UNIT - II: DISCRETE PROBABILITY DISTRIBUTIONS

- Binomial Distributions
- Poisson Distributions
- Evaluation of statistical parameters for these distributions
- Poisson approximation to the binomial distribution.

COSM: UNIT-II

PROBABILITY DISTRIBUTIONS

Discrete

Binomial ***Possion*** ***Geometric***

Bernoulli's distribution:

In this distribution there are the probability of success denoted by p , failure q with x denoting the number of success for which only a single trial is considered.

Definition:

A random variable X is set to follow the Bernoulli's distribution if its mass function is given by $\boxed{P(x) = p^x . q^{1-x}}$ $x = 0,1,2, \dots$

$If\ x = 0 , P(0) = p^0 . q^{1-0} = p^0 . q^1 = q$

$If\ x = 1 , P(1) = p^1 . q^{1-1} = p^1 . q^0 = p$

$\therefore P(0) = q \text{ and } P(1) = p$

Binomial distributions:

There are $n -$ *trials which are considered with* p, q *as the probability of success and failure with n to be finite.*

Definition:

A random variable X is said to follow the binomial distribution if its probability mass function is given by $\boxed{P(x) = n_{c_x} . p^x . q^{n-x}}$ *for* $x = 0,1,2, \dots \dots . n$

where $x = no.\ of\ success$

$p = probability\ of\ success$

$q = probability\ of\ failure$

$n = no.of\ trials$

$\boxed{p + q = 1}$

Verification of $\boldsymbol{p(x)}$ ***as the probability mass function***:

1) *since* x, p, q, n *are all non* $-$ *negative* $\rightarrow P(x) \geq 0$

2) *Consider binomial expansion*

$$(q+p)^n = n_{c_0}q^n p^0 + n_{c_1}q^{n-1}p^1 + n_{c_2}q^{n-2}p^2 + \cdots \ldots \ldots + n_{c_n}q^0 p^n \ \ldots\ldots (1)$$

$$Consider \sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} n_{c_x} p^x . q^{n-x}$$

$$= n_{c_0}p^0q^n + n_{c_1}p^1q^{n-1} + n_{c_2}p^2q^{n-2} + \cdots \ldots \ldots + n_{c_n}p^nq^0 \ .$$

$$\sum_{x=0}^{n} P(x) = n_{c_0}q^n + n_{c_1}p^1q^{n-1} + n_{c_2}p^2q^{n-2} + \cdots \ldots \ldots + n_{c_n}p^n \ \ldots\ldots (2)$$

From (1) *and* (2) *we get*

$$(q+p)^n = \sum_{x=0}^{n} P(x)$$

$$Thus \ \boxed{\sum_{x=0}^{n} P(x) = 1}$$

Assumptions under which binomial distribution works:

1) *There are* $n-$ *trials which are repeated under identical conditions.*

2) *Each trial has two possible outcomes i.e., success or failure.*

3) *The probability of success or failure remains constant for all the trials.*

4) *The trials are independent i.e. the probability of success or failure of one trial will not effect the other.*

Mean:

$$Mean = E[X] = \sum x.P(x)$$

$$= \sum_{x=0}^{n} x.n_{c_x} p^x . q^{n-x}$$

$$= 0.n_{c_0}p^0q^n + 1.n_{c_1}p^1q^{n-1} + 2.n_{c_2}p^2q^{n-2} + \cdots \ldots \ldots + n.n_{c_n}p^nq^{n-n}$$

$$= n.p.q^{n-1} + 2.\frac{n(n-1)}{2!} + \ldots\ldots + n.p^n$$

$$= np\left[(n-1)_{c_0}q^{n-1} + (n-1)_{c_1}p^1q^{(n-1)-1} + \ldots\ldots + (n-1)_{c_{(n-1)}}p^{n-1}\right]$$

$$= np[q+p]^{n-1}$$

$$\therefore \boxed{Mean = np}$$

Variance:

$$Variance = E[X^2] - (E[X])^2$$

$$= \sum_{x=0}^{n} x^2.P(x) - (np)^2$$

$$= \sum_{x=0}^{n}[x^2 + x - x]\ P(x) - n^2p^2$$

$$= \sum_{x=0}^{n}[x(x-1) + x]\,P(x) - n^2p^2$$

$$= \sum_{x=0}^{n} x(x-1)P(x) + \sum_{x=0}^{n} x\,P(x)\ - n^2p^2$$

$$= \sum_{x=2}^{n} x(x-1)\ n_{c_x} p^x.q^{n-x} + np - n^2p^2$$

$$= [2.1\ n_{c_2}p^2q^{n-2} + 3.2\ n_{c_3}p^3q^{n-3} + .\ \ldots\ldots + n(n-1)\ n_{c_n}p^nq^{n-n}] + np - n^2p^2$$

$$= [2.\frac{n(n-1)}{1.2}p^2q^{n-2} + 3.2\ \frac{n(n-1)(n-2)}{1.2.3}p^3q^{n-3} + \ldots\ldots + n(n-1)p^n] + np - n^2p^2$$

$$= n(n-1)p^2[q^{n-2} + (n-2)p^1q^{(n-2)-1} + \ldots\ldots + p^{n-2}] + np - n^2p^2$$

$$= n(n-1)p^2[q+p]^{n-2} + np - n^2p^2$$

$$= [n^2p^2 - np^2][q+p]^{n-2} + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p) = npq$$

$$\therefore \boxed{Variance = npq}$$

Moment Generating Function:

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x} e^{tx}\,P(x)$$

$$= \sum_{x} e^{tx}\,n_{c_x}.p^x.q^{n-x}$$

$$= \sum_{x} n_{c_x}\,(pe^t)^x q^{n-x}$$

$$\therefore M_X(t) = (q + pe^t)^n$$

Problems:

1) *The incidents of a disease in an industry is such that the workers have* 20% *of chance of suffering from the disease. What is the probability that out of* 6 *workers choosen at random* 4 *are more will suffer with the disease.*

Sol: we have $p = \frac{20}{100} = 0.2$

we have $p + q = 1$

$\Rightarrow q = 1 - p = 1 - 0.2 = 0.8$

$n = 6, x = 4,5,6$

$\therefore P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$

since $P(x) = n_{c_x}.\, p^x.\, q^{n-x}$

$= 6_{c_4}(0.2)^4(0.8)^{6-4} + 6_{c_5}(0.2)^5(0.8)^{6-5} + 6_{c_6}(0.2)^6(0.8)^{6-6}$

$= \frac{6 \times 5}{1 \times 2}(0.2)^4(0.8)^2 + 6(0.2)^5(0.8)^1 + (0.2)^6$

$= 0.01696$

2) *For a binomial variate the mean is* 4 *and variance is* 3. *Determine the parameters of the distribution.*

Sol: $\boxed{X \sim b(n,p)}$

Given that $np = 4$..........(1)

$npq = 3$ ... ...(2)

$\frac{(2)}{(1)} \Rightarrow \frac{npq}{nq} = \frac{3}{4}$

$\Rightarrow \boxed{q = \frac{3}{4}}$

we have $p + q = 1 \Rightarrow p = 1 - q$

$p = 1 - \frac{3}{4}$

$\Rightarrow \boxed{p = \frac{1}{4}}$

substitute in (1) we get

$n\,[\frac{1}{4}] = 4 \Rightarrow \boxed{n = 16}$

$\therefore \boxed{X \sim b(16, \frac{1}{4})}$

3) *A coin which is fair or unbiosed tossed for 6 times. Find the probability of getting atmost 4*

Sol: Given that $n = 6, p = q = \frac{1}{2}$, $P[X \leq 4]$

$P(X \leq 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)(or)$

$P(X \leq 4) = 1 - P[X > 4]$

$= 1 - P(X = 5) + P(X = 6)$

$= 1 - [6_{c_5}[\frac{1}{2}]^5 [\frac{1}{2}]^{6-5} + 6_{c_6}[\frac{1}{2}]^6]$

$= 1 - [6[\frac{1}{2}]^6 + [\frac{1}{2}]^6]$

$= 1 - [[\frac{1}{2}]^6 (7)]$

$\therefore \boxed{P(X \leq 4) = 0.8906}$

4) *It has been claimed that 60% of all solar installations leads to the reduction in utility bill by $\frac{1}{3}$rd of its value. Accordingly what are the probabilities for the utility bill to be reduced by $\frac{1}{3}$rd of its values in 4 or ≥ 4 out of 5 installations.*

Sol: we have $p = \frac{60}{100} = 0.6$

we have $p + q = 1$

$\Rightarrow q = 1 - p = 1 - 0.6 = 0.4$

we know that $P(x) = n_{c_x} . p^x . q^{n-x}$

$n = 5$

at $x = 4$ at 5

$\therefore P(X = 4) + P(X = 5)$

$= 5_{c_4}(0.6)^4(0.4)^1 + 5_{c_5}(0.6)^5(0.4)^0$

$= 5(0.05184) + 0.0777$

$\therefore P(X = 4) + P(X = 5) = 0.33696$

5) *If the probability of defective bolt is 0.2. Find the mean, variance and the S. D for*

$the distribution\ in\ which\ 40\ bolts\ are\ choosen\ for\ simple.$

$Sol: Given\ p = 0.2$

$we\ have\ p + q = 1$

$\Rightarrow q = 1 - p = 1 - 0.2 = 0.8$

$n = 40$

$Mean = np = 40 \times (0.2) = 8$

$Variance = npq = 40 \times (0.2) \times (0.8) = 6.4$

$\therefore S.D = \sqrt{npq} = \sqrt{6.4} = 2.5298$

Recurrence relation between probabilities:

$P(x) = n_{c_x}.p^x.q^{n-x}$

$P(x+1) = n_{c_{x+1}}.p^{x+1}.q^{n-(x+1)}$

$$\frac{P(x+1)}{P(x)} = \frac{n_{c_{x+1}}.p^{x+1}.q^{n-(x+1)}}{n_{c_x}.p^x.q^{n-x}}$$

$$= \frac{n_{c_{x+1}}.p^x.p.q^{n-x}.q^{-1}}{n_{c_x}.p^x.q^{n-x}}$$

$$= \frac{n_{c_{x+1}}}{n_{c_x}}.\frac{p}{q}$$

$$= \frac{\frac{n!}{(x+1)!\,(n-(x+1))!}}{\frac{n!}{x!\,(n-x)!}}.\frac{p}{q}$$

$$\frac{P(x+1)}{P(x)} = \frac{n-x}{x+1}.\frac{p}{q}$$

$$\therefore \boxed{P(x+1) = \frac{n-x}{x+1}.\frac{p}{q}.P(x)}\ where\ x = 0,1,2,\ldots\ldots$$

$$At\ x = 0\ , P(1) = \frac{n-0}{0+1}.\frac{p}{q}.P(0)$$

1) $For\ a\ binomial\ distribution\ with\ p = 0.4\ and\ n = 5.\ Evaluate\ all\ the\ probabilities$

$using\ the\ recurrence\ relationship.$

$Sol: Given\ p = 0.4, q = 1 - p = 1 - 0.4 = 0.6$

$n = 5$

$P(x) = n_{c_x}.p^x.q^{n-x}$

$P(0) = 5_{c_0}.(0.4)^0.(0.6)^{5-0} = (0.6)^5$

$= 0.0778$

$P(x+1) = \frac{n-x}{x+1}.\frac{p}{q}.P(x)$

$At\ x = 0, P(1) = \frac{5-0}{0+1}.\frac{(0.4)}{0.6}.(0.0778)$

$\therefore P(1) = 0.2593$

$At\ x = 1, P(2) = \frac{5-1}{1+1}.\frac{(0.4)}{(0.6)}.(0.2593)$

$= 2.\frac{(0.4)}{(0.6)}.(0.2593)$

$= 0.3457$

$At\ x = 2, P(3) = \frac{5-2}{2+1}.\frac{(0.4)}{(0.6)}.(0.3457)$

$= 1.\frac{(0.4)}{(0.6)}.(0.3457)$

$= 0.2305$

$At\ x = 3, P(4) = \frac{5-3}{3+1}.\frac{(0.4)}{(0.6)}.(0.2305)$

$= \frac{2}{4}.\frac{(0.4)}{0.6}.(0.2305)$

$= 0.0768$

$At\ x = 4, P(5) = \frac{5-4}{4+1}.\frac{(0.4)}{(0.6)}.(0.0768)$

$= \frac{1}{5}.\frac{(0.4)}{0.6}.(0.0768)$

$= 0.0102$

2) *For a binomial variate the mean is 4 and variance is* $\frac{4}{3}$. *Find the probability of* $P(X \geq 1)$

Sol: Given that $np = 4$ *and* $npq = \frac{4}{3}$

$\frac{npq}{np} = \frac{\frac{4}{3}}{4} = \frac{4}{12} = \frac{1}{3}$

$\therefore \boxed{q = \frac{1}{3}}$

we have $p + q = 1 \Rightarrow p = 1 - q$

$= 1 - \frac{1}{3}$

$\therefore \boxed{p = \frac{2}{3}}$

Since $np = 4$

$\Rightarrow n\left[\frac{2}{3}\right] = 4$

$\Rightarrow n = \frac{3 \times 4}{2}$

$\therefore \boxed{n = 6}$

$X \sim b\left(6, \frac{2}{3}\right)$

$P(X \geq 1) = 1 - P[X < 1] = 1 - P[X = 0]$

$= 1 - 6_{c_0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{6-0}$

$= 1 - \left(\frac{1}{3}\right)^6 = 1 - 0.0014$

$\therefore P(X \geq 1) = 0.9986$

3) *Consider a random experiment of throwing* 2 *dice at a time for* 120 *times.*

Find the average no. of times, the no. on the first die exceeds the no. of the 2*nd die*

if it follows the binomial distribution.

Sol: Given that $n = 120$

{ (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)

(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)

(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

$p = \frac{15}{36} = 0.4167$

$\therefore Average = Mean = np = 120 \times 0.4167 = 50$

4) *Out of* 800 *families with* 5 *childern each how many would you expect to have*

the following i) 3 *boys* ii) 5 *girls* iii) *either* 2B *or* 3B iv) *atleast* 1 *boy*

Sol: $X \rightarrow$ *Getting a male child*

$p = \frac{1}{2} = q$

$i)\ 3\ boys\ and\ 2\ girls$

$x = 3, n = 5$

$P[X = 3] = 5_{c_3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3}$

$= 10.\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

$= 0.3125$

$\therefore No.\ of\ families\ out\ of\ 800\ with\ this\ condition$

$= 800 \times 0.3125$

$= 250$

$ii)\ 5\ girls\ and\ 0\ boys$

$x = 0\ , n = 5$

$P[X = 0] = 5_{c_0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5$

$= \left(\frac{1}{2}\right)^5$

$= 0.03125$

$\therefore No.\ of\ families = 800 \times 0.03125 = 25$

$iii)\ either\ 2B\ (or)\ 3B$

$2B\ \&\ 3G\ \ (or)\ 3B\ \&\ 2G$

$X = 2\ (or)\ X = 3$

$P[X = 2] = 5_{c_2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = 0.3125$

$P[X = 3] = 5_{c_3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 0.3125$

$P[X = 2] + P[X = 3] = 0.3125 + 0.3125$

$= 0.6250$

$\therefore No.\ of\ families = 800 \times 0.6250 = 500$

$iv)\ atleast\ 1\ boy$

$means\ 1B\&4G, 2B\&3G\ , 3B\&2G\ , 4B\&1G\ , 5B\&0G$

(or)

$1 - 0B\&5G$

$P[X = 0] = 5_{c_0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 0.03125$

$1 - P[X = 0] = 1 - 0.03125 = 0.9688$

$\therefore No.\,of\ families = 800 \times 0.9688 = 775$

Fitting of Binomial Distribution:

1) *Fit the binomial distribution to the following data*

	0	1	2	3	4	5	6
	13	25	52	58	32	16	4

$Sol{:}\ Given\ n = 6$

$Mean = \frac{\sum f_i x_i}{\sum f_i}$

$= \frac{(13 \times 0) + (25 \times 1) + (52 \times 2) + (58 \times 3) + (32 \times 4) + (16 \times 5) + (4 \times 6)}{13 + 25 + 52 + 58 + 32 + 16 + 4}$

$= \frac{25 + 104 + 174 + 128 + 80 + 24}{200}$

$= \frac{535}{200} = 2.675$

$np = 2.675$

$\Rightarrow 6p = 2.675$

$p = \frac{2.675}{6} = 0.4458$

$q = 1 - p = 1 - 0.4458 = 0.5542$

$N = \sum f_i = 200$

$Expected\ frequency{:}\ P(x) = n_{c_x}.\,p^x.\,q^{n-x}$

1) $P(X = 0) = 6_{c_0}(0.4458)^0(0.5542)^{6-0}$

$= 0.0290$

$Frequency = 200 \times 0.0290 = 5.795 \cong 6$

2) $P(X = 1) = 6_{c_1}(0.4458)^1(0.5542)^{6-1}$

$= 0.1398$

$Frequency = 200 \times 0.1398 = 27.96 \cong 28$

3) $P(X=2)=6_{c_2}(0.4458)^2(0.5542)^{6-2}$

$=0.2812$

$Frequency = 200 \times 0.2812 = 56.42 \cong 56$

4) $P(X=3)=6_{c_3}(0.4458)^3(0.5542)^{6-3}$

$=0.3016$

$Frequency = 200 \times 0.3016 = 60.3 \cong 60$

5) $P(X=4)=6_{c_4}(0.4458)^4(0.5542)^{6-4}$

$=0.1820$

$Frequency = 200 \times 0.1820 = 36.400 \cong 36$

6) $P(X=5)=6_{c_5}(0.4458)^5(0.5542)^{6-5}$

$=0.0585$

$Frequency = 200 \times 0.0585 = 11.7 \cong 12$

7) $P(X=6)=6_{c_6}(0.4458)^6(0.5542)^{6-6}$

$=0.0078$

$Frequency = 200 \times 0.0078 = 1.56 \cong 2$

The binomial distribution is

	0	1	2	3	4	5	6
	13	25	52	58	32	16	4
	6	28	56	60	36	12	2

2) *Fit a binomial distribution to the following data*

	0	1	2	3	4	5	6	7
	7	6	19	35	30	23	7	1

Calculate the probabilities using recurrence relationship.

Sol: Here $n = 7$

$$Mean = \frac{\sum f_i x_i}{\sum f_i}$$

$$= \frac{(0\times 7)+(1\times 6)+(2\times 19)+(3\times 35)+(4\times 30)+(5\times 23)+(6\times 7)+(7\times 1)}{7+6+19+35+30+23+7+1}$$

$= \frac{433}{128} = 3.3828$

$we\ have\ np = 3.3828$

$\Rightarrow 7p = 3.3828 \Rightarrow p = \frac{3.3828}{7}$

$\Rightarrow \boxed{p = 0.4833}$

$q = 1 - p = 1 - 0.4833$

$\Rightarrow \boxed{q = 0.5167}$

$P(x) = n_{c_x}.p^x.q^{n-x}\ \ldots\ldots.(1)$

$P(x+1) = \frac{n-x}{x+1}.\frac{p}{q}.p(x)\ldots\ldots(2)$

Expected Frequencies:

1) $put\ x = 0\ in\ (1)$

$p(0) = 7_{c_0}.p^0.q^{7-0}$

$= 7_{c_0}.(0.4833)^0.(0.5167)^7$

$= 0.0098$

$Frequency = 128 \times 0.0098 = 1.2544 \cong 1$

2) $put\ x = 0\ in\ (2)$

$p(1) = \frac{7-0}{0+1}.\frac{0.4833}{0.5167} \times 0.0098$

$= 0.0641$

$Frequency = 128 \times 0.0641 = 8.2132 \cong 8$

3) $put\ x = 1\ in\ (2)$

$p(2) = \frac{7-1}{1+1}.\frac{0.4833}{0.5167} \times 0.0641$

$= 0.1802$

$Frequency = 128 \times 0.1802 = 23.0656 \cong 23$

4) $put\ x = 2\ in\ (2)$

$p(3) = \frac{7-2}{2+1}.\frac{0.4833}{0.5167} \times 0.1802$

$= 0.2810$

$Frequency = 128 \times 0.2810 = 35.968 \cong 36$

5) $put\ x = 3\ in\ (2)$

$$p(4) = \frac{7-3}{3+1} \cdot \frac{0.4833}{0.5167} \times 0.2809$$

$= 0.2628$

$Frequency = 128 \times 0.2628 = 33.6384 \cong 34$

6) $put\ x = 4\ in\ (2)$

$$p(5) = \frac{7-4}{4+1} \cdot \frac{0.4833}{0.5167} \times 0.2628$$

$= 0.1475$

$Frequency = 128 \times 0.1475 = 18.88 \cong 19$

7) $put\ x = 5\ in\ (2)$

$$p(6) = \frac{7-5}{5+1} \cdot \frac{0.4833}{0.5167} \times 0.1475$$

$= 0.0456$

$Frequency = 128 \times 0.0456 = 5.8368 \cong 6$

8) $put\ x = 6\ in\ (2)$

$$p(7) = \frac{7-6}{6+1} \cdot \frac{0.4833}{0.5167} \times 0.0456$$

$= 0.0061$

$Frequency = 128 \times 0.0061 = 0.7808 \cong 1$

	0	1	2	3	4	5	6	7
	7	6	19	35	30	23	7	1
	1	8	23	36	34	19	6	1

3) *Consider a random experiment of tossing 7 coins at a time which are unbaised.*

Fit the binomial distribution if the no. of heads follows the following frequencies.

	0	1	2	3	4	5	6	7
	8	7	19	36	32	28	8	2

$Sol: n = 7$

$\sum f_i = 8 + 7 + 19 + 36 + 32 + 28 + 8 + 2$

$N = 140$

$p = q = \frac{1}{2}$

$P(x) = n_{c_x} . p^x . q^{n-x}$

1) $P(0) = 7_{c_0} . (\frac{1}{2})^0 . (\frac{1}{2})^{7-0} = 0.0078$

$Frequency = 140 \times 0.0078 = 1.0920 \cong 1$

2) $P(1) = 7_{c_1} . (\frac{1}{2})^1 . (\frac{1}{2})^{7-1}$

$= 7_{c_1} . (\frac{1}{2})^1 . (\frac{1}{2})^6$

$= 0.0547$

$Frequency = 140 \times 0.0547 = 7.6563 \cong 8$

3) $P(2) = 7_{c_2} . (\frac{1}{2})^2 . (\frac{1}{2})^5$

$= \frac{7 \times 6}{2 \times 1} . (\frac{1}{2})^7$

$= 0.1641$

$Frequency = 140 \times 0.1641 = 22.967 \cong 23$

4) $P(3) = 7_{c_3} . (\frac{1}{2})^3 . (\frac{1}{2})^{7-3}$

$= 0.2734$

$Frequency = 140 \times 0.2734 = 38.28 \cong 38$

5) $P(4) = 7_{c_4} . (\frac{1}{2})^4 . (\frac{1}{2})^{7-4}$

$= 0.2734$

$Frequency = 140 \times 0.2734 = 38.25 \cong 38$

6) $P(5) = 7_{c_5} . (\frac{1}{2})^5 . (\frac{1}{2})^{7-5}$

$= 0.1641$

$Frequency = 140 \times 0.1641 = 23.016 \cong 23$

7) $P(6) = 7_{c_6} . (\frac{1}{2})^6 . (\frac{1}{2})^{7-6}$

$= 0.0548$

$Frequency = 140 \times 0.0548 = 7.6720 \cong 8$

8) $P(7) = 7_{c_7}.(\frac{1}{2})^7.(\frac{1}{2})^{7-7}$

$= 0.0078$

$Frequency = 140 \times 0.0078 = 1.0938 \cong 1$

$\therefore$ *The binomial distribution is*

	0	1	2	3	4	5	6	7
	8	7	19	36	32	28	8	2
	1	8	23	38	38	23	8	1

4) *In a binomial distribution containing* 5 *independent trials, the probability of* 1 *and* 2 *are given by* 0.4096 *and* 0.2048 *respectively. FInd the parameters of the distribution.*

Sol: $P(x) = n_{c_x}.p^x.q^{n-x}$

$P(1) = 5_{c_1}.p^1.q^{5-1} = 5pq^4 = 0.4096 \ \ldots\ldots (1)$

$P(2) = 5_{c_2}.p^2.q^{5-2} = 10p^2q^3 = 0.2048 \ldots\ldots (2)$

$\frac{(1)}{(2)} \Rightarrow \frac{5pq^4}{10p^2q^3} = \frac{0.4096}{0.2048}$

$\frac{q}{2p} = 2 \Rightarrow q = 4p$

$\Rightarrow 1 - p = 4p$

$\Rightarrow 5p = 1 \Rightarrow \boxed{p = \frac{1}{5}}$

$\Rightarrow q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

$\boxed{q = \frac{4}{5}}$

$\therefore X \sim b\,(5, \frac{1}{5})$

Poisson Distribution:

$\boxed{(\ ) = \frac{e^{-\lambda}.\lambda^x}{!}}$ *where* $x = 0,1,2, \ldots\ldots$

Definition: A random variable X is said to follow the Poisson distribution if its probability mass function is given by $P(x) = \frac{e^{-\lambda}.\lambda^x}{x!}$ *for* $x = 0,1,2, \ldots.$

Assumptions under which the binomial distribution tends to poisson

1) *The number of trials are very high i.e* $n \to \infty$

2) *The probability of success p is very small i.e* $p \to 0$

3) $np = \lambda$ *is always finite.*

Mean:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots..$$

$$Mean = E[X]$$

$$= \sum_x x\, P(x)$$

$$= \sum_{x=0}^{\infty} x . \frac{e^{-\lambda} . \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} x . \frac{e^{-\lambda} . \lambda^x}{x . (x-1)!}$$

$$Mean = \sum_{x=1}^{\infty} \frac{e^{-\lambda} . \lambda^x}{(x-1)!}$$

$Let\ x - 1 = y\ then\ x = y + 1$

$when\ x = 1\ then\ y = 0\ and\ x = \infty\ then\ y = \infty$

$$Mean = \sum_{y=0}^{\infty} \frac{e^{-\lambda} . \lambda^{y+1}}{y!}$$

$$= e^{-\lambda} . \lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$= e^{-\lambda} . \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\ldots\right]$$

$$= e^{-\lambda} . \lambda . e^{\lambda}$$

$$\therefore \boxed{Mean = \lambda}$$

Variance:

$$\sigma^2 = E[X^2] - [E[X]]^2$$

$$= \sum_{x=0}^{\infty} x^2\, P(x) - \lambda^2$$

$$= \sum_x x^2 \frac{e^{-\lambda}.\lambda^x}{x!} - \lambda^2$$

$$= \sum_x x \frac{e^{-\lambda}.\lambda^x}{(x-1)!} - \lambda^2$$

$$= \sum_x [(x-1)+1] \ \frac{e^{-\lambda}.\lambda^x}{(x-1).(x-2)!} - \lambda^2$$

$$= \sum_x (x-1) \ \frac{e^{-\lambda}.\lambda^x}{(x-1).(x-2)!} + \sum_x \frac{e^{-\lambda}.\lambda^x}{(x-1)!} - \lambda^2$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}.\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda}.\lambda^x}{(x-1)!} - \lambda^2$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}.\lambda^x}{(x-2)!} + \lambda - \lambda^2$$

$put\ x-2 = y\ then\ x = y+2$

$when\ x = 2 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$variance = \sum_{y=0}^{\infty} \frac{e^{-\lambda}.\lambda^{y+2}}{y!} + \lambda - \lambda^2$$

$$= e^{-\lambda}.\lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} + \lambda - \lambda^2$$

$$= e^{-\lambda}.\lambda^2.e^{\lambda} + \lambda - \lambda^2$$

$\therefore \boxed{Variance = \lambda}$

$Thus\ \boxed{Mean = Variance = \lambda}$

***Moment Generating function*:**

$M_X(t) = E[e^{tX}]$

$E[e^{tX}]\ for\ discrete\ \sum x.P(x)$

$$= \sum_x e^{tx}.P(x)$$

$$= \sum_x e^{tx}.\frac{e^{-\lambda}.\lambda^x}{x!}$$

$$M_X(t) = \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left[1+\frac{\lambda e^t}{1!}+\frac{(\lambda e^t)^2}{2!}+\frac{(\lambda e^t)^3}{3!}+\ldots\ldots\right]$$

$$= e^{-\lambda}[e^{\lambda e^t}] = e^{-\lambda+\lambda e^t}$$

$$= e^{-\lambda[1-e^t]}$$

$$\therefore \boxed{M_X(t) = e^{-\lambda[1-e^t]}}$$

Problems:

1) *Consider the example of phone calls received at a specific point in a given timespan. The average of no. of phone calls per minute coming into a switch board between 2PM to 4PM is 2.5. Determine the probability that in that particular time the no. of phone calls are 1) 4 (or) fewer 2) More than 6 calls coming into the switch board.*

Sol: Given that $\lambda = 2.5$

We have average = mean = λ

1) $P[X \le 4] = P[X=0]+P[X=1]+P[X=2]+P[X=3]+P[X=4]$

$$= \frac{e^{-2.5}(2.5)^0}{0!}+\frac{e^{-2.5}(2.5)^1}{1!}+\frac{e^{-2.5}(2.5)^2}{2!}+\frac{e^{-2.5}(2.5)^3}{3!}+\frac{e^{-2.5}(2.5)^4}{4!}$$

$$= e^{-2.5}\left[1+2.5+\frac{(2.5)^2}{2!}+\frac{(2.5)^3}{3!}+\frac{(2.5)^4}{4!}\right]$$

$= 0.8912$

$\therefore P[X \le 4] = 0.8912$

2) $P[X > 6] = 1 - P[X \le 6]$

$$= 1-[P[X=0]+P[X=1]+P[X=2]+P[X=3]+P[X=4]+P[X=5]+P[X=6]]$$

$$= 1-\left[\frac{e^{-2.5}(2.5)^0}{0!}+\frac{e^{-2.5}(2.5)^1}{1!}+\frac{e^{-2.5}(2.5)^2}{2!}+\frac{e^{-2.5}(2.5)^3}{3!}+\frac{e^{-2.5}(2.5)^4}{4!}+\frac{e^{-2.5}(2.5)^5}{5!}+\frac{e^{-2.5}(2.5)^6}{6!}\right]$$

$$= 1-\left[0.8912+\frac{e^{-\lambda}\lambda^5}{5!}+\frac{e^{-\lambda}\lambda^6}{6!}\right]$$

$= 0.1088 - e^{-\lambda}[0.8138 + 0.3391]$

$= 0.1088 - e^{-2.5}[1.1529]$

$= 0.1088 - 0.0946$

$= 0.0142$

$\therefore P[X > 6] = 0.0142$

Recurrence Relationship between Probabilities:

$P(x) = \frac{e^{-\lambda}.\lambda^x}{x!} \ldots\ldots(1)$

$P(x+1) = \frac{e^{-\lambda}.\lambda^{x+1}}{(x+1)!} \ldots\ldots(2)$

$\frac{(2)}{(1)} \Rightarrow \frac{P(x+1)}{P(x)} = \frac{\frac{e^{-\lambda}.\lambda^{x+1}}{(x+1)!}}{\frac{e^{-\lambda}.\lambda^x}{x!}}$

$\frac{P(x+1)}{P(x)} = \frac{e^{-\lambda}.\lambda^x.\lambda}{x!(x+1)} \cdot \frac{x!}{e^{-\lambda}.\lambda^x}$

$\Rightarrow \frac{P(x+1)}{P(x)} = \frac{\lambda}{x+1}$

$\therefore \boxed{P(x+1) = \frac{\lambda}{x+1} P(x)\,, x = 0,1,2,3,\ldots\ldots}$

1) *If the probability that an individual suffers with a bad reaction from an injection is 0.001. Determine the probability that out of 2000 individuals i) exactly 3 ii) More than 2 iii) None iv) More than 1 individual suffers with bad reaction of the injection*

Sol: Since $p = 0.001$ *very small and* $n = 2000$ *is very large by assumption*

$Mean\ \lambda = np$

$= 0.001 \times 2000$

$\Rightarrow \boxed{\lambda = 2}$

$P(x) = \frac{e^{-\lambda}.\lambda^x}{x!}$

1) $P[X = 3] = \frac{e^{-2}.(2)^3}{3!} = 0.1804$

2) $P[X > 2] = 1 - P[X \leq 2]$

$= 1 - [P[X = 0] + P[X = 1] + P[X = 2]]$

$= 1 - [\frac{e^{-2}.(2)^0}{0!} + \frac{e^{-2}.(2)^1}{1!} + \frac{e^{-2}.(2)^2}{2!}]$

$= 0.3233$

3) $P[X = 0] = \frac{e^{-2}.(2)^0}{0!} = 0.1353$

4) $P[X > 1] = 1 - P[X \leq 1]$

$= 1 - [[P[X = 0] + P[X = 1]]$

$= 1 - [\frac{e^{-2}.(2)^0}{0!} + \frac{e^{-2}.(2)^1}{1!}]$

$= 0.5940$

2) *A car hire firm has 2 cars which hit hires out everyday the no. of demands on each day are distributed with poison having a mean* 1.5. *Calculate the proportion of days in a year on which i) there is no demand ii) on which the deamnd is rejected*

Sol: Let X represents the no. of demands required on a day

Given $\lambda = 1.5$

i) $P[X = 0] = \frac{e^{-1.5}.(1.5)^0}{0!} = 0.2231$

proportion of days in a year on which there is no demand $= 0.2231 \times 365$

$= 81.4425 \cong 81$ *days*

ii) $P[X > 2] = 1 - P[X \leq 2]$

$= 1 - [[P[X = 0] + P[X = 1] + P[X = 2]]$

$= 1 - [0.2231 + \frac{e^{-1.5}.(1.5)^1}{1!} + \frac{e^{-1.5}.(1.5)^2}{2!}]$

$= 0.1912$

proportion of days on which the demand is rejected $= 0.1912 \times 365$

$= 69.7819 \cong 70$ *days*

3) *Average of accidents on a road is* 1.8. *Determine the probability that the no. of accidents are i) atleast* 1 *ii) atmost* 1

Sol: Given that $\lambda = 1.8$

i) $P[X \geq 1] = 1 - P[X < 1]$

$= 1 - P[X < 0]$

$= 1 - \frac{e^{-1.8}.(1.8)^0}{0!}$

$\therefore P[X \geq 1] = 0.8347$

ii) $P[X \leq 1] = P[X = 0] + P[X = 1]$

$= \frac{e^{-1.8}.(1.8)^0}{0!} + \frac{e^{-1.8}.(1.8)^1}{1!}$

$\therefore P[X \leq 1] = 0.4628$

4) $If\ X\ is\ a\ poisson\ variate\ such\ that\ P[X=0]=P[X=1]\ then\ i)\ find\ the\ parameter\ of\ the\ distribution\ ii)\ Find\ the\ P[X=0]\ iii)\ Use\ recurrence\ relation\ ship\ for\ finding\ the\ probabilities\ with\ x=0,1,2,3,4$

$Sol: i)\ Given\ that\ P[X=0]=P[X=1]$

$$P[X=0]=\frac{e^{-\lambda}.\lambda^0}{0!}\ .....(1)\ ,P[X=1]=\frac{e^{-\lambda}.\lambda^1}{1!}\ .....(2)$$

$$(1)=(2)\ \Rightarrow \frac{e^{-\lambda}.\lambda^0}{0!}=\frac{e^{-\lambda}.\lambda^1}{1!}$$

$$\Rightarrow \boxed{\lambda=1}$$

$$ii)\ P[X=0]=\frac{e^{-1}.1^0}{0!}=0.3679$$

$$iii)\ using\ recurrence\ relation\ P(x+1)=\frac{\lambda}{x+1}P(x)$$

$$at\ x=0\ ,P(1)=\frac{1}{0+1}P(0)=0.3679$$

$$at\ x=1\ ,P(2)=\frac{1}{1+1}P(1)=\frac{1}{2}(0.3679)=0.1840$$

$$at\ x=2\ ,P(3)=\frac{1}{2+1}P(2)=\frac{1}{3}(0.1840)=0.0613$$

$$at\ x=3\ ,P(4)=\frac{1}{3+1}P(3)=\frac{1}{4}(0.0613)=0.0153$$

$$at\ x=4\ ,P(5)=\frac{1}{4+1}P(4)=\frac{1}{5}(0.0153)=0.0031$$

5) $If\ X\ is\ a\ poisson\ variate\ such\ that\ \frac{3}{2}P[X=1]=P[X=3].Find\ i)\ \lambda\ and\ hence\ find\ the\ probability\ of\ [X\geq 1]\ i.e\ P[X\geq 1]\ \ iii)\ P[X\leq 3]\ \ iv)\ P[2\leq X\leq 5]$

$Sol: i)\ Given\ that\ \frac{3}{2}P[X=1]=P[X=3]$

$$\frac{3}{2}\frac{e^{-\lambda}.\lambda^1}{1!}=\frac{e^{-\lambda}.\lambda^3}{3!}$$

$$\Rightarrow \lambda^2=9$$

$$\Rightarrow \lambda=\pm 3$$

$$\therefore\ \boxed{\lambda=3}$$

$$ii)\ P[X\geq 1]=1-P[X<1]$$

$$=1-P[X=0]$$

$= 1 - \frac{e^{-3}.3^0}{0!}$

$\therefore P[X \geq 1] = 0.9502$

$iii) P[X \leq 3] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3]$

$= \frac{e^{-3}.3^0}{0!} + \frac{e^{-3}.3^1}{1!} + \frac{e^{-3}.3^2}{2!} + \frac{e^{-3}.3^3}{3!}$

$= e^{-3} + e^{-3}.3 + \frac{e^{-3}.9}{2} + \frac{e^{-3}.27}{6}$

$= 0.6472$

$iv) P[2 \leq X \leq 5] = P[X = 2] + P[X = 3] + P[X = 4] + P[X = 5]$

$= \frac{e^{-3}.3^2}{2!} + \frac{e^{-3}.3^3}{3!} + \frac{e^{-3}.3^4}{4!} + \frac{e^{-3}.3^5}{5!}$

$= 0.7169$

6) *2 cards are drawn from a well shuffled pack of 52 cards which are diamonds using possion distribution find the probability of getting 2 diamonds atleast 3 times in 51 consecutive trials.*

Sol: The no. of trials $n = 51$

$p = \frac{13_{C_2}}{52_{C_2}} = \frac{\frac{13 \times 12}{1 \times 2}}{\frac{52 \times 51}{1 \times 2}} = 0.0588$

$\lambda = np = 51 \times 0.0588 = 3$

$\therefore \boxed{\lambda = 3}$

Atleast $3: P[X \geq 3] = 1 - P[X < 3]$

$= 1 - [P[X = 0] + P[X = 1] + P[X = 2]]$

$= 1 - [\frac{e^{-3}.3^0}{0!} + \frac{e^{-3}.3^1}{1!} + \frac{e^{-3}.3^2}{2!}]$

$= 1 - [e^{-3} + e^{-3}.3 + \frac{e^{-3}.9}{2}]$

$= 0.5768$

$\therefore P[X \geq 3] = 0.5768$

Fitting of poisson distribution:

1) *Fit the poisson distribution to the following data*

	0	1	2	3	4	5
	142	156	69	27	5	1

Sol: $\sum f_i = 400$

$$Mean\ (\lambda) = \frac{\sum f_i\, x_i}{\sum f_i} = \frac{0\times 142 + 1\times 156 + 2\times 69 + 3\times 27 + 4\times 5 + 5\times 1}{400} = 1$$

$\therefore\ \boxed{\lambda = 1}$

$$P(x) = \frac{e^{-\lambda}.\,\lambda^x}{x!}$$

$\boldsymbol{Expected\ frequencies}$:

1) $P[X = 0] = \dfrac{e^{-1}.\,1^0}{0!} = 0.3679$

$Expected\ frequency = 0.3679 \times 400 = 147.15 \cong 147$

2) $P[X = 1] = \dfrac{e^{-1}.\,1^1}{1!} = 0.3679$

$Expected\ frequency = 0.3679 \times 400 = 147.15 \cong 147$

3) $P[X = 2] = \dfrac{e^{-1}.\,1^2}{2!} = 0.1839$

$Expected\ frequency = 0.1839 \times 400 = 73.57 \cong 74$

4) $P[X = 3] = \dfrac{e^{-1}.\,1^3}{3!} = 0.0613$

$Expected\ frequency = 0.0613 \times 400 = 24.52 \cong 25$

5) $P[X = 4] = \dfrac{e^{-1}.\,1^4}{4!} = 0.0153$

$Expected\ frequency = 0.0153 \times 400 = 6.13 \cong 6$

6) $P[X = 5] = \dfrac{e^{-1}.\,1^5}{5!} = 0.0031$

$Expected\ frequency = 0.0031 \times 400 = 1.24 \cong 1$

	0	1	2	3	4	5
	142	156	69	27	5	1
	147	147	74	25	6	1

2) $Fit\ the\ poisson\ distribution\ to\ the\ following\ table\ by\ calculating\ the\ probabilities$

using recurrence relation.

	0	1	2	3	4
	109	65	22	3	1

Sol: $\sum f_i = 200$

$$Mean\ (\lambda) = \frac{\sum f_i\, x_i}{\sum f_i} = \frac{0\times 109 + 1\times 65 + 2\times 22 + 3\times 3 + 4\times 1}{200} = 0.6100$$

$\therefore\ \boxed{\lambda = 0.61}$

$$P(x) = \frac{e^{-\lambda}.\,\lambda^x}{x!}$$

Expected frequencies:

1) $P[X=0] = \dfrac{e^{-0.61}.\,(0.61)^0}{0!} = 0.5434$

$Expected\ frequency = 0.5434 \times 200 = 108.68 \cong 109$

$we\ have\ P(x+1) = \dfrac{\lambda}{x+1} P(x)$

$put\ x = 0\ in\ recurrence\ relation$

2) $P[X=1] = \dfrac{0.61}{0+1}(0.5434) = 0.3314$

$Expected\ frequency = 0.3314 \times 200 = 66.29 \cong 66$

$put\ x = 1\ in\ recurrence\ relation$

3) $P[X=2] = \dfrac{0.61}{1+1}(0.3314) = 0.1010$

$Expected\ frequency = 0.1010 \times 200 = 20.2 \cong 20$

$put\ x = 2\ in\ recurrence\ relation$

4) $P[X=3] = \dfrac{0.61}{2+1}(0.1010) = 0.0205$

$Expected\ frequency = 0.0205 \times 200 = 4.1 \cong 4$

$put\ x = 3\ in\ recurrence\ relation$

5) $P[X=4] = \dfrac{0.61}{3+1}(0.0205) = 0.0031$

$Expected\ frequency = 0.0031 \times 200 = 0.6253 \cong 1$

	0	1	2	3	4
	109	65	22	3	1
	109	65	20	4	1

Poisson Distribution as a limiting case of Binomial Distribution:

$Let\ np = \lambda \Rightarrow n = \frac{\lambda}{p}\ and\ p = \frac{\lambda}{n} \ldots\ldots (1)$

$Consider\ p(x) = n_{C_x} p^x q^{n-x}$

$$= \frac{n!}{x!\,(n-x)!} p^x [1-p]^{n-x}$$

$$= \frac{1.2.3\ldots\ldots(n-x)(n-x+1)\ldots\ldots n}{x!\,1.2.3\ldots\ldots(n-x)} p^x [1-p]^{n-x}$$

$$= \frac{n(n-1)\ldots\ldots(n-x+1)}{x!} p^x [1-p]^{n-x}$$

$$= \frac{\frac{\lambda}{p}\left(\frac{\lambda}{p}-1\right)\ldots\ldots\left(\frac{\lambda}{p}-(x-1)\right)}{x!} \frac{p^x\left[1-\frac{\lambda}{n}\right]^n}{[1-p]^x}$$

$$= \frac{\frac{\lambda}{p}\left[\frac{\lambda-p}{p}\right]\ldots\ldots\left[\frac{\lambda-(x-1)p}{p}\right]}{x!} \frac{p^x\left[1-\frac{\lambda}{n}\right]^n}{[1-p]^x}$$

$$= \frac{\lambda[\lambda-p]\ldots\ldots[\lambda-(x-1)p]}{p^x.\,x!} \frac{p^x\left[1-\frac{\lambda}{n}\right]^n}{[1-p]^x}$$

$$\lim_{p\to 0} p(x) = \lim_{p\to 0} \frac{\lambda[\lambda-p]\ldots\ldots[\lambda-(x-1)p]}{x!} \frac{\left[1-\frac{\lambda}{n}\right]^n}{[1-p]^x}$$

$$= \frac{\lambda^x}{x!}\left[1-\frac{\lambda}{n}\right]^n$$

$$\lim_{\substack{p\to 0\\ n\to\infty}} p(x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left[1-\frac{\lambda}{n}\right]^n$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[1-\frac{\lambda}{n}\right]^{-\frac{n}{\lambda}(-\lambda)}$$

$$\lim_{\substack{p\to 0\\ n\to\infty}} p(x) = \frac{\lambda^x}{x!}.e^{-\lambda} = \frac{e^{-\lambda}\lambda^x}{x!} = p(x)$$