Probability, Statistics and Complex Variables

(25MA302)

II B.Tech, I Semester-Mech Academic Year: 2026-27

COURSE CONTENTS

UNIT-V

COMPLEX INTEGRATION

Introduction

Complex integration forms the cornerstone of complex variable theory. The key results in Chapter are the Cauchy–Goursat theorem and the Cauchy integral formulas. Other interesting results include (Gauss' mean value theorem, Liouville's theorem and the maximum modulus theorem). The link of analytic functions and complex integration with the study of conservative fields is considered. Complex variable methods are seen to be effective analytical tools to solve conservation field models in potential flows, gravitational potentials and electrostatics.

Simple curve: Simple curve is a curve having no self intersections i.e., no two distinct values of 't' correspond to the same point (x, y)

Closed curve: Closed curve is one in which end point coincide i.e.,$\emptyset(a) = \emptyset(b)\ and\ \varphi(a) = \varphi(b)$.

Simple closed curve: Simple closed curve is a curve having no self-intersection and with coincident end point.

Contour: Contour is a continuous chain of a finite number of smooth arcs

Closed contour is a piecewise smooth closed curve without point of self intersection.

Examples: Boundaries of circle, ellipse, rectangle, triangle.

Line Integral: Definite integral or complex line integral or simply line integral of a complex function $f(z)$

From $z_1 to z_2$ along a curve 'C' is defined as $\int_c f(z)dz$ here 'C' is known as the path of integration .If it is a closed curve, the line integral is denoted by $\oint_c$.

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy)$$

Basic Properties of Line Integrals

1. Linearity: $\int_c (c_1 f(z) + cl2g(z))dz = c_1\int_c f(z)dz + c_2\int_c g(z)dz$
2. Sense reversal: $\int_A^B f(z)dz = -\int_B^A f(z)dz$
3. Partitioning of path: $\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{C_2} f(z)dz$

Where curve c consists of the curves $c_1\ and c_2$

Evaluation of complex line integral

1. By indefinite integration (of analytic function): If $f(z)$ is analytic in a simply connected domain 'D' then $\int_{z_2}^{Z_1} f(z)dz = F(z_2) - F(z_1)$ where $F^{'}(z) = \frac{dF}{dz} = f(z)$ in D.
2. By use of the path: If curve 'C' is represented by $z = z(t), a \le t \le b$ then $\int_c f(z)dz = \int_a^b f(z(t))\frac{dz}{dt}dt$ i.e. the integral is converted to an ordinary integral in t by making use of the property of the curve c.

<u>Problems on line integral</u>

1. Evaluate $\int_0^{1+i} (x^2 + iy)dz$ along the path (i) $y = x$ (ii) $y = x^2$

Sol: We known that $z = x + iy \Rightarrow dz = dx + idy$

(i) Along the line $y = x \Rightarrow dy = dx$ and x varies from 0 to 1.

$$\int_0^{1+i} (x^2 + iy)dz = \int_{(0,0)}^{(1,1)} (x^2 + iy)(dx + idy)$$

$$= \int_0^1 (x^2 + ix)(dx + idx) \quad (\text{since } y = x)$$

$$= (1+i)\int_0^1 (x^2 + ix)dx \Rightarrow (1+i)\left(\left(\frac{x^3}{3} + i\frac{x^2}{2}\right)\right)_0^1$$

$$= (1+i)(\frac{1}{3} + \frac{i}{2})$$

(ii) Along the parabola $y = x^2 \Rightarrow dy = 2xdx$ and x varies from 0 to 1

$$\int_0^{1+i} (x^2 + iy)dz = \int_{(0,0)}^{(1,1)} (x^2 + iy)(dx + idy)$$

$$= \int_0^1 (x^2 + ix^2)(dx + i2xdx) \quad (\text{since } y = x^2)$$

$$= \int_0^1 (x^2(1+i)(1+2xi)dx \Rightarrow (1+i)\int_0^1 x^2(1+2xi)dx$$

$$= (1+i)\int_0^1 (x^2 + 2x^3 i)dx$$

$$= (1+i)\left(\left(\frac{x^3}{3} + 2i\frac{x^4}{4}\right)\right)_0^1$$

$$= (1+i)(\frac{1}{3} + \frac{i}{2})$$

2. Evaluate $\int (2y + x^2)dx + (3x - y)dy$ along the parabola $x = 2t, y = t^2 + 3$ joining $(0,3)$ and $(2,4)$.

Sol: Given that $x = 0, y = 3, t = 0$ and at $x = 2, y = 4, t = 1$

Substituting for x and y in terms of t, i.e., $x = 2t,\ y = t^2 + 3 \Rightarrow dx = 2dt,\ dy = 2t\,dt$ we get

$$I = \int_{t=0}^{1} [2(t^2+3)+4t^2]2dt + \int_{t=0}^{1} [6t - t^2 - 3]2t\,dt$$

$$= \int_{0}^{1} (24t^2 - 2t^3 - 6t + 12)dt$$

$$= \left(\left(\frac{24t^3}{3} - \frac{2t^4}{4} - \frac{6t^2}{2} + 12t\right)\right)_0^1$$

$$= \frac{33}{2}$$

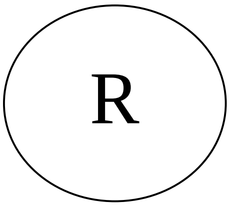
Exercise problems

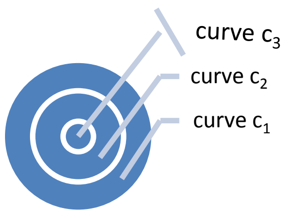
1. Evaluate $\oint_c |z^2| dz$ around the square with vertices at $(0,0), (1,0), (1,1)$ and $(0,1)$.
2. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)dz$ along the straight line joining $(1,-i)$ and $(2,i)$.
3. Evaluate $\int_{0}^{1+i} (x - y + ix^2)dz$ (i) along the straight line from $z = 0$ to $z = 1 + i$

   (ii) Along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 0$ to $z = 1 + i$

4. Evaluate $\int_{0}^{3+i} z^2 dz$ along parabola $x = 3y^2$

Simple curve: A curve which does not intersect itself. A simple curve which is closed is called simple closed curve**.**
Multiple curves: A curve which crosses itself.
Connected region: A region is said to be connected region if any two points of the region can be connected by a curve which lies entirely within the region.
Multiply connected region: A region which is bounded by more than one curve is called a **multiply connected region.**





R-- Simple connected region, multiply connected region by more than one curve

Cauchy- Goursat Theorem: A function $f(z)$ is analytic at each point inside and on a simple closed curve 'c', then $\int_c f(z)dz = 0$.

Cauchy's (Integral) Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be analytic on and within a simple closed curve contour 'c' and let $f^{'}(z)$ be continuous there. Then, $\int_c f(z)dz = 0.$

Problems on Cauchy integral theorem

1. Evaluate $\int_c \frac{e^{2z}}{z-2}dz$ where C is $|z|=1$

   Sol: Given that $|z|=1$ means the circle having radius one with centre (0, 0)

   $\therefore$ The point $z=2$ lies outside C.

   But by Cauchy integral theorem it is analytic within and on C

   Hence by Cauchy's theorem, $\int_c \frac{e^{2z}}{z-2}dz = 0$.

2. Verify Cauchy's theorem for the function $f(z) = z^2 + 3z - i2$ if C is the circle $|z|=1$

   Sol: Let C: $z = re^{i\theta}$ where $0 \le \theta \le 2\pi$.

   $$z = e^{i\theta}, dz = ie^{i\theta}d\theta \because r = 1$$

   $$\therefore \quad \int_c f(z)dz = \int_0^{2\pi} (e^{2i\theta} + 3e^{i\theta} - 2i)(ie^{i\theta})d\theta$$

   $$= i\int_0^{2\pi} (e^{3i\theta} + 3e^{2i\theta} - 2ie^{i\theta})d\theta$$

   $$= \quad 0 \qquad\qquad (\because \int_0^{2\pi} e^{in\theta} = 0 if n \neq 0)$$

Exercise problems:

1. Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ if C is the square with vertices at $1 \pm$ □ □□□ $-1 \pm$ □ .
2. Verify Cauchy's theorem for the integral of □^{3} taken over the boundary of the rectangle with vertices $-1, 1, 1+$□$, -1+$□ .

Cauchy's integral formula: Let $f(z)$ be an analytic function everywhere on and within a closed contour 'C'. If $z = a$ is any point within 'C', then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

Where the integral is taken in the positive sense around 'C'.

Cauchy's theorem for doubly connected regions: If $f(z)$ is analytic in the doubly connected Region bounded by the curve $c_1 and c_2$ then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

Generalized Cauchy's integral formula: If $f(z)$ is on analytic on and within a simple closed curve 'C' and if 'a' is any point within 'C', then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz.$$

Problems on Cauchy's integral formula (including generalization)

1. Let 'C' be the circle $|z|$=3 described in positive sense. Let $f(a)=\int_c \frac{2z^2-z-2}{z-a}dz$

   (|a|≠3)

Show that $f(2)=8\pi i$.what the value is of $f(a) if$ |□| > *3*.

Sol: Given that |□| = *3* is the circle with centre at (0,0) and radius equal to 3 units.

Consider $f(a)$ is analytic everywhere except at □ = □

This point maybe (i) within the circle or (ii) on the circle or(iii) outside the circle

Since |a|≠3, □ = □ is not on the circle.

Case I: If □ = □ is within the circle, $f(a)$is analytic everywhere except at □ = □

Take $f(z)=2z^2-z-2$

$$f(a)=\int_c \frac{2z^2-z-2}{z-a}dz$$

$$=\int_c \frac{f(z)}{z-a}dz$$

$$=2\pi i f(a)$$ (By Cauchy's integral formula)

$$=2\pi i(2a^2-a-2)$$

$$\therefore f(2)=2\pi i(8-2-2)=8\pi i.$$

Case II: If|□| > *3*, □ = □ is outside the circle|□| = *3*.

$\therefore f(a)$ Is analytic everywhere on and within 'C'.

Hence, $\int_c \frac{2z^2-z-2}{z-a}dz=0$ by Cauchy's theorem

2. Evaluate $\int_c (z-a)^n dz$ where C is a simple closed curve and the point $z=a$ is (i) inside C (ii) outside C (n is any integer≠ -1).

Sol: The parametric equation of C is

$$z-a=re^{i\theta} \Rightarrow dz=ire^{i\theta}d\theta$$

And □ varies from 0 to 2□

(i) $\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta}d\theta = i\int_0^{2\pi} d\theta$ =□(2□ − 0) = 2□□

(ii) If a lies inside C

$$\int_c (z-a)^n dz = \int_0^{2\pi} r^n e^{in\theta} ire^{i\theta}d\theta$$

$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta$$

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - e^{0}\right]$$

$$= \frac{r^{n+1}}{n+1}\left[\cos(n+1)2\pi - 1\right]$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - e^{0}\right]$$

$$= \frac{r^{n+1}}{n+1}\left[\cos(n+1)2\pi - 1\right]$$

$$= \frac{r^{n+1}}{n+1}\left[1-1\right] = 0 \qquad \text{if } □ \neq -1$$

If □ $= -1$ then

$$\int_c (z-a)^n\,dz = \int_c (z-a)^{-1}\,dz = \int_c \frac{dz}{z-a} = 2\pi i(by(i))$$

3. Evaluate $\int_c \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3}dz$ with □: |□| $= 2$ Using Cauchy's integral formula.

Sol: Given that $f(z) = z^3 - \sin 3z$ is analytic everywhere.

$\therefore f(z)$ Is analytic within □: |□| $= 2$

Here $z = \frac{\pi}{2}$ is a singular point and lies inside c $\quad (\because z = \frac{\pi}{2} \prec 2)$

Since by Cauchy's integral formula, $f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz$

Here □ $= \frac{□}{2}$, □ $= 2$

$$\int_c \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3}dz = \frac{2\pi i}{2!} f''(\frac{\pi}{2})$$

$$= \pi i\left[\frac{d^2}{dz^2}(z^3 - \sin 3z)\right]_{z=\frac{\pi}{2}}$$

$$= \pi i\left[\frac{d}{dz}(3z^2 - 3\cos 3z)\right]_{z=\frac{\pi}{2}}$$

$$= \pi i\left[(6z + 9\sin 3z)\right]_{z=\frac{\pi}{2}}$$

$$= \pi i\left[6\frac{\pi}{2} + 9\sin 3\frac{\pi}{2})\right]$$

$$= 3\pi i(\pi - 3)$$

4. Evaluate $\int_c \frac{dz}{z^3(z+4)}$ where □: |□| $= 2$ Using Cauchy's integral formula.

Sol: Given that $f(z) = \frac{1}{z^3(z+4)}$

$f(z)$ is not analytic at □ $= 0$, □ $= -4$

But here □ $= 0$ is lies inside 'c' and □ $= -4$ is lies outside' c'

Let $g(z)=\dfrac{1}{z+4}$, $g(z)$is analytic on and within C.

By Cauchy's integral formula, we have $g^{(n)}(a)=\dfrac{n!}{2\pi i}\displaystyle\int_c \frac{g(z)}{(z-a)^{n+1}}\,dz$

Taking □ = 0, □ = 2

$$g^{''}(a)=\frac{2!}{2\pi i}\int_c \frac{\frac{1}{z+4}}{(z-0)^{2+1}}\,dz$$

$$\int_c \frac{dz}{z^3(z+4)} \;=\; \pi i g^{''}(0)$$

$$= \pi i\left[\frac{2}{(z+4)^3}\right]_{z=0}$$

$$=\left[\frac{\pi i}{(0+4)^3}\right]=\frac{\pi i}{32}$$

Introduction

Complex power series is the main theme in Complex field. We introduce different types of convergence of series of complex functions. The various tests that examine the convergence of complex series are discussed. The Taylor series theorem and Laurent series theorem show that a convergent power series is an analytic function within its disk or annulus of convergence, respectively. The notion of analytic continuation of a complex function is discussed. As an application, the solution to the potential flow over a perturbed circle is obtained as a power series in a perturbation parameter.

Definitions:

- A Power series in power of $(z-z_0)$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots \qquad \_\_\_\_\_\_(1)$$

Here $a_0, a_1, a_2, \ldots$ are complex (or real) constants know as coefficients of the series (1).z Is complex variable and z_0 is called the centre of the series. (1) is also known as power series about the point z_0.

Note: power series in powers of z is

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots$$

This is obtained as a particular case with $z_0 = 0$ in (1)

Region of convergence of a series is the set of all points z for which the series converges.

If $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$,then R is called the Radius of convergence of the power series and $|z| = R$ is called the circle of convergence of the power series.

<u>Taylor's Theorem:</u> A function $f(z)$ which is analytic at all points within a circle c_0 with centre at z_0 and of radius R can be represented uniquely as a convergent power series in c_0 known as the Taylor's series given by

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n \; where a_n = \frac{f^{(n)}(z_0)}{n!} \;\Rightarrow\; f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

i.e. $$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)+...$$

Corollary: Maclaurin's series

If we take $z_0=0$(i.e. the centre is origin) in Taylor's series expansion we get

$$f(z)=f(0)+\frac{f'(0)}{1!}(z)+\frac{f''(0)}{2!}(z)+...$$

This is known as Maclaurin's series expansion of $f(z)$.

Problems on Taylor's series:

1. Expand e^z as Taylor's series about $z=a$

Sol: Direct method: Here $z=z_0=a$

Taylor's series of $f(z)$is $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n$

Here$f(z)=e^z$, $f'(z)=e^z,\dots\ f^{(n)}(z)=e^z \Rightarrow f^{(n)}(a)=e^a$

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)+...$$

$$e^z \quad =e^a+\frac{e^a}{1!}(z-a)+\frac{e^a}{2!}(z-a)^2+...$$

$$=e^a\sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}$$

Indirect method: Take $z-a=w\Rightarrow z=w+a$

Given that $f(z)=e^z\Rightarrow e^{w+a}$

$$\Rightarrow e^a e^w \Rightarrow e^a(1+w+\frac{w^2}{2!}+\frac{w^3}{3!}+...)$$

$$\Rightarrow e^a(1+(z-a)+\frac{(z-a)^2}{2!}+\frac{(z-a)^3}{3!}+....)$$

$$\Rightarrow e^a\sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}$$

2. Expand $f(z)=\frac{1}{z^2-z-6}$ as Taylor's series about (i) $z=-1$ (ii) $z=1$

Sol: Given that $f(z)=\frac{1}{z^2-z-6}$

By using partial fractions, we get

$$f(z)=\frac{1}{(z-3)(z+2)}$$

$$=\frac{1}{5}\left[\frac{1}{z-3}-\frac{1}{z+2}\right]$$

(i) About $z=-1$ i.e., in powers of $(z+1)$

$$f(z)=\frac{1}{5}\left[\frac{1}{z-3}-\frac{1}{z+2}\right]$$

$$=\frac{1}{5}\left[\frac{1}{z+1-4}-\frac{1}{1+(z+1)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-4\left[1-\left(\frac{z+1}{4}\right)\right]} - \frac{1}{1+(z+1)}\right]$$

$$= \frac{-1}{20}\left[1-\left(\frac{z+1}{4}\right)\right]^{-1} - \frac{1}{5}\left[1+(z+1)\right]^{-1}$$

Expanding by binomial series, we get

$$= \frac{-1}{20}\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n (z+1)^n$$

This is valid for $|z+1| < 1$ i.e., region of convergence is the interior of circle with Centre at $z = -1$ and radius 1.

(ii) About $z = 1$ i.e., in powers of $(z-1)$

$$f(z) = \frac{1}{5}\left[\frac{1}{z-3} - \frac{1}{z+2}\right]$$

$$= \frac{1}{5}\left[\frac{1}{z-1-2} - \frac{1}{z-1+3)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-2\left[1-\left(\frac{z-1}{2}\right)\right]} - \frac{1}{3\left[1+\left(\frac{z-1}{3}\right)\right]}\right]$$

$$= \frac{-1}{10}\left[1-\left(\frac{z-1}{2}\right)\right]^{-1} - \frac{1}{15}\left[\left[1+\left(\frac{z-1}{3}\right)\right]\right]^{-1}$$

Expanding by binomial series, we get

$$= \frac{-1}{10}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z-1}{3}\right)^n$$

This is valid for $|\frac{z-1}{2}| < 1$ and $|\frac{z-1}{3}| < 1$ i.e., region of convergence is $|z-1| < 2$

Exercise problems:

1. Expand e^z as Taylor's series about $z = 1$
2. Expand $\frac{1}{z^2}$ as Taylor's series about $|z+1| < 1$
3. Expand $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ as Taylor's series about (i) $z < 2$ (ii) $z > 3$
4. Expand $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ as Taylor's series about $2 < |z| < 3$

Laurent's series: Let $f(z)$ be analytic on two concentric circles C_1 and C_2 with centre Z_0 and radii R_1 and R_2 and in the annulus region $R_1 < |Z - Z_0| < R_2$. Then $f(z)$ is uniquely represented by a convergent Laurent series given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Problems on Laurent series

1.Find the Laurent series of $f(z)=\frac{1}{z^2+1}$ about its singular points. Determine the region of convergence.

Sol: Given that $f(z)=\frac{1}{z^2+1}$

$f(z)$ Have two singular points $z=\pm i$

(i) Laurent series about $z=i$

$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$$

$$=\frac{1}{(z-i)}\cdot\frac{1}{z-i+2i}$$

$$=\frac{1}{(z-i)}\cdot\frac{1}{2i\left[1+\left(\frac{z-i}{2i}\right)\right]} \qquad \text{Provided } \left|\frac{z-i}{2i}\right|\prec 1$$

$$=\frac{1}{2i}\frac{1}{(z-i)}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n \qquad \text{If } |z-i|\prec|2i|=2$$

$$f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^{n-1}}{(2i)^{n+1}} \qquad |z-i|\prec 2$$

Region of convergence is $|z-i|\prec 2$

(ii) Laurent series about $z=-i$

$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$$

$$=\frac{1}{(z+i)}\cdot\frac{1}{z+i-2i}$$

$$=\frac{1}{(z+i)}\cdot\frac{1}{-2i\left[1-\left(\frac{z+i}{2i}\right)\right]} \qquad \text{Provided } \left|\frac{z+i}{2i}\right|\prec 1$$

$$=\frac{-1}{2i}\frac{1}{(z+i)}\cdot\sum_{n=0}^{\infty}\left(\frac{z+i}{2i}\right)^n \qquad \text{If } |z+i|\prec|2i|=2$$

$$f(z)=-\sum_{n=0}^{\infty}\frac{(z+i)^{n-1}}{(2i)^{n+1}} \qquad \text{If } |z+i|\prec 2$$

Region of convergence is $|z+i|\prec 2$

1. Expand $f(z)=\frac{e^{2z}}{(z-1)^3}$ about $z=1$ as a Laurent series also find the region of convergence.

Sol: Given that $f(z)=\frac{e^{2z}}{(z-1)^3}$

We want the Laurent series expansion of $f(z)$ about$z=1$.

Put $z-1=w\Rightarrow z=w+1$

$$f(z)=\frac{e^{2z}}{(z-1)^3}=\frac{e^{2(w+1)}}{w^3}$$

$$f(z)=e^2.\frac{e^{2w}}{w^3}$$

$$=e^2\,\frac{1}{w^3}\left[1+2w+\frac{(2w)^2}{2!}+\frac{(2w)^3}{3!}+...\right]\quad \text{If } w\neq 0$$

$$=e^2\,\frac{1}{w^3}\sum_{n=0}^{\infty}\frac{2^n}{n!}w^n \qquad \text{If } w\neq 0$$

$$=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}w^{n-3} \qquad \text{If } w\neq 0$$

$$=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-3} \qquad \text{If } z-1\neq 0$$

$$=e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-3} \qquad \text{If } |z-1|>0$$

Exercise problems:

1. Expand $f(z)=\frac{1}{(z-1)(z-2)}$ as Laurent's series about $0<|z-1|<1$
2. Expand $f(z)=\frac{z+1}{z-1}$ using Laurent's series for the domain $z>1$

Zeros of an Analytic function: If a function $f(z)$ analytic in a region R, is zero at a point z_0 in R, then z_0 is called a zero of $f(z)$

If $f(z_0)=0$ but $f'(z_0)\neq 0$, z_0 is called a simple zero (or) a zero of the first order.

If $f(z_0)=f'(z_0)=0$ ... $=f^{(n-1)}(z_0)=0$ and $f^{(n)}(z_0)\neq 0$, then z_0 is called a zero of order n.

Note: zeros of an analytic function is isolated.

Singularities

1. **Singularity:** A point at which a function $f(z)$ is not analytic is known as a singular point (or) singularity of $f(z)$.
2. **Isolated and non-isolated singularity:** If $z=a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z=a$ then $z=a$ is said to be an isolated singularity of the function $f(z)$. Otherwise it is called non-isolated.

Examples: 1) $f(z)=\frac{z^2+1}{(z-1)^2(z^2+4)}$ the singular points of $f(z)$ are $z=1, z=2i \text{ and } z=-2i$ are isolated singular points.

Types of singularities

- If $f(z)$ is analytic and hence if z is any point of region , then by Laurent's theorem, we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n} \quad ,0<|z-a|<R$$

The term $\sum_{n=1}^{\infty}b_n(z-a)^{-n}$ on the right hand side is called the principal part of $f(z)$ at the isolated Singularity at$z=a$.

In relation to the principal part, there are three possibilities

(i) All the coefficients b_n (principal part)are zero. i.e., no term in the principal part, then 'a' is called removable singularity of $f(z)$. (or) If $\lim_{z\to a} f(z)$ exists finitely, then $z = a$ is a removable singularity.

Example: 1) The function $f(z) = \frac{\sin z}{z}$ is removable singularity of $f(z)$ at $z = 0$. Thus $f(z)$ have no negative powers of z.

2) Consider the function $f(z) = \frac{z^2 - a^2}{z - a}$ $\Rightarrow \lim_{z\to a} f(z) = 2a$

So, $z = a$ is a removable singularity.

(ii) If the principal part of $f(z)$ at $z = a$ contains an infinite number of terms, then 'a' is called <u>Essential singularity (or) isolated essential singularity of $f(z)$.</u>

(Or) If $\lim_{z\to a} f(z)$ does not exist, then $z = a$ is an Essential singularity Example: 1) the function $f(z) = e^{\frac{1}{z}}$ is Essential singularity.

Thus $f(z)$ has infinite number of negative power of 'z' so $z = 0$ is called Essential singularity

2) Consider the function $f(z) = e^{\frac{1}{z-a}}$ $\Rightarrow \lim_{z\to a} f(z) = \lim_{z\to a} e^{\frac{1}{z-a}}$ does not exist.

So, $z = a$ is an Essential Singularity

(iii) If the principal part of $f(z)$ at $z = a$ consists of a finite number of terms, say 'm' we say that 'a' is a Pole of order m of $f(z)$. In other words the principal part of $f(z)$ at $z = a$ contains finite number of terms (Or) If $\lim_{z\to a} f(z) = \infty$ then $z = a$ is a Pole of order m of $f(z)$.

Example: Consider the function $f(z) = \frac{z^2 + a^2}{z - a}$ $\Rightarrow \lim_{z\to a} f(z) = \infty$

So, $z = a$ is a pole of $f(z)$.

<u>Evaluation of Residue by formula and Laurent series – Residue Theorem without Proof; Evaluation of integrals of type</u>

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta;\ \int_{-\infty}^{\infty} f(x)dx;\ and\ \int_{-\infty}^{\infty} e^{imx} f(x)dx$$

The coefficient of $\frac{1}{z-a}$ in the Laurent series expansion of f(x) about the isolated singularity z = a is called the residue of f(z) at that point, denoted with b_1 from the Laurent series b_1 is given by $b_1 = \frac{1}{2\pi i}\int_C f(z)dz$

$$\int f(z)dz = 2\pi i \times b_1 = 2\pi i \times [\text{Residue of } f(z)]$$

Where C is a closed curve containing the point z = a

Residue at a pole

Suppose z = a is an isolated singularity of f(z)

Let us expand f(z) in a Laurent series about z = a

$$\therefore\ f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

$$(I)\qquad\qquad (II)$$

$(I)\Rightarrow$ is called analytical part of f(z)

(II) $\Rightarrow$ is called principal part of f(z)

If the principal part of f(z) contains only a finite number of terms say $\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+......+\frac{b_m}{(z-a)^n}$

Then z = a is called pole of order m

If m=1 then z = a is called a simple pole.

If the principal part contains of f(z) contains an infinite number of terms, then z = a is called an essential singularity of f(z).

If z = a is an isolated singularity of analytic function of f(z) then b_1, the coefficient of $\frac{1}{z-a}$ is called the residue of f(z) and is written as Res f(z), b_1 = Res f(z) $_{z=a}$

Theorems:

1. If f(x) has a simple pole at z = a then Res f(z) = $_{z=a}$

$$=\underset{z\to a}{Lt}(z-a)f(z)$$

2. Suppose f(z) = $\frac{\phi(z)}{\psi(z)}$ where $\psi(z)=(z-a)F(z)$

   Where $f(a)\neq 0$ *the* $\underset{z=a}{\operatorname{Res} f(z)}=\frac{\phi(a)}{\psi'(a)}$

3. Let z = a be a pole of f(z) of order "m" then

$$\underset{z=a}{\operatorname{Res} f(z)}=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

Problems:

1. F(z) = $\frac{z^2}{(z-1)^2(z+2)}$ determine the poles and residues at each pole?

   Sol: z = 1 is a zero of denominator of order 2

   And = -2 is a zero of denominator of order 1

   z = 1 is a pole of order 2

   z = 2 is a pole of order 1

$$\therefore \underset{z=-2}{\operatorname{Res} f(z)}=\underset{z\to -2}{Lt}(z+2)\frac{z^2}{(z-1)^2(z+2)}=\frac{(-2)^2}{(-2-1)^2}=4/9$$

$$\therefore \underset{z=1}{\operatorname{Res} f(z)}=\underset{z\to 1}{Lt}\frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\times\frac{z^2}{(z-1)^2(z+2)}\right]$$

$$\underset{z\to 1}{Lt}\frac{d}{dz}\left[\frac{z^2}{z+2}\right]=\underset{z\to 1}{Lt}\left[\frac{(z+2)^2z-z^2}{(z+2)^2}\right]$$

$$= \frac{3\times 2-1}{3^2} = \frac{5}{9} = \frac{5}{9}$$

2. Find the Residue of $\frac{z^2}{z^4-1}$ at those singular points which lie inside the circle |z| = 2

Sol: $f(z) = \frac{z^2}{z^4-1} \Rightarrow z^4 - 1 = 0 \Rightarrow (z^2+1)(z-1)(z+1) = 0 \; z = \pm i$ of order 1

Sin |z| = 2 is a circle with centre at the origin and redius 2, Hence all these points we inside the circle |z|=2

$$\underset{z=-1}{\operatorname{Re}sf(z)} = \underset{z\to -1}{Lim}(z+1)f(z) = \underset{z\to}{Lt}(z+1)\frac{z^2}{(z^2+1)(z-1)(z+1)}$$

$$= \frac{(-1)^2}{(1+1)(-1-1)} = \frac{1}{-4} = -\frac{1}{4}$$

$$\underset{z=1}{\operatorname{Re}sf(z)} = \underset{z\to 1}{Lim}(z-1)f(z) = \underset{z\to}{Lt}(z-1)\frac{z^2}{(z^2+1)(z-1)(z+1)}$$

$$= \frac{(1)^2}{(1+1)(1+1)} = \frac{1}{4}$$

$$\underset{z=-i}{\operatorname{Re}sf(z)} = \underset{z\to -i}{Lim}(z+i)\frac{z^2}{(z+i)(z-i)(z+1)(z-1)}$$

$$= \frac{(-i)^2}{(-i-i)(-i+1)(-i-1)} = \frac{i^2}{(-2i)\left[(-i)^2-1\right]}$$

$$= \frac{-1}{(-2i)(-1-1)} = \frac{-1}{4i} = \frac{i^2}{4i} = \frac{i}{4}$$

$$\underset{z=i}{\operatorname{Re}sf(z)} = \underset{z\to i}{Lt}(z-i)\frac{z^2}{(z+i)(z-i)(z^2-1)}$$

$$= \frac{i^2}{(i+i)[-1-1]} = \frac{i^2}{(2i)(-2)} = \frac{-i}{4}$$

3. Let f(z) = $\frac{ze^z}{(z-1)^3}$

$\therefore$ z = 1 is a pole of f(z) of order 3.

We know that if f(z) has a pole of order m at z = a, then

$$\underset{z=a}{\operatorname{Re}sf(z)} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

Here a = 1, m= 3

$$\therefore \underset{z=1}{\operatorname{Re}s}\, f(z) = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3 f(z)\right] = \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[ze^z\right]$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d}{dz}\left[ze^z + e^z\right] = \frac{1}{2}\lim_{z\to 1}\left[ze^z + e^z + e^z\right]$$

$$=\frac{1}{2!}\lim_{z\to 1} e^{z}(z+2)=\frac{1}{2}.e(3)\frac{3e}{2}$$

1. Determine the poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ and residues at each poles?

Sol: Z = 1 is a zero of the order denominator of order 2 and z = -2 is a zero of the denominator of order 2 and z = -2 is a zero of the denominator of order 1

Z = 1 is a pole of order 2

Z = 2 is a pole of order 1 of f(z)

$$\therefore \underset{Z=-2}{\operatorname{Res} f(Z)} = \underset{Z\to -2}{Lim}(Z+2) \qquad f(Z)=\underset{Z\to -2}{Lt}(Z+2)\frac{Z^2}{(Z-1)^2(Z+2)}$$

$$=\frac{(-2)^2}{(-2-1)^2}=4/9$$

$$\operatorname{Res} f(z)=\underset{Z\to 1}{Lim}\frac{1}{1!}\frac{d}{dz}\left[(Z-1)^2 f(Z)\right]=\underset{Z\to 1}{Lim}\left[(Z-1)^2\frac{Z^2}{(Z-1)^2(Z+2)}\right]$$

$$=\underset{Z\to 1}{Lim}\frac{d}{dz}\left[\frac{Z^2}{Z+2}\right]=\underset{Z\to 1}{Lim}\left[\frac{(Z+2)2Z-Z^2}{(Z+2)^2}\right]$$

$$=\frac{3\times 2-1}{3^2}=\frac{5}{9}$$

2. Expand f(z) = $\frac{e^z}{(z-1)^2}$ in a Laurent series and obtain the residue at its singular point?

$$f(Z)=\frac{e^Z}{(Z-1)^2} \qquad \begin{array}{l} Lt\ u=Z-1 \\ u+1=Z \end{array}$$

$$\therefore\ f(Z)=\frac{e^{u+1}}{u^2}=\frac{e^u e}{u^2}=\frac{e}{u^2}\left[1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+....\right]$$

$$=\frac{e}{u^2}+\frac{e}{u}+\frac{1}{2!}+\frac{4e}{3!}+......$$

$$=\frac{e}{(Z-1)^2}+\frac{e}{(Z-1)}+\frac{1}{2!}+\frac{(Z-1)e}{Z!}+.......$$

$$\therefore \underset{Z=1}{\operatorname{Res} f(Z)}=Coefficien\,t\,of\ \frac{1}{Z-1}\Rightarrow e.$$

3. Find the Residue of $f(Z)=\frac{Z^2-2Z}{(Z+1)^2(Z^2+1)}$ at each pole.

Sol: The poles of f(z) are given by $(Z+1)^2\,(Z^2+1) = 0$.

f(Z) has pole at Z = $\pm$ i of order one and Z = -1 of order 2

4.

$$\underset{Z=i}{\operatorname{Res} f(Z)}=\underset{Z\to i}{Lt}\left[\frac{Z^2-2Z}{(Z+1)^2(Z-i)(Z+I)}\right]=\frac{i^2-2i}{(i+1)^2(i+i)}$$

$$=\frac{-1-2i}{(-1+1+2i)(i+1)}=\frac{-1-2i}{(2i)(2i)}=\frac{-(1+2i)}{-4}=\frac{1+2i}{4}$$

$$\underset{Z=-i}{\operatorname{Res} f(Z)} = \underset{Z\to -i}{Lt}\left[(Z+i)\frac{Z^2-2Z}{(Z+1)^2(Z-i)(Z+i)}\right] = \frac{(-i)^2-2(-i)}{(-i+1)^2(-i-i)}$$

$$= \frac{-1+2i}{(1-1-2i)}(-2i) = \frac{2i-1}{-2i\times -2i} = \frac{2i-1}{4} = \frac{1-2i}{4}$$

$$\underset{Z=-1}{\operatorname{Res} f(Z)} = \frac{1}{1^2-1}\underset{Z\to -1}{Lt}\frac{d}{dZ}\left[(Z+1)^2\frac{Z^2-2Z}{(Z+1)^2(Z^2+1)}\right]$$

$$= \underset{Z\to -1}{Lt}\frac{d}{dZ}\left[\frac{Z^2-2Z}{Z^2+1}\right]$$

$$= \underset{Z\to -1}{Lt}\frac{(Z^2+1)[2Z-2]-(Z^2-2Z)\times 2Z}{(Z^2+1)^2}$$

$$= \frac{(1+1)(-2-2)-(1+2)\times 2(-1)}{(2)2} = \frac{(2\times -4)+6}{4}$$

$$= \frac{-8+6}{4} = \frac{-2}{4} = -\frac{1}{2}$$

Find the Residues of the function f(Z) = $\frac{1-e^{2Z}}{Z^4}$ at each pole.

Sol: Expanding f(z) $= \frac{1-e^{2Z}}{Z^4} = \frac{1-\left[1+\frac{2Z}{1!}+\frac{(2Z)^2}{2!} = 1+\frac{2Z}{1!}+\frac{(2Z)^2}{2!}+\frac{(2Z)^3}{3!}+......\right]}{Z^4}$

$$= \frac{1}{Z^4}\left[1-1\frac{2Z}{1!}-\frac{(2Z)^2}{2!}-\frac{(2Z)^3}{3!}-.......\right]$$

$$= -\left[\frac{2}{Z^3}+\frac{2}{Z^2}+\frac{4}{3}\times\frac{1}{Z}+\frac{2}{Z}+\frac{4Z}{15}+......\right]$$

Which is Laurent series.

Hence $\frac{1}{Z^3}$ is highest negative power of (Z-0)

$\therefore$ Z = 0 is a pole of order 3

$\therefore$ The Residue of f(Z) at Z = 0 is -4/3.

Find the poles and Residues of Tan Z?

$$f(Z) = Tan\ Z = \frac{\sin Z}{\cos Z}$$

The Poles of f(Z) an given by cos Z = 0

$\therefore$ Z = (2n+1) $\pi/2$ where $n \in$ I

F(Z) = Tan Z = $\frac{\sin Z}{\cos Z} = \frac{\varphi(Z)}{\psi(Z)}$

$\therefore \varphi(Z) = \sin Z$ $\qquad\qquad$ $\psi(Z) = \cos Z$

$\phi^1(Z) = \cos Z$ $\qquad\qquad$ $\psi(Z) = -\sin Z$

$$\underset{Z=a}{\operatorname{Res}\ f(Z)} = \frac{\varphi(a)}{\psi(a)} = \frac{\sin a}{-\sin a} = -1$$

<u>Cauchy’s Residue Theorem:-</u>

If f(z) is analytic at all points inside and on a simple closed contour C Except at a finite number of isolated singular points $a_1, a_2 \ldots\ldots a_n$ then $\int_c f(z)dz = 2\pi i\sum e_i$.

Where $e_i = \underset{z=a_i}{\mathrm{Re}s}\, f(z)$ where I = 1,2,3,………..n

Proof:-

Draw small circle $C_1, C_2 \ldots C_n$ with centres $a_1, a_2, \ldots a_n$ and very small radii so that their circles $C_1, C_2 \ldots C_n$ lie wholly inside C.

Since f(Z) is analytic in the multiply connected region bounded by the curve C and the circles $C_1, C_2 \ldots C_n$; Hence from Cauchy's Theorem for multiply connected region we have

$$\oint_C (Z)dZ = \oint_{C_1} f\Big((Z)/dZ + \oint_{C_2} f(Z)dZ + \ldots\ldots \oint_{C_n} f(Z)dZ\Big) ------(1)$$

But from the definition of Residue

$$\oint_{C_1} f(Z)dZ = 2\pi \mathrm{Re}s f(Z) = 2\pi i r,\ Z = a$$

Similarly $\oint_{C2} f(Z)dZ = 2\pi i r_2$

$$\oint_{Cn} f(Z)dZ = 2\pi i r_n$$

Substituting in (1)

$$\oint f(Z)dz = 2\pi i r_1 + 2\pi i r_2 + 2\pi i r_3 + \ldots + 2\pi i r_n$$

$$= 2\pi i \sum_{i=1}^{n} l_i$$

<u>Working Rule</u>

1. Find the poles of f(Z)
2. Calculate the Residues at the poles which are inside the given contour C.
3. Substitute in the formula

$$\oint_C f(Z)dZ = 2\pi^i \left[Sum of\ \mathrm{Re}siduces\ at\ Poles\right]$$

1. $\int_C \frac{\cot hz}{Z-i} dZ$ where C is the circle |Z| = 2

$f(Z) = \frac{\cot hZ}{(Z-i)} = \frac{\cosh Z}{(Z-i)\sinh Z}$; the poles of f(Z) are Z = i and Z = 0. These poles inside the circle |Z|=2.

$$r_1 = \underset{Z=i}{\mathrm{Re}s\, f(Z)} = \underset{Z\to i}{Lt(Z-i)} f(Z) = \underset{Z\to i}{Lim(Z-i)} \frac{\cos hZ}{(Z-i)\sinh Z} = \cot hi$$

$$r_2 = \underset{Z=0}{\mathrm{Re}s\, f(Z)} = \frac{\varphi(0)}{\psi^1(0)}$$

$\varphi(Z) = \cos hZ$ $\qquad\qquad \therefore \varphi(0) = \cos ho = 1$

$\psi(Z) = (Z-i)\sin hZ$ $\qquad\qquad r_2 = -\frac{1}{i}$

$\psi^1(Z) = (Z-i)\cos hZ + (\sin hZ)\times 1$

$\psi^1(0) = (0-i)\cos ho + \sin ho$

= - i cos ho

= -i

$\therefore$ By residue theorem $\therefore \oint \frac{\cot hZ}{(Z-i)} dZ = 2\pi i(r_1 + r_2) = 2\pi\left[\cot hi - \frac{1}{i}\right]$

$\oint_C Evaluate \oint_C \frac{\sin^2 Z}{\left(Z - \frac{\pi}{6}\right)} dZ$ where C is the circle |Z|= 2

$$f(Z) = \frac{\sin^2 Z}{\left(Z - \frac{\pi}{6}\right)^2}$$

$Z = \frac{\pi}{6}$ is a pole of f(Z) of order 2 and it lies within the circle |Z|=2

$$\underset{Z=\pi/6}{\operatorname{Res} f(Z)} f(Z) = \lim_{Z \to \frac{\pi}{6}} \frac{d}{dZ}\left[\left(Z - \frac{\pi}{6}\right)^2 f(Z)\right]$$

$$= \lim_{Z \to \frac{\pi}{6}} \frac{d}{dZ}\left[\left(Z - \frac{\pi}{6}\right)^2 \times \frac{\sin^2 Z}{\left(Z - \frac{\pi}{6}\right)^2}\right] = \lim_{2 \to \frac{\pi}{6}} \frac{d}{dZ}\left[\sin^2 Z\right]$$

$$= \lim_{Z \to \frac{\pi}{6}} 2\sin Z \cos Z = 2 \sin \frac{\pi}{6} \cos \frac{\pi}{6} = 2\frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

By cauchy's Residue theorem we have

$$\oint_C \frac{\sin^2 Z}{\left(Z = \frac{\pi}{6}\right)^2} dZ = 2\pi i r_1 = 2\pi i \frac{\sqrt{3}}{2} = \sqrt{3}\pi i$$

Evaluate $\int_C \frac{Z-3}{Z^2+2Z+5} dZ$ where C is the circle

(i) |Z|=1 (ii) |Z+1-i|=2 (iii) |Z+1+i|=2

(i) Let f(Z) = $\frac{Z-3}{Z^2+2Z+5} \Rightarrow Z^2 + 2Z + 5 = 0 \Rightarrow Z = -1 \pm 2i$

The poles are given by Z = -1 $\pm$ 2i

The poles Z = -1+2i and z = -1 -2i ia outside the circle |Z|=1 $\Rightarrow$ Hence f(Z) is analytic within and on C.

By couchy's theorem we have $\oint_C \frac{Z-3}{Z^2+2Z+5} dZ = 0$

(ii) Only one pole Z = -1+2i lies inside the circle |Z+1-i|=2$\Rightarrow$ Hence f(Z) is analytic every where except at the point Z = -1+2i

$$\underset{Z=-1+2i}{\operatorname{Res} f(Z)} = \underset{Z \to -1+2i}{Lt}\left[Z - (1+2i) f(Z)\right]$$

$$= \left(\underset{Z \to -1+2i,}{Lt} \frac{Z-3}{Z+1+2i}\right) = \frac{-1+2i-3}{-1+2i+1+2i} = \frac{2i-4}{4i}$$

$$= \frac{2i}{4i} - \frac{4}{4i} = \frac{1}{2} - \frac{1}{i} \times \frac{i}{i} = \frac{1}{2} - \frac{i}{-1} = \frac{1}{2} + i$$

By Residue theorem $\oint_C \frac{Z-3}{Z^2+2Z+5} dZ = 2\pi i\left[\frac{1}{2} + i\right]$

$$= \pi i - 2\pi = \pi(i-2)$$

(iii) Only Z = -1-2i lies inside circle |Z+1+i|=2

$$\underset{Z=-1-2i}{\operatorname{Res} f(Z)} = \underset{Z \to -1-2i}{Lt}\left[\frac{(Z+1+2i)(Z-3)}{(Z+1-2i)(Z+1+2i)}\right] = \frac{-1-2i-3}{-1-2i+1-2i}$$

$$= \frac{-2i-4}{-4i} = \frac{+2i}{+4i} + \frac{4}{4i} = \frac{1}{i} + \frac{1}{2}$$
$$= \frac{1}{2} - i$$

By Residue theorem has hence

$$\oint_C \frac{Z-3}{Z^2+2Z+5} dZ = 2\pi i\left[\frac{1}{2} - i\right] = \pi i + 2\pi$$
$$= \pi(i+2)$$

Evaluation of Some Definite Integrals

–

Evaluate the integrals of the **Type 1:** $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$

Sol: $Z = e^{i\theta}$ $\quad \cos\theta = \frac{1}{2}\left[Z + \frac{1}{Z}\right]$ $\quad$ dZ= $i e^{i\theta} d\theta$

$\frac{1}{Z} = e^{-i\theta}$ $\quad$ sin $\theta = \frac{1}{2i}\left[Z - \frac{1}{Z}\right]$ $\quad \frac{dZ}{iZ} = d\theta$

In the unit circle θ varies from 0 to 2 π

Problems

$$\int_o^{2\pi} \frac{do}{a+b+\cos\theta} = \int_o^{2\pi} \frac{do}{a+b+\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad a > b > o$$

$$Let\ I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} \qquad \text{I} = \int_C \frac{1}{a + \frac{b}{2i}\left(Z - \frac{1}{Z}\right)} \frac{dZ}{iZ}$$

$$Sin\theta = \frac{1}{2i}\left[Z - \frac{1}{Z}\right] \Rightarrow = \int \frac{2i}{2ia + b\left(Z - \frac{1}{Z}\right)} \times \frac{dZ}{iZ} = \int \frac{2iZ}{\left(2iaZ + bZ^2 - b\right)^{iZ}} dZ$$

$$= 2\int \frac{dZ}{bZ^2 + 2iaZ - b} = \frac{2}{b}\int_C \frac{dZ}{Z^2 + \frac{2ia}{b} - 1}$$

$$Let\ f(Z) = \frac{1}{Z^2 + \frac{2ia}{b} - 1}$$

$$= \frac{1}{2}\left[\frac{-2ai}{b} \pm \sqrt{\frac{-4a^2}{b^2} + 4}\right] = \frac{1}{2}\left[\frac{-2ai}{b} \pm 2i\sqrt{\frac{a^2b^2}{b^2}}\right]$$

$$= +\frac{i}{b}\left[-a \pm \sqrt{a^2-b^2}\right]$$

The poles of f(Z) and $\alpha = \frac{i}{b}\left[-a+\sqrt{a^2-b^2}\right], \beta = \frac{i}{b}\left[-a-\sqrt{a^2-b^2}\right]$

Where $\alpha + \beta = \frac{-2ai}{b}, \alpha\beta = -1$

Sin a>b>o | β |>,

Also $|\alpha\beta| = |\frac{i^2}{b^2}|\left[a^2-\left(a^2-b^2\right)\right] = 1$

$\therefore |\alpha| < 1$, Z = α lies inside the unit circle.

$\therefore \phi Z = \alpha$ line inside the unit circle?

$$\therefore r_1 = \underset{Z=\alpha}{\operatorname{Res}} f(Z) = \underset{Z \to \alpha}{Lim}(Z-\alpha) \times \frac{1}{(Z-\alpha)(Z-\beta)} = \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\sqrt{\left(\alpha^2+p\right)^2 - 4\alpha\beta}} = \frac{1}{\sqrt{\frac{-4a^2}{b^2}+4}} = \frac{1}{2i\frac{\sqrt{a^2-b^2}}{b}}$$

$$r_1 = \frac{b}{2i\sqrt{a^2-b^2}}$$

By couchy's residue theorem $I = \frac{2}{b} \times 2\pi i r_1 = \frac{4\pi}{b} i \times \frac{b}{i\sqrt{a^2-b^2}}$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Similarly we can show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Type –II

Integrals of the type $\int_{-\infty}^{\infty} f(x)dx \Rightarrow$ Integration around a semi circle.

If $f(z)$ is analytic in the upper half of the circle z-plane except at 'a' finite number of poles in it and having no pole on the real axis and if further $zf(z) \to 0$ as $z \to \infty$ then by contour integration $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \in R^+$ Where R^+ represents the sum of residues at the poles in the upper half the z-plane.

$$\int_c f(z)dz = 2\pi i \in R^+$$

$$\int_{-R}^{R} f(z)dz + \int_{\Gamma} f(z)dz = 2\pi i \in R^+$$

Now as z f(z)$\to 0$ as $z \to \infty$, $\quad \therefore \int_{\Gamma} f(z)dz = 0$

$\therefore As\ z \to \infty,\ R \to \infty\ and\ along\ real\ axis\ z = x$

$\therefore$ from (1) and (2)

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^{+}$$

Let $f(z) = \dfrac{P(z)}{Q(z)}$ where P (z) and Q(z) are polynomials such that

(i) Q(z) = 0 has no real roots and

(ii) Deg P(z) is at least two less than the deg Q(z) so that z f(z)$\to$0 as z$\to \infty$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi \sum R^{+}$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

Solution Given $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4}$

The corresponding complex integral is

$$\int_C f(z)\,dz = \int_C \frac{dz}{1+z^4}$$

Here, $f(z) = 1/1+z^4 = \dfrac{\phi(z)}{\psi(z)}$ the poles of $f(z)$ are given by $1 + z^4 = 0$

$\therefore$ The poles of $f(z)$, therefore, are 4th roots of unity. They are

$$e^{\frac{i\pi}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$$

which are all simple poles.

Here C is the contour consisting of the real axis $[-R, R]$ and the upper semicircle $|z| = R$ donated by C_R.

$$\therefore \int_C f(z)dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)dz \qquad \text{(i)}$$

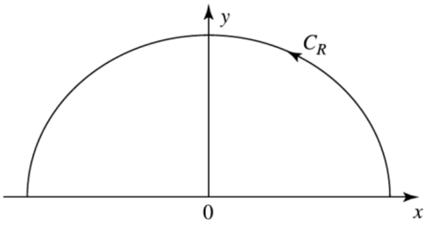


Fig. 6.6

$\therefore$ The poles which lies inside the contour C_R are $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$.

Hence we have

$$\text{Res}\left[f(z), z = e^{\frac{i\pi}{4}}\right] = \frac{\phi\left(e^{\frac{i\pi}{4}}\right)}{\psi'\left(e^{\frac{i\pi}{4}}\right)}$$

where $\phi(z) = 1, \psi'(z) = 4z^3$

$$\therefore \quad \text{Res}\left[f(z), z = e^{\frac{i\pi}{4}}\right] = \frac{1}{4e^{\frac{i3\pi}{4}}} = \frac{1}{4}e^{\frac{-i3\pi}{4}}$$

$$\text{Similarly, Res}\left[f(z), e^{\frac{i3\pi}{4}}\right] = \frac{\phi\left(e^{\frac{i3\pi}{4}}\right)}{\psi'\left(e^{\frac{i3\pi}{4}}\right)} = \frac{e^{\frac{-i9\pi}{4}}}{4}$$

Hence, by residue theorem, we have $\int_C f(z)dz = 2\pi i$ (sum of the residues at the poles).

$$= \frac{2\pi i}{4}\left[\left(e^{\frac{-i3\pi}{4}} + e^{\frac{-i9\pi}{4}}\right)\right]$$

$$= \frac{\pi i}{2}\left[\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + \left\{\cos\left(\frac{9\pi}{4}\right) - i\sin\left(\frac{9\pi}{4}\right)\right\}\right]$$

$$= \frac{\pi i}{4}\left[\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

Thus $\int_C f(z)\,dz = \dfrac{\pi}{\sqrt{2}}$ (ii)

From (i) and (ii), we get

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(z)\,dx + \int_{C_R} f(z)\,dz$$

Now let R tend to ∞ and $\int_{C_R} f(z)\,dz$ tend to zero

$$\text{Hence } \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Type III

Evaluation of the integrals of the form $\int_{-\infty}^{\infty} f(x)\cos mx dx, \int_{-\infty}^{\infty} f(x)\sin mx\,dx$

The above integral can be evaluated by integrating f(z) e^{imx} around the contous "C"

By Residue theorem $\int e^{imz} f(z)dz = 2\pi i\Sigma r^{+}$

$$\int_{-R}^{R} e^{imz} f(z)dz + \int_{\tau} e^{imz} f(z)dz = 2\pi i\Sigma R^{+}$$

As Z f(z) → o as z → ∞ $\Rightarrow \int_{\Pi} e^{imz} f(z)dz = 0$

As $z \to \infty, R \to \infty$ and along the real axis z = x, $\int_{-\infty}^{\infty} e^{imx} f(x)dx = 2\pi i\Sigma R^{+}$

Or

$$\int_{-\infty}^{\infty} f(x)\cos mx dx + i\int_{-\infty}^{\infty} f(x)\sin mx dx = 2\pi i\Sigma R^{+}$$

Equating real and imaginary parts we get the values of given integrals.

Evaluate $\int_{0}^{\infty} \frac{\cos ax}{x^2+1} dx;\ a \geq 0$

Solution Given $\int_0^\infty \frac{\cos ax}{x^2+1} dx.$

We know that $\cos ax$ is the real part of $\exp\{iax\}$. Hence we write the corresponding complex integral as

$$\int_C \frac{e^{iaz}}{z^2+1} dz, \text{ where } f(z) = \frac{\phi(z)}{\psi(z)} = \frac{e^{iaz}}{z^2+1}$$

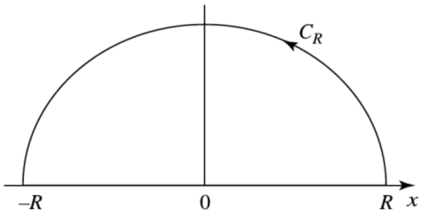


Fig. 6.7

where $\phi(z) = e^{iaz}$, $\psi(z) = z^2 + 1$.

The poles of $f(z)$ are given by

$\psi(z) = z^2 + 1 = 0$, $z = i, -i$ are the simple poles. The contour C consists $-R$ to R real axis and the upper semicircle C_R (Fig. 6.7). Out of the poles $z = i, -i$ only $z = i$ lies in the upper half of z plane.

$$\text{Hence} \quad \operatorname{Res}[f(z), z = i] = \left[\frac{\phi(z)}{\psi'(z)}\right]_{z=i}$$

$$= \left(\frac{e^{iaz}}{2z}\right)_{z=i} = \frac{e^{-a}}{2i}$$

Hence, by residence theorem, we have

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_C f(z)dz = 2\pi i \text{ (Sum of residues at poles)}$$

$$= \frac{2\pi i e^{-a}}{2i}$$

Let R tend to ∞ so that $\int_{C_R} f(z)dz$ goes to zero. Then

$$\int_C f(z)\, dz = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx = \pi e^{-a}$$

$$\therefore \quad 2\int_0^\infty \frac{\cos ax\, dx}{x^2+1} = \pi e^{-a}$$

$$\therefore \quad \int_0^\infty \frac{\cos ax}{x^2+1} dx = \frac{\pi}{2} e^{-a}$$

Type IV

<u>Indenting contours having poles on real axis</u>

If the integrand has a simple pole on real axis we delete it by indenting the contour which consists of drawing a small semi-circle having the pole as centre.

Jordhn's Lemma:- If f(x) → 0 uniformly as Z → ∞ then $\lim_{R \to \infty} \int_\tau e^{imz} f(z)dz = 0, m > 0$

Where τ denotes the semi-circle |Z|=R, here R is taken so large such that all the singularities lie within the semi-circle π and no singularities are lie on its boundary.

Theorem:- If AB is the are $\alpha \le \theta \le \beta$ of the circle |z-a| =2, then if $\lim_{Z \to a} (Z-a) f(Z) = k$ then

$$\lim_{l \to 0} \int_{AB} f(z)dz = i(\beta - \alpha)k$$

Where K = $\operatorname*{Res}_{Z=a} f(z) = 0$ is a pole of f(z)

Theorem:- If AB is the on C $\alpha \le \theta \le \beta$ of the circle |Z|=R and if $\lim_{\theta \to \infty} Zf(Z) = k$ then

$$\lim_{R \to \infty} \int_{AB} f(z)dz = i(\beta - \alpha)k$$

Prob:- Evaluate $\int_0^\infty \frac{\sin mx}{x} dx$, where m>0, hence show that $\int_{-\infty}^\infty \frac{\cos mx}{x} dx = 0$

Solution We shall evaluate these two integrals by complex integration.

(i) Given real definite integral is $\int_0^\infty \frac{\sin mx \, dx}{x}$

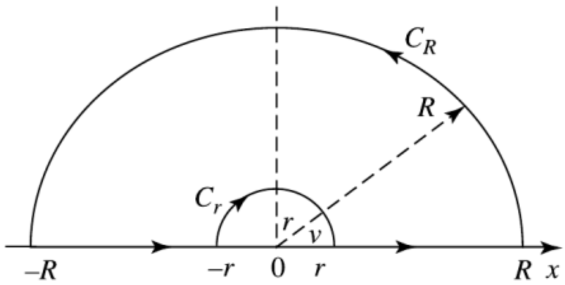


Fig. 6.9

The corresponding complex integral is $\int_C \frac{e^{miz}}{z} dz = \int_C f(z)\, dz$

where C consists of

(i) The real axis from r to R

(ii) The upper half of the circle $C_R : |z| = R$ (Fig. 6.9).

(iii) The real axis $-R$ to $-r$

(iv) The upper half of the circle C_r: $|z| = r$

The only pole which is $z = 0$ is deleted by drawing a small semicircle $C_r, f(z)$ has no singularly inside C.

i.e., we have deleted the pole at $z = 0$ by indentation.

Hence, by Cauchy's theorem we have

$$\int_C f(z)\,dz = \int_r^R f(x)\ dx + \int_{C_R} f(z)\,dz + \int_{-R}^{-r} f(x)\ dx + \int_{C_r} f(z)\,dz \qquad \text{(i)}$$

But $z = re^{i\theta}$ $\quad\therefore\quad \int_{C_r} f(z)\,dz = i\int e^{imr(\cos\theta+\sin\theta)}d\theta$

Now making $R \to \infty$, we observe that $r \to 0$, and

$$\int_{C_R} f(z)\ dz \to 0 \text{ and} \int_{C_r} f(z)\,dz \to -\pi i$$

So that from **(i)** we have

$$\int_C f(z)\,dz = \int_r^R f(x)\,dx + \int_{-R}^{-r} f(x)\,dx$$

$$= \int_0^\infty f(x)\ dx + \int_{-\infty}^0 f(x)\ dx$$

$$= \int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^\infty \frac{e^{imx}}{x}dx = i\pi$$

$\therefore$ equating imaginary parts on both sides, we get

$$\int_{-\infty}^\infty \frac{\sin mx}{x}dx = \pi$$

$$\therefore\quad 2\int_0^\infty \frac{\sin mx}{x}dx = \pi$$

$$\therefore\quad \int_0^\infty \frac{\sin mx}{x}dx = \pi/2$$

Equating real part of both sides of (2) we get

$$\int_{-\infty}^\infty \frac{\cos mx}{x}dx = 0$$