25MA101 : MATRICES AND CALCULUS

I B.Tech, I Semester,

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UNIT-I
Matrices

Introduction

Matrices - Introduction

Matrix algebra has at least two advantages:

- Reduces complicated systems of equations to simple expressions
- Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

Properties:

- **A specified number of rows and a specified number of columns**
- **Two numbers (rows x columns) describe the dimensions or size of the matrix.**

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Matrices - Introduction

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{ij} & a_{in} \\ a_{21} & a_{22} \cdots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

Matrices - Introduction

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Matrices - Introduction

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $A_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Matrices - Introduction

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

7. Null (zero) matrix - O

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_{ij} = 0$ For all i,j

Matrices - Introduction

TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

Matrices - Introduction

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Matrices – Introduction

TYPES OF MATRICES

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = a$ for all $i = j$

Matrices - Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad A = B$$

Matrices - Operations

Some properties of equality:

- If **A** = **B**, then **B** = **A** for all **A** and **B**
- If **A** = **B**, and **B** = **C**, then **A** = **C** for all **A**, **B** and **C**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If **A** = **B** then $a_{ij} = b_{ij}$

Matrices - Operations

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Matrices - Operations

Commutative Law:

A + **B** = **B** + **A**

Associative Law:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A 2x3   **B** 2x3   **C** 2x3

Matrices - Operations

$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$

$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$ (where $-\mathbf{A}$ is the matrix composed of $-a_{ij}$ as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Matrices - Operations

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Matrices - Operations

$$4\times\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}\times 4=\begin{bmatrix}12 & -4\\ 8 & 4\\ 8 & -12\\ 16 & 4\end{bmatrix}$$

Properties:

- $k(\mathbf{A}+\mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
- $(k+g)\mathbf{A} = k\mathbf{A} + g\mathbf{A}$
- $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k)\mathbf{B}$

Matrices - Operations

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \quad x \quad \mathbf{B} \; = \quad \mathbf{C}$$

$$(1x3) \quad (3x1) \quad (1x1)$$

Matrices - Operations

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible!

(6x2) (6x3)

Example

A x **B** = **C**

(2x3) (3x2) (2x2)

Matrices - Operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$

$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$

$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$

$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** row by column multiplication

Matrices - Operations

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Matrices - Operations

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI** = **IA** = **A**
2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)
3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)
4. (**A**+**B**)**C** = **AC** + **BC** - (second distributive law)

NOTE:

1. **AB** not generally equal to **BA**, **BA** may not be conformable
2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**
3. If **AB** = **AC**, **B** not necessarily = **C**

Matrices - Operations

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

Matrices - Operations

If **AB** = **0**, neither **A** nor **B** necessarily = **0**

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices - Operations

TRANSPOSE OF A MATRIX

If :

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2x3

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T$$ For all i and j

Matrices - Operations

To transpose:

Interchange rows and columns

The dimensions of $\mathbf{A}^T$ are the reverse of the dimensions of **A**

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2 x 3

$$A^T = {}_3A^{T^2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

3 x 2

Matrices - Operations

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Matrices - Operations

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

Matrices - Operations

$$(\mathbf{AB})^T = \mathbf{B}^T\,\mathbf{A}^T$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

Matrices - Operations

SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Matrices - Operations

When the original matrix is square, transposition does not affect the elements of the main diagonal

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The identity matrix, **I**, a diagonal matrix **D**, and a scalar matrix, **K**, are equal to their transpose since the diagonal is unaffected.

Matrices - Operations

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

k=7 the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be **AB** = **AC** while **B** $\neq$ **C**

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Matrices - Operations

Example:

$$A = {}_2 A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices - Operations

- Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

- A square matrix that has an inverse is called a nonsingular matrix
- A matrix that does not have an inverse is called a singular matrix
- Square matrices have inverses except when the determinant is zero
- When the determinant of a matrix is zero the matrix is singular

Matrices - Operations

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or |**A**|

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

Matrices - Operations

If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

Then |**A**| =det **A** = a_{11}

If **A** is (n x n), its determinant may be defined in terms of order (n-1) or less.

Matrices - Operations

MINORS

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where *i* and *j* correspond to the deleted

row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

Matrices - Operations

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Matrices - Operations

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices - Operations

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i+j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1,\ j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1,\ j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1,\ j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Matrices - Operations

DETERMINANTS CONTINUED

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Matrices - Operations

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices - Operations

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Matrices - Operations

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Matrices - Operations

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Matrices - Operations

Example :

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Matrices - Operations

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If $$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is $$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

Matrices - Operations

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj}\mathbf{A})\ \mathbf{A} = |\mathbf{A}|\ \mathbf{I}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Matrices - Operations

$$A(adjA) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adjA)\,A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

Matrices - Operations

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj }\mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Matrices - Operations

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices - Operations

Example:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

$$|\mathbf{A}| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2$$

The elements of the cofactor matrix are

$$c_{11} = +(-1), \quad c_{12} = -(-2), \quad c_{13} = +(3),$$
$$c_{21} = -(-1), \quad c_{22} = +(-4), \quad c_{23} = -(7),$$
$$c_{31} = +(-1), \quad c_{32} = -(-2), \quad c_{33} = +(5),$$

Matrices - Operations

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Matrices - Operations

The result can be checked using

$$\mathbf{AA^{-1} = A^{-1} A = I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

Linear Equations

- Linear equations are common and important for survey problems
- Matrices can be used to express these linear equations and aid in the computation of unknown values
- n equations in n unknowns, the a_{ij} are numerical coefficients, the b_i are constants and the x_j are unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Linear Equations

The equations may be expressed in the form

$$\mathbf{AX} = \mathbf{B}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n1} \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

n x n   n x 1   n x 1

Number of unknowns = number of equations = n

Linear Equations

If the determinant is nonzero, the equation can be solved to produce n numerical values for x that satisfy all the simultaneous equations

To solve, premultiply both sides of the equation by $\mathbf{A}^{-1}$ which exists because $|\mathbf{A}| \neq \mathbf{0}$

$$\mathbf{A}^{-1}\,\mathbf{AX} = \mathbf{A}^{-1}\,\mathbf{B}$$

Now since

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

We get

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

So if the inverse of the coefficient matrix is found, the unknowns, **X** would be determined

Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$

$$2x_1 + x_2 = 1$$

$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linear Equations

When $\mathbf{A}^{-1}$ is computed the equation becomes

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore

$$x_1 = 2,$$

$$x_2 = -3,$$

$$x_3 = -7$$

Linear Equations

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2, \quad x_2 = -3, \quad x_3 = -7$$

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

$$3\times(2) - (-3) + (-7) = 2$$
$$2\times(2) + (-3) = 1$$
$$(2) + 2\times(-3) - (-7) = 3$$

Rank of a matrix:

Let A is be an matrix .If A is null matrix , we define its rank to be 0 (zero).

- If A is non zero matrix ,we say that ‘r’ is the rank of A if
- (i) every (r+1)th order minor of A is 0(zero) and
- (ii) there exists at least one rth order minor of A which is not zero
- Rank of A is denoted by ρ(A)
- **Note:**
- **1) E**very matrix will have rank
- 2) Rank of a matrix is unique
- 3) ρ (A)= 1 when A is a non-zero matrix
- 4)If A is a matrix of order rank of A= ρ (A) min(m,n)
- 5)If ρ (A) = r then every minor of A of order r+1 or more is zero
- 6)Rank of the identity matrix In is n
- 7)If A is a matrix of order ‘n’ and A is non-singular (i.e; det A 0) then ρ (A)=n.
- 8)The rank of the transpose of a matrix is the some as that of the original matrix(i.e; ρ (A)= ρ (AT))
- 9) If A and B are two equivalent matrices then rank A= rank B
- 10)if A and B are two equivalent matrixes then rank A = rank B.

2) Find rank of the matrix $\begin{bmatrix} 1 & -2 & -1 \\ -3 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}$

Sol:- det|A = (A) = 1(12-0) – (-2) (-12-0) -1(-6-6)

= 12-24+12=0

$\therefore$ A is singular

Let us take a submatrix of given matrix

$B=\begin{bmatrix} 1 & -2 \\ -3 & 3 \end{bmatrix} \Rightarrow \{B\} = 3-6 = -3 \neq 0$

Rank of given matrix = submatrix rank = P(A) = 2

Find the rank of the matrix $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3x3}$

ol: Det A of given matrix (A) = -1(18-1) – 0(9+5)+(3+30) = -17-0+198

= 181≠0

A is non – singular third order matrix

rank of A = ρ (A) = 3 = order of given matrix.

Echelon form:-

The Echelon form of a matrix A is an equivalent matrix, obtained by finite number of elementary operations on A by the following way.

1) The zero rows, if any, are below a nonzero row

2) The first nonzero entry in each nonzero row is one (1)

3) The number of zeros before the first nonzero entry in a row is lessthan the number of such zeros in the next row immediately below it.

Note:- (i) Condition (2) is optinal

(ii) The rank of A is equal to the number of nonzero rows in its echelon form.

Solved Problems:

1) Find the rank of the matrix by echelon form $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Sol:- Given A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A)$ = Rank of A = number of non zero rows = 2

2) Find the rank of the matrix $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -15 \end{bmatrix}$

Sol :- Given A = $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -15 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to 2R_3 + R_1$

$\sim \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ Rank of A = $\rho(A)$ = Number of non zero rows = 1

3) Find the value of K such that the rank of A = $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is 2

Sol:- Given A = $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & +1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & +3 & -2 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & 0 & -k+2 & 0 \end{bmatrix}$

Give rank of A is 2, there will be only two non zero rows

$\therefore$ Third row must be zero row $\Rightarrow$ 2-K=0

$\Rightarrow$ K = 2

Normal form:

Every m x n matrix of rank r can be reduced to the for [Ir 0] or Ir or (3) $\begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}$ by a finale number of elementary row or column transformations. Here 'r' indicates rank of the matrix.

Solved Problems:

1) Find the rank of the matrix by using normal form where A= $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Sol:- Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1$; $R_3 \rightarrow R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$C_2 \rightarrow C_2 + 3C_1$; $C_3 \rightarrow C_3 + C_1$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$R_2 \rightarrow R_2 \frac{1}{7}$, $R_3 \rightarrow R_3 . \frac{1}{9}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of A = ρ (A) = r = 2 = unit matrix order

2) Find the rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ by using normal form.

Sol: Given A = $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$C_3 \to C_3 - 2C, C_4 \to C_4 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & 6 \\ 0 & 2 & -3 & 3 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 \cdot \frac{1}{4}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 6C_2, C_4 \to C_4 - 6C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of A = ρ (A) = r = 2

Inverse of Non-singular matrix by Gauss – Jordan method:-

We can find the inverse of a non-singular square matrix using elementary row operations only.

Suppose A is a nonsingular square matrix of order n we write $A = I_n A$

Now we apply elementary row operations only to the matrix A and the prefactor I_n of the R.H.S. We will do this till we get an equation of the form $I_n = BA$. Then abviously B is the inverse of A.

1) Find the inverse of the Matrix $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ by using Gaus – Jordan Method

Sol:- Given $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Write $A = I_n A$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A$$

$R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} . A$$

$R_2 \rightarrow R_2 . (\frac{-1}{3})$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & -1 & 1 \end{bmatrix} . A$$

$R_1 \to R_1 - R_2$; $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & -2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -2/3 & 1/3 & 1 \end{bmatrix}$$

$R_3 \to R3(-3/2)$

$$\begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 1 & -1/2 & -3/2 \end{bmatrix} . A$$

$R_1 \to R_1 - 4/3.R_3$; $R_2 \to R_2 + 1/3.R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} . A$$

$I_{3x3} = B.A$ where $B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$ is the inverse of given matrix.

Exercise:

Find the inverse of the following matrixes by using Gaugs – Jordan method.

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

2) $\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

3) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Solution of linear System of equations:

An equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots + a_nx_n = b$..............(1)

Where $x_1, x_2, \ldots, x_n$ are unknowns and $a_1, a_2, \ldots, a_n$, b are constants is called a linear equations in n unknowns consider the system of m linear equations in n unknowns .

$x_1, x_2, \ldots, x_n$ as given below

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b2$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = bm \quad \ldots\ldots\ldots(2)$$

where aij's and $b_1, b_2, \ldots, b_m$ are constants. An ordered n- tuple $(x_1, x_2, \ldots, x_n)$ satisfying all equations in (2) is called a solution of the system (2).

The System of equations in (2) can be written in matrix from A X= B ....(3)

Where A = [aij], $x = (x_1, x_2, \ldots, x_n)^T$, $B = (b_1, b_2, \ldots, b_m)^T$

The Matrix [A/B] is called the augmented matrix of the system(2)

If B=0 in (3), the system is said to be Homogeneous otherwise the system is said to be non – homogeneous.

* The system AX = 0 is always consistent since X = 0 (i.e., $x_1=0$, $x_2=0$, ... $X_n=0$) is always a solution of AX = 0 This solution is called Trival solution of the system.

* Given AX = 0, we try to decide whether it has a solution $X \neq 0$. Such a solution, if exists, is called a non-Trival solution.

* If there is a least one solution for the given system is said to consistent, if the system does not have any solution, the system is said to be inconsistent.

Solution of Non-homogeneous system of equations:

The system AX=B is consistent i.e., it has a solution (unique or infinite) if and only if rank A = rank [A/B]

i) If rank of A = rank of [A/B] = r<n then the system is consistent and it has infinitely many solutions. There r = rank, n = number of unknowns in the system.

ii) If rant of A = rank of [A/B] = r = n then the system has unique solution.

iii) If rank of A≠ rank [A/B] then the system is inconsistent i.e., It has no solution.

Solved Problems:

1) Solve the system of equations $x+2y+3z=1$; $2x+37+8z=2$; $x+y+z = 3$

Sol: Given system can be written in matrix form

as $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

A X = B

Augmented matrix of the given system

$[A/B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix}$

$\therefore$ rank of A = rank [A/B] = r = 3 = number of unknowns = n

$\therefore$ n = r = 3

$\therefore$ The given system is consistent and it has unique solution. The solution is as follows from the last augmented matrix we can write as

$-4z = 2$	$-y+2z = 0$	$x+2y+3z = 1$
$z = \frac{-1}{2}$	$2z = y$	$x = 1-2y-3z$
	$2(\frac{-1}{2}) = y$	$= 1-2(-1)-3(\frac{-1}{2})$
	$Y=-1$	$= 1+2+\frac{3}{2}$
		$X= 9/2$

$\therefore$ The solution of given system : $x=9/2$; $y= -1$, $z=-1/2$

2) Solve the system of equations $x+2y+z=14$

$3x+4y+z=11$

$2x+3y+z=11$

Sol:- Given system can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$$

A X = B

The augmented matrix of the given system as

$$[A/B] = \begin{bmatrix} 1 & 2 & 1 & 14 \\ 3 & 4 & 1 & 11 \\ 2 & 3 & 1 & 11 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 14 \\ 0 & -2 & -2 & -31 \\ 0 & -1 & -1 & -17 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 14 \\ 0 & -2 & -2 & -31 \\ 0 & 0 & -0 & -3 \end{bmatrix}$$

Rank of A = 2≠3 = rank of AB

∴ The given system has no solution, i.e., the system is inconsistent

3) Show that the system $x+y+z=6$; $x+2y+3z = 14$; $x+4y+7z = 30$ are consistent and solve them.

Sol:- Given system can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Augmented matrix

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A = rank of AB = r = 2 < 3 = n = number of unknowns

∴ The system has consistent and it has infinitely many solutions.

Here $x+y+z = 6$

$Y+2z = 8$

Let $z = k$

Now $y = 8-2z = 8-2k$

Now $x = 6-y-z$

$= 6-(8-2k)-k$

$x = 6-8+2k-k$

$x = k-2$

∴ The system has infinitely many solutions $x=k-2$; $y=8-2k$; $z=k$

5) For what values of λ and μ the system of equations

$2x+3y+5z = 9$ have (i) no solution

$7x+3y-2x = 8$ (ii) unique solution

$2x+3y+lz = \mu$ (iii) infinitely many solutions

The matrix form of given system of equations

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The augmented matrix of given system

$$[A/B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \lambda \end{bmatrix}$$

$R_2 \rightarrow 2R_2 - 7R1;\ R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$$

$R_1 \rightarrow R1(\frac{1}{2})$

$$\sim \begin{bmatrix} 1 & 3/2 & 5/2 & 9/2 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$$

Case 1 : $\lambda=5,\ \mu\neq 9$

Then $\rho(A) = 2,\ \rho(AB) = 3$

$\rho(A) = 2 \neq 3 = \rho(AB)$

The system has no solution

Case 2:- $\lambda\neq 5,\ \mu\neq 9$

Then $\rho(A) = \rho(A/B) = r = n = 3$

$\therefore$ The system has unique solution

Case 3: $\lambda=5,\ \mu=9$

Then $\rho(A) = \rho(A/B) = r = 2 < 3 = n =$ number of unknowns

$\therefore$ The system has infinitely many solutions.

Consistency of system of homogeneous linear equations:

Consider of system of homogeneous linear equations in n unknowns namely

$a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{1n}x_n=0$

$a_{21}x_1+a_{22}x_2+\ldots\ldots+a_{2n}x_n=0$

....

....

$a_{m1}x_1+a_{m2}x_2+\ldots\ldots+a_{mn}x_n=0$

This system can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ a_{m1} & a_{m1} & \cdots & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$A \quad X = 0$$

1. If rank of A = n (number of variables)

$\Rightarrow$ The system of equations have only trivial solution (i.e., zero solution)

2. If r<n then the system have an infinitive number of non trivial solutions.

Solved Problems:

1) Find all the solutions of the system of equations

X+2y-z=0, 2x+y+z=0, x-4y+5z=0

Sol. Given system can be written in matrix form

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$[A/B] = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A = rank of AB = r = number of non zero rows = 2<3=n= number of variables

∴ The system has infinitely many solutions from the above matrix

-3y+3z=0 x+2y-z=0

⇒ y=z

Let us consider n-r=3-2=1 arbitrary constants

Let z=k, then y = k

Since x+2y-z=0

⇒ x=z-2y

= z-2y

=k-2k

x=-k

∴ x=-k, y=z=k

2) Solve the system of equations $x+y+w=0$; $y+z=0$, $x+y+z+w=0$, $x+y+2z=0$

Sol: Given system can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$[A/B] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$; $R_4 \rightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Rank of A = Rank of AB = r =4=n= number of unknowns

∴ Therefore there is no non=zero solution

∴ $x=y=z=w=0$ is only the trivial solution.

Gauss Seidel iteration method:

We will consider the system of equations

$a_{11}x_1+a_{12}x_2+a_{13}x_3 = b_1$ ....(1)

$a_{21}x_1 + a_{22}x_2+ a_{23}x_3 = b_2$ .... (2)

$a_{31}x_1 + a_{32}x_2+ a_{33}x_3 = b_3$ .... (3)

Where the diagonal coefficients are not zero and are large compared to other coefficients such a systen is called a "diagonally dominant system".

The system of equations (1) can be written as

$x_1=\frac{1}{a_{11}}[b_1-a_{12}x_2-a_{13}x_3]$............(4)

$x_2=\frac{1}{a_{22}}[b_2-a_{21}x_1-a_{23}x_3]$......... (5)

$x_3=\frac{1}{a_{33}}[b_3-a_{31}x_1-a_{32}x_2]$..........(6)

Let the initial approximate solution be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ are zero Substitute x_2^0, x_3^0 in (4) we get

$x_1^1 = 1/a_{11} [b_1-a_{12}x_2^0-a_{13}x_3^0]$ this is taken as first approximation of x_1

Substitute x_1^1, x_3^0 in (5) we get $x_2^1 = 1/a_{22}[b_2-a_{21}x_1^1-a_{23}x_3^0]$

This is taken as first approximation of x_2 now substitute x_1^1, x_2^1 in (6), we get

$x_3^1 = 1/a_{33}\ b_3-a_{31}x_1^1-a_{32}x_2^1]$

This is taken as first approximation of x_3 continue the same procedure until the desired order of approximation is reached or two successive iterations are nearly same. The final values of x_1, x_2, x_3 obtained an approximate solution of the given system