Probability, Statistics and Complex Variables

(25MA302)

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COURSE CONTENTS

UNIT-IV

COMPLEX DIFFERENTIATION

Introduction

Complex analysis traditionally known as the theory of functions of a complex variable is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatory, applied mathematics as well as in physics, including hydrodynamics and thermodynamics and also in engineering fields such as nuclear, aerospace, mechanical and electrical engineering. We have used complex numbers in a number of situations, and in general, complex analysis is used in many areas of electrical engineering including Circuit theory (impedance, transfer functions, *etc*.),Electromagnetism (time-harmonic fields),Electrostatics (Solutions to Laplace's equation), and Electromagnetic.

Functions of Complex Variables:- If $z=x+iy$ and $w=u+iv$ are two complex variables and w be related to z in such a way that for each value of z in complex plane, there corresponding one or more values of w, then w is said to be a function of z and we write $w=f(z)=u+iv$.

Hence in general we write $w=f(z)=u(x,y)+iv(x,y)$ where u and v are called real and imaginary parts of $w=f(z)$.

Ex: $f(z)=z^2$

$=(x+iy)^2$

$f(z)=x^2-y^2+2xiy$

$u(x,y)=x^2-y^2$ & $v(x,y)=2xy$

Limits of a Complex Function:- In complex plain if a variable z of a function $w=f(z)$ approaches to a point z_0 irrespective of direction, then the function is said to be tent to limit (l). Mathematically limit of a function is written as

$$\lim_{z\to z_0} f(z)=l$$

Continuity of a Complex Function:- A function $f(z)$ is said to be continuous at a point $z=z_0$ if

Case (i):

$\lim_{z\to z_0} f(z)$ exists

Case (ii):

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Derivative of f(z):- A function $f(z)$ is said to be differentiable at a point $z = z_0$ if

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ tends to unique limit as $z \to z_0$ along any path. This unique limit is called derivative of $f(z)$ at $z = z_0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

It can also be written as

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0)$$

Analytic function [entire]:- Let a function $f(z)$ is derivable / differentiable at every point of z in an ε neighborhood of z_0 (i.e.) $f'(z)$ exists for all z. such that $|z - z_0| < \varepsilon$ where $\varepsilon > 0$ then $f(z)$ is said to be analytic at z_0.

$\Rightarrow$ it is also called as Holomorphic / regular function.

Properties of Analytic functions

1. If $f(z)$ and $g(z)$ are analytic functions then $f \pm g,\ f.g,\ \frac{f}{g}$ are also analytic functions [if $g(z) \neq 0$]

2. Analytic function of analytic function is also analytic.

3. An entire function of an entire function is also entire.

4. Derivative of analytic function is itself analytic function.

Harmonic function – Laplace equation

A function $f(x, y)$ is said to be harmonic function, if

i) It possesses second order continuous partial derivatives.
ii) It satisfies Laplace equation

(or)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Cauchy – Riemann equations [C – R equations]

The necessary and sufficient conditions for the functions $w = f(z) = u(x, y) + iv(x, y)$ to be analytic in the given region R are

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region of R.
2. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

   The equations in point 2 are called C – R equations

Polar form of C – R equations $u_r = \frac{1}{r} v_\theta,\ v_r = -\frac{1}{r} u_\theta$

Let $x = r\cos\theta$ & $y = r\sin\theta$

$z = x + iy$

$z = r\cos\theta + i r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$

$f(z) = u + iv$

$f(re^{i\theta}) = u + iv$ ⟶ (1)

Differentiating equation (1) w.r.t. 'r' partially

$f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$ ⟶ (2)

Again differentiating equation (1) w.r.t. 'θ' partially

$f'(re^{i\theta})rie^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$ ⟶ (3)

From equation (2) and (3)

$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) ri$

$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ri\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$

Now equating the real and imaginary parts on both sides

$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ & $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$

Hence $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta};\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

There are called C – R equations in polar form

Problem:-

1) If $w = \log z$ then find $\frac{dw}{dz}$ and determine where w is non – analytic?

Sol:-

$w = \log z$

$w = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$ $[\because \log e = 1]$

$= \log r + i\theta$

taking $x = r\cos\theta, y = r\sin\theta$

where $z = x + iy$

$z = r\cos\theta + ir\sin\theta$

$z = re^{i\theta}$

where $r = \sqrt{x^2 + y^2}$

$\& \theta = \tan^{-1}\left[\frac{y}{x}\right]$

$w = \frac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left[\frac{y}{x}\right]$

We know that $w = f(z) = u + iv$

$\therefore u = \frac{1}{2}\log\left(x^2 + y^2\right)$ & $v = \tan^{-1}\left[\frac{y}{x}\right]$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2 + y^2}(2x) \Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2 + y^2}(2y) \Rightarrow \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$

$\frac{\partial v}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2 + y^2}\left(\frac{-y}{x^2}\right) \Rightarrow \frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}$

$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x^2}{x^2 + y^2}\left(\frac{1}{x}\right) \Rightarrow \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$

$\therefore u_x = v_y$ & $u_y = -v_x$

The C – R equations are satisfied and also partial derivatives are continuous except at z = 0

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$

$= \frac{x - iy}{(x + iy)(x - iy)}$

$f'(z) = \frac{1}{x + iy} = \frac{1}{z}\ [z \neq 0]$

$\therefore$ The function is non – analytic (or) non – differentiable at z = 0

2) Find whether $f(z) = \sin x \sin y - i\cos x \cos y$ is analytic or not?

Sol:-

Given that $u = \sin x \sin y$ & $v = -\cos x \cos y$

$\frac{\partial u}{\partial x} = \cos x \sin y$

$\frac{\partial u}{\partial y} = \sin x \cos y$

$\frac{\partial v}{\partial x} = \sin x \cos y$

$$\frac{\partial v}{\partial y} = \cos x \sin y$$

$u_x = v_y$ & $u_y \neq -v_x$

$\therefore$ It is non – analytic

3) Find the values of k such that $f(z) = e^x(\cos ky + i \sin ky)$ is analytic

Sol:-

Given that $f(z) = e^x(\cos ky + i \sin ky)$

$u = e^x \cos ky$ & $v = e^x \sin ky$

$u_x = e^x \cos ky$; $v_x = e^x \sin ky$

$u_y = -ke^x \sin ky$; $v_y = ke^x \cos ky$

Given that the function is analytic

$\therefore$ C – R equations are satisfied

i.e. $u_x = v_y$	$u_y = -v_x$
$e^x \cos ky = ke^x \cos ky$	$-ke^x \sin ky = -e^x \sin ky$

There equations are true for $k = 1$

4) Show that $f(z) = \sin z$ is analytic at everywhere in complex plane and find $f'(z)$

Sol:-

$f(z) = \sin z$

$f(z) = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$f(z) = u + iv = \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y$ & $v = \cos x \sinh y$

$u_x = \cos x \cosh y$; $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$; $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ & $u_y = -v_x$

The C – R equations are satisfied. Hence it is analytic

$f'(z) = u_x + iv_x$

$= \cos x \cosh y - i \sin x \sinh y$

$f'(z) = \cos(x + iy) \Rightarrow f'(z) = \cos z$

5) Show that $f(z) = \cos z$ is analytic at everywhere in complex plane and find $f'(z)$

Sol:-

$f(z) = \cos z$

$f(z) = \cos(x + iy)$

$$= \cos x \cos iy - \sin x \sin iy$$

$$f(z) = u + iv = \cos x \cosh y - i \sin x \sinh y$$

$$\therefore u = \cos x \cosh y \ \& \ v = -\sin x \sinh y$$

$$u_x = -\sin x \cosh y; \ v_x = -\cos x \sinh y$$

$$u_y = \cos x \sinh y; \ v_y = -\sin x \cosh y$$

$$\therefore u_x = v_y \ \& \ u_y = -v_x$$

The C – R equations are satisfied. Hence it is analytic

$$f'(z) = u_x + iv_x$$

$$= -\sin x \cosh y - i \cos x \sinh y$$

$$= -(\sin x \cosh y + i \cos x \sinh y)$$

$$f'(z) = -\sin(x + iy) \Rightarrow f'(z) = -\sin z$$

6) If $f(z) = \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\left[\frac{px}{y}\right]$ is analytic then find p

Sol:-

Given $f(z) = \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\left[\frac{px}{y}\right]$

$$\therefore u = \frac{1}{2}\log(x^2 + y^2) \ \& \ v = \tan^{-1}\left[\frac{px}{y}\right]$$

$$u_x = \frac{x}{x^2 + y^2}; \ v_x = \frac{1}{1 + \frac{p^2x^2}{y^2}}\left[\frac{p}{y}\right] = \frac{py}{y^2 + p^2x^2}$$

$$u_y = \frac{y}{x^2 + y^2}; \ v_y = \frac{1}{1 + \frac{p^2x^2}{y^2}}\left[\frac{-px}{y^2}\right] = \frac{-px}{y^2 + p^2x^2}$$

Given that $f(z)$ is analytic. Therefore C – R equations are satisfied.

i.e. $u_x = v_y$	$u_y = -v_x$
$\frac{x}{x^2 + y^2} = \frac{-px}{y^2 + p^2x^2}$	$\frac{y}{x^2 + y^2} = \frac{py}{y^2 + p^2x^2}$

There equations are true for $p = -1$

7) Find where the functions

i) $w = \frac{1}{z}$ **ii)** $w = \frac{z}{z-1}$ **iii)** $w = z^3 - 4z + 1$ **iv)** $w = \frac{z+2}{z(z^2+1)}$

v) $z = e^{-v}(\cos u + i \sin u)$ fails to be analytic

Sol:-

i) Given $w = \frac{1}{z}$

$$\frac{dw}{dz}=\frac{-1}{z^2}\ [z\neq 0]$$

The function if $z=0$; $\frac{dw}{dz}$ does not exists

$\therefore$ w is analytic everywhere except at $z=0$.

[Here $z=0$ is called singular point of $f(z)$]

ii) Given $w=\frac{z}{z-1}$

$$\frac{dw}{dz}=\frac{(z-1)-z}{(z-1)^2}$$

$$\frac{dw}{dz}=\frac{-1}{(z-1)^2}\ [z\neq 1]$$

The function if $z=1$; $\frac{dw}{dz}$ does not exists

$\therefore$ w is analytic everywhere except at $z=1$.

iii) Given $w=z^3-4z+1$

$$\frac{dw}{dz}=3z^2-4$$

The function is analytic everywhere in the complex plane.

iv) Given $w=\frac{z+2}{z(z^2+1)}$

$$\frac{dw}{dz}=\frac{z(z^2+1)-(z+2)(3z^2+1)}{z^2(z^2+1)^2}$$

$$\frac{dw}{dz}=\frac{z^3+z-3z^3-z-6z^2-2}{z^2(z^2+1)^2}$$

$$\frac{dw}{dz}=\frac{-2(z^3+3z^2+1)}{z^2(z^2+1)^2}\ [z\neq 0, z^2+1\neq 0]$$

The function if $z=0,\pm i$ then $\frac{dw}{dz}$ does not exists

$\therefore$ w is analytic everywhere except at $z=0,i,-i$.

v) Given $z=e^{-v}(\cos u+i\sin u)$

$$z=e^{-v}e^{iu}=e^{iu-v}$$

$$z=e^{i(u+iv)}$$

$$z=e^{iw}\Rightarrow iw=\log z$$

$$w=\frac{1}{i}\log z$$

$$\frac{dw}{dz}=\frac{1}{iz}\ [z\neq 0]$$

The function if $z=0;\ \dfrac{dw}{dz}$ does not exists

$\therefore$ w is analytic everywhere except at $z=0$.

8) Prove that $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\left|\text{Real } f(z)\right|^2 = 2\left|f'(z)\right|^2$ where w is analytic function

Sol:-

$f(z)=u+iv$ $\qquad$ $f'(z)=u_x+iv_x$

Real $f(z)=u$ $\qquad$ $\left|f'(z)\right|=\sqrt{u_x{}^2+v_x{}^2}$

$\left|\text{Real } f(z)\right|=\sqrt{u^2}$ $\qquad$ $\left|f'(z)\right|^2=u_x{}^2+v_x{}^2$

$\left|\text{Real } f(z)\right|^2=u^2$

$\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)u^2$

$\dfrac{\partial}{\partial x}u^2=2u\dfrac{\partial u}{\partial x}$

$\dfrac{\partial^2}{\partial x^2}u^2=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u^2}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(2u\dfrac{\partial u}{\partial x}\right)=2\left[u\dfrac{\partial^2 u}{\partial x^2}+\left(\dfrac{\partial u}{\partial x}\right)^2\right]$

$\therefore \dfrac{\partial^2}{\partial x^2}u^2=2\left[u\dfrac{\partial^2 u}{\partial x^2}+\left(\dfrac{\partial u}{\partial x}\right)^2\right]$

Similarly $\dfrac{\partial^2}{\partial y^2}u^2=2\left[u\dfrac{\partial^2 u}{\partial y^2}+\left(\dfrac{\partial u}{\partial y}\right)^2\right]$

Adding the above 2 equations

$\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)u^2=2\left[u\dfrac{\partial^2 u}{\partial x^2}+\left(\dfrac{\partial u}{\partial x}\right)^2\right]+2\left[u\dfrac{\partial^2 u}{\partial y^2}+\left(\dfrac{\partial u}{\partial y}\right)^2\right]$

$=2\left[\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2+u\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)\right]$

Given that $f(z)$ is analytic.

$u_x=v_y$ & $u_y=-v_x$ and $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ [harmonic condition]

$=2\left[\left(\dfrac{\partial u}{\partial x}\right)^2+\left(-\dfrac{\partial v}{\partial x}\right)^2\right]$

$=2\left[u_x{}^2+v_x{}^2\right]$

$\therefore\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\left|\text{Real } f(z)\right|^2=2\left|f'(z)\right|^2$

9) If $f(z)$ is analytic function of z then Prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

where w is analytic function

Sol:-

$f(z)=u+iv$ $\qquad$ $f'(z)=u_x+iv_x$

$|f(z)|=\sqrt{u^2+v^2}$ $\qquad$ $|f'(z)|=\sqrt{u_x{}^2+v_x{}^2}$

$|f(z)|^2=u^2+v^2$ $\qquad$ $|f'(z)|^2=u_x{}^2+v_x{}^2$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(u^2+v^2\right)$$

$$\frac{\partial^2}{\partial x^2}\left(u^2+v^2\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(u^2+v^2\right)\right)$$

$$=\frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)$$

$$=2\left(\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right)\right)$$

$$\frac{\partial^2}{\partial x^2}\left(u^2+v^2\right)=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly $\frac{\partial^2}{\partial y^2}\left(u^2+v^2\right)=2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$

Adding the above 2 equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(u^2+v^2\right)=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]+$$

$$2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]+2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]$$

Given that $f(z)$ is analytic.

$u_x=v_y$ & $u_y=-v_x$ and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ & $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$ [harmonic functions]

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(-\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 2\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2 \right]$$

$$= 4\left[u_x{}^2 + v_x{}^2 \right]$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left|f(z)\right|^2 = 4\left|f'(z)\right|^2$$

10) Show that $f(z) = \frac{\bar{z}}{z}$ is not continuous at $z = 0$

Sol:-

Given that $f(z) = \frac{\bar{z}}{z}$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} \frac{x - iy}{x + iy}$$

i) Along X – axis; $y = 0$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{x}{x} = 1$$

ii) Along Y – axis; $x = 0$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{-iy}{iy} = -1$$

Limit is not unique; $\therefore$ Limit does not exists

Hence $f(z)$ is not continuous at $z = 0$

11) Show that the function $f(z) = z^n$, where n is the +ve integer. This function is Differentiable for all values of z

Sol:-

From the definition of derivative, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^n} + nz^{n-1}\Delta z + \frac{n(n+1)}{2} z^{n-2}(\Delta z)^2 + ... - \cancel{z^n}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[nz^{n-1} + \frac{n(n+1)}{2} z^{n-2}\Delta z + ... \right]$$

$$\therefore f'(z) = nz^{n-1}$$

$\therefore f'(z)$ can exists for all values of z

12) If $f(z) = |z|^2$ is a function which is continuous for all z but it is not derivable at any $z \neq 0$?

Sol:-

Given that $f(z)=|z|^2=z\bar{z}$

and $f(z_0)=|z_0|^2=z_0\bar{z_0}$

$$\lim_{z\to z_0} f(z)=\lim_{z\to 0} z_0\bar{z_0}$$

$$\therefore \lim_{z\to z_0} f(z)=f(z_0)$$

$\therefore$ The function is continuous for all values of z

From the definition of derivative, we have

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$f(z)=|z|^2=z\bar{z}$$

$$f(z+\Delta z)=|z+\Delta z|^2=(z+\Delta z)(\bar{z}+\Delta\bar{z})$$

$$f(z+\Delta z)=z\bar{z}+z\Delta\bar{z}+\Delta z\bar{z}+\Delta z\Delta\bar{z}$$

$$f'(z)=\lim_{\Delta z\to 0}\frac{\cancel{z\bar{z}}+z\Delta\bar{z}+\Delta z\bar{z}+\Delta z\Delta\bar{z}-\cancel{z\bar{z}}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{z\Delta\bar{z}+\Delta z\bar{z}+\Delta z\Delta\bar{z}}{\Delta z}$$

i) Along X – axis; $\Delta y=0$

$$=\lim_{\Delta x\to 0}\frac{z\Delta x+\bar{z}\Delta x+\Delta x\Delta x}{\Delta x}$$

$$=\lim_{\Delta x\to 0} z+\bar{z}+\Delta x$$

$$=z+\bar{z} \longrightarrow (1)$$

ii) Along Y – axis; $\Delta x=0$

$$=\lim_{\Delta y\to 0}\frac{z(-i\Delta y)+\bar{z}(i\Delta y)+(i\Delta y)(-i\Delta y)}{i\Delta y}$$

$$=\lim_{\Delta y\to 0} -z+\bar{z}-i\Delta y$$

$$=-z+\bar{z} \longrightarrow (2)$$

From (1) and (2)

$$z+\cancel{\bar{z}}=-z+\cancel{\bar{z}}$$

$$2z=0\Rightarrow z=0 \qquad [\text{But } z\neq 0]$$

$\therefore f'(z)$ does not exists even though $f(z)=|z|^2$ is continuous

13) Show that $f(z)=\bar{z}$ is not derivable anywhere

Sol:-

From the definition of derivative, we have

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$f'(z)=\lim_{\Delta z\to 0}\frac{\cancel{\bar{z}}+\Delta\bar{z}+\cancel{-\bar{z}}}{\Delta z}\Rightarrow f'(z)=\lim_{\Delta z\to 0}\frac{\Delta\bar{z}}{\Delta z}$$

i) Along X – axis; $\Delta y=0$

$$f'(z)=\lim_{\Delta x\to 0}\frac{\cancel{\Delta x}}{\cancel{\Delta x}}=1$$

ii) Along Y – axis; $\Delta x=0$

$$f'(z)=\lim_{\Delta y\to 0}\frac{-i\Delta y}{i\Delta y}=-1$$

$\therefore$ The limits are not unique

Hence the function $f(z)$ is not differentiable.

14) Prove that the function $f(z)$ defined by $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$ $[z\neq 0]$

is continuous and C – R equations are satisfied at the origin, yet $f'(z)$ does not exists.

Sol:-

Case (i) Along X – axis; $y=0$

$$\lim_{z\to 0}f(z)=\lim_{x\to 0}\frac{\cancel{x^3}(1+i)}{\cancel{x^2}}=\lim_{x\to 0}x(1+i)=0$$

Case ii) Along Y – axis; $x=0$

$$\lim_{z\to 0}f(z)=\lim_{y\to 0}\frac{-\cancel{y^3}(1-i)}{\cancel{y^2}}=\lim_{y\to 0}-y(1-i)=0$$

Case iii) Along the line $y=mx$

$$\lim_{z\to 0}f(z)=\lim_{x\to 0}\frac{x^3(1+i)-m^3x^3(1-i)}{x^2+m^2x^2}$$

$$\lim_{z\to 0}f(z)=\lim_{x\to 0}\frac{x(1+i)-m^3x(1-i)}{1+m^2}=0$$

Hence $\lim_{z\to 0}f(z)=0=f(0)$

$\therefore$ $f(z)$ is continuous at the origin

$\rightarrow$ Given that $f(z)=u+iv$

$$u=\frac{x^3-y^3}{x^2+y^2},\ v=\frac{x^3+y^3}{x^2+y^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{at(0,0)}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{x^3/x^2}{x}=1$$

$$\left(\frac{\partial u}{\partial y}\right)_{at(0,0)} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{-y^3/y^2}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{at(0,0)} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{x^3/x^2}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{at(0,0)} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{y^3/y^2}{y} = 1$$

Hence the C – R equations are satisfied at the origin

$\rightarrow$From the definition of derivative, we have

$$f'(z) = \lim_{z\to 0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to 0}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)} \qquad \left[\because f(z_0)=f(0),\ \therefore z_0=0\right]$$

Along $y = mx$

$$= \lim_{x\to 0}\frac{x^3(1+i)-m^3x^3(1-i)}{(x^2+m^2x^2)(x+ix)}$$

$$= \lim_{x\to 0}\frac{(1+i)-m^3(1-i)}{(1+m^2)(1+i)} = \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

This limit value is different for different values of 'm'. So $f'(z)$ is not unique at the origin i.e. $f'(0)$ doesn't exists

Hence $f(z)$ is not analytic at the origin even though it is continuous and satisfying the C – R equations at the origin

15) Show that both real and imaginary parts of an analytic function satisfy the Laplace equation

Sol:-

Given that the function is analytic. Therefore C – R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow (1) \qquad \& \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (2)$$

Differentiating equation (1) w.r.t. 'x'

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

Differentiating equation (1) w.r.t. 'y'

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \qquad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right]$$

Differentiating equation (1) w.r.t. 'y'

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$$

Differentiating equation (1) w.r.t. 'x'

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

Subtracting equations then

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$$

$\therefore$ u & v satisfies Laplace equation then u & v said to be Harmonic functions

16) Every Analytic function $f(z) = u + iv$ defines 2 families of curves.
$u(x,y) = c_1$ & $v(x,y) = c_2$ forming an orthogonal system

Sol:-

Given $u(x,y) = c_1$

Differentiating w.r.t. 'x'

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$$

& $v(x,y) = c_2$

Differentiating w.r.t. 'x'

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dv}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = m_2 \qquad \left[\because u_x = v_y,\ u_y = -v_x\right]$$

$$m_1 \times m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1$$

Hence the curves $u(x,y) = c_1$ & $v(x,y) = c_2$ cut each other orthogonally i.e. they form an orthogonal system

17) If $f(z) = u + iv = \frac{1}{z}$ then show that the curves $u(x,y) = c_1$ & $v(x,y) = c_2$ intersect orthogonally

Sol:-

Given that $f(z)=\frac{1}{z}=u+iv$

$$f(z)=\frac{1}{z}=\frac{1}{x+iy}\times\frac{x-iy}{x-iy}$$

$$f(z)=\frac{x-iy}{x^2+y^2}$$

$$u(x,y)=\frac{x}{x^2+y^2},\ v(x,y)=\frac{-y}{x^2+y^2}$$

$$u(x,y)=c_1$$

$$\frac{x}{x^2+y^2}=c_1$$

Differentiating w.r.t. 'x'

$$\frac{\left(x^2+y^2\right)-x\left(2x+2y\frac{dy}{dx}\right)}{\left(x^2+y^2\right)^2}=0$$

$$x^2+y^2-2x^2-2xy\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}=m_1 \longrightarrow (1)$$

& $v(x,y)=c_2$

$$\frac{-y}{x^2+y^2}=c_2$$

Differentiating w.r.t. 'x'

$$\frac{\left(x^2+y^2\right)\left(-\frac{dy}{dx}\right)-y\left(2y\frac{dy}{dx}+2x\right)}{\left(x^2+y^2\right)^2}=0$$

$$\left(x^2+y^2\right)\left(-\frac{dy}{dx}\right)+2xy+2y^2\frac{dy}{dx}=0$$

$$\left(y^2-x^2\right)\left(\frac{dy}{dx}\right)=-2xy$$

$$\frac{dy}{dx}=\frac{-2xy}{y^2-x^2}=m_2 \longrightarrow (2)$$

$$m_1\times m_2=\frac{\cancel{y^2-x^2}}{\cancel{2xy}}\times\frac{\cancel{-2xy}}{\cancel{y^2-x^2}}=-1$$

$\therefore$ It forms orthogonal / the curves intersect orthogonal

18) $w=\exp\left(z^2\right)$ then find u & v and prove that the curves $u(x,y)=c_1$ & $v(x,y)=c_2$ where are constants and cuts orthogonally

Sol:-

Given that $w=\exp\left(z^2\right)$

$$w = e^{z^2} = e^{(x+iy)^2}$$

$$w = e^{x^2-y^2+2ixy}$$

$$w = e^{x^2-y^2}e^{2ixy}$$

$$w = u + iv = e^{x^2-y^2}\left(\cos 2xy + i\sin 2xy\right)$$

$$u(x,y) = e^{x^2-y^2}\cos 2xy,\ v(x,y) = e^{x^2-y^2}\sin 2xy$$

We know that Analytic function of Analytic function is Analytic.

Since e^z is Analytic & z^2 is also Analytic

$\therefore e^{z^2}$ is also Analytic

We know that Analytic function $w = u + iv$ where $u(x,y) = c_1$ & $v(x,y) = c_2$ forms an orthogonal function

19) Show that $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally

Sol:-

Given $r^n = \alpha \sec n\theta$ & $r^n = \beta \operatorname{cosec} n\theta$

$$r^n \cos n\theta = \alpha \ \&\ r^n \sin n\theta = \beta$$

$$w = \alpha + i\beta$$

$$w = r^n \cos n\theta + r^n \sin n\theta$$

$$= r^n\left(\cos n\theta + \sin n\theta\right)$$

$$= r^n e^{in\theta} = \left(re^{i\theta}\right)^n = z^n$$

$$\therefore w = z^n$$

We know that z^n is Analytic function. Hence the family of curves form an orthogonal system.

20) Prove that z^n [where n is positive integer] is Analytic and find its derivative.

Sol:-

$$f(z) = z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

$$f(z) = r^n\left(\cos n\theta + i\sin n\theta\right)$$

$u(r,\theta) = r^n \cos n\theta$	$v(r,\theta) = r^n \sin n\theta$
$u_r = nr^{n-1}\cos n\theta$	$v_r = nr^{n-1}\sin n\theta$
$u_\theta = -nr^n \sin n\theta$	$v_\theta = nr^n \cos n\theta$

$$u_r = \frac{1}{r}v_\theta = \frac{1}{r}\left(nr^n\cos n\theta\right) = nr^{n-1}\cos n\theta$$

$$v_r = -\frac{1}{r}u_\theta = -\frac{1}{r}\left(-nr^n\sin n\theta\right) = nr^{n-1}\sin n\theta$$

$$\therefore u_r = \frac{1}{r} v_\theta \quad \& \quad v_r = -\frac{1}{r} u_\theta$$

$\therefore$ The C – R equations are satisfied. Hence the function is Analytic.

$$f'(z) = \frac{1}{e^{i\theta}}(u_r + iv_r)$$

$$f'(z) = \frac{1}{e^{i\theta}}\left(nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta\right)$$

$$= \frac{nr^{n-1}}{e^{i\theta}}(\cos n\theta + i\sin n\theta)$$

$$= nr^{n-1}e^{in\theta}.e^{-i\theta}$$

$$f'(z) = nr^{n-1}e^{i\theta(n-1)}$$

$$f'(z) = n\left(re^{i\theta}\right)^{n-1} = nz^{n-1}$$

21) If $u(x,y)$ and $v(x,y)$ are harmonic functions in region R then prove that $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an Analytic function

Practice Problem

1) Show that $f(z) = z^3$ is Analytic for all z

   [put $z = x + iy$ find $u(x,y)$ and $v(x,y)$, check C – R equations are satisfied or not]
2) Show that z^2 is Analytic for all z
3) Show that $f(z) = z + \bar{z}$ is not Analytic anywhere in complex plane
4) Show that $f(z) = z + 2\bar{z}$ is not Analytic anywhere in complex plane
5) $\frac{x - iy}{x^2 + y^2}$ is Analytic or not
6) Find all values of 'k' such that $f(z) = e^x(\cos ky + i\sin ky)$ is Analytic

   [Ans: k = 1]
7) Show that $(i)\ f(z) = e^z$ $(ii)\ f(z) = e^{-z}$ is Analytic everywhere in complex plane and find $f'(z)$

   [Hint: $w = f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos iy + i\sin iy)$ & $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$]
8) Prove that the function $f(z) = \bar{z}$ is not Analytic at any point
9) Determine whether the function $2xy + i(x^2 - y^2)$ is Analytic or not
10) Show that the function $f(z) = \sqrt{|xy|}$ is not Analytic at the origin.

    [Hint: $u = \sqrt{|xy|}$ & $v = 0$]

22) Show that xy^2 can't be real part of an Analytic function

Sol:-

$$u = xy^2$$

$$\frac{\partial u}{\partial x} = u_x = y^2$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = 0$$

$$\frac{\partial u}{\partial y} = u_y = 2xy$$

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = 2x$$

$$u_{xx} + u_{yy} = 0 + 2x = 2x \neq 0$$

$\therefore$ Harmonic function is not satisfied

Hence xy^2 can't be real part of an analytic function

23) If u is Harmonic function then show that $w = u^2$ is not a Harmonic function unless u is constant

Sol:-

Given that u is harmonic function

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

& Given $w = u^2$

$$\frac{\partial w}{\partial x} = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 w}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

Similarly $\frac{\partial^2 w}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

Given that u is harmonic function

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

If u is constant

$$\frac{\partial u}{\partial x} = 0 \quad \& \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

$\therefore w$ is harmonic when u is constant

Hence $w=u^2$ is analytic function

$\Rightarrow w=u^2$ is not a Harmonic function unless u is constant

24) Verify $u=e^x\cos y$ is harmonic. Determine it's harmonic conjugate $v(x,y)$ and find $f(z)=u+iv$

Sol:-

Given $u=e^x\cos y$

$u_x=e^x\cos y$

$u_{xx}=e^x\cos y$

$u_y=-e^x\sin y$

$u_{yy}=-e^x\cos y$

$u_{xx}+u_{yy}=e^x\cos y-e^x\cos y=0$

$\therefore$ It is harmonic function

Given that $u_x=e^x\cos y$ & $u_y=-e^x\sin y$

From C – R equations $u_x=v_y$ & $u_y=-v_x$

$v_y=u_x=e^x\cos y$

$v_y=e^x\cos y$

Integrating w.r.t. 'y'

$v=\int e^x\cos y dy+c(x)$

$v=e^x\sin y+c(x)$

Differentiating w.r.t. 'x'

$v_x=e^x\sin y+c'(x)$

From C – R equations $u_y=-v_x$

$-u_y=v_x=e^x\sin y+c'(x)$

$e^x\sin y=e^x\sin y+c'(x)$

$c'(x)=0$

$c(x)$ is constant

$v(x,y)=e^x\sin y+c$

$f(z)=u+iv=e^x\cos y+ie^x\sin y+c$

$=e^x(\cos y+i\sin y)+c$

$=e^xe^{iy}+c$

$=e^{x+iy}+c$

$f(z)=u+iv=e^z+c$

25) Find the orthogonal trajectories of the family of curves $r^2\cos 2\theta=c$

Sol:-

Given $r^2\cos 2\theta = c$

$u = r^2\cos 2\theta$

$u_r = 2r\cos 2\theta \quad \& \quad u_\theta = -2r^2\sin 2\theta$

From C – R equations of polar form

$$u_r = \frac{1}{r}v_\theta \quad \& \quad v_r = -\frac{1}{r}u_\theta$$

$$u_r = \frac{1}{r}v_\theta \Rightarrow v_\theta = ru_r$$

$v_\theta = 2r^2\cos 2\theta$

Integrating w.r.t. 'θ'

$$v = \int 2r^2\cos 2\theta d\theta + c(r)$$

$$v = \frac{2r^2}{2}\sin 2\theta + c(r)$$

$$v = r^2\sin 2\theta + c(r)$$

Differentiating w.r.t. 'r'

$$v_r = 2r\sin 2\theta + c'(r)$$

$$-\frac{1}{r}u_\theta = 2r\sin 2\theta + c'(r) \qquad \left[\because v_r = -\frac{1}{r}u_\theta\right]$$

$$2r\sin 2\theta = 2r\sin 2\theta + c'(r)$$

$c'(r) = 0 \Rightarrow c(r) = \text{constant}$

$v = r^2\sin 2\theta$

Practice Problem:-

1) Prove that if $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ both u and v satisfy Laplace equation, but u+iv is not regular (analytic) function of z.
   [Hint: Prove $\nabla^2 u = 0, \nabla^2 v = 0$ check C – R equations for analytic]
2) Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic
3) Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find its conjugate harmonic [Ans: $v = 2xy + x + c$]
4) Find 'k' such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic and find its conjugate
   [Ans: $k = -1,\ v = 3x^2y - y^3 + c$]
5) Show that $u = 2\log(x^2+y^2)$ is harmonic and find its conjugate
   [Ans: $v = 4\tan^{-1}\left(y/x\right) + c$]
6) Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its conjugate

[Ans: $v=\tan^{-1}\left(y/x\right)+c$]

7) Prove that $u=x^2-y^2-2xy-2x+3y$ is harmonic and find $f(z)=u+iv$

[Ans: $f(z)=x^2-y^2-2xy-2x+3y+i\left(x^2-y^2+2xy-3x-2y+c\right)$]

8) Prove that $e^{x^2-y^2}$ is harmonic and find its harmonic conjugate

[Ans: $u=e^{x^2-y^2}$ is not harmonic conjugate]

9) Find the analytic function $f(z)=u+iv$ if $u=a(1+\cos\theta)$

[Ans: $v=2a\sin\theta\log r+c$]

10) Find the analytic function whose real part is $r^2\cos 2\theta+r\sin\theta$ and also find $f(z)$

[Ans: $v=r^2\sin 2\theta r\cos\theta+c$ & $f(z)=z^2-iz+ic$]

11) If $v(r,\theta)=\left(r-\frac{1}{r}\right)\sin\theta\ [r\neq 0]$ then find u [Ans: $u(r,\theta)=\left(r+\frac{1}{r}\right)\cos\theta$]

12) If $v(r,\theta)=r^2\cos 2\theta-r\cos\theta+2$ then find u & $f(z)$

<u>Milne – Thomson method – Construction of regular function</u>

Let $f(z)=u(x,y)+iv(x,y)$ & $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

Case (i): If u is given

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

From C – R equations

$$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

$$f'(z)=\phi_1(x,y)-i\phi_2(x,y)$$

Replacing 'x' by 'z' & 'y' by 0

$$f'(z)=\phi_1(z,0)-i\phi_2(z,0)$$

Integrating on both sides.

$$f(z)=\int\left[\phi_1(z,0)-i\phi_2(z,0)\right]dz+c$$

Case (i): If v is given

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

From C – R equations

$$f'(z)=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

$$f'(z)=\psi_1(x,y)+i\psi_2(x,y)$$

Replacing 'x' by 'z' & 'y' by 0

$$f'(z)=\psi_1(z,0)+i\psi_2(z,0)$$

Integrating on both sides.

$$f(z)=\int\left[\psi_1(z,0)+i\psi_2(z,0)\right]dz+c$$

<u>Problem:-</u>

1) Find the analytic function whose real part is

(i) $\frac{x}{x^2+y^2}$ **(ii)** $\frac{y}{x^2+y^2}$

Sol:-

(i) $\frac{x}{x^2+y^2}$

$u(x,y)=\frac{x}{x^2+y^2}$

$u_x=\frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$

$u_y=x\left[\frac{-1}{\left(x^2+y^2\right)^2}(2y)\right]=\frac{-2xy}{\left(x^2+y^2\right)^2}$

$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ [From C – R equations $u_y=-v_x$]

$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\Rightarrow f'(z)=\frac{\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}+i\frac{2xy}{\left(x^2+y^2\right)^2}$

By Milne – Thomson method

Replacing ‘x’ by ‘z’ & ‘y’ by 0

$f'(z)=\frac{-z^2}{z^4}=\frac{-1}{z^2}$

Integrating on both sides.

$f(z)=\int\frac{-1}{z^2}dz+c$

$=\frac{-z^{-2+1}}{-2+1}+c$

$\therefore f(z)=\frac{1}{z}+c$

(ii) $\frac{y}{x^2+y^2}$

$u(x,y)=\frac{y}{x^2+y^2}$

$u_x=y\left[\frac{-1}{\left(x^2+y^2\right)^2}(2x)\right]=\frac{-2xy}{\left(x^2+y^2\right)^2}$

$u_y=\frac{\left(x^2+y^2\right)-y(2y)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$

$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ [From C – R equations $u_y=-v_x$]

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{-2xy}{\left(x^2+y^2\right)^2} - i\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

By Milne – Thomson method

Replacing 'x' by 'z' & 'y' by 0

$$f'(z) = -i\frac{z^2}{z^4}$$

$$f'(z) = -i\frac{1}{z^2}$$

Integrating on both sides.

$$f(z) = -i\int \frac{1}{z^2}dz + c$$

$$= \frac{-z^{-2+1}}{-2+1}(i) + c$$

$$\therefore f(z) = \frac{i}{z} + c$$

2) If $f(z) = u + iv$ is an analytic function of z and if $u - v = e^x(\cos y - \sin y)$ then find $f(z)$ in terms of z

Sol:-

Given $u - v = e^x(\cos y - \sin y)$

$f(z) = u + iv$

$if\, f(z) = iu - v$

$(1+i) f(z) = (u-v) + i(u+v)$

Let $F(z) = (1+i) f(z) = (u-v) + i(u+v) = U + iV$

$F(z) = U + iV$ where $U = u - v$ & $V = u + v$

$U = u - v = e^x(\cos y - \sin y)$

$$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$ [From C – R equations $V_x = -U_y$]

$$F'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$F'(z) = e^x(\cos y - \sin y) + ie^x(\sin y + \cos y)$$

$$(1+i) f'(z) = e^x(\cos y - \sin y) + ie^x(\sin y + \cos y)$$

By Milne – Thomson method

Replacing 'x' by 'z' & 'y' by 0

$$(1+i) f'(z) = e^z(1) + ie^z(1)$$

$$\cancel{(1+i)} f'(z) = \cancel{(1+i)} e^z$$

$f'(z)=e^z$

Integrating on both sides.

$f(z)=\int e^z dz+c$

$\therefore f(z)=e^z+c$

3) $u+v=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$ then find $f(z)$

Sol:-

Given $u+v=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$

$f(z)=u+iv$

$if(z)=iu-v$

$(1+i)f(z)=(u-v)+i(u+v)$

Let $F(z)=(1+i)f(z)=(u-v)+i(u+v)=U+iV$

$F(z)=U+iV$ where $U=u-v$ & $V=u+v$

$V=u+v=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$

$\dfrac{\partial V}{\partial x}=\dfrac{2\cos 2x(\cosh 2y-\cos 2x)-\sin 2x(+\sin 2x)2}{(\cosh 2y-\cos 2x)^2}$

$\dfrac{\partial V}{\partial x}=\dfrac{2\cos 2x\cosh 2y-2\cos^2 2x-2\sin^2 2x}{(\cosh 2y-\cos 2x)^2}$

$\dfrac{\partial V}{\partial x}=\dfrac{2\cos 2x\cosh 2y-2(\cos^2 2x+\sin^2 2x)}{(\cosh 2y-\cos 2x)^2}$

$\dfrac{\partial V}{\partial x}=\dfrac{2\cos 2x\cosh 2y-2}{(\cosh 2y-\cos 2x)^2}$

$\dfrac{\partial V}{\partial y}=\sin 2x\left[\dfrac{-1}{(\cosh 2y-\cos 2x)^2}(2\sinh 2y)\right]$

$\dfrac{\partial V}{\partial y}=\dfrac{-2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}$

$F'(z)=\dfrac{\partial U}{\partial x}+i\dfrac{\partial V}{\partial x}$ [From C – R equations $U_x=V_y$]

$F'(z)=\dfrac{\partial V}{\partial y}+i\dfrac{\partial V}{\partial x}$

$F'(z)=\dfrac{-2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}+i\left[\dfrac{2\cos 2x\cosh 2y-2}{(\cosh 2y-\cos 2x)^2}\right]$

By Milne – Thomson method

Replacing ‘x’ by ‘z’ & ‘y’ by 0

$$(1+i)f'(z)=0+i\left[\frac{2\cos 2z-2}{(1-\cos 2z)^2}\right]$$

$$(1+i)f'(z)=0+i\left[\frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}\right]$$

$$=-2i\left[\frac{1}{1-\cos 2z}\right]$$

$$=\frac{-2i}{2\sin^2 z}$$

$$(1+i)f'(z)=-i\cos ec^2 z$$

Integrating on both sides.

$$f(z)=\frac{-i}{1+i}\int \cos ec^2 z dz+c$$

$$=\frac{-i}{1+i}[-\cot z]+\frac{c}{1+i}$$

$$=\frac{i\cot z}{1+i}+\frac{c}{1+i}$$

4) If $f(z)=u+iv$ is an analytic function then find $f(z)$ **if** $2u+v=e^{2x}\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]$

Sol:-

Given $2u+v=e^{2x}\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]$

$f(z)=u+iv$

$2if(z)=2iu-2v$

$(1+2i)f(z)=(u-2v)+i(2u+v)$

Let $F(z)=(1+2i)f(z)=(u-2v)+i(2u+v)=U+iV$

$F(z)=U+iV$ where $U=u-2v$ & $V=2u+v$

$V=e^{2x}\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]$

$$\frac{\partial V}{\partial x}=e^{2x}(2\cos 2y+\sin 2y)+\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]2e^{2x}$$

$$\frac{\partial V}{\partial y}=e^{2x}\left[\cos 2y+(2x+y)(-2\sin 2y)\right]+\left[(x-2y)2\cos 2y-2\sin 2y\right]$$

$$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}$$ [From C – R equations $U_x=V_y$]

$$F'(z)=\frac{\partial V}{\partial y}+i\frac{\partial V}{\partial x}$$

By Milne – Thomson method

Replacing ‘x’ by ‘z’ & ‘y’ by 0

$F'(z)=e^{2z}[1+2z]+i\left[2e^{2z}+(2z)2e^{2z}\right]$

$(1+2i)f'(z)=e^{2z}+2ze^{2z}+i2e^{2z}+4zie^{2z}$

$(1+2i)f'(z)=(1+2i)e^{2z}+2ze^{2z}(1+2i)$

$f'(z)=e^{2z}+2ze^{2z}$

Integrating on both sides.

$$f(z)=\int(1+2z)e^{2z}dz+c$$

$$=(1+2z)\frac{e^{2z}}{2}-\frac{e^{2z}}{4}(2)+c$$

$$=\frac{e^{2z}}{2}+\frac{2ze^{2z}}{2}-\frac{e^{2z}}{2}+c$$

$$f(z)=ze^{2z}+c$$

5) Determine the analytic function $f(z)=u+iv$ given that $3u+2v=y^2-x^2+16x$

Sol:-

Given $3u+2v=y^2-x^2+16x$ ⟶ (A)

Differentiating w.r.t. 'x' partially

$$3\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial x}=-2x+16 \longrightarrow (1)$$

Differentiating (A) w.r.t. 'y' partially

$$3\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}=2y$$

Given that $f(z)$ is analytic, from C – R equations $u_x=v_y$ & $u_y=-v_x$

$$-3\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial x}=2y$$

$$2\frac{\partial u}{\partial x}-3\frac{\partial v}{\partial x}=2y \longrightarrow (2)$$

$$2\times eq^n(1)\Rightarrow 6\frac{\partial u}{\partial x}+4\frac{\partial v}{\partial x}=-4x+32$$

$$3\times eq^n(2)\Rightarrow 6\frac{\partial u}{\partial x}-9\frac{\partial v}{\partial x}=6y$$

$$-\quad +\quad -$$

$$13\frac{\partial v}{\partial x}=-4x-6y+32$$

$$\frac{\partial v}{\partial x}=\frac{1}{13}[-4x-6y+32]$$

$$3\times eq^n(1)\Rightarrow 9\frac{\partial u}{\partial x}+6\frac{\partial v}{\partial x}=-6x+48$$

$$2\times eq^n(2)\Rightarrow 4\frac{\partial u}{\partial x}-6\frac{\partial v}{\partial x}=4y$$

$$13\frac{\partial u}{\partial x}=-6x+4y+48$$

$$\frac{\partial u}{\partial x}=\frac{1}{13}[-6x+4y+48]$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z)=\frac{1}{13}[-6x+4y+48]+i\frac{1}{13}[-4x-6y+32]$$

By Milne – Thomson method

Replacing 'x' by 'z' & 'y' by 0

$$f'(z)=\frac{1}{13}[-6z+48]+i\frac{1}{13}[-4z+32]$$

Integrating on both sides.

$$f(z)=\frac{1}{13}\int\{[-6z+48]+i[-4z+32]\}dz+c$$

$$=\frac{1}{13}\left[\frac{-6z^2}{2}+48z+i\left(\frac{-4z^2}{2}+32z\right)\right]$$

$$f(z)==\frac{1}{13}\left[-3z^2+48z+i\left(-2z^2+32z\right)\right]+\frac{c}{13}$$

6) Find an analytic function $f(z)$ such that $\mathrm{Re}[f'(z)]=3x^2-4y-3y^2$ and $f(1+i)=0$

Sol:-

Given that $\mathrm{Re}[f'(z)]=3x^2-4y-3y^2$

Let $F(z)=f'(z)=u+iv$

$u=3x^2-4y-3y^2$

$\frac{\partial u}{\partial x}=6x,\ \frac{\partial u}{\partial y}=-4-6y$

$F'(z)=\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}$ [From C – R equations $u_y=-v_x$]

$F'(z)=\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}$

$F'(z)=6x+i(4+6y)$

By Milne – Thomson method

Replacing 'x' by 'z' & 'y' by 0

$F'(z)=6z+i4$

$f''(z)=6z+i4$

Integrating on both sides.

$f'(z)=\int(6z+i4)dz+c_1$

$f'(z)=\frac{6z^2}{2}+i4z+c_1$

Again integrating on both sides.

$f(z)=\int(3z^2+i4z+c_1)dz+c_2$

$f(z)=\frac{3z^3}{3}+i\frac{4z^2}{2}+c_1z+c_2$

$f(z)=z^3+i2z^2+c_1z+c_2$

Again $f(1+i)=0$

$f(1+i)=(1+i)^3+i2(1+i)^2+c_1(1+i)+c_2$

$0=1-i+3i-3+2i[1-1+2i]+c_1(1+i)+c_2$

$0=1-i+3i-3-4i^2+c_1(1+i)+c_2$

$0 = -6 + 2i + c_1(1+i) + c_2$

$c_2 = 6 - 2i - c_1(1+i)$

Sub in $f(z)$

$f(z) = z^3 + i2z^2 + c_1 z + 6 - 2i - c_1(1+i)$

Practice Problem

1) If $e^x\left[\left(x^2 - y^2\right)\cos y - 2xy\sin y\right]$ is real part of an analytic function, find analytic function

[Ans: $f(z) = e^z z^2 + c$]

2) Find an analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

[Ans: $f(z) = \cot z + ic$]

3) Prove that the function $v(x, y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

4) satisfies Laplace equation. Determine the corresponding analytic function $u + iv$

[Ans: $f(z) = 2\sin z + 2z^2 + 2i\left(\sin z + z^2\right) + c$]

5) Find an analytic function whose imaginary part is $\dfrac{2\sin x \sin y}{\cos 2x + \cosh 2y}$

[Ans: $f(z) = \sec z + c$]

6) Find an analytic function whose real part is $u = e^{2x}\left(x\cos 2y - y\sin 2y\right)$

(or)

$u(x, y) = e^{2x}\left(x\cos 2y - y\sin 2y\right)$ is a harmonic function, find its harmonic conjugate

[Hint: put $z = x + iy$ in $f(z)$ then you will get v]

[Ans: $f(z) = ze^{2z} + c,\ v = e^{2x}\left(x\sin 2y + y\cos 2y + c\right)$]

7) Find the conjugate harmonic of $u = e^{x^2 - y^2}\cos 2xy$. Hence find $f(z)$ in terms of z

[Ans: $f(z) = e^{z^2} + c,\ v = e^{x^2 - y^2}\sin 2xy + c$]

8) Show that $u = e^{-2xy}\sin\left(x^2 - y^2\right)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z

[Ans: $f(z) = -ie^{iz^2} + c,\ v = e^{-2xy}\cos\left(x^2 + y^2\right) + c$]

9) Find the function w such that $w = u + iv$ is an analytic function given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ [Ans: $f(z) = z^2 + 3z + 1 + ic$]

10) Find a & b if $f(z) = \left(x^2 - 2xy + ay^2\right) + i\left(bx^2 - y^2 + 2xy\right)$ is analytic. Hence find $f(z)$ in terms of [C – R equations $u_x = v_y$ & $u_y = -v_x$. check for which value of a & b C – R equations are satisfied. Next after substituting a & b write $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ use Milne – Thomson method, then we get $f(z) = (1+i)z^2 + c$]

[Ans: a = -1 & b = 1]

PS&CV

Multiple Choice Questions

1. An Analytic function $f(z) = u(x,y) + iv(x,y)$ then $u(x,y)$& $v(x,y)$must satisfy- [ ]

   a. $u_x = v_y$ & $u_y = -v_x$ b. $u_x = v_y$ & $u_y = v_x$ c. None d.none [Ans.a]

2. Let $f_1(z) = z^2$ & $f_2(z) = \bar{z}$ be two complex variable functions. …….[ ] [Ans.b]

   a. Both are analytic b. only $f_1(z)$ is analytic c. only $f_2(z)$ is analytic d.none

3. If $f(z) = (x^2 + ay^2) + ibxy$ is analytic function then---------- [ ] [Ans.b]

   a. $a = -1$ & $b = -1$ b. $a = -1$ & $b = 2$ c. $a = 1$ & $b = 2$ d. $a = 1$ & $b = -2$

4. Let $z = x + iy$ then $\overline{\cos z}$ =---------- [ ] [Ans.b]

   a. $\cos z$ b. $\cos \bar{z}$ c. $\sin z$ d. None

5. $\lim_{z \to i} \frac{z^2+1}{z^3+2z-i(z^2+2)}$ = ---------- [ ] [Ans.d]

   a. $-2i$ b. $-i$ c. i d. $2i$

6. If $2x - x^2 + ay^2$ is to be harmonic , then a Should be---------- [ ] [Ans.a]

   a. 1 b. -3 c. -2 d. -1

7. Solutions of Laplace's equation having continuous II order partial derivatives are called [Ans.b]

   a. Biharmonic functions b. harmonic functions c. Conjugate harmonic functions d.none

8. $w = \frac{1}{z}$ is analytic everywhere except at ---------- [ ] [Ans.b]

   a. $z = \infty$ b. $z = 0$ c. $z = -1$ d. None

9. Real part of $w = e^{z^2}$---------- [ ] [Ans.a]

   a. $u = e^{x^2-y^2} \cos 2xy$ b. $u = e^{x^2} \cos 2xy$ c. $u = e^{x^2-y^2} \sin 2xy$ d.none

10. Imaginary part of $\sin \bar{z}$ is ---------- [ ] [Ans.c]

    a. $\sin x \sinh y$ b. $\cos x \sinh y$ c. $-\cos x \sinh y$ d. None

Fill in the blanks

1. Analytic function of an analytic function is ---------- [Ans: Analytic]

2. $f(z) = \bar{z}$ is not ---------- anywhere . [Ans: Differentiable]
3. $\lim_{z\to 1} \frac{z^2-1}{z-1} =$ ---------- [Ans: 2]
4. Polar form C – R Equations ---------- [Ans: $u_r = \frac{1}{r} v_\theta,\ v_r = -\frac{1}{r} u_\theta$]
5. If $f(z) = z\bar{z}$ then $u =$----------, $v =$ ----------[Ans: $x^2 + y^2, 0$]
6. $w = z^3 - 4z + 1$ is analytic ----------[Ans: Everywhere]
7. Real part of e^z ----------[Ans: $e^x \cos y$]
8. $w = \frac{z+2}{z(z^2+1)}$ is not analytic at---------- [Ans: $0, \pm i$]
9. $u(x, y)$ is said be harmonic if ---------- [Ans: $\nabla^2 u = 0$]
10. Imaginary part of e^{iz} ----------[Ans: $e^{-y} \sin x$]

Short Answer Questions

1. Find the value of a if $\cos ax \sin hy$ is harmonic.
2. Find the points at which $f(z) = \frac{z}{(z^2-z)}$ is not analytic.
3. Show that the function $f(z) = e^z$ is entire.
4. Find the analytic function whose real part is xy.
5. Show that $f(z) = z|z|$ is not analytic anywhere.
6. Find the harmonic conjugate of $u = 2xy + 3y$.
7. Write the Cauchy's Riemann equations in polar form.
8. Show that the function $f(z) = \sin x \cos hy + i \cos x \sin hy$ is continuous and analytic everywhere.
9. Show that $f(z) = z + \bar{z}$ is not analytic anywhere in the complex plane.
10. Show that $u = \frac{x}{x^2+y^2}$ is harmonic.
11. State the necessary and sufficient conditions for a function $f(z)$ to be analytic.
12. Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic.
13. Show that $f(z) = |z|^2$ is not analytic at any point.

Long Answer Questions

1. Find the analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$.
2. Show that both the real part and the imaginary part of any analytic function satisfy Laplace's equation.

3. Find the analytic function $f(z) = u + iv$ where $u = \frac{\sin 2x}{(\cos h2y - \cos 2x)}$.
4. Prove that the function of $f(z)$ defined by $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is continuous and C-R equations are satisfied at the origin, yet $f'(0)$ does not exists.
5. If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$
6. Find the analytic function whose real part is $\left(r - \frac{1}{r}\right) Sin\,\theta$.
7. Determine p so that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + iTan^{-1}\left(\frac{px}{y}\right)$ is analytic.
8. Find the analytic function $f(z) = u + iv$ if $u - v = e^x(\cos y - \sin y)$.
9. Determine the analytical function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 = 1$. Also find the harmonic conjugate of this real part.
10. Discuss the continuity of $f(x,y) = \begin{cases} \frac{2xy(x+y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$
11. If $u(x,y)$ & $v(x,y)$ are harmonic functions in a region R, Prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\delta v}{\delta x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\delta v}{\delta y}\right)$ is an analytic function.
12. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be an analytic. If $u = -r^3 \sin 3\theta$, then construct the corresponding analytic function (z) in terms of z.
13. Find analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ where $v(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$.
14. If $f(z) = u + iv$ be an analytic function of z and if $u - v = (x-y)(x^2 + 4xy + y^2)$ find $f(z)$ in terms of z.
15. Define analyticity of a function. Show that the function defined by $f(z) = \sqrt{|xy|}$is not analytic at the origin although the C-R equations are satisfied at that point.
16. Determine the analytic function whose real part is $\log\sqrt{x^2 + y^2}$.
17. Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic, find the conjugate function $'v'$ and express $f(z) = u + iv$ is an analytic function of z.
18. If $f(z) = u + iv$ is an analytic function in a region R. prove that the curves $u(x,y) = c_1, v(x,y) = c_2$ form two orthogonal families.
19. Find the analytic function $f(z)$whose real part is e^{-x}(x sin y − y cos y).
20. Find the analytic function $f(z)$whose real part is e^{x}(x sin y − y cos y).

GATE PREVIOUS QUESTIONS

1. The function w = u+iv $= \frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\frac{y}{x}$ is not analytic at the point

[GATE 2005]

(a) (0,0) (b) (0,1) (c) (1,0) (d) (2,0)

2. The function w=u+iv $= \frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\frac{y}{x}$ is not analytic at the point

[GATE 2005]

(a) (0,0) (b) (0,1) (c) (1,0) (d) (2,0)

3. An analytic function of a complex variable z = x + iy is expressed as
f(z) = u (x, y) + i v(x, y) where $i = -1$. If u = xy, the expression for v should be

[GATE-2009]

(a) $\frac{(x+y)^2}{2}+k$ (b) $\frac{(x-y)^2}{2}+k$ (c) $\frac{\left(y^2-x^2\right)}{2}+k$ (d) $\frac{\left(x^2-y^2\right)}{2}+k$

4. The modulus of the complex number $\frac{(3+4i)}{(1-2i)}$ is. [GATE-2010]

(a) 5 (b) $\sqrt{5}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{1}{5}$

5. For an analytic function, $f(x+iy)=u(x,y)+iv(x,y)$,u is given by $u=x^3-3x^2y^2$.
The expression for v, [GATE-2010]

6. For an analytic function, $f(x+iy)=u(x,y)+iv(x,y)$,u is given by $u=3x^2-3y^2$.
The expression for v, considering K to be constant is [GATE-2011]

(a) $u=3y^2-3x^2+k$ (b) $u=6x-6y+k$ (c) $u=6y-6x+k$ (d) $u=6xy+k$

7. For an analytic function, $f(x+iy)=u(x,y)+iv(x,y)$,u is given by $u=x^2y^2$. The expression for v, _____ [GATE-2014]

8. For an analytic function, $f(x+iy)=u(x+iy)+v(x+iy)$,u is given by $u=e^{-y}\cos x$.
The expression for v, _____ [GATE-2014]