UNIT-III
CAlCULUS

CALCULUS

Continuous function

Let y=f(x) be a function Continuous in the closed interval a, b this means that if [a<b<c a<c<b]< t ,f(x)=f(c) ,where c€(b,a)

Differentiable

let y=f(x) be a differentiable function in the interval (a,b) this means that if a<c<b the derivative of f(x) at x=c exist

$$\underset{x \to c}{lt} \frac{f(x)-f(c)}{xc}$$

Note

1. Every Polynomial is a Continuous function.
2. Every logarithm function is Continuous function.
3. Every exponential function is continuous function.

Rolles Theorem

Rolle's Theorem states that:

If a function **f(x)** satisfies the following three conditions:

- **Continuous** on the closed interval **[a, b]**
- **Differentiable** on the open interval **(a, b)**
- **f(a) = f(b)**,

then **there exists at least one point** *c* in the interval (ளக) such that:

ந′(உ) = 0.

- Verify Rolle’s Theorem for $f(x) = x^2 - 4x + 3$ on the interval $[1, 3]$. Find the point c.
- Check continuity: $f(x)$ is a polynomial → continuous everywhere ✅
- Check differentiability: $f(x)$ is a polynomial → differentiable everywhere ✅
- Check $f(a) = f(b)$:
- $f(1) = 1^2 - 4(1) + 3 = 0$
- $f(3) = 3^2 - 4(3) + 3 = 0$
- ✅ condition satisfied
- Find c using $f'(c) = 0$:
- $f'(x) = 2x - 4$
- $2c - 4 = 0 \implies c = 2$

GEOMETRICAL INTERPRETATION:

- Rolle's Theorem says that if a smooth curve starts and ends at the **same height**, then somewhere between those two points the curve must have a **horizontal tangent line**.
- Consider the graph of a function f(஌)on the interval [a , b]
- If the points (ளௐ(ள))and (கௐ(க))lie **on the same horizontal line**,
- and the curve is smooth (continuous and differentiable),
- then **the curve must rise and fall smoothly between these points**, and at some point in between, there will be a **highest or lowest point**.
- At this point, the tangent to the curve is **horizontal**, which means:
- ௐ′(உ) = 0
- for some $c \in$ (ளக).

Lagrange s Mean value Theorem

Lagrange's Mean Value Theorem states that:

If a function **f(x)**

- is **continuous** on the closed interval **[a, b]**,
- is **differentiable** on the open interval **(a, b)**,
- then **there exists at least one point** *c*in (ளக)such that:
- ரு′(உ) = (ரு(க)−ரு(ள)) / (க−ள) .

Verify Lagrange's Mean Value Theorem for $f(x) = x^2$ on the interval $[1,3]$. Find the point c

- **Check continuity:** $f(x) = x^2$ is continuous everywhere .
- **Check differentiability:** $f(x) = x^2$ is differentiable everywhere .
- **Find slope of secant line:**
- $\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{9-1}{2} = \frac{8}{2} = 4$
- **Set derivative equal to slope of secant line:**
- $f'(x) = 2x$
- $2c = 4 \implies c = 2$

Geometrical Interpretation :

- Lagrange's Mean Value Theorem states that for a smooth curve, there is at least one point where the **tangent line** is **parallel** to the **secant line** joining the endpoints of the interval.
- Consider the curve $y = f($ൗ$)$ on the interval [ளக].
- Draw a **secant line** joining the points (ளரு(ள))and (கரு(க)).
- This secant line has slope= (ரு(க)−ரு(ள)) / (க−ள)
- LMVT guarantees that there is at least one point ∈ (ளக)where the **tangent to the curve** has the **same slope** as this secant line.

CAUCHY'S MEAN VALUE THEOREM

- **f(x)** and **g(x)** are two functions that are **continuous** on the closed interval **[a, b].**
- **differentiable** on the open interval **(a, b)**, and **g′(x) ≠ 0** for all **x ∈ (a, b).**

then there exists at least one point **c ∈ (a, b)** such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Let $f(x) = x^2$ and $g(x) = x$ on $[1,3]$. Find c that satisfies Cauchy's Mean Value Theorem.

- $f(x)$ and $g(x)$ are polynomials → continuous & differentiable
- $g'(x) = 1 \neq 0$
- $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$
- $f'(x) = 2x,\ g'(x) = 1$
- **Compute slope ratio:** $\frac{f(3)-f(1)}{g(3)-g(1)} = \frac{9-1}{3-1} = \frac{8}{2} = 4$
- **Set derivative ratio equal to slope ratio:**
- $\frac{f'(c)}{g'(c)} = \frac{2c}{1} = 2c = 4 \implies c = 2$

Taylor's Theorem:

Taylor's Theorem states that:

- If a function $f(x)$ has derivatives up to the n^{th} order in an interval containing a point a, then for any point x in that interval:

- $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n$

- where R_n is the **remainder term** (or error term) of the Taylor series.

Find the Taylor series of $f(x) = \cos x$ about $a = 0$ up to x^4 term.

- **Compute derivatives at $x = 0$:**
- $f(x) = \cos x$
- $f(0) = \cos 0 = 1$
- $f'(x) = -\sin x \implies f'(0) = 0$
- $f''(x) = -\cos x \implies f''(0) = -1$
- $f'''(x) = \sin x \implies f'''(0) = 0$
- $f^{(4)}(x) = \cos x \implies f^{(4)}(0) = 1$
- $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$
- $\cos x \approx 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{24}$
- $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$

MACLAURIN’S THEOREM

- **Maclaurin’s Theorem is a special case of Taylor’s Theorem, in which a function is expanded in a power series about $x = 0$.**
- If a function $f(x)$ has derivatives of all orders at $x = 0$, then it can be expressed as:
- $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + R_n$
- where R_n is the remainder term.

Expand $f(x) = e^x$ into a Maclaurin series up to the x^3 term.

- Compute derivatives at $x = 0$:
- $f(x) = e^x$
- $f(0) = e^0 = 1$
- $f'(x) = e^x \implies f'(0) = 1$
- $f''(x) = e^x \implies f''(0) = 1$
- $f'''(x) = e^x \implies f'''(0) = 1$
- Apply Maclaurin series formula:
- $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$
- $f(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$
- $f(x) \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
- $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

Curve Tracing

- **Curve tracing is the process of studying and analyzing the properties of a curve to sketch its shape accurately without plotting many points.**
- It involves examining important features of the curve such as:
- **Domain and range**
- **Symmetry**
- **Intercepts**
- **Asymptotes**
- **Maxima and minima**
- **Curvature and concavity**
- **Behavior at infinity**

Trace the curve $y = x^2 - 4$ and find:

- x-intercepts
- y-intercept
- Symmetry
- Turning point

- **x-intercepts:**
  Set $y = 0$:
- $x^2 - 4 = 0 \implies x^2 = 4 \implies x = \pm 2$
- So, x-intercepts are $(2, 0)$ and $(-2, 0)$

Symmetry About the Y-Axis – Definition

- A curve is said to be **symmetric about the y-axis** if replacing $x$with $-x$leaves the equation unchanged.

Condition:

- $f(x) = f(-x)$
- **Meaning:**
- The left side of the graph is a mirror image of the right side with respect to the **y-axis**.

Example:

- $y = x^2$, $y = \cos x$

Symmetry About the X-Axis – Definition

- A curve is said to be **symmetric about the x-axis** if replacing $y$with $-y$leaves the equation unchanged.

Condition:

- If $(x,y)$is on the curve, then $(x,-y)$is also on the curve.
- **Meaning:**
- The upper half of the graph is a mirror image of the lower half with respect to the **x-axis**.
- **Example:**
- $x = y^2 y^2$ = 4ax,

Symmetry About the Origin – Definition

- A curve is said to be **symmetric about the origin** if for every point (x, y) on the curve, the point $(-x, -y)$ is also on the curve.
- **Mathematically:**
- Replace x with $-x$ and y with $-y$.
  If the equation remains unchanged, the curve is symmetric about the **origin**.
- **Meaning:**
- The curve is rotated 180° around the origin and looks the same.
- **Example:**
- $y = x^3$, $xy = 1$