# PROBABILITY AND STATISTICS COURSE CONTENT

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# PROBABILITY

## UNIT - I: RANDOM VARIABLES

- Introduction to Basic Probability
- Random variables: Discrete Random Variable; Continuous Random variables
- Expectation of Random Variables
- Variance of Random variables
- Moments.

**Unit-I notes**

***Trial***:

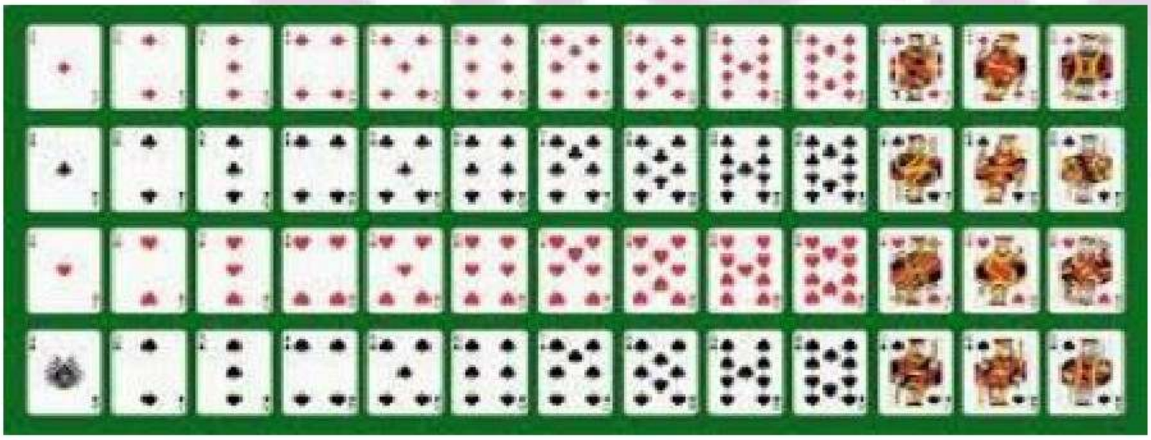
*A single performance of any action or any activity is called a Trial.*

*Ex*: 1. *Tossing a coin*

2. *Throwing a die*

3. *Drawing a card from a pack of cards*

4. *Drawing a ball from a bag of balls*



***Random Experiment***:

A single trial or a collection of trials together defines a Random Experiment.

The experiment is based on the users choice.

Ex: Tossing two coins successively or simultaneously.

**Event(or)Outcome:**

The result of a trial or a random experiment is called an Event(or) an Outcome.

*Ex: In tossing a coin the possible events are getting a head or a tail.*

**Definition:-**

If an 'experiment' is conducted, any number of times, under essentially identical conditions, there is a set of possible outcomes associated with it. If the result is not certain and is anyone of the several possible outcomes, the experiment is called a random trail or a random experiment.

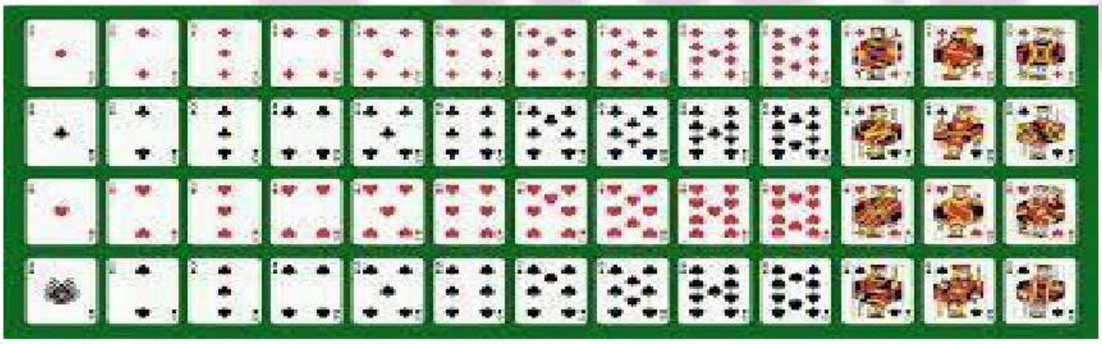
The outcomes are known as elementary events and set of outcomes is an event. Thus the elementary event also known as event.

***Classification of Events*:**

**Equally likely events:**

Events are said to be equally likely when there is no reason to expect any one of them rather than anyone of the others.

Ex: when a card is drawn from pack, any card may be obtained. In this tail, all the 52 elementary events are equally likely.



***Exhaustive Events*:**

*The collection of all possible outcomes of a random experiment are called collectively exhaustive.*

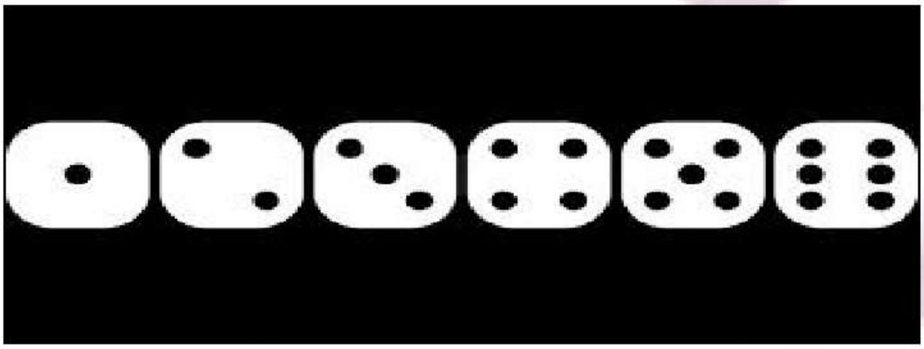
*Ex: In tossing a coin the outcomes of getting a head anda tail are collectively exhaustive.*


***Exclusive Events***:

*Events are said to be mutually exclusive if the happening of one will not allow the happening of other.*

*Ex*:1. *In the random experiment of a tossing a coin, getting a head or tail are mutualy exclusive.*

2. *In throwing a die all the six faces are mutually exclusive*



***Sample Space***:

*The collection of all possible outcomes of a random experiment defines a Sample Space. It is denoted by* $S$.

*Ex*: 1. *In tossing a coin* $S = \{H, T\}$

2. *In tossing two coins at a time that is simultaneously* $S = \{HH, HT, TH, TT\}$

3. *In throwing a die* $S = \{1,2,3,4,5,6\}$

## ***Probability***

In a random experiment, let there be n mutually exclusive and equally likely elementary events. Let E be an event of the experiment. If m elementary events from event E (are favorable to E), then the probability of E is

$$P(E) = \frac{No.of\ events\ favourable\ to\ E}{Total\ no.of\ events}$$

$$P(E) = \frac{m}{n}$$

Ex: $In\ tossing\ a\ coin\ S = \{H, T\},\ if\ E\ denotes\ the\ event\ of\ getting\ a\ head(or)$ $a\ tail,\ then$

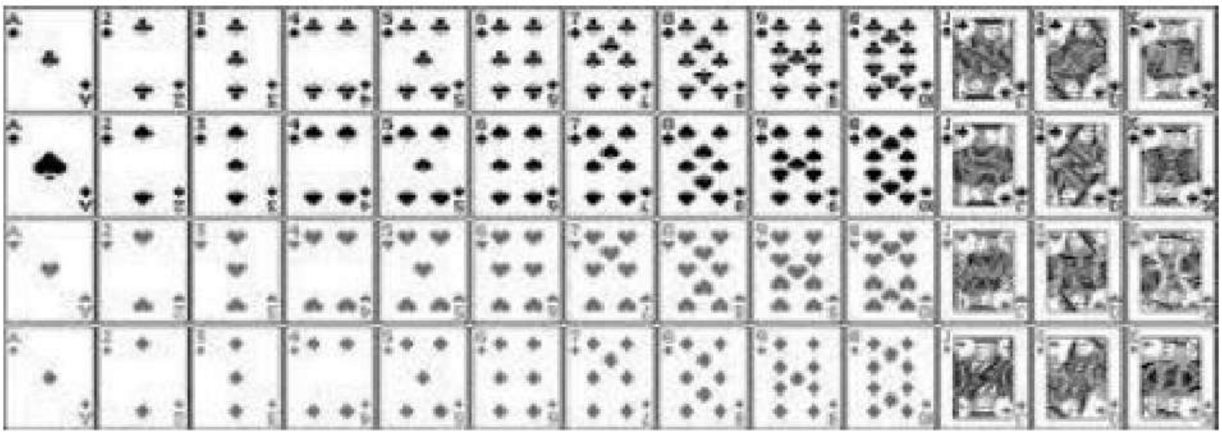
$$P(E) = 1/2$$

**Example1:**A coin is thrown 3 times .what is the probability that at least one head is obtained?

**Sol:** Sample space = [HHH, HHT, HTH, THH, TTH, THT, HTT, TTT]
Total number of ways = 2 × 2 × 2 = 8.

Fav. Cases = 7
P (E) = 7/8

**Example2:** Find the probability of getting a numbered card when a card is drawn from the pack of 52 cards.



Sol: Total Cards n = 52.

Numbered Cards = (2, 3, 4, 5, 6, 7, 8, 9, 10) 9 from each suit $4 \times 9 = m = 36$

P (E) = m/n=36/52 = 9/13

**Example3:** What is the probability of getting a sum of 7 when two dice are thrown?

**Sol:**

$$A = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Total number of ways = $6 \times 6 = 36$ ways.

Favorable cases = (1, 6) (6, 1) (2, 5) (5, 2) (3, 4) (4, 3) --- 6 ways.

P (E) = 6/36 = 1/6

**Example4:** There are 5 green 7 red balls. Two balls are selected one by one without replacement. Find the probability that first is green and second is red.

**Sol:** P(G) =5/12

P(R) =7/11

P (G) × P (R) = (5/12) x (7/11) = 35/132

**Example5:** There are 6 beads in a bag, 3 are red, 2 are yellow and 1 is blue. What is the probability of picking a yellow?

Solu:-The probability is the number of yellows in the bag divided by the total number of balls

i.e. 2/6 = 1/3.

**Example6**. *There are 10 boys and 5 girls in a class evaluate the following probabilities:*

*4 members are to be choosen as a committe with the following conditions*

1) *All 4 boys*:

$$P(4\ boys) = \frac{10_{C_4}}{15_{C_4}} = \frac{2}{3} = 0.1538$$

2) *All 4 girls*:

$$P(4\ girls) = \frac{5_{C_4}}{15_{C_4}} = \frac{1}{273} = 0.0037$$

3) *2 boys and 2 girls*:

$$P(2B\ and\ 2G) = \frac{10_{C_2} \times 5_{C_2}}{15_{C_4}} = \frac{30}{91}$$

4) *Atleast one boy*:

$1B$ *and* $3G$ (*or*) $2B$ *and* $2G$ (*or*) $3B$ *and* $1G$(*or*) $4B$

$$= \frac{10C_1 \times 5C_3}{15C_4} + \frac{10C_2 \times 5C_2}{15C_4} + \frac{10C_3 \times 5C_1}{15C_4} + \frac{10C_4}{15C_4}$$

$$= \frac{272}{273}$$

5) *Atmost 2 Girls*:

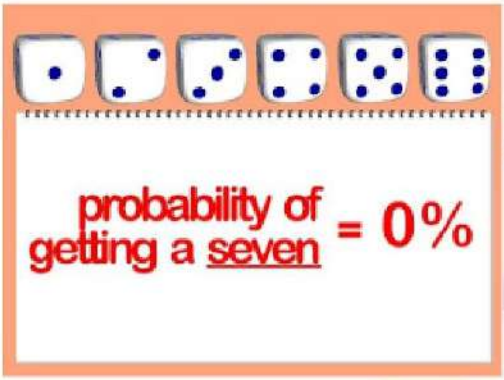
$2G$ *and* $2B$ (*or*) $1G$ *and* $3B$ (*or*) $0G$ *and* $4B$

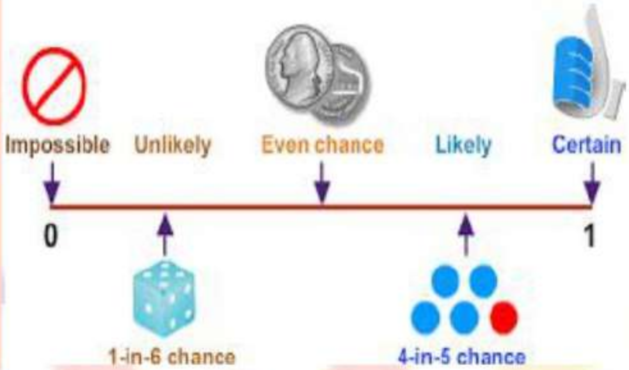
$$= \frac{10_{C_2} \times 5_{C_2}}{15_{C_4}} + \frac{10_{C_3} \times 5_{C_1}}{15_{C_4}} + \frac{10_{C_4}}{15_{C_4}} = \frac{12}{13}$$

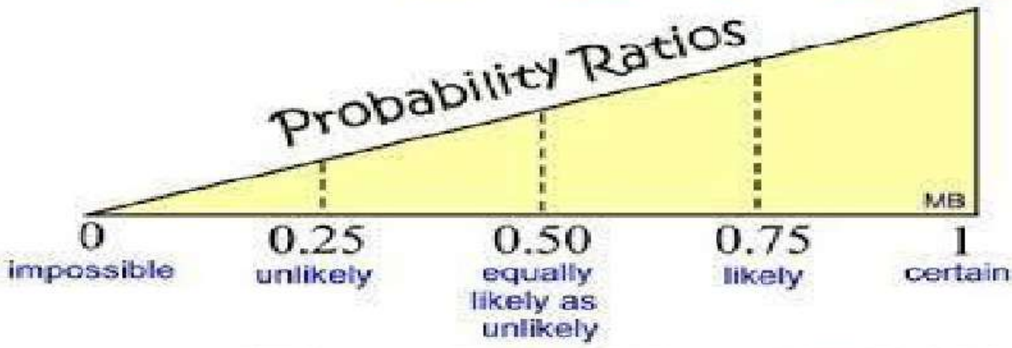
**Example7**: *Two dice are drawn at once find the probability for the following*

1) *The sum is* 8
2) *Sum is atleast* 9
3) *sum is either* 5 *or* 6

*Sol*: *As two dice are thrown the total outcomes are* $6^2 = 36$



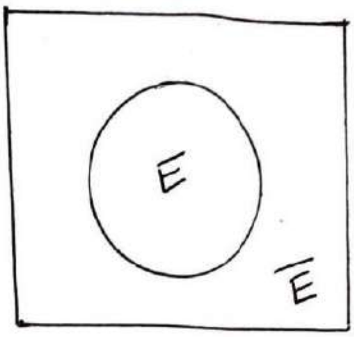




***<u>Complementary Event</u>:***

*The sample space S is given by* $S = E \cup \overline{E}$

$\therefore P(S) = P(E \cup \overline{E})$




*By the axiom of uncertainity, we have* $P(S) = 1$

$\therefore 1 = P(E) + P(\overline{E}) \Rightarrow \boxed{P(E) = 1 - P(\overline{E})}$

EX:- A teacher chooses a student at random from a class of 30 girls. What is the probability that the student chosen is a girl?

Probability: Since all the students in the class are girls, the teacher is certain to choose a girl.

$$P(\text{girl}) = \frac{30}{30} = 1$$

Ex:- A single card is chosen at random from a standard deck of 52 playing cards. What is the probability that the card chosen is a joker card?

Probability: It is impossible to choose a joker card since a standard deck of cards does not contain any jokers. This is an impossible event.

$$P(\text{joker}) = \frac{0}{52} = 0$$

Note: Experiment will involve **a standard deck of 52 playing cards**, which consists of 4 suits: hearts, clubs, diamonds and spades. Each suit has 13 cards as follows: ace, deuce, three, four, five, six, seven, eight, nine, ten, jack, queen, and king. Picture cards include jacks, queens and kings. There are no joker cards. There are only 4 of a kind, for example, 4 tens.

Ex: Determine the probability for each of the following events: A non-defective bolt will be found if out of 600 bolts already examined 12 were defective

Solu: The probability of defective bolts ,P(D) $=\frac{12}{600} = \frac{1}{50}$

$\therefore$ The probability of finding a non-defective bolt $P(\bar{D})$=1- P(D)

$$=1- \frac{1}{50} = \frac{49}{50} = 0.98$$

**<u>Mutually Exclusive events:</u>**

Two Events $E_1, E_2$ of a sample space S are said to be mutually exclusive if they have no sample points in common i. e $E_1 \cap E_2 = \emptyset$.

Mutually exclusive events are also called as **disjoint events**

## *Axioms of Probability*:

**1. *Axiom of Positivity*:** *It states that for any event* $E, P(E) \geq 0$

**2. *Axiom of Uncertainity*:** *For the sample space S that contains all possible outcomes,* $P(S) = 1$

**3. *Axiom of Addition*:** *For any two mutually exclusive events* $E_1, E_2$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

***Problems***:

1. *What is the probability that a card drawn at random from the pack of playing card may be either a Queen or a king?*

**Solu**: *Let S be the sample space associated with drawing of a card*

$n(s) = 52_{c_1} = 52$

*Let* $E_1$ *be the event of the card draw being a queen.*

$$n(E_1) = 4_{c_1} = 4$$

*Let* $E_2$ *be the event of the card draw being a king.*

$$n(E_2) = 4_{c_1} = 4$$

But are $E_1, E_2$ mutually exclusive events.

Since $E_1 \cup E_2$ *is the event of drawing eithe a queen or a king, we have*

$$P(_1 \cup E_2) = P(_1) + P(_2) = \frac{n(E_1)}{n(s)} + \frac{n(E_2)}{n(s)} = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$$

2. *Three students are running in a race the probability of winning in the race is equal for both A and B. The probability of C to win in the race is equal to twice the probability of either A or B. Find the probability that either A or C wins the race.*

$Sol$: $Let\ E_1 \rightarrow represents\ the\ event\ of\ A\ to\ win\ in\ the\ race$

$E_2 \rightarrow represents\ the\ event\ of\ B\ to\ win\ in\ the\ race$

$E_3 \rightarrow represents\ the\ event\ of\ C\ to\ win\ in\ the\ race$

$P(E_1) = P(E_2)$

$P(E_3) = 2P(E_1) = 2P(E_2)$

$Sample\ space\ S = E_1 \cup E_2 \cup E_3$

$By\ the\ axiom\ of\ uncertainity\ P(S) = P(E_1) + P(E_2) + P(E_3)$

$\Rightarrow 1 = 2P(E_1) + 2P(E_1)$

$\Rightarrow 1 = 4P(E_1)$

$\Rightarrow P(E_1) = \frac{1}{4} = P(E_2)$

$and\ P(E_3) = 2P(E_1) = 2 \times \frac{1}{4} = \frac{1}{2}$

$\therefore P(E_1 \cup E_3) = P(E_1) + P(E_3)$

$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ $[since\ E_1 \&\ E_3\ are\ mutually\ exclusive]$

**Practices problems**

1. A card is drawn at random from a pack of 52 cards. Find the probability that the drawn card is either a club or an ace of diamond.

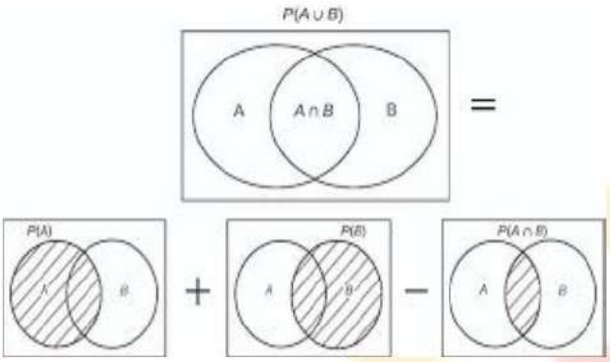
2. A herd contains 30 cows numbered from 1 to 30. One cow is selected at random. Find the probability that number of the selected cow is a multiple of 5 or 8.

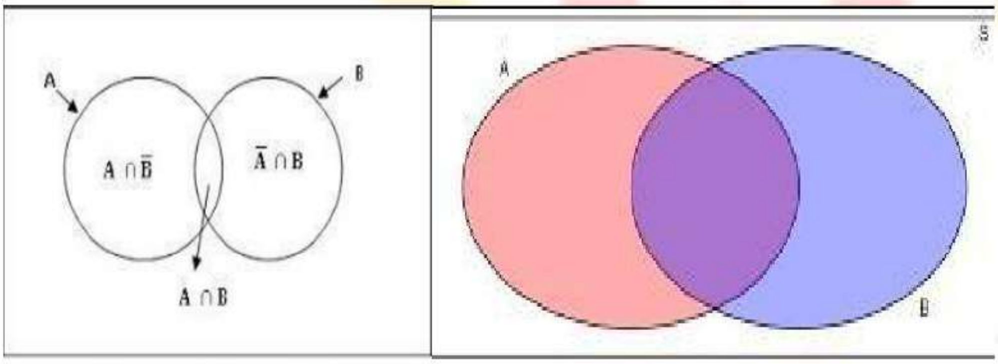
## ***Addition theorem on Probability***:

If S is a sample space, and $\boldsymbol{A}, \boldsymbol{B}$ are any events in S then

$$P(A\ or\ B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof:-




Using venn diagram, we observe that the events A and $\bar{A} \cap B$ are disjoint

$\therefore A \cup B$ = AU $(\bar{A} \cap B)$ [ Since P$(E_1 \cup E_2)$ =P$(E_1)$+P$(E_2)$]

P$(A \cup B)$ =P(A) + P$(\bar{A} \cap B)$

P$(A \cup B)$ =P(A) + P$(\bar{A} \cap B)$ + P(A∩B) - P(A∩B)

P$(A \cup B)$ =P(A)+{P$(\bar{A} \cap B)$+ P(A∩B)} - P(A∩B)

Since $\bar{A} \cap B$ and A∩B disjoint events , the union of these two give B

**P$(A \cup B)$ =P(A)+P(B) - P(A∩B)**

**Problems:**

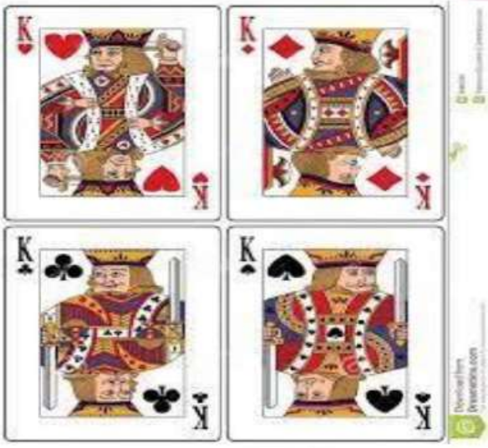
**PROBLEM 1**: **ONE CARD IS DRAWN FROM A REGULAR DECK OF 52 CARDS.WHAT IS THE PROBABILITY OF THE CARD BEING EITHER RED OR A KING.**

SOLUTION: Let A represents an event that the card is red and Let B represents an event that the card is king.

For a card to either red or king, it is required to find AUB

P (A) =26/52

P (B) =4/52



There are two red coloured king cards, Hence

P (A∩B) =2/52

Therefore P(AUB) = P (A)+P(B)- P(A∩B)

$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52}$$

$$= \frac{7}{13}$$

**<u>Example 2.</u> A Bag contains 12 balls numbered from 1 to 12.If a ball is taken at random, what is the probability of having a ball with a number which is multiple of either 2 or 3**

**Solution:** Let A be the event of number being a multiple of 2 and B be the event of number being a multiple of 3.

Favorable cases for event A are {2, 4, 6, 8, 10, and 12}
Similarly favorable cases for event B are {3, 6, 9, 12}
The probability of the number being a multiple of 2 is P (A) = $\frac{6}{12}$

The probability of the number being a multiple of 3 is P (B) = $\frac{4}{12}$

Therefore the events are not mutually exclusive

P (A∩B) = 2/12

The probability that the selected number is a multiple of 2 or 3 is

Therefore P(AUB)=P(A)+P(B)- P(A∩B)

$=\frac{6}{12}+\frac{4}{12}-\frac{2}{12}=2/3$

**Example 3: A** card is drawn at random from a pack of 52 cards. Find the probability that the drawn card is either a spade or a king.

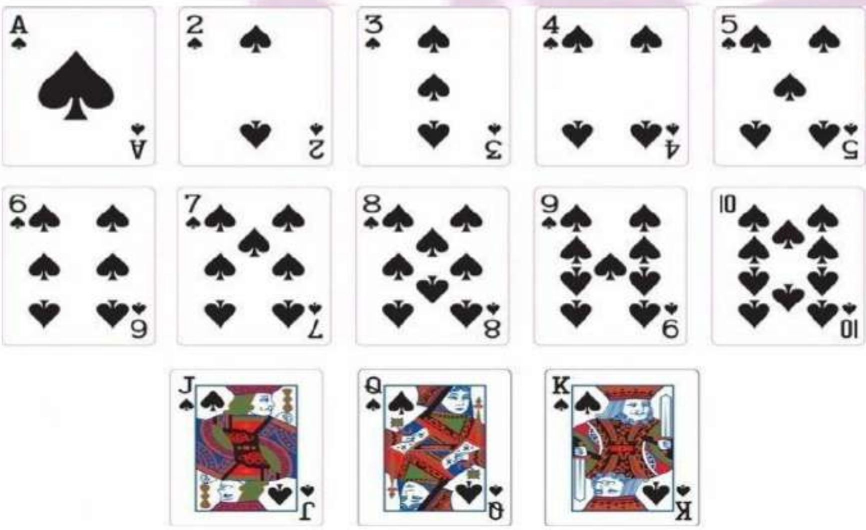
**Solution:** Let A: Event of drawing a card of spade andB:

Event of drawing a king card

The probability of drawing a card of spade

$P(A)=\frac{13}{52}$

The probability of drawing a king card $P(B)=\frac{4}{52}$



The probability of drawing a king of spade is

$P(A\cap B)=\frac{1}{52}$

So, the probability of the drawing a spade or king card is:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$$

**Example 4. A herd contains 30 cows numbered from 1 to 30. One cow is selected at random. Find the probability that the number of the selected cow is a multiple of 5 or 6.**

**Solution:** Let A be the event of number being a multiple of 5 within 30 and B be the event of number being a multiple of 6 within 30.

Favorable cases for event A are {5, 10, 15, 20, 25, 30}

Similarly favorable cases for event B are {6, 12, 18, 24, 30}

The probability of the number being a multiple of 5 within 30 is P(A)= $\frac{6}{30}$

The probability of the number being a multiple of 6within 30 is P(B)= $\frac{5}{30}$

Since 30 is a multiple of 5 as well as 6, therefore the events are not mutually exclusive

$$P(A \cap B) = P(A \text{ and } B) = \frac{1}{30}$$

The probability that the number of the selected cow is a multiple of 5 or 6 is :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{6}{30} + \frac{5}{30} - \frac{1}{30} = \frac{10}{30} = \frac{1}{10}$$

**Example 5.** A number was drawn at random from the numbers 1 to 50. What is the probability that it will be a multiple of 2 or 3 or 10?

**Solution:** Probability of getting a multiple of 2: $P(A) = \frac{25}{50}$

Probability of getting a multiple of 3: $P(B) = \frac{16}{50}$

Probability of getting a multiple of 10: $P(C) = \frac{5}{50}$

Common Probability of getting a multiple of 2 and 3:

$$P(A \cap B) = \frac{8}{50}$$

Common Probability of getting a multiple of 3 and 10:

$$P(B \cap C) = \frac{1}{50}$$

Common Probability of getting a multiple of 2 and 10:

$$P(A \cap C) = \frac{5}{50}$$

Common Probability of getting a multiple of 2, 3 and 10:

$P(A \cap B \cap C) = \frac{1}{50}$

Probability that it is a multiple of 2 or 3 or 10:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$= \frac{25}{50} + \frac{16}{50} + \frac{5}{50} - \frac{8}{50} - \frac{1}{50} - \frac{5}{50} + \frac{1}{50} = \frac{33}{50}$$

**Example 6**. *In a city there are three news papers A, B, C. A is read by 20% , B is read by 16%, C is read by 14% . Both A and B are read by 8% , B and C read by 4% & A and C readby 5% , A , B and C is read by 2% . Find the probability of the people that read atleastany one of them.*

*Sol*: *Let* $E_1 \rightarrow$ *represents the event of reading the news paper A*

$E_2 \rightarrow$ *represents the event of reading the news paper B*

$E_3 \rightarrow$ *represents the event of reading the news paper C*

$$\text{we have } P(E_1) = \frac{20}{100}\ ;\ P(E_2) = \frac{16}{100};\ P(E_3) = \frac{14}{100}$$

$$P(E_1 \cap E_2) = \frac{8}{100}\ ;\ P(E_2 \cap E_3) = \frac{4}{100};\ P(E_1 \cap E_3) = \frac{5}{100} \text{ and } P(E_1 \cap E_2 \cap E_3) = \frac{2}{100}$$

$\text{then } P(E_1 \cup E_2 \cup E_3) = ?$

$\text{we have } P(E_1 \cup E_2 \cup E_3)$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) - P(E_1 \cap E_2 \cap E_3)$$

$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{4}{100} - \frac{5}{100} + \frac{2}{100}$$

$$= \frac{52 - 17}{100} = \frac{35}{100}$$

$\therefore 35\%$ *of people read either A or B or C*

***Conditional event*:**

*If A, B are events of sample space S and if B occurs after the occurrence of A, then the event of occurrence of B after the event A is called conditional event of B given by A.*

*It is denoted by* $\frac{B}{A}$. Similarly we define $\frac{A}{B}$

Ex: A die is thrown 3 times. The event of getting the sum of the numbers thrown is 15 when it is known that the first throw was a 5 is a conditional event.

***Conditional Probability*:**

*Let A and B are two events in a sample space S and* $P(A) \neq 0$, *then the probability of B, after the event A has occured, is called the conditional probability of the event of B given A and is denoted by* $P(\frac{B}{A})$ and is defined by

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

Similarly we define

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

***Multiplication theorem of probability*:**

In a random experiment if A, B are two events such that $P(A) \neq 0$ and $P(B) \neq 0$, then

$$P(A \cap B) = P(A).P\left(\frac{B}{A}\right)$$

$$P(B \cap A) = P(B).P\left(\frac{A}{B}\right)$$

***Compound event***:

When two or more events occur in conjunction with each other, their joint occurrence is called compound event

EX: When a die and a coin are tossed the event of getting at most 4 on the die and head on the coin is a compound event and separately they are independent events.

***Independent Events***:

*Events are said to be independent if the chance of happening of one will not effect the chance of happening of the other.*

i.e If the occurrence of the event B is not affected by the occurrence or non- occurrence of the event A, then the event B is said to be independent of A and

$$P\left(\frac{B}{A}\right) = P(B)$$

NOTE: From multiplication theorem

$$\boxed{\boldsymbol{P(A \cap B) = P(A).P(B)}}$$

If the occurrence of the event B is effected by the occurrence of A, then the events A,B, are dependent and

$P\left(\frac{B}{A}\right) \neq P(B)$

Problems:

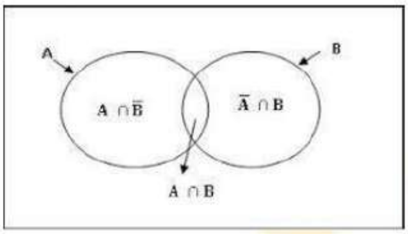
1. Determine (i) $(\frac{\ }{A})$ (ii) $P(\frac{\ }{B^C})$ if A and B are events with $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$, P(A B)= $\frac{1}{2}$.

**Solu:** Given $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$, $P(AUB)=\frac{1}{2}$

From Addition Theorem $P(AUB)=P(A)+P(B)- P(A \cap B)= \frac{1}{3}+\frac{1}{4}-\frac{1}{2}=\frac{1}{12}$

(i) $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/3} = \frac{1}{4}$

(ii) $P(B^c) = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$




$P(A \cap B^C) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$

$P\left(\frac{A}{B^C}\right) = \frac{P(A \cap B^C)}{P(B^c)} = \frac{1/4}{3/4} = \frac{1}{3}$

2. *What is the probability that the total of two dice will be greater than* 9, *given*
3. *that the first die is a* 5?

***Solution***:

*Let* $A =$ *first die is* 5

*Let* $B =$ *total of two dice is greater than* 9

$P(A) = \frac{1}{6}$

*Possible outcomes for A and B*: (5, 5), (5, 6)

$P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{1/18}{1/6} = \frac{1}{3}$

4. ***Box A contains* 5 *red and* 3 *white marbles and box B contains* 2 *red and* 6 *white marbles. If a marble is dram from each box, what is the Probability that they***

***are both of same colour.***

***Solu:***

*Let* $E_1 \rightarrow$ *represents the event that the marble is drawn from box A and is red.*

$$P(E_1) = \frac{1}{2}.\frac{5}{8} = \frac{5}{16}$$

*Let* $E_2 \rightarrow$ *represents the event that the marble is drawn from box Band is red.*

$$P(E_2) = \frac{1}{2}.\frac{2}{8} = \frac{1}{8}$$

The Probability that both the marbles are red is

$$P(E_1 \cap E_2) = P(E_1).P(E_2) = \frac{5}{16} . \frac{1}{8} = \frac{5}{128}$$

*Let* $E_3 \rightarrow$ *represents the event that the marble is drawn from box A and is white.*

$$P(E_3) = \frac{1}{2}.\frac{3}{8} = \frac{3}{16}$$

*Let* $E_4 \rightarrow$ *represents the event that the marble is drawn from box Band is white*

$$P(E_4) = \frac{1}{2}.\frac{6}{8} = \frac{3}{8}$$

The Probability that both the marbles are white is

$$P(E_3 \cap E_4) = P(E_3).P(E_4) = \frac{3}{16} . \frac{3}{8} = \frac{9}{128}$$

*The probability that the morbles are of the same colour*
$= P(E_1 \cap E_2) + P(E_3 \cap E_4)$

$$= \frac{5}{128} + \frac{9}{128} = \frac{14}{128}$$

$$= \frac{7}{64} = 0.109$$

4. ***Find the probability of drawing 2 red balls in succession from a bag containing 4 red balls and 5 black balls when the ball drawn first is 1) not replaced 2) replaced***

*Sol: Let* $E_1 \rightarrow$ *represents the event of drawing a red ball in the first draw*

$E_2 \rightarrow$ *represents the event of drawing a red ball in the second draw we have*

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$P(E_1 \cap E_2) = P(E_1).P\left(\frac{E_2}{E_1}\right)$

1) $P(E_1 \cap E_2) = \frac{4c_1}{9c_1}.\frac{3c_1}{8c_1} = \frac{12}{72} = \frac{1}{6}$
2) $P(E_1 \cap E_2) = \frac{4c_1}{9c_1}.\frac{4c_1}{9c_1} = \frac{16}{81}$

**BAYE'S THEOREM:**

**Statement:** $E_1, E_2, E_3$............ $E_n$ are n mutually exclusive and exhaustive events such that

$P(E_i) > 0$ (i = 1,2,…..,n) in a sample space S and A is any other event in S intersecting with

Every $E_i$(i.e A can occur in combination with any one of the events $E_1, E_2, E_3$............ $E_n$ ) such

that P(A) > 0

If $E_i$ is any of the events of $E_1, E_2, E_3 \ldots\ldots\ldots E_n$ where $P(E_1), P(E_2)$................$P(E_n)$ and $P(A/E_1)$,$P(A/E_2)$.............$P(A/E_n)$ are known, then

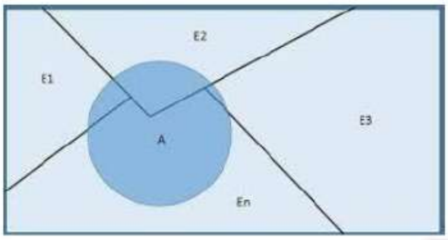
$$P\left(\frac{}{A}\right) = \frac{P(E_K).P\left(\frac{A}{E_K}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+\cdots+P(E_n).P\left(\frac{A}{E_n}\right)}$$

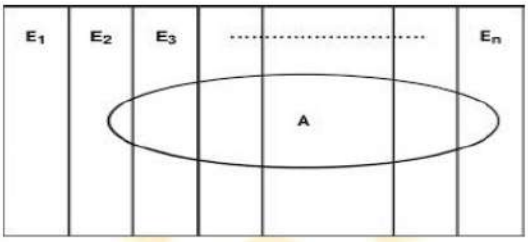
## Proof:-

$E_1, E_2, E_3$............ $E_n$ are n mutually exclusive and exhaustive events in a sample space S

$S = E_1 \cup E_2 \cup E_3 \cup \ldots\ldots\ldots \cup E_n$

A is any other event in S intersecting with every $E_i$






$A = A \cap S$

$A = A \cap (E_1 \cup E_2 \cup E_3 \cup \ldots\ldots\ldots\ldots \cup E_n)$

$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots\ldots\ldots\ldots \cup (A \cap E_n)$

Hence $(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots\ldots\ldots\ldots \cup (A \cap E_n)$ are mutually exclusive events. Then

$$P\left(\frac{E_K}{A}\right) = \frac{P(E_K \cap A)}{P(A)}$$

$$= \frac{P(E_K \cap A)}{P((A\cap E_1)\cup(A\cap E_2)\cup(A\cap E_3)\cup\ldots\ldots\ldots\cup(A\cap E_n))}$$

$$= \frac{P(E_K \cap A)}{P(A\cap E_1)+P(A\cap E_2)+P(A\cap E_3)+\cdots+P(A\cap E_n)}$$

($A \cap E_1, A \cap E_2, \ldots, A \cap E_n$ Disjoint sets)

$$= \frac{P(E_K).P\left(\frac{A}{E_K}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+\cdots+P(E_n).P\left(\frac{A}{E_n}\right)}$$

*A random variable is defined as a real valued function for which every point of sample space is mapped to a real value.*

*It is denoted by* $X: S \rightarrow R$ *where* $S$ *represents the sample space* *and $R$ is the real field (or) real line*

***Definition***: *A real variable $X$ whose value is determined by the outcome of a random experiment is called a random variable.*

*Ex*: 1) *Consider a random experiment of tossing a coin, then* $S = \{T, H\}$ *then* $X = \{0,1\}$ *that represents the number of heads or tails.*

2) *If two coins are tossed simultaneously then* $S = \{TT, TH, HT, HH\}$ *then* $X = \{0,1,2\}$ *that represents the number of heads or tails.*

*There are two types of random variables*

1) ***Discrete Random Variable***

2) ***Continuous Random Variable***

1) ***Discrete Random Variable***: *A random variable $X$ which can take only a finite number of discrite values in an interval of domain is called a discrete random variable.*

OR

*If the random variable takes the values only on the set* $\{0,1,2, \dots . n\}$ *is called a discrete random variable*

*Ex*: *Marks of students, no. of pages in a book, no. of accidents on a road, no. of mistakes printing in a book etc.*

2) ***Continuous Random Variable***: *A Random Variable $X$ which can take values continuously i.e which takes all possible values in a given interval is called a Continuous Random Variable.*

*Ex*: *heights and weights of individuals, temperature and time are continous random variables.*

:

***Probability mass function of a Discrete Random Variable***: −

*If for a Discrete Random Variable X, the real valued function P(x) is such that* $P(X = x) = P(x)$ *then P(x) is called Probability mass function or Probability Density function of a Discrete Random Variable X.*
*Probability function p(x) gives the measure of probability for different values of X*

**Properties**:

1) $p(x) \geq 0$ *for all x*

(2) $\sum_{i=1}^{n} p(x) = 1$

(3) $p(x)$ *cannot be negative for any value of x*

**Probability Distribution Function**:

*Let X be a Random Variable, then the Probability Distribution function associated with X is*

$F(x) = P(X \leq x), -\infty < x < \infty$ *is called the Distribution function of X.*

**Discrete probability distribution or Probability Mass Function (P.M.F)**:

*Let X be a discrete random variable that takes n number of values given by*

$X = \{x_1, x_2, \dots \dots, x_n\}$

*The Probability Mass Function of X is denoted by* $p(x_i) = p_i = P[X = x_i]$

*for* $i = 1, 2, \dots\dots, n$

*(i.e) It is defined as the probability that X takes a value of* $x_i$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| X | 1 | 2 | - | - | | - | |
| P(X) | 1 | 2 | - | - | | | |

$Ex$: 1) $For\ the\ random\ experiment\ of\ tossing\ a\ coin\ S = \{T, H\}\ and\ X = \{0,1\}$

$The\ Probability\ distribution\ is$

| | | |
|---|---|---|
| | $0(\ _1)$ | $1(\ _2)$ |
| $(\ )$ | $P[x = 0] = 1/2$ | $P[x = 1] = 1/2$ |

2) $For\ the\ random\ experiment\ of\ a\ tossing\ two\ coins\ at\ a\ time\ S = \{TT, TH, HT, HH\}$

$and\ X = \{0,1,2\}$

$then\ Probability\ distribution\ is$

| | | | |
|---|---|---|---|
| | 0 | 1 | 2 |
| $(\ ) =$ | $P[x = 0] = 1/4$ | $P[x = 1] = 2/4$ | $P[x = 2] = 1/4$ |

***Distribution Function***:

$It\ is\ also\ called\ as\ cumulative\ distribution\ function\ denoted\ by$

$F(x) = P[X \leq x]$

$Ex$: $For\ the\ random\ experiment\ of\ a\ tossing\ two\ coins\ at\ a\ time\ for\ which$
$X = \{0,1,2\}\ then\ P.M.F\ is$

| | | | |
|---|---|---|---|
| | 0 | 1 | 2 |
| $(\ ) =$ | $P[x = 0] = 1/4$ | $P[x = 1] = 2/4$ | $P[x = 2] = 1/4$ |

$F(x_1) = F(0) = P[X \leq 0] = \frac{1}{4}$

$F(x_2) = F(1) = P[X \leq 1]$

$= P[X = 0] + P[X = 1]$

$= \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$

$F(x_3) = F(2) = P[X \leq 2]$

$= P[X = 0] + P[X = 1] + P[X = 2]$

$= \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$

***es***:

1) $0 \leq F(x) \leq 1$

2) $If\ x < y\ then\ F(x) < F(y)$

3) $F(-\infty) = 0\ and\ F(\infty) = 1$

***Problems***:

1) *Consider the random experiment of thrown two dice at a time. Let the random variable represents the sum on both the faces. Prepare a table that represents the probability mass function and hence evaluate the cumulative distributive function.*

*Sol: The random variable X is given by*

$X \rightarrow sum\ on\ both\ the\ faces$

$X = \{2,3,4,5,6,7,8,9,10,11,12\}$

*The sample space is given by*

$S = \{ (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6), (2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$

$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6), (4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$

$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6), (6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$

*Probability Mass Function*

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

*Cumulative Distribution*

| | ( ) |
|---|---|
| 2 | $\frac{1}{36}$ |

| | |
|---|---|
| 3 | $\frac{1}{36}+\frac{2}{36}=\frac{3}{36}$ |
| 4 | $\frac{6}{36}$ |
| 5 | $\frac{10}{36}$ |
| 6 | $\frac{15}{36}$ |
| 7 | $\frac{21}{36}$ |
| 8 | $\frac{26}{36}$ |
| 9 | $\frac{30}{36}$ |
| 10 | $\frac{33}{36}$ |
| 11 | $\frac{35}{36}$ |
| 12 | $\frac{36}{36}=1$ |

***Mathematical Expectation***:

*Let $X$ be a discrete random variable that takes $n-$ values given by $X=\{x_1,x_2,\dots,x_n\}$ with respect to the probabilities $P=\{p_1,p_2,\dots,p_n\}$. The mathematical expectation of $X$ is given by*

$$\boxed{E(x)=\sum_{i=1}^{n} x_i p_i}$$

*Similarly* $E(x^2)=\sum_{i=1}^{n}(x_i)^2 p_i$

*Ex: For the random experiment of tossing a coin we have*

| | | |
|---|---|---|
| | 0 | 1 |
| | 1/2 | 1/2 |

$we\ have\ E(x) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

***Properties***:

1) $E[aX] = a.E[X], a \neq 0$ , *a constant*

2) $E[aX + b] = aE[X] + b$ , *a & b are constants.*

3) $E[X + Y] = E[X] + E[Y]$

4) $E[X.Y] = E[X].E[Y]$

***Joint Probability Mass Function***:

*Consider two variables X and Y where X takes n − values given by*

$X = \{x_1, x_2, \dots, x_n\}$ *Y takes m − values given by* $Y = \{y_1, y_2, \dots, y_m\}$.

$E(X) = \sum_{i=1}^{n} p_i x_i$ *and* $E(Y) = \sum_{j=1}^{m} p_j y_j$

*The joint probability is denoted by* $p_{ij} = P[X = x_i \cap Y = y_j] = p(x_i, y_j)$

$p_{12}$ *means* $= P[X = x_1 \cap Y = y_2]$

*<u>Properties:</u>*

1. $\sum_{i=1}^{n} p_{ij} = p_j$ *for some j*

2. $\sum_{j=1}^{m} p_{ij} = p_i$ *for some i*

***Parameters***:

1) $\boxed{Mean\ \mu = E[X]}$

2) $\boxed{Variance\ (\sigma^2) = E[X^2] - (E[X])^2}$

3) $\boxed{Standard\ deviation\ (\sigma) = \sqrt{variance} = \sqrt{E[X^2] - (E[X])^2}}$

*Results on Variance*:

1. $V(k) = 0$

2. $V(kX) = k^2\ V(X)$

3. $V(X + k) = V(X)$

4. $V(ax + b) = a^2 V(x)$

5. *If x and y are 2 independent random variables then* $V(X \pm Y) = V(X) \pm V(Y)$

***Problems***:

1. *A discrete random variable X has the following probability mass function*

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| | k | 2k | 2k | 3k | 7k | k |

1) *Find k* 2) *Find* $P(X > 3)$ 3) $P(X \leq 4)$ 4) *Mean* 5) *Variance*

1) $\sum_{i=1}^{6} p_i = 1$

$\Rightarrow k + 2k + 2k + 3k + 7k + k = 1$

$\Rightarrow 16k = 1 \Rightarrow \boxed{=\frac{1}{16}}$

2)

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| | 1/16 | 2/16 | 2/16 | 3/16 | 7/16 | 1/16 |

$P(X > 3) = P(X = 4) + P(X = 5)$

$$= \frac{7}{16} + \frac{1}{16} = \frac{8}{16}$$

$$= \frac{1}{2}$$

3) $P(X \leq 4) = 1 - P(X > 4)$

$= 1 - P(X = 5)$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

4) $Mean\ \mu = E[x] = \sum_{i=1}^{6} p_i x_i$

$$= 0 \times \frac{1}{16} + 1 \times \frac{2}{16} + 2 \times \frac{2}{16} + 3 \times \frac{3}{16} + 4 \times \frac{7}{16} + 5 \times \frac{1}{16}$$

$$= 0 + \frac{2}{16} + \frac{4}{16} + \frac{9}{16} + \frac{28}{16} + \frac{5}{16}$$

$$= \frac{48}{16} = 3$$

5) $Variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$$E[X^2] = \sum_{i=1}^{6} (x_i)^2 p_i$$

$$= 0^2 \times \frac{1}{16} + 1^2 \times \frac{2}{16} + 2^2 \times \frac{2}{16} + 3^2 \times \frac{3}{16} + 4^2 \times \frac{7}{16} + 5^2 \times \frac{1}{16}$$

$$= 0 + \frac{2}{16} + \frac{8}{16} + \frac{27}{16} + \frac{112}{16} + \frac{25}{16}$$

$$E[X^2] = \frac{174}{16}$$

$Variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$$= \frac{174}{16} - (3)^2 = \frac{174}{16} - 9$$

$\therefore \sigma^2 = 1.875$

*Note: The alternative definition of variance is* $\boxed{\sigma^2 = E\,[X - E[x]]^2}$

2) *Let X be a discrete random variable with the folowing probability mass function*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| | 0 | k | 2k | 2k | 3k | 2 | 2 2 | 7 2 + |

*Evaluate the following*

1) $k$ 2) $P(X \geq 6)$ 3) $P(X \leq 6)$ 4) $P(0 < X < 5)$ 5) $P(0 \leq X \leq 4)$

6) *mean* 7) *variance* 8) *Distribution function* 9) *find k forwhich* $P(X \leq k) > \dfrac{1}{2}$

*Sol*: 1) $\sum_{i=1}^{8} p_i = 1$

$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 10k^2 + 9k = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

*By solving we get* $k = \dfrac{1}{10}, -1$

*probability mass function*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| ( ) | 0 | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{3}{10}$ | $\frac{1}{100}$ | $\frac{2}{100}$ | $\frac{17}{100}$ |

2) $P(X \geq 6) = P(X = 7) = \dfrac{17}{100}$

3) $P(X \leq 6) = 1 - P(X > 6)$

$= 1 - \frac{17}{100} = \frac{83}{100}$

4) $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10}$$

5) $P(0 \leq X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10}$$

6) $Mean\ \mu = E[x] = \sum_{i=1}^{8} p_i x_i$

$$= 0 + \frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10} + \frac{5}{100} + \frac{12}{100} + \frac{119}{100}$$

$$= \frac{23}{10} + \frac{136}{100}$$

$$= \frac{336}{100} = 3.66$$

7) $variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$$E[X^2] = \sum_{i=1}^{8} (x_i)^2 p_i$$

$$= 0 + 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{2}{10} + 4^2 \times \frac{3}{10} + 5^2 \times \frac{1}{100} + 6^2 \times \frac{2}{100} + 7^2 \times \frac{17}{100}$$

$$= \frac{1}{10} + \frac{8}{10} + \frac{18}{10} + \frac{48}{10} + \frac{25}{100} + \frac{72}{100} + \frac{833}{100}$$

$$= \frac{75}{10} + \frac{930}{100} = \frac{168}{10} = 16.8$$

$\therefore\ variance\ (\sigma^2) = 16.8 - (3.66)^2 = 3.4044$

$Standard\ deviation\ (\sigma) = \sqrt{3.41} = \pm 1.84$

8) $Distribution\ function$

| | ( ) |
|---|---|
| 0 | 0 |

| | |
|---|---|
| 1 | $\frac{1}{10}$ |
| 2 | $\frac{3}{10}$ |
| 3 | $\frac{5}{10}$ |
| 4 | $\frac{8}{10}$ |
| 5 | $\frac{81}{100}$ |
| 6 | $\frac{83}{100}$ |
| 7 | $\frac{100}{100} = 1$ |

9) $P(X \leq k) > \frac{1}{2}$

$\Rightarrow P(X \leq x) > \frac{1}{2}$

$\Rightarrow F[x] > \frac{1}{2}$

$\Rightarrow k = 4$

$\therefore 4 \textit{ is the maximum value of } k \textit{ at which } P(X \leq x) > \frac{1}{2}$

3. *Four coins are tossed simultaneously at once. Let the random variable X represents the number of heads. Prepare the probability mass function and hence calculate mean variance and the standard deviation.*

*Sol: The probability mass function is below*

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| ( ) | $\frac{1}{16}$ | $\frac{4}{16}$ | $\frac{6}{16}$ | $\frac{4}{16}$ | $\frac{1}{16}$ |

$$Mean\ \mu = E[x] = \sum_{i=1}^{5} p_i x_i$$

$$= 0 \times \frac{1}{16} + 1 \times \frac{4}{16} + 2 \times \frac{6}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}$$

$$= 0 + \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = \frac{32}{16}$$

$$= 2$$

$$variance\ (\sigma^2) = E[X^2] - (E[X])^2$$

$$E[X^2] = 0^2 \times \frac{1}{16} + 1^2 \times \frac{4}{16} + 2^2 \times \frac{6}{16} + 3^2 \times \frac{4}{16} + 4^2 \times \frac{1}{16}$$

$$= \frac{4}{16} + \frac{24}{16} + \frac{36}{16} + \frac{16}{16}$$

$$= \frac{80}{16} = 5$$

$$\sigma^2 = 5 - 2^2 = 5 - 4 = 1$$

$$S.D = \sigma = \sqrt{E[X^2] - (E[X])^2} = \sqrt{1} = \pm 1$$

4. *The following table represents the probability mass function of a discrete random variable X. Evaluate k and hence mean and variance.*

| | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| ( ) | k | 0.1 | k | 0.2 | 2k | 0.4 | 2k |

$$we\ have\ \sum_{i=1}^{7} p_i = 1$$

$$\Rightarrow k + 0.1 + k + 0.2 + 2k + 0.4 + 2k = 1$$

$$\Rightarrow 6k + 0.7 = 1$$

$$\Rightarrow 6k = 0.3 = \frac{3}{10}$$

$\Rightarrow k = \frac{13}{60} = \frac{1}{20} = 0.05$

*probability mass function*

| | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| ( ) | 0.05 | 0.1 | 0.05 | 0.2 | 0.1 | 0.4 | 0.1 |

$Mean(\mu) = E[X] = \sum_{i=1}^{7} p_i x_i$

$= (-3) \times 0.05 + (-2) \times 0.1 + (-1) \times 0.05 + 0 \times 0.2 + 1 \times 0.1 + 2 \times 0.4 + 3 \times 0.1$

$\mu = 0.8$

$variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$E[X^2] = (-3)^2 \times 0.05 + (-2)^2 \times 0.1 + (-1)^2 \times 0.05 + 0^2 \times 0.2 + 1^2 \times 0.1 + 2^2 \times 0.4 + 3^2 \times 0.1$

$= 3.5$

$\sigma^2 = 3.5 - (0.8)^2 = 2.86$

$S.D = \sqrt{\sigma} = \sqrt{2.86} = \pm 1.65$

*4. Consider the random experiment of throwing two dice at a time. Let the random variable X represents the maximum value from both the faces. Prepare the p.m.f and hence mean and variance.*

*Sol:* $Let\ X = \{1,2,3,4,5,6\}$

*Probability mass function*

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| ( ) | $\frac{1}{36}$ | $\frac{3}{36}$ | $\frac{5}{36}$ | $\frac{7}{36}$ | $\frac{9}{36}$ | $\frac{11}{36}$ |

$Mean(\mu) = E[X] = \sum_{i=1}^{6} p_i x_i$

$$= 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36}$$

$$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36}$$

$$\therefore \mu = \frac{161}{36} = 4.472$$

$variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$$E[X^2] = 1^2 \times \frac{1}{36} + 2^2 \times \frac{3}{36} + 3^2 \times \frac{5}{36} + 4^2 \times \frac{7}{36} + 5^2 \times \frac{9}{36} + 6^2 \times \frac{11}{36}$$

$$= \frac{1}{36} + \frac{12}{36} + \frac{45}{36} + \frac{112}{36} + \frac{225}{36} + \frac{396}{36}$$

$$= \frac{791}{36} = 21.972$$

$(\sigma^2) = E[X^2] - (E[X])^2$

$= 21.972 - (4.472)^2$

$= 21.972 - 19.998$

$\sigma^2 = 1.974$

$S.D = \sigma = \sqrt{1.974} = \pm 1.404$

5. *Let the discrete random variable X be represented by the following P.M.F.*

*Find the mean and variance*

| | | | | |
|---|---|---|---|---|
| | 0 | 1 | ………… | n |
| ( ) | $1/n$ | $1/n$ | ………… | $1/n$ |

Solution: $Mean(\mu) = E[X] = \sum_{i=1}^{n} p_i x_i$

$$= 1 \times \frac{1}{n} + 2 \times \frac{1}{n} + \cdots \ldots + n \times \frac{1}{n}$$

$$= \frac{1}{n}[1 + 2 + 3 + \cdots \ldots + n]$$

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right]$$

$$\therefore \mu = \frac{n+1}{2}$$

$$variance\ (\sigma^2) = E[X^2] - (E[X])^2$$

$$E[X^2] = \sum_{i=1}^{n} p_i x_i^2$$

$$= 1^2 \times \frac{1}{n} + 2^2 \times \frac{1}{n} + \cdots \ldots . + n^2 \times \frac{1}{n}$$

$$= \frac{1}{n}[1^2 + 2^2 + \cdots + n^2]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$E[X^2] = \frac{(n+1)(2n+1)}{6}$$

$$\therefore variance\ (\sigma^2) = E[X^2] - (E[X])^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right]$$

$$= (n+1)\left[\frac{4n+2-3n-3}{12}\right]$$

$$= (n+1)\left[\frac{n-1}{12}\right]$$

$$= \frac{n^2-1}{12}$$

6. $A\ box\ contains$ 10 $items\ out\ of\ which$ 3 $were\ found\ to\ be\ defective. A\ sample\ of$ 4

*items were drawn. Let the random variable X denotes the number of defects in the sample space. Prepare the probability mass function and hence evaluate the mean and variance.*

$Sol: Here\ X = \{0,1,2,3\}$

1) $X = 0$

$P(0) = P[All\ are\ good]$

$= \frac{7_{C_4}}{10_{C_4}}$

$= \frac{\left(\frac{7\times6\times5\times4}{1\times2\times3\times4}\right)}{\left(\frac{10\times9\times8\times7}{1\times2\times3\times4}\right)}$

$= \frac{7\times6\times5\times4}{10\times9\times8\times7}$

$\therefore P(0) = \frac{1}{6}$

2) $X = 1$

$P(1) = P[1\ defect\ and\ 3\ good]$

$= \frac{3_{C_1}\times7_{C_3}}{10_{C_4}}$

$\therefore P(1) = \frac{1}{2}$

3) $X = 2$

$P(2) = P[2\ defect\ and\ 2\ good]$

$= \frac{3_{C_2}\times7_{C_2}}{10_{C_4}}$

$\therefore P(2) = \frac{3}{10}$

4) $X = 3$

$P(3) = P[3\ defect\ and\ 1\ good]$

$= \frac{3_{C_3} \times 7_{c1}}{10_{C_4}}$

$\therefore P(3) = \frac{1}{30}$

$P.M.F$

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| ( ) | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{30}$ |

$Mean(\mu) = E[X] = \sum_{i=1}^{4} p_i x_i$

$= 0 \times \frac{1}{6} + 1 \times \frac{1}{2} + 2 \times \frac{3}{10} + 3 \times \frac{1}{30}$

$= 0 + \frac{1}{2} + \frac{6}{10} + \frac{1}{10}$

$= \frac{12}{10} = \frac{6}{5}$

$\therefore \mu = 1.2$

$E[X^2] = \sum_{i=1}^{4} p_i x_i^2$

$= 0^2 \times \frac{1}{6} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{3}{10} + 3^2 \times \frac{1}{30}$

$= \frac{1}{2} + \frac{12}{10} + \frac{9}{30}$

$= \frac{60}{30} = 2$

$variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$= 2 - (1.2)^2 = 2 - 1.44$

$\sigma^2 = 0.56$

***<u>Continuous Random Variables</u>:***

***Probability Density Function*:**

*A continuous random variable X is identified with a function called as Probability Density Function denoted by* $f(x)$*and given by*

$$f(x) = P\left[x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right]$$

***Properties*:**

1. $f(x) \geq 0$

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

3. $P[E] = \int_E f(x)\, dx$

*Ex*: 1) $P[1 \leq X \leq 2] = \int_1^2 f(x)\, dx$

2) $P[X \leq 5] = \int_{-\infty}^{5} f(x)\, dx$

3) $P[X \geq 5] = \int_5^{\infty} f(x)\, dx$

***Cumulative Distributive Function $F(x)$*:**

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(x)\, dx$$

$$\Rightarrow \boxed{\frac{d}{dx}[F(x)] = f(x)}$$

***Properties*:**

1) $0 \leq F(x) \leq 1$

2) $F[-\infty] = 0, F[\infty] = 1$

3) $\frac{d}{dx}[F(x)] = f(x)$

***Expectation***:

$E[X] = \int_{-\infty}^{\infty} x\, f(x)\; dx$

$E[X^2] = \int_{-\infty}^{\infty} x^2\; f(x)dx$

$\therefore Mean(\mu) = E[X]$

$variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$Standard\ deviation(\sigma) = \sqrt{E[X^2] - (E[X])^2}$

***Median***:

$M$ is the median that divide the interval $-\infty$ to $\infty$ into 2 equal parts.

The condition to solve Median is $\int_{-\infty}^{M} f(x)\, dx = \int_{M}^{\infty} f(x)\; dx$=1/2

***Mode***:

Mode is the value of $x$ at which $f(x)$is maximum. i.e., $f'(x) = 0\ \&\ f''(x) < 0$

*Problems*:

1. Let $X$ be a continuous random variable whose probability density function is

given by $f(x) = \begin{cases} 2e^{-2x} & for\ x > 0 \\ 0 & for\ x \leq 0 \end{cases}$

Evaluate the following 1) probability between 1 and 2 2) greater then 0.5

3) between $-1$ to 1

*Sol*: 1) $P[1 \leq X \leq 2] = \int_{1}^{2} f(x)\; dx$

$= \int_1^2 2e^{-2x} \ dx$

$= 2 \left(\frac{e^{-2x}}{-2}\right)_1^2$

$= -[e^{-4} - e^{-2}]$

$= e^{-2} - e^{-4}$

2) $P[X > 0.5] = \int_{0.5}^{\infty} f(x) \ dx$

$= \int_{0.5}^{\infty} 2e^{-2x} \ dx$

$= 2 \left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty}$

$= -[e^{-\infty} - e^{-1}] = e^{-1}$

3) $P[-1 \le X \le 1] = \int_{-1}^{1} f(x) \ dx$

$= \int_{-1}^{0} f(x) \ dx + \int_{0}^{1} f(x) \ dx$

$= 0 + \int_0^1 2e^{-2x} \ dx$

$= 2 \left(\frac{e^{-2x}}{-2}\right)_0^1$

$= -[e^{-2} - e^{0}]$

$= 1 - e^{-2}$

2. *For a continuous random variable the probability density function is given by*

$$f(x) = \begin{cases} k(1 - x^2) & for\ 0 < x < 1 \\ 0 & otherwise \end{cases}$$

*Find the value of* 1) $k$ *and also the probabilities* 2) *between* 0.1 *and* 0.2

3) *and greater than* 0.4

*Sol: consider* $\int_{-\infty}^{\infty} f(x)\ dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\ dx + \int_{0}^{1} f(x)\ dx + \int_{1}^{\infty} f(x)\ dx = 1$$

$$\Rightarrow 0 + \int_{0}^{1} k(1 - x^2)\ dx + 0 = 1$$

$$\Rightarrow k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow k\left[1 - \frac{1}{3}\right] = 1$$

$$\Rightarrow k\left[\frac{2}{3}\right] = 1$$

$$\Rightarrow k = \frac{3}{2} = 1.5$$

2) $P[0.1 < X < 0.2]$

$$= \int_{0.1}^{0.2} f(x)\ dx$$

$$= \int_{0.1}^{0.2} k(1 - x^2)\ dx$$

$$= 1.5\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= 1.5\left[\left(0.2 - \frac{(0.2)^3}{3}\right) - \left(0.1 - \frac{(0.1)^3}{3}\right)\right]$$

$$= 1.5(0.097)$$

$$\therefore P[0.1 < X < 0.2] = 0.1465$$

3) $P(X > 0.4) = \int_{0.4}^{\infty} f(x)\ dx$

$= \int_{0.4}^{1} f(x)\ dx + \int_{1}^{\infty} f(x)\ dx$

$= \int_{0.4}^{1} k(1 - x^2)\ dx + 0$

$= 1.5\left[x - \frac{x^3}{3}\right]_{0.4}^{1}$

$= 1.5\left[\left(1 - \frac{(1)^3}{3}\right) - \left(0.4 - \frac{(0.4)^3}{3}\right)\right]$

$= 1.5(0.288)$

$\therefore P(X > 0.4) = 0.432$

3. *For a continuous random variable the probability density function is given by*

$f(x) = \begin{cases} kxe^{-\lambda x} & for\ x \geq 0, \lambda > 0 \\ 0 & otherwise \end{cases}$ *.Find k also find the mean and variance of x*

*Sol*: 1) *consider* $\int_{-\infty}^{\infty} f(x)\ dx = 1$

$\Rightarrow \int_{-\infty}^{0} f(x)\ dx + \int_{0}^{\infty} f(x)\ dx = 1$

$\Rightarrow 0 + \int_{0}^{\infty} kxe^{-\lambda x}\ \ dx = 1$

$\Rightarrow k\left[x\frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right]_{0}^{\infty} = 1$ (*by using integral by patrs*)

$\Rightarrow k\left[0 - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$

$\Rightarrow k\left[\frac{1}{\lambda^2}\right] = 1$

$\Rightarrow \boxed{k = \lambda^2}$

$\therefore f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & x > 0 \\ 0 & otherwise \end{cases}$

2) $Mean(\mu) = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$\Rightarrow \mu = \int_{-\infty}^{0} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx$

$\Rightarrow 0 + \int_{0}^{\infty} x[\lambda^2 x e^{-\lambda x}]\, dx$

$= \lambda^2 \int_{0}^{\infty} x^2 e^{-\lambda x}\, dx$

$= \lambda^2 \left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \frac{e^{-\lambda x}}{\lambda^2} + 2 \frac{e^{-\lambda x}}{-\lambda^3}\right]_0^{\infty}$

$Mean(\mu) = \lambda^2 \left[(0) - [0 - 0 - \frac{2}{\lambda^3}]\right]$

$= \lambda^2 \left(\frac{2}{\lambda^3}\right)$

$Mean = \frac{2}{\lambda}$

3) $Variance(\sigma^2)$

$E[X^2] = \int_{-\infty}^{\infty} x^2\, f(x) dx$

$= \int_{-\infty}^{0} x^2 f(x)\, dx + \int_{0}^{\infty} x^2 f(x)\, dx$

$= 0 + \int_{0}^{\infty} x^2 [\lambda^2 x e^{-\lambda x}]\, dx$

$= \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x}\, dx$

$= \lambda^2 \left[x^3 \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \frac{e^{-\lambda x}}{\lambda^2} + 6x \frac{e^{-\lambda x}}{-\lambda^3} - 6 \frac{e^{-\lambda x}}{\lambda^4}\right]_0^{\infty}$

$= \lambda^2 \left[(0) - \frac{-6}{\lambda^4}\right] - \lambda^2 \left(\frac{6}{\lambda^4}\right)$

$E[X^2] = \frac{6}{\lambda^2}$

$variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$= \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2$

$\therefore variance = \frac{2}{\lambda^2}$

4. *For a continuous random variable the probability density function is given by*

$f(x) = \begin{cases} \frac{1}{2}\sin x & for\ 0 \le x \le \pi \\ 0 & otherwise \end{cases}$ *.Find the mean, mode and also the probabilities between* 0 *and* $\pi/2$.

*Sol*:

$Median = \int_0^M f(x)\, dx = \int_M^{\pi} f(x)\, dx = \frac{1}{2}$

$consider \int_0^M f(x)\, dx = \frac{1}{2}$

$\int_0^M \frac{1}{2}\sin x\ dx = 1/2$

$\frac{1}{2}[-\cos x]_0^M = 1/2$

$-[\cos M - \cos 0] = 1$

$-[\cos M - 1] = 1$

$\Rightarrow -\cos M = 0$

$\therefore \boxed{\cos M = 0}$

$M = \frac{\pi}{2}$

$Mode: f'(x) = 0$

$\Rightarrow \frac{d}{dx}[\frac{1}{2}\sin x] = 0$

$\Rightarrow \frac{1}{2}\cos x = 0 \Rightarrow \cos x = 0$

$\Rightarrow \boxed{x = \frac{\pi}{2}}$

$f''(x) = -\frac{1}{2}\sin x$

$at\ x = \frac{\pi}{2}, f''(x) = -\frac{1}{2}\sin\frac{\pi}{2} = -\frac{1}{2}(1)$

$= -\frac{1}{2} < 0$

$\therefore Mode\ (x) = \frac{\pi}{2}$

$P[0 \le X \le \frac{\pi}{2}] = \int_0^{\pi/2} f(x)\, dx$

$= \int_0^{\pi/2} \frac{1}{2}\sin x\ dx$

$= \frac{1}{2}[-\cos x]_0^{\pi/2}$

$= -\frac{1}{2}[\cos \pi/2 - \cos 0]$

$= -\frac{1}{2}[0 - 1] = \frac{1}{2}$

5. *For a continuous random variable the probability density function is given by*

$f(x) = \begin{cases} kx^2 & for\ 0 < x < 3 \\ 0 & otherwise \end{cases}$ .Find the value of $k$ and hence mean & variance.

Sol: 1) consider $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\Rightarrow \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)\, dx = 1$

$\Rightarrow 0 + \int_{0}^{3} kx^2 dx + 0 = 1$

$\Rightarrow k\left[\frac{x^3}{3}\right]_0^3 = 1$

$\Rightarrow k[9-0] = 1$

$\Rightarrow k = \frac{1}{9}$

$\therefore f(x) = \begin{cases} \frac{x^2}{9} & for\ 0 < x < 3 \\ 0 & otherwise \end{cases}$

2) $Mean(\mu) = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$\Rightarrow \mu = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)\, dx$

$= 0 + \int_{0}^{3} x.\frac{x^2}{9} dx + 0$

$= \frac{1}{9}\int_{0}^{3} x^3\, dx$

$= \frac{1}{9}\left[\frac{x^4}{4}\right]_0^3$

$= \frac{1}{9}\left[\frac{81}{4} - 0\right]$

$\therefore Mean(\mu) = \frac{9}{4}$

3) $Variance(\sigma^2)$

$E[X^2] = \int_{-\infty}^{\infty} x^2 \ f(x)dx$

$= \int_{-\infty}^{0} x^2 f(x) \ dx + \int_{0}^{3} x^2 f(x) \ dx + \int_{3}^{\infty} x^2 f(x) \ dx$

$= 0 + \int_{0}^{3} x^2 \left[\frac{x^2}{9}\right] dx + 0$

$= \frac{1}{9}\left[\frac{x^5}{5}\right]_0^3 = \frac{1}{9}\left[\frac{243}{5}\right]$

$= \frac{27}{5}$

$variance \ (\sigma^2) = E[X^2] - (E[X])^2$

$= \frac{27}{5} - \left(\frac{9}{4}\right)^2 = \frac{27}{5} - \frac{81}{16}$

$= \frac{27}{80} = 0.3375$

$\therefore \sigma^2 = 0.3375$

6. *For a continuous random variable the probability density function is given by*

$f(x) = \begin{cases} ce^{-|x|} & for -\infty < x < \infty \\ 0 & otherwise \end{cases}$ *.Find the value of c and hence mean & variance.*

$Sol$: 1) $Consider \int_{-\infty}^{\infty} f(x) \ dx = 1$

( $f(-x) = f(x)$ *then* $f(x)$ *is even,* $f(-x) = -f(x)$ *then* $f(x)$ *is odd*)

$Given \ f(x) = ce^{-|x|}$

$f(-x) = ce^{-|-x|} = ce^{-|x|} = f(x)$

$i.e., f(x)$ *is an even function.*

$now \int_{-\infty}^{\infty} f(x)\ dx = 2\int_{0}^{\infty} f(x)\ dx$

$\Rightarrow 2\int_{0}^{\infty} ce^{-|x|}\ \ dx = 1$

$\Rightarrow c[-e^{-x}]_{0}^{\infty} = \frac{1}{2}$

$\Rightarrow -c[0-1] = \frac{1}{2}$

$\Rightarrow \boxed{c = \frac{1}{2}}$

2) $Mean(\mu) = E[X] = \int_{-\infty}^{\infty} x\ f(x)\, dx$

$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2} e^{-|x|}\ dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} x\, e^{-|x|}\ dx$ $\quad (\int_{-a}^{a} f(x)\ dx = 0\ if\ f(x)\ is\ odd$

$let\ g(x) = x\, e^{-|x|}$

$g(-x) = -x\, e^{-|-x|} = -x\, e^{-|x|} = -g(x)$

$i.e., g(x)$ *is an odd function*

$\therefore Mean\ \mu = 0$

3) $Variance(\sigma^2)$

$E[X^2] = \int_{-\infty}^{\infty} x^2\ f(x)dx$

$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|}\ dx$

$$=\frac{1}{2}\int_{-\infty}^{\infty} x^2 \,.\, e^{-|x|}\ dx$$

$Since\ x^2\ .\ e^{-|x|}\ is\ an\ even\ function\ so\ x^2\ .\ e^{-|x|} = x^2\ .\ e^{-x}$

$$E[X^2] = \frac{1}{2}(2)\int_{0}^{\infty} x^2\ .\ e^{x}\ \ dx$$

$$= \left[x^2\frac{e^{-x}}{-1} - 2x\frac{e^{-x}}{1} + 2\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= [(0) - (0 - 0 - 2)] = 2$$

$variance\ (\sigma^2) = E[X^2] - (E[X])^2$

$= 2 - 0 = 2$

$7. To\ the\ folowing\ definition\ for\ f(x), verify\ its\ existence\ as\ the\ probability\ density\ function\ and\ hence\ evaluate\ the\ mean$

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2\ for -3 \le x \le -1 \\ \frac{1}{16}(6-2x^2)\ for -1 \le x \le 1 \\ \frac{1}{16}(3-x)^2\ for\ 1 \le x < 3 \\ 0 \quad\quad otherwise \end{cases}$$

$Sol$: 1) $f(x) \ge 0$

2) $To\ verify \int_{-\infty}^{\infty} f(x)\ dx = 1$

$$Consider \int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{-3} f(x)\ dx + \int_{-3}^{-1} f(x)dx + \int_{-1}^{1} f(x)\ dx$$

$$+\int_{1}^{3} f(x)\ dx + \int_{3}^{\infty} f(x)\ dx$$

$$= 0 + \int_{-3}^{-1}\frac{1}{16}(3+x)^2 dx + \int_{-1}^{1}\frac{1}{16}(6-2x^2)\ dx + \int_{1}^{3}\frac{1}{16}(3-x)^2\ dx + 0$$

$$= \int_{-3}^{-1} \frac{1}{16}(9 + x^2 + 6x)\, dx + \int_{-1}^{1} \frac{1}{16}(6 - 2x^2)\, dx + \int_{1}^{3} \frac{1}{16}(9 + x^2 - 6x)\, dx$$

$$= \frac{1}{16}\left[9x + \frac{x^3}{3} + \frac{6x^2}{2}\right]_{-3}^{-1} + \frac{1}{16}\left[6x - \frac{2x^3}{3}\right]_{-1}^{1} + \frac{1}{16}\left[9x + \frac{x^3}{3} - \frac{6x^2}{2}\right]_{1}^{3}$$

$$= \frac{1}{16}\left(\left[9(-1) + \frac{(-1)^3}{3} + \frac{6(-1)^2}{2}\right] - \left[9(-3) + \frac{(-3)^3}{3} + \frac{6(-3)^2}{2}\right]\right)$$

$$+ \frac{1}{16}\left(\left[6(1) - \frac{2(1)^3}{3}\right] - \left[6(-1) - \frac{2(-1)^3}{3}\right]\right)$$

$$+ \frac{1}{16}\left(\left[9(3) + \frac{(3)^3}{3} - \frac{6(3)^2}{2}\right] - \left[9(1) + \frac{(1)^3}{3} - \frac{6(1)^2}{2}\right]\right) = \frac{1}{16}\left[\frac{48}{3}\right] = 1$$

$\therefore f(x)$ *is the probability density function*

2) $Mean(\mu) = E[X] = \int_{-\infty}^{\infty} x\ f(x)\, dx$

$$= \int_{-\infty}^{-3} x.f(x)\ dx + \int_{-3}^{-1} x.f(x)dx + \int_{-1}^{1} x.f(x)\ dx + \int_{1}^{3} x.f(x)\ dx + \int_{3}^{\infty} x.f(x)\ dx$$

$$= 0 + \int_{-3}^{-1} x.\frac{1}{16}(3 + x)^2 dx + \int_{-1}^{1} x.\frac{1}{16}(6 - 2x^2)\ dx + \int_{1}^{3} x.\frac{1}{16}(3 - x)^2\ dx + 0$$

$$= \frac{1}{16}\int_{-3}^{-1} x.(9 + x^2 + 6x)\, dx + \frac{1}{16}\int_{-1}^{1} x.(6 - 2x^2)\, dx + \frac{1}{16}\int_{1}^{3} x.(9 + x^2 - 6x)\, dx$$

$$= \frac{1}{16}\int_{-3}^{-1} (9x + x^3 + 6x^2)\, dx + \frac{1}{16}\int_{-1}^{1} (6x - 2x^3)\, dx + \frac{1}{16}\int_{1}^{3} (9x + x^3 - 6x^2)\, dx$$

$$= \frac{1}{16}\left(\left[\frac{9x^2}{2} + \frac{x^4}{4} + \frac{6x^3}{3}\right]_{-3}^{-1} + \left[\frac{6x^2}{2} - \frac{2x^4}{4}\right]_{-1}^{1} + \left[\frac{9x^2}{2} + \frac{x^4}{4} - \frac{6x^3}{3}\right]_{1}^{3}\right)$$

$$= \frac{1}{16}\left[\left(\frac{9}{2} + \frac{1}{4} - \frac{6}{3}\right) - \left(\frac{81}{2} + \frac{81}{4} - 54\right)\right] + \left[\left(3 - \frac{1}{2}\right) - \left(3 - \frac{1}{2}\right)\right]$$

$$+ \left[\left(\frac{81}{2} + \frac{81}{4} - 54\right) - \left(\frac{9}{2} + \frac{1}{4} - \frac{6}{3}\right)\right]$$

$\therefore Mean\ (\mu) = 0$

Moment (Definition): The arithmetic mean of the various powers of the deviations of items from their mean (assumed or actual) will give the required power of moment of the distribution.

Moments are of two types:

***1. Central moment and 2.Raw moment***

***1. Central moment***: If the deviations of the items are taken from the arithmetic mean of the distribution it is known as Central moment denoted by $\mu_r$.

r=1 $\Rightarrow$ $\mu_1$ called as first central moment

r=2 $\Rightarrow$ $\mu_2$ called as second central moment

r=3 $\Rightarrow$ $\mu_3$ called as third central moment

r=4 $\Rightarrow$ $\mu_4$ called as fourth central moment so on

$\mu_r$ denotes $r^{th}$ central moment.

**Calculating Central moments about actual mean:**

1. Central moments for individual series is given by $\mu = \sum \frac{x^r}{N}$ where $x = X - \bar{X}$; $\bar{X} = \sum \frac{x}{N}$, N denotes the total number of observations and $x = X - \bar{X}$ denotes the deviations of x from $\bar{X}$

For r=1$\Rightarrow$ $\mu_1 = \sum \frac{x}{N}$

$For\ r = 2 \Rightarrow \mu_2 = \sum \frac{x^2}{N}$ and so on.

2. central moments for Frequency Distribution is given by $\mu_r=\sum\frac{f_i x_i^r}{N}$

For r=1 $\Rightarrow \mu_1=\sum\frac{f_1 x_i}{N}$

For r=2$\Rightarrow$ $\mu_2=\sum\frac{f_i x_i^2}{N}$ so on.

**Properties of central moments:**

1. The first Moment is always zero i.e, $\mu_1=0$.

2. The second Moment about mean measures Variance i.e, $\mu_2=\sigma^2$ .

3. The third Moment measures Skewness of a given distribution:

(i)If $\mu_3 > 0 \Rightarrow$the distribution is positively skewed.

(i)If $\mu_3 < 0 \Rightarrow$the distribution is negatively skewed.

(i)If $\mu_3 = 0 \Rightarrow$the distribution is symmetrical.

4. The fourth Moment about mean measures Kurtosis of a frequency distribution.

5. Skewness and Kurtosis of a distribution is given by skewness= $_1=\frac{\mu_3^2}{\mu_2^3}$ and Kurtosis = $_2=\frac{\mu_4}{\mu_2^2}$

***2. Raw moments about Arbitrary origin:***

Definition: Moments about the arbitrary origin are called raw moments denoted by $\mu'_r$

Therefore raw moments given by formula $\mu'_r = \frac{1}{N}\sum f_i(\ \ _i)^r$ where

$d_i$=$x_i$-A

And in the case of class interval frequency distribution, $\mu'_r = \frac{1}{N}\sum f_i(d_i)^r$ where $d_i = \frac{X-A}{C}$ where C=length of interval.

For r=1 $\Rightarrow \mu'_1 = \frac{1}{N}\sum f_i(d_i)'$ is called first raw moment.

For r=2 $\Rightarrow \mu'_2 = \frac{1}{N}\sum f_i(d_i)^2$ is called second raw moment.

For r=3 $\Rightarrow \mu'_3 = \frac{1}{N}\sum f_i(d_i)^3$ is called third raw moment.

For r=4 $\Rightarrow \mu'_4 = \frac{1}{N}\sum f_i(d_i)^4$ is called fourth raw moment and so on.

***<u>Relation between Central moments and Raw moments:</u>***

1. When raw moments are given, we can compute central moments as

$\mu_1$=0

$\mu_2=\mu'_2-(\mu'_1)^2$

$\mu_3=\mu'_3-3\mu'_2\mu'_1+2(\mu'_1)^3$

$\mu_4=\mu'_4-4\mu'_3\mu'_1+6\mu'_2(\mu'_1)^2-3(\mu'_1)^4$

2. When central moments are given, we can compute raw moments as

$\mu'_1=\bar{X}A$

$\mu'_2=\mu_2+-(\mu'_1)^2$

$\mu'_3 = \mu_3 + 3\mu_2\mu'_1 + (\mu'_1)^3$

$\mu'_4 = \mu_4 + 4\mu_3\mu_1 + 6\mu_2(\mu'_1)^2 + (\mu'_1)^4$

Problem 1. Find the first four moments for the set of numbers 2,4,6,8.

Sol: Let N=Number of values=4

$\bar{X}=\frac{2+4+6+8}{4}=5$

Let $d_i=x_i-\bar{X}=x_i-5$

| | = --5 | 2 | 3 | 4 |
|---|---|---|---|---|
| 2 | -3 | 9 | -27 | 81 |
| 4 | -1 | 1 | -1 | 1 |
| 6 | 1 | 1 | 1 | 1 |
| 8 | 3 | 9 | 27 | 81 |
| $\sum x_i$=20 | $\sum d_i$=0 | $\sum d_i^{2}$=20 | $\sum d_i^{3}$=0 | $\sum d_i^{4}$=164 |

Moments about mean are

$\mu_1=\frac{\sum d_i}{N}=\frac{0}{4}=0$

$\mu_2=\frac{\sum d_i^{2}}{N}=\frac{20}{4}=5$

$\mu_3=\frac{\sum d_i^{3}}{N}{}_i=\frac{0}{4}=0$

$\mu_4=\frac{\sum d_i^{4}}{N}=\frac{164}{4}=41$

Hence the first four moments are 0, 5,0,41

**Problem 2:**

Calculate the first four moments of the following distribution about the mean:

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| f | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |

Also evaluate $\beta_1$ and $\beta_2$.

Sol: Let A=4 then d=x-A=x-4

| x | f | d=x-4 | fd | $^2$ | $^3$ | $^4$ |
|---|---|---|---|---|---|---|
| 0 | 1 | -4 | -4 | 16 | -64 | 256 |
| 1 | 8 | -3 | -24 | 72 | -216 | 648 |
| 2 | 28 | -2 | -56 | 112 | -224 | 448 |
| 3 | 56 | -1 | -56 | 56 | -56 | 56 |
| 4 | 70 | 0 | 0 | 0 | 0 | 0 |
| 5 | 56 | 1 | 56 | 56 | 56 | 56 |
| 6 | 28 | 2 | 56 | 112 | 224 | 448 |
| 7 | 8 | 3 | 24 | 72 | 216 | 648 |
| 8 | 1 | 4 | 4 | 16 | 64 | 256 |
| | 256 | 0 | 0 | 512 | 0 | 2816 |

Moments about the point A=4 are

$\mu_1' = \frac{1}{N}\sum f_i d_i = \frac{0}{256} = 0$

$\mu_2' = \frac{1}{N}\sum f_i d_i^2 = \frac{512}{256} = 2$ $\quad\quad$ $\mu_3' = \frac{1}{N}\sum f_i d_i^3 = \frac{0}{256} = 0$

$\mu_4' = \frac{1}{N}\sum f_i d_i^4 = \frac{2816}{256} = 11$

Moments about the actual mean are

$\mu_1=0$

$\mu_2=\mu_2'-(\mu_1')^2=2-0=2$

$\mu_3=\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3=-3(2)(0)+2(0)=0$

$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4=11-4(0)(0)+6(2)(0)-3(0)=11$

skewness= $\beta_1=\frac{\mu_3^2}{\mu_2^3}=0$and Kurtosis = $\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{11}{4}=2.75(<3)$

$\Rightarrow$The distribution is Platykurtic.

Hence the given distribution is symmetrical(since skewness=0) and Platykurtic.

**Problem 3:**

Calculate the first four moments about the mean of the following distribution:

| Marks | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 | 50-60 | 60-70 |
|---|---|---|---|---|---|---|---|
| No. of Students | 8 | 12 | 20 | 30 | 15 | 10 | 5 |

Also evaluate $\beta_1$ and $\beta_2$.

Sol: Let A=midvalue of 30-40=35

C=class size=10

$d=\frac{X-A}{C}=\frac{X-35}{10}$

| Marks | No. of students | Mid value(X) | d=$\frac{X-35}{10}$ | fd | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| 0-10 | 8 | 5 | -3 | -24 | 72 | -216 | 648 |
| 10-20 | 12 | 15 | -2 | -24 | 48 | -96 | 192 |
| 20-30 | 20 | 25 | -1 | -20 | 20 | -20 | 20 |
| 30-40 | 30 | 35 | 0 | 0 | 0 | 0 | 0 |
| 40-50 | 15 | 45 | 1 | 15 | 15 | 15 | 15 |
| 50-60 | 10 | 55 | 2 | 20 | 80 | 80 | 160 |
| 60-70 | 5 | 65 | 3 | 15 | 135 | 135 | 405 |
| | 100 | | | -18 | 240 | -102 | 1440 |

Moments about the arbitrary origin are

$$\mu_1' = \frac{1}{N}\sum f_i d_i * C = \frac{-18}{100} * 10 = -1.8$$

$$\mu_2' = \frac{1}{N}\sum f_i d_i^2 * C^2 = \frac{240}{100} * 100 = 240$$

$$\mu_3' = \frac{1}{N}\sum f_i d_i^3 * C^3 = \frac{-102}{100} * 1000 = -1020$$

$$\mu_4' = \frac{1}{N}\sum f_i d_i^4 * C^4 = \frac{1440}{100} * 10000 = 144000$$

Moments about the actual mean are

$\mu_1=0$, $\mu_2=\mu_2'-(\mu_1')^2=240-(-1.8)^2=236.76$

$\mu_3=\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3=-1020+1296-11.664=264.336$

$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4=144000-7344+4665-31.49=141289.51$

skewness= $\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{(264.336)^2}{(236.76)^3} = 0.005$ and Kurtosis $=\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{141289.51}{(236.76)^2}=2.52(<3)$