UNIT-IV

MULTY VARIABLE CAlCULUS
(PARTIAL DIFFERENCTIAL EQUATION)

Definition of Limit (Two Variables)

Let $f(x,y)$ be a function of two variables.

We say the **limit of $f(x,y)$ as (x,y) approaches (a,b)** is L if:

👉**For every path approaching (a,b),**

the value of $f(x,y)$ gets closer and closer to L.

Mathematically:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

This means:

When (x,y) moves very close to (a,b),

The output $f(x,y)$ becomes very close to L,

No matter which **direction or path** you follow to approach the point.

Euler's Theorem (Definition)
If a function $u = f(x, y)$ is **homogeneous of degree** n, then it satisfies:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

⭑ **Meaning:**
If all terms of the function have the **same total power**, the function is homogeneous, and Euler's theorem applies.

1. Verify Euler's theorem for u = $x^3 + y^3$

Step 1: Evaluate partial derivatives

$$u_x = 3x^2, \quad u_y = 3y^2$$

Step 2: Apply Euler's Theorem

$$xu_x + yu_y = x(3x^2) + y(3y^2) = 3x^3 + 3y^3 = 3(x^3 + y^3) = 3u$$

✔ **Final Result:**

$$xu_x + yu_y = 3u$$

Thus, Euler's theorem is **verified**.

Definition of Chain Rule :

If a function $u$depends on variables $x$and y, and each of them depends on another variable t, then the **total derivative** of $u$with respect to $t$is:

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

1. If $u = x^2 + y^2$, x=t^2 ,y=t+1 then find $\frac{du}{dt}$

Step 1: Find partial derivatives of u

$$u_x = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

$$u_y = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

Step 2: Find derivat.ives of $x$and $y$with respect to t

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t + 1 \Rightarrow \frac{dy}{dt} = 1$$

Step 3: Apply Chain Rule

$$\frac{dz}{dt} = z_x \cdot \frac{dx}{dt} + z_y \cdot \frac{dy}{dt}$$

Substitute all values:

$$\frac{dz}{dt} = (2x)(2t) + (2y)(1)$$

Now substitute $x = t^2$ and $y = t + 1$:

$$\frac{dz}{dt} = (2t^2)(2t) + 2(t + 1)$$

$$= 4t^3 + 2t + 2$$

Final Answer:

$$\boxed{\frac{dz}{dt} = 4t^3 + 2t + 2}$$

Total Derivative:

If $u = f(x, y)$ is a function of two variables, and each variable depends on another variable t, that is:

$$x = x(t),\ y = y(t)$$

then the **total derivative** of u with respect to t is:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

u=x2y+y2,x=t2,y=t+1 Find the **total derivative** $\frac{du}{dt}$.

✨ Step-by-Step Solution

Step 1: Find partial derivatives of u

$$u = x^2 y + y^2$$

$$\frac{\partial u}{\partial x} = 2xy$$

$$\frac{\partial u}{\partial y} = x^2 + 2y$$

Step 2: Differentiate $x$and $y$with respect to t

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t + 1 \Rightarrow \frac{dy}{dt} = 1$$

Step 3: Apply Total Derivative Formula

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

Substitute values:

$$\frac{dz}{dt} = (2xy)(2t) + (x^2 + 2y)(1)$$

Now substitute $x = t^2$ and $y = t + 1$:

$$= (2(t^2)(t+1))((2t)) + t^4 + 2(t+1)($$

Simplify:

First part:

$$2t^2(t+1) \cdot 2t = 4t^3(t+1) = 4t^4 + 4t^3$$

Second part:

$$t^4 + 2t + 2$$

Add both:

$$\frac{dz}{dt} = (4t^4 + 4t^3) + (t^4 + 2t + 2)$$

$$= 5t^4 + 4t^3 + 2t + 2$$

🎉 **Final Answer:** $\boxed{\frac{dz}{dt} = 5t^4 + 4t^3 + 2t + 2}$

JACOBIAN:

$$u = u(x, y),\ v = v(x, y),$$

the **Jacobian** is the determinant:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

It shows *how variables transform* when we change from (x, y) to (u, v).

1. Find $\frac{\partial(u,v)}{\partial(x,y)}$ where $u = x + y,\ v = x - y$.

Solution:

$$u_x = 1,\ u_y = 1,\ v_x = 1,\ v_y = -1$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1)$$

$$= -1 - 1 = -2$$

FUNCTIONAL DEPENDENCE:

Two functions $u(x,y)$ and $v(x,y)$ are **functionally dependent** if:

$$F(u,v) = 0$$

i.e., they satisfy a relation linking them.

A test for dependence:

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \Rightarrow Dependent$$

If u=x2+y2,v=tan−1(xy) verify **functionally independent** are not ?

Jacobian Test:

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2}, \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

Compute Jacobian:

$$J = \begin{vmatrix} 2x & 2y \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix}$$

$$J = \frac{2x^2 + 2y^2}{x^2 + y^2} = 2$$

Since

$$J \neq 0,$$

the functions are **functionally independent**.

Maxima and Minima :

Definition:

Let $f(x,y)$ be a function of two variables. A point (a,b) is said to be a **maximum** or **minimum** if:

The first partial derivatives vanish:

$$f_x(a,b) = 0, \quad f_y(a,b) = 0$$

The second-order derivatives satisfy the test using the determinant:

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2$$

Then:

If $D > 0$ and $f_{xx} < 0$ → **Local Minimum**

If $D > 0$ and $f_{xx} > 0$ → **Local Maximum**

If $D < 0$ → **Saddle Point**

If $D = 0$ → **Test fails**

1. Find the minimum and maximum value of $f(x, y) = x^2 + y^2 - 4x - 6y$

GIven

$$f(x, y) = x^2 + y^2 - 4x - 6y$$

STEP 1: Compute first derivatives

$$f_x = 2x - 4, \quad f_y = 2y - 6$$

Set them equal to 0:

$$2x - 4 = 0 \Rightarrow x = 2$$
$$2y - 6 = 0 \Rightarrow y = 3$$

So the point is (2, 3).

This is how you **find the point**.

◈ STEP 2: Second partial derivatives

$$f_{xx} = 2, \; f_{yy} = 2, \; f_{xy} = 0$$

◈ STEP 3: Hessian determinant

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2 = (2)(2) - 0 = 4$$

Since

$$D > 0, \; f_{xx} < 0$$

✔ It is a **Minimum**.

🎯 Final Answer:

Stationary point: $(2,3)$

Type: Minimum value

Lagrange's Multipliers:

Definition:

To find the **extrema** of a function

$$f(x,y)$$

subject to a constraint

$$g(x,y) = 0,$$

we form the function

$$F(x,y,\lambda) = f(x,y) + \lambda g(x,y),$$

where λ is called the **Lagrange multiplier**.

The stationary points satisfy:

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial \lambda} = 0.$$

These points give the constrained maxima or minima.

1. Find the maximum and minimum values of

$$f(x,y) = x^2 + y^2$$

subject to the constraint

$$x + y = 10.$$

Solution:

Form the Lagrangian:

$$F = x^2 + y^2 + \lambda(10 - x - y)$$

Take partial derivatives:

$$F_x = 2x - \lambda = 0 \Rightarrow 2x = \lambda$$

$$F_y = 2y - \lambda = 0 \Rightarrow 2y = \lambda$$

So,

$$2x = 2y \Rightarrow x = y$$

Use the constraint:

$$x + x = 10 \Rightarrow 2x = 10 \Rightarrow x = 5$$

$$\therefore y = 5$$

Now evaluate:

$$f(5,5) = 5^2 + 5^2 = 25 + 25 = 50$$

✔ **Extremum value = 50 at (5, 5)**

Since the function is convex (positive coefficients), it is a **minimum**.


Thank you