q = 0.2$

The Binomial Distribution
Overview

- Imagine that a trial is repeated *n* times

- Examples
  - A coin is tossed 5 times
  - A die is rolled 25 times
  - 50 chicken eggs are examined

- Assume *p* remains constant from trial to trial and that the trials are statistically independent of each other

The Binomial Distribution
Overview

- What is the probability of obtaining *x* successes in *n* trials?

- Example
  - What is the probability of obtaining 2 heads from a coin that was tossed 5 times?

$P(HHTTT) = (1/2)^5 = 1/32$

The Binomial Distribution
Overview

- But there are more possibilities:

HHTTT HTHTT HTTHT HTTTH

THHTT THTHT THTTH

TTHHT TTHTH

TTTHH

$P(2 \text{ heads}) = 10 \times 1/32 = 10/32$

The Binomial Distribution
Overview

- In general, if trials result in a series of success and failures,

$$FFSFFFFSFSFSSFFFFFSF\ldots$$

Then the probability of x successes <u>in that order</u> is

$$P(x) = q \cdot q \cdot p \cdot q \cdot \ldots$$
$$= p^x \cdot q^{n-x}$$

The Binomial Distribution
Overview

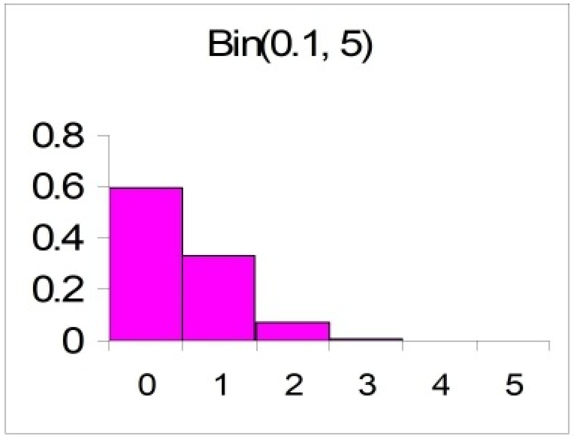
- However, if order is not important, then

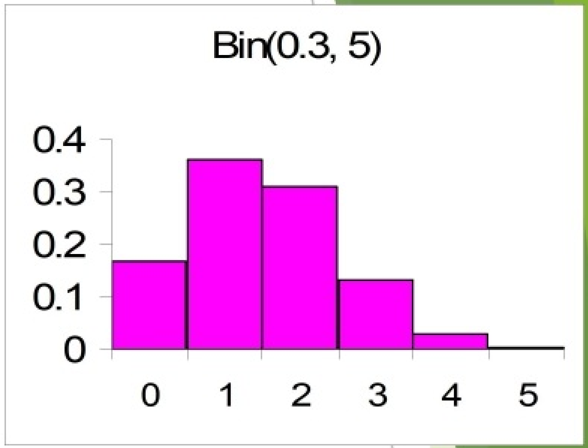
$$P(x) = \frac{n!}{x!(n-x)!} p^x \cdot q^{n-x}$$

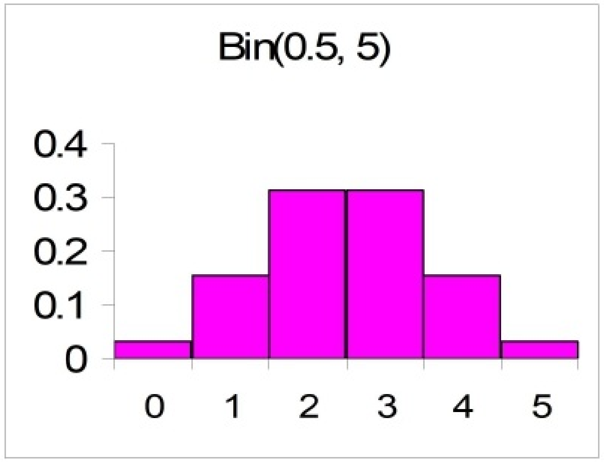
where $\frac{n!}{x!(n-x)!}$ is the number of ways to obtain x successes

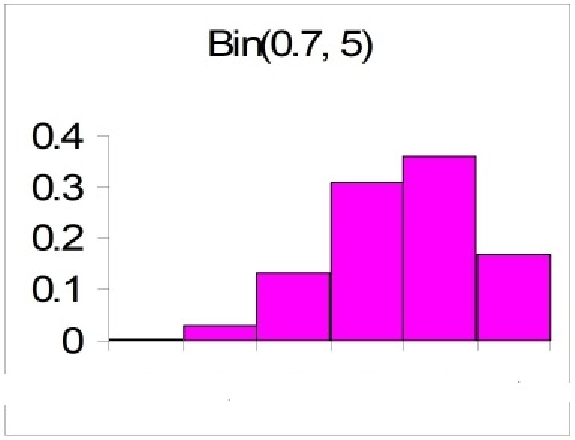
in n trials, and $i! = i \cdot (i-1) \cdot (i-2) \cdot \ldots \cdot 2 \cdot 1$

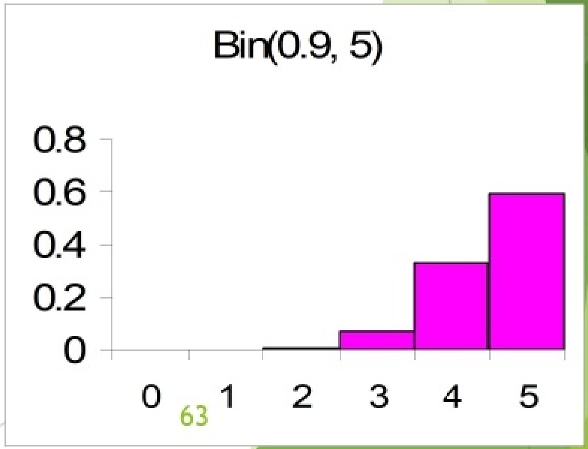
The Binomial Distribution
Overview











The Poisson Distribution
Overview

- When there is a large number of trials, but a small probability of success, binomial calculation becomes impractical
  - Example: Number of deaths from horse kicks in the Army in different years

Simeon D. Poisson (1781-1840)

- The mean number of successes from n trials is $\mu = np$
  - Example: 64 deaths in 20 years from thousands of soldiers

The Poisson Distribution

Overview

- If we substitute μ / n for p, and let n tend to infinity, the binomial distribution becomes the Poisson distribution:

$$P(x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

The Poisson Distribution

Overview

- Poisson distribution is applied where random events in space or time are expected to occur

- Deviation from Poisson distribution may indicate some degree of non-randomness in the events under study

- Investigation of cause may be of interest

The Poisson Distribution
Emission of α-particles

- Rutherford, Geiger, and Bateman (1910) counted the number of α-particles emitted by a film of polonium in 2608 successive intervals of one-eighth of a minute
  - What is *n*?
  - What is *p*?

- Do their data follow a Poisson distribution?

The Poisson Distribution

Emission of α-particles

- Calculation of μ:

  μ = No. of particles per interval

  = 10097/2608

  = 3.87

- Expected values:

$$2680 \times P(x) = 2608 \times \frac{e^{-3.87}(3.87)^x}{x!}$$

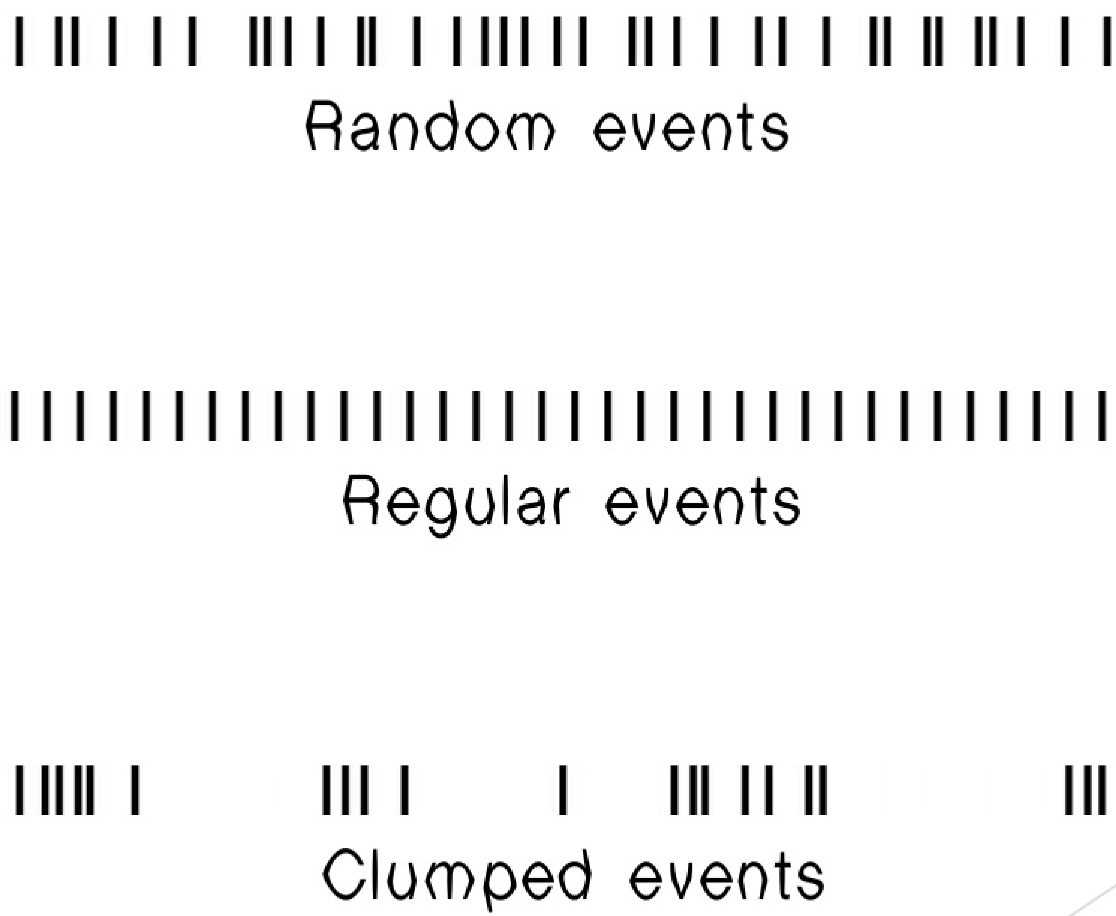
No. α-particles	Observed
0	57
1	203
2	383
3	525
4	532
5	408
6	273
7	139
8	45
9	27
10	10
11	4
12	0
13	1
14	1
Over 14	0
Total	2608

The Poisson Distribution

Emission of α-particles

No. α-particles	Observed	Expected
0	57	54
1	203	210
2	383	407
3	525	525
4	532	508
5	408	394
6	273	254
7	139	140
8	45	68
9	27	29
10	10	11
11	4	4
12	0	1
13	1	1
14	1	1
Over 14	0	0
Total	2608	2680

The Poisson Distribution
Emission of α-particles



The Poisson Distribution