UNIT-V

MULTIPLE CALCULUS (INTEGRATION)

DOUBLE INTEGRALS

Definition:

The expression:

$$\int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} f(x,y)dx.dy$$

is called a double integral and provided the four limits on the integral are all constant the order in which the integrations are performed does not matter.

If the limits on one of the integrals involve the other variable then the order in which the integrations are performed is crucial.

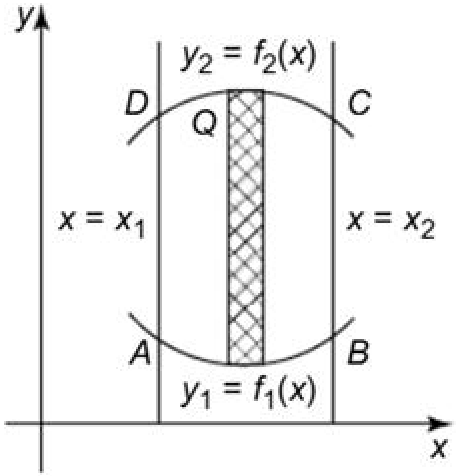
EVALUATION OF DOUBLE INTEGRAL IN CARESION CO ORDINATES:

We shall discuss the three cases.

CASE (i)

When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$ is first integrated w.r.to y keeping x constant between limits $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and then the resulting expression is integrated w.r.to x within the limits x_1, x_2;

The region of integration is the area $ABCD$, shown in the figure. Here the strip PQ moves along the direction of x axis. Thus,

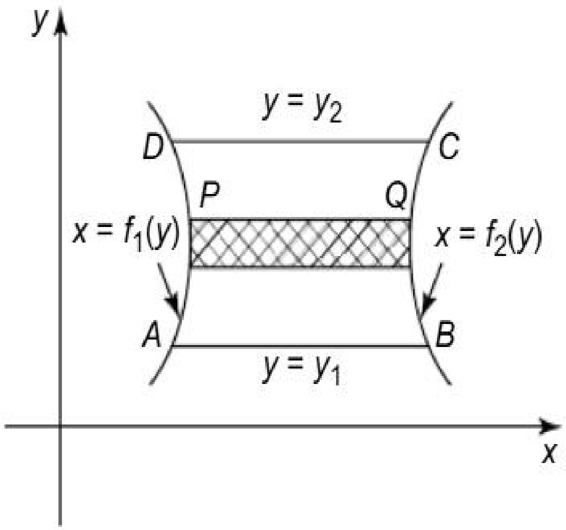


$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dxdy = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\, dy}\, dx}$$

CASE (ii)

When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated w.r.to x keeping y fixed, within the limits $x_1 = f_1(y)$ and $x_2 = f_2(y)$ and the resulting expression is integrated w.r.to y. The region of the integration $ABCD$ is shown in the figure.

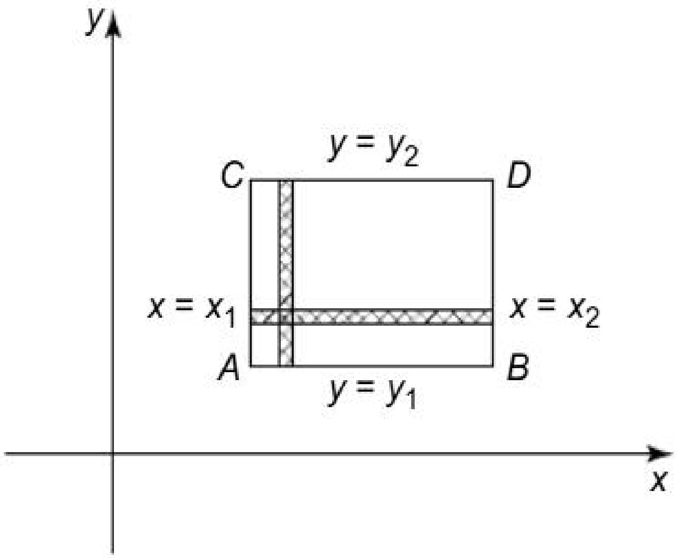
The horizontal strip PQ moves along the direction of y axis. Thus, we have



$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \boxed{\int_{y_1}^{y_2}\boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy}$$

CASE (iii)

When both pairs of limits are constants, the region of integration is the rectangle *ABDC*, shown in figure. The order of integration is immaterial in this case. We may first integrate $f(x, y)$ w.r.to x keeping y constant and the result is integrated w.r.to y or we may integrate $f(x, y)$ first w.r.to y keeping x constant and the result w.r.to x.



It is clear that in the integral $\int_c^d \int_a^b f(x,y)\,dy\,dx$, the limits for y are a to b and limits for x are c to d.

$$\textbf{Evaluate} \int_0^1 \int_0^2 xy\,dy\,dx$$

SOLUTiO

Integrate w.r.to y keeping x as constant, so $\int_0^1 x\left(\frac{y^2}{2}\right)_0^2 dx = \int_0^1 (2)\,x\,dx = 2\left(\frac{x^2}{2}\right)_0^1 = 1$

$$\textbf{Evaluate} \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

SOLUTiO

$$\int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy = \int_0^a \frac{a}{2} y^2\, dy = \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a = \frac{a^4}{6}$$

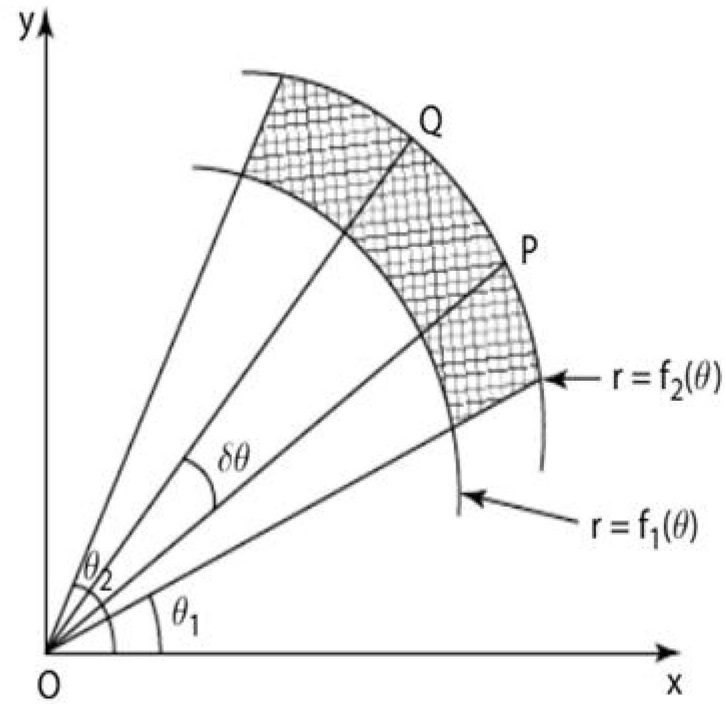
$$\text{Evaluate} \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dy\,dx$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 (y\,dy)\,dx = \int_0^a x^2 \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a (a^2x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[a^2\left(\frac{x^3}{3}\right)_0^a - \left(\frac{x^5}{5}\right)_0^a\right] = \frac{1}{2}\left[\frac{a^5}{3} - \frac{a^5}{5}\right] = \frac{a^5}{15}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

In polar coordinates, the double integral $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)\, dr d\theta$ gives the integral of f (r, θ) over the region A, shaded and is as shown in the figure.



This region is bounded by the curves $r = r_1 = f_1(\theta)$, $r = r_2 = f_2(\theta)$ and the straight lines $\theta = \theta_1$ and $\theta = \theta_2$. To cover this region, we first move over the sector OPQ from $r = r_1$ to $r = r_2$ and then rotate this sector from $\theta = \theta_1$ and $\theta = \theta_2$

$\therefore$ To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) dr d\theta$, we first integrate w.r.to r between the limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.to θ from θ_1 to θ_2

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\theta} r^2 \, dr \, d\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\theta} r^2 \, dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^3}{3}\right)_0^{2\cos\theta} d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8\cos^3\theta - 0)\, d\theta$$

$$= \frac{8}{3} \times 2\int_{0}^{\frac{\pi}{2}} \cos^3\theta d\theta$$

(using Reduction Formula)

$$= \frac{16}{3} \cdot \frac{2}{3} = \frac{16}{3} \times \frac{2}{3} = \frac{32}{9}$$

Evaluate $\iint_R xy\,dx\,dy$ R is the first quadrant of the circle

$$x^2 + y^2 = a^2 \,(x \geq 0, y \geq 0)$$

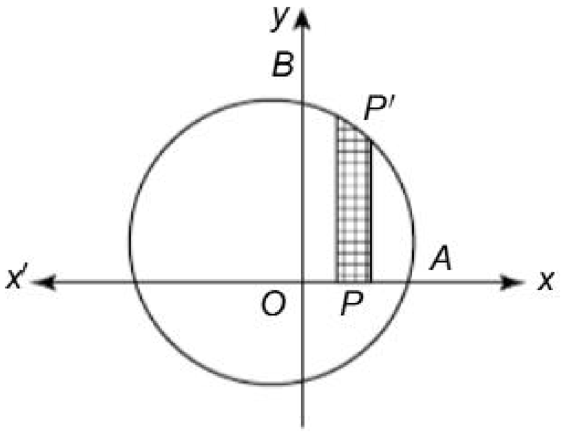
SOLUTIO

The region of integration is the first quadrant of the circle. *OAB* is the region of integration.

Let us assume the strip parallel to *y*-axis, so *y* should have variable limit $y = \pm\sqrt{a^2 - x^2}$;

As in the first quadrant, take $y = \sqrt{a^2 - x^2}$

Limits: *x*: 0 to *a*. $y : 0$ to $\sqrt{a^2 - x^2}$



$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a (xa^2 - x^3)\,dx$$

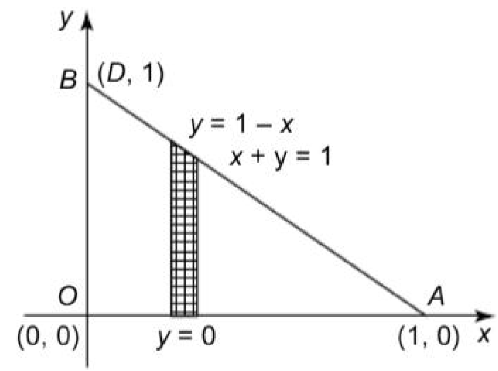
$$= \frac{1}{2}\left[a^2\left(\frac{x^2}{2}\right)_0^a - \left(\frac{x^4}{4}\right)_0^a\right] = \frac{a^4}{8}$$

Evaluate $\iint x^2 y\, dxdy$ over the region in the positive quadrant for which $x + y \leq 1$

SOLUTiO

The region of integration is the triangle *OAB*. Draw strip parallel to y axis; y should have variable limit.

Limits: $x : 0$ to 1; $y : 0$ to $(1 - x)$



$$I = \int_0^1 \int_0^{1-x} x^2 (y)\, dydx$$

$$= \int_0^1 x^2 \left(\frac{y^2}{2} \right)_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 x^2 (1 - x)^2 dx$$

$$= \frac{1}{2} \int_0^1 x^2 (1 + x^2 - 2x) dx$$

$$=\frac{1}{2}\left(\frac{x^3}{3}+\frac{x^5}{5}-\frac{2x^4}{4}\right)_0^1$$

$$=\frac{1}{2}\times\frac{1}{30}=\frac{1}{60}$$

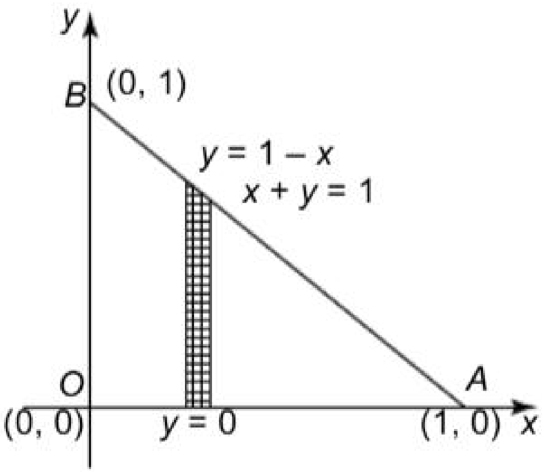
Evaluate $\iint (x^2+y^2)\,dxdy$ over the region for which x, y each ≥ 0 and $x+y\leq 1$

SOLUTiO

The region of integration is the triangle bounded by the lines $x=0$, $y=0$ and $x+y=1$

Limits of y : 0 to $1 - x$

Limits of x : 0 to 1



$$\therefore \iint (x^2 + y^2)\,dxdy = \int_0^1 \int_0^{1-x} (x^2 + y^2)\,dydx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)_0^{1-x} dx$$

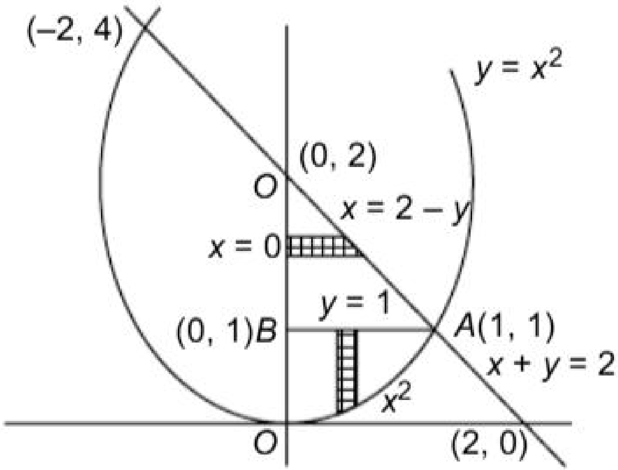
$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}$$

Evaluate $\iint ydxdy$ over the region between the parabola $x^2 = y$ and the line $x + y = 2$, in the positive quadrant

SOLUTIO



Points of intersection: $x^2 = 2 - x$

$\therefore \qquad x^2 + x - 2 = 0$

$\Rightarrow \qquad (x+2)(x-1) = 0$

$\qquad x = 1, -2$

$\therefore$ The point A is $(1, 1)$

$$\therefore \qquad \iint ydxdy = \iint_{OAB} ydxdy + \iint_{ABC} ydxdy$$

$$= \int_0^1 \int_{x^2}^1 ydydx + \int_1^2 \int_0^{2-y} ydxdy = \int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^1 dx + \int_1^2 y(x)_0^{2-y} dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{x^4}{2}\right) dx + \int_1^2 y(2-y)dy = \left(\frac{x}{2} - \frac{x^5}{10}\right)_0^1 + \left(y^2 - \frac{y^3}{3}\right)_1^2$$

$$=\frac{1}{2}-\frac{1}{10}+3-\frac{7}{3} \quad =\frac{2}{5}+\frac{2}{3}=2\left[\frac{8}{15}\right]=\frac{16}{15}$$

Evaluate $\iint r^3 \sin^2\theta\, dr\, d\theta$ over the circle $r = a\cos\theta$

SOLUTiO

The equation of the circle is $r = a\cos\theta$

$\therefore$ r varies from 0 to $a\cos\theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\therefore \quad \iint r^3 \sin^2\theta\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{a\cos\theta} r^3 \sin^2\theta\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \left(\frac{r^4}{4}\right)_0^{a\cos\theta} d\theta$$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, a^4 \cos^4\theta\, d\theta = \frac{a^4}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2\theta)\cos^4\theta\, d\theta$$

$$=\frac{a^4}{2}\left[\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta - \int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta\right]$$

$$\frac{a^4}{2}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$\frac{a^4\cdot 3\pi}{32}\left[1-\frac{5}{6}\right]=\frac{3\pi a^4}{64}\cdot\left[\frac{1}{6}\right]=\frac{\pi a^4}{64}$$

Evaluate $\int_0^1\int_{x^2}^{x}\frac{xdydx}{x^2+y^2}$ **and indicate the region of integration**

SOLUTiO

$$\int_0^1\int_{x^2}^{x}\frac{xdydx}{x^2+y^2}=\int_0^1\left[\int_{x^2}^{x}\frac{xdy}{x^2+y^2}\right]dx=\int_0^1\left[\tan^{-1}\frac{y}{x}\right]_{x^2}^{x}dx$$

$$=\int_0^1[\tan^{-1}(1)-\tan^{-1}x]dx=\int_0^1\left[\frac{\pi}{4}-\tan^{-1}x\right]dx$$

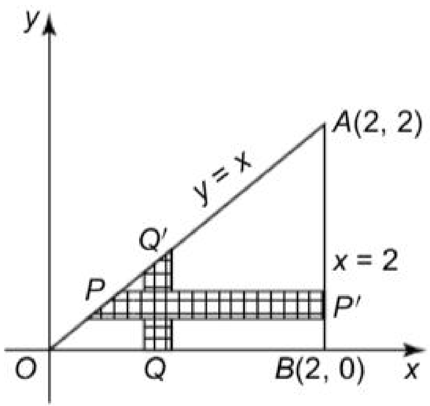
$$=\frac{\pi}{4}(x)_0^1-\int_0^1\tan^{-1}xdx=\frac{\pi}{4}-(x\tan^{-1}x)_0^1+\int_0^1\frac{x}{1+x^2}dx$$

$$=\frac{\pi}{4}-\frac{\pi}{4}+\left[\frac{1}{2}\log(1+x^2)\right]_0^1=\frac{\pi}{4}-\frac{\pi}{4}+\left[\frac{1}{2}\log(1+x^2)\right]_0^1=\frac{1}{2}\log 2$$

CHANGE OF ORDER OF INTEGRATION

Evaluate $\int_0^2 \int_y^2 \frac{x}{x^2+y^2}dxdy$ by changing the order of integration

SOLUTIO



Given: First integrate w.r.to x, then w.r.to y

To change the order of integration, first determine the region of integration

Limits - x: $x = y$ to $x = 2$

y: $y = 0$ to $y = 2$

which are straight lines, the region of integration is shown in figure.

The region is covered by first moving along the horizontal strip PP' and then sliding this strip parallel to itself to from $y = 0$ and $y = 2$.

By changing the order of integration, first move along the vertical strip QQ' on which y varies from $y = 0$ to $y = x$ then slide from $x = 0$, to $x = 2$.

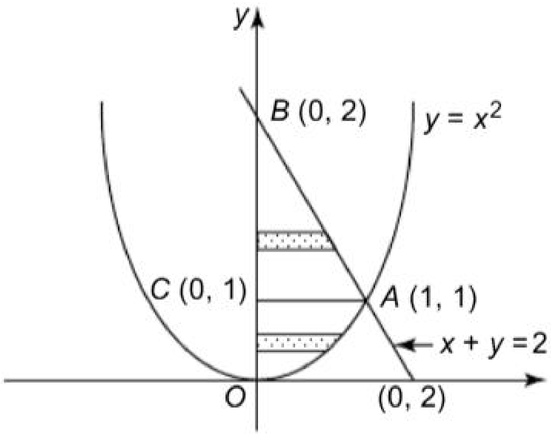
$$I = \int_0^2 \int_0^x \frac{x}{(x^2+y^2)}dydx = \int_0^2 \left(\tan^{-1}\frac{y}{x}\right)_0^x dx = \int_0^2 \frac{\pi}{4}dx = (x)_0^2 = \frac{\pi}{2}$$

Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ **and hence evaluate the same** **(AU., 2012)**

SOLUTIO

$I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$. Here first integrate w.r.to y and the then w.r.to x.

By changing the order of integration, integrate first w.r.to x, then w.r.to y.



Region of integration is $y = x^2$ (i)

$y = 2 - x, x + y = 2$ (ii)

$x = 0, x = 1.$

Solve (i) and (ii), the point of intersection is $A(1, 1)$

The region of integration is $OABCO$

The region is divided into two parts: OAC, ABC

$$I = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_1^{2-y} xy\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 (x^2 y)_0^{\sqrt{y}}\,dy + \frac{1}{2}\int_1^2 (x^2 y)_0^{2-y}\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\left(2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right)_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left(10 - \frac{28}{3} - \frac{1}{4}\right) = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

CHANGE OF VARIABLE FROM CARTESION TO POLAR CO-ORDINATES

Let the polar coordinates of point P whose cartesian coordinates are (x, y) be (r, θ)

Put $x = r\cos\theta$ and $y = r\sin\theta$

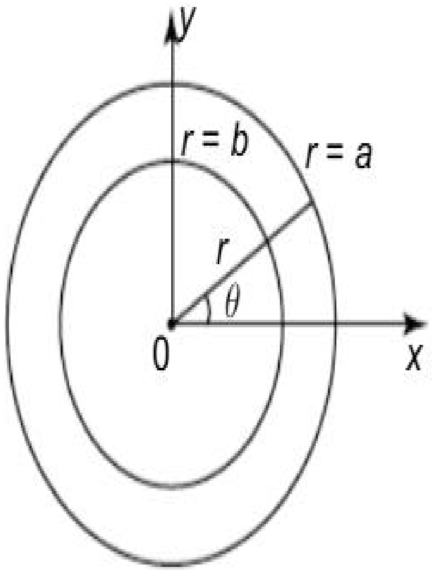
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \qquad \int_R F(x, y)\, dxdy = \int_R \int f(r,\theta) r\, dr d\theta$$

Hence change $dxdy$ into $rdrd\theta$ using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$

EXAMPLE 1 By transforming into polar coordinates evaluate $\iint \frac{x^2 \cdot y^2}{x^2+y^2} dxdy$ over the annular region bounded the circle $x^2+y^2=a^2$ and $x^2+y^2=a^2 (a>b)$

SOLUTiO



By transforming into polar coordinates, the two circles become $r=a$ and $r=b$

$$\iint_R \frac{x^2 \cdot y^2}{x^2+y^2} dxdy = \iint \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \cdot rdrd\theta$$

$$= \iint r^3\cos^2\theta\sin^2\theta\, drd\theta$$

Here r varies from b to a and θ varies from 0 to 2π

$$\therefore \quad \iint_R \frac{x^2 y^2 dxdy}{x^2 + y^2} = \int_0^{2\pi}\int_b^a r^2 \cos^2\theta \sin^2\theta \, dr d\theta$$

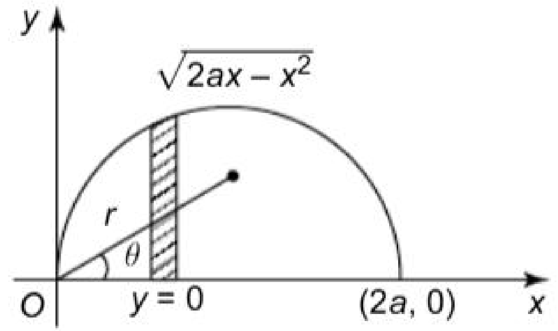
$$= \left(\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta\right)\left(\int_b^a r^3 dr\right)$$

$$= 4\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta \cdot \left[\frac{r^4}{4}\right]_b^a$$

$$= \left[4 \cdot \frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}\right]\left(\frac{a^4 - b^4}{4}\right) = \pi\left[\frac{a^4 - b^4}{16}\right]$$

Example 2 By changing into polar coordinates evaluate the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dxdy$

SOlUtiOn



y varies from 0 to $\sqrt{2ax-x^2}$

$$y = \sqrt{2ax-x^2}$$

$$\Rightarrow \quad y^2 = 2ax - x^2$$

$$\therefore \quad x^2 + y^2 - 2ax = 0 \Rightarrow (x-a)^2 + y^2 = a^2$$

This is a circle with centre $(a, 0)$ and radius $= a$

The region of integration is the semicircle $x^2 + y^2 = 2ax$ above the x-axis.

Changing into polars, the region becomes $r = 2a\cos\theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

Hence the required integral,

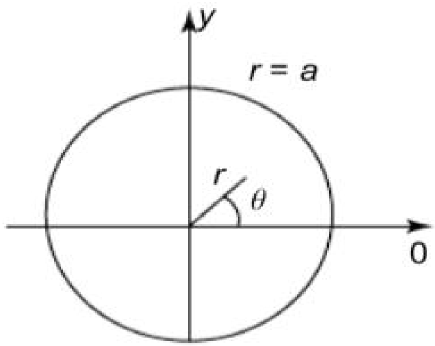
$$\therefore \quad I = \int_0^{\pi/2}\int_0^{2a\cos\theta} (r^2\cos^2\theta + r^2\sin^2\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta = 4a^4 \int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{3\pi a^4}{4}$$

Example 7 **By changing into polar coordinates, evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y + y^3)\,dxdy$

y varies from 0 to $\sqrt{a^2-x^2}$

$$y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2=a^2$$



The region of integration is the positive quadrant of the circle $x^2 + y^2 = a^2$.

Put $x = r\cos\theta$, $y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and $dxdy = r\,drd\theta$

$\therefore$ r varies from 0 to a: θ varies from 0 to $\frac{\pi}{2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)\,dxdy = \int_0^{\pi/2}\int_0^a r^3 \sin\theta\,(r\,drd\theta)$$

$$= \left(\int_0^{\pi/2} \sin\theta\,d\theta\right)\left(\int_0^a r^4\,dr\right)$$

$$= (-\cos\theta)_0^{\pi/2} \left(\frac{r^5}{5}\right)_0^5 = \frac{a^5}{5}$$

Triple Integrals

Consider a function $f(x, y, z)$ defined at all points of a given volume V. Divide V into 'n' elementary volumes.

Let $\delta x\, \delta y\, \delta z$ be the volume of an element surrounding the point (x, y, z)

Then $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum\sum\sum \delta x \delta y \delta z$ is called the *triple integral* of $f(x, y, z)$ over the volume V and is written as $\iiint_V f(x, y, z)\, dxdydz$

It can be expressed as a repeated integral of the form $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dzdydx$

If x_1, x_2 are constants; y_1, y_2 are functions of x and z_1, z_2 are functions of x and y, then this integral is evaluated as follows:

First $f(x, y, z)$ is integrated w.r.to z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.to y between the limits y_1 and y_2 keeping x constant. Finally, the result is obtained by integrating the expression w.r.to x from x_1 to x_2.

The order of integration may be different for different types of limits.

VOLUME OF TRIPLE INTEGRALS

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x \delta y \delta z$

$$\text{Then the total volume} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum\sum\sum \delta x\, \delta y\, \delta z$$

$$\Rightarrow \quad \text{volume of the solid} = \int\int\int dx\, dy\, dz$$

The above volume integral is evaluated with appropriate limits of integration

Example 1 Evaluate $\iiint\limits_V \frac{dzdydx}{(x+y+z+1)^3}$ where V is the volume of the region bounded by $x=0, y=0, z=0, x+y+z=1$ (AU., Dec. 2013)

The volume V is bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

Limits of z: 0 to $1-x-y$ $\quad [x+y+z=1 \Rightarrow z=1-x-y]$

Limits of y: 0 to $1-x$ $\quad [x+y=1 \Rightarrow y=1-x]$

Limits of x: 0 to 1 $\quad [x=1]$

$$\therefore \quad \iiint\limits_V \frac{dzdydx}{(1+x+y+z)^3} = \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}(1+x+y+z)^{-2}\right]_0^{1-x-y} dydx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(1+x+y)^2}\right] dydx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x}\right] dx$$

$$= -\frac{1}{8}\int_0^1 (1-x)\,dx - \frac{1}{4}\int_0^1 dx + \frac{1}{2}\int_0^1 \frac{dx}{1+x}$$

$$= \frac{1}{8}\left[\frac{(1-x)^2}{2}\right]_0^1 - \left(\frac{x}{4}\right)_0^1 + \frac{1}{2}[\log(1+x)]_0^1$$

$$= -\frac{1}{16} - \frac{1}{4} + \frac{1}{2}\log_e 2 = \frac{1}{2}\log_e 2 - \frac{5}{16}$$

$$= \frac{1}{16}[8\log_e 2 - 5]$$

Example 2 **Evaluate** $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

SOlUtiOn

Let

$$I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}(e^{z})_0^{x+\log y}\,dy\,dx$$

$$= \int_{0}^{\log 2} \int_{0}^{x} e^{x+y}(ye^{x} - 1)\,dy\,dx$$

$$= \int_{0}^{\log 2} \left[e^{2x} \int_{0}^{x} ye^{y}\,dy - \int_{0}^{x} e^{x+y}\,dy \right] dx$$

$$= \int_{0}^{\log 2} [e^{2x}[ye^{y} - e^{y}]_{0}^{x} - e^{x}(e^{y})_{0}^{x}]\,dx$$

$$= \int_{0}^{\log 2} \{e^{2x}[xe^{x} - e^{x} + 1] - e^{x}(e^{x} - 1)\}\,dx$$

$$= \int_{0}^{\log 2} \{xe^{3x} - e^{3x} + e^{2x} - e^{2x} + e^{x}\}\,dx$$

$$= \int_{0}^{\log 2} [xe^{3x} - e^{3x} + e^{x}]\,dx$$

$$= \int_{0}^{\log 2} xe^{3x}\,dx - \int_{0}^{\log 2} e^{3x}\,dx + \int_{0}^{\log 2} e^{x}\,dx$$

$$=\left[x\frac{e^{3x}}{3}-\frac{e^{3x}}{9}\right]_0^{\log 2}-\left(\frac{e^{3x}}{3}\right)_0^{\log 2}+(e^x)_0^{\log 2}$$

$$=\frac{\log 2}{3}e^{3\log 2}-\frac{e^{3\log 2}}{9}+\frac{1}{9}-\frac{1}{3}e^{3\log 2}+\frac{1}{3}+e^{\log 2}-1$$

$$=\frac{\log 2}{3}e^{\log 8}-\frac{e^{\log 8}}{9}+\frac{1}{9}-\frac{1}{3}e^{\log 8}+\frac{1}{3}+2-1$$

$$=\frac{8}{3}\log 2-\frac{8}{9}+\frac{1}{9}-\frac{8}{3}+\frac{1}{3}+1=\frac{8}{3}\log 2-\frac{7}{9}-\frac{4}{3}$$

$$=\frac{8}{3}\log_e 2-\frac{19}{9}$$

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{(a^2-r^2)}{a}} r\,dr\,d\theta\,dz$

$$\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{(a^2-r^2)}{a}} r\,dr\,d\theta\,dz = \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} r(z)_0^{\frac{a^2-r^2}{a}}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} r\left(\frac{a^2-r^2}{a}\right)dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\left(a^2\frac{r^2}{2}-\frac{r^4}{4}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\left(\frac{a^4\sin^2\theta}{2}-\frac{a^4\sin^4\theta}{4}\right)d\theta$$

$$= \frac{a^3}{4}\left[2\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta - \int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta\right]$$

$$= \frac{a^3}{4}\left[\frac{1}{2}\cdot\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] \quad \text{using Reduction formula}$$

$$= \frac{a^3}{4}\cdot\frac{\pi}{2}\left[1-\frac{3}{8}\right] = \frac{5\pi a^3}{64}$$

Example 10 Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$ (or) Evaluate $\iiint xyz\,dx\,dy\,dz$ taken over the positive octant of the sphere $x^2 + y^2 + z^2 = 1$

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$$\text{Given integral} = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{\sqrt{1-x^2}} y(1-x^2-y^2)\,dy\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{y^2}{2} - \frac{y^4}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x^2)^2}{2} - \frac{(1-x^2)^2}{4}\right]dx = \frac{1}{8}\int_0^1 x(1-x^2)^2 \cdot dx$$

$$= -\frac{1}{16}\int_0^1 (1-x^2)^2(-2x\,dx)$$

$$= -\frac{1}{16}\left[\frac{(1-x^2)^3}{3}\right]_0^1 = -\frac{1}{16}\left[0-\frac{1}{3}\right] = \frac{1}{48}$$

Example 11 **Evaluate** $\int_0^2 \int_1^3 \int_1^2 xy^2 z\,dz\,dy\,dx$

SOlUtiOn

Given integral $= \left(\int_1^3 y^2\,dy\right)\left(\int_0^2 x\,dx\right)\left(\int_1^2 z\,dz\right) = \left(\frac{y^3}{3}\right)_1^3 \left(\frac{x^2}{2}\right)_0^2 \left(\frac{z^2}{2}\right)_1^2$

$$= \left(\frac{26}{3}\right)\cdot(2)\cdot\left(\frac{3}{2}\right) = 26$$

Example 14 **Evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2-z^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

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Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1} \frac{z}{\sqrt{a^2-x^2-y^2}} \right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx \quad \text{Since } \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\frac{\pi}{2}\right) dy\,dx = \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \frac{\pi a^2}{4}\left(\frac{\pi}{2}\right) = \frac{\pi^2 a^2}{8}$$

Example 2 **Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes**

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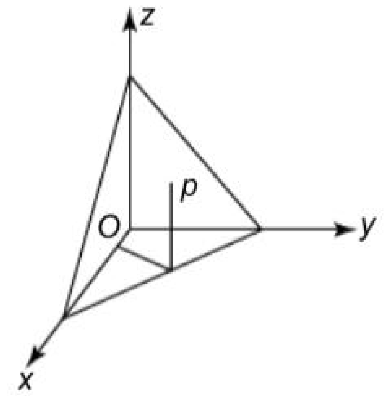

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \Rightarrow z=c\left[1-\frac{x}{a}-\frac{y}{b}\right]$$

When $z=0$, $\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow y=b\left[1-\frac{x}{a}\right]$

When $y=0, z=0$, $\frac{x}{a}=1 \Rightarrow x=a$

Over the volume of the tetrahedron,

z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$ and x varies from 0 to a

The required volume $= \iiint_v dxdydz$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx \quad = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left[\left(1-\frac{x}{a}\right)-\frac{y}{b}\right] dydx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_{y=0}^{b\left(1-\frac{x}{a}\right)} dx$$

$$= bc\int_0^a \left[\left(1-\frac{x}{a}\right)^2-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{abc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 \frac{dx}{a} \quad = \frac{abc}{2}\int_1^0 t^2(-dt) \text{ where } t = 1-\frac{x}{a}$$

$$= \frac{abc}{2}\int_0^1 t^2\,dt = \frac{abc}{2}\left[\frac{t^3}{3}\right]_0^1 = \frac{abc}{6}$$

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Example 3 **Find the volume of that portion of the ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **which lies in the first octant using triple integration** **(AU., 2007)**

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Required volume $=\iiint_V dxdydz$

where V is the region specified by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1$

Hence z varies from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$ and x varies from 0 to a

$\therefore$ Required volume $=\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}\, dydx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\, dydx$$

Taking $B^2 = b^2\left(1-\frac{x^2}{a^2}\right)$, we find Volume $= \frac{c}{b}\int_0^a \int_0^B \sqrt{B^2 - y^2}\, dydx$

$$\text{Volume} = \frac{c}{b}\int_0^a \left[\frac{y}{2}\sqrt{B^2 - y^2} + \frac{B^2}{2}\sin^{-1}\left(\frac{y}{B}\right)\right]_{y=0}^{B} dx$$

$$= \frac{c}{b}\int_0^a \frac{B^2}{2}[\sin^{-1}(1) - \sin^{-1}(0)]\, dx$$

$$= \frac{\pi c}{4b}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx \quad = \frac{\pi bc}{4}\int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{\pi bc}{4}\left[x - \frac{x^3}{3a^2}\right]_{x=0}^{a} \quad = \frac{\pi bc}{4}\left[a - \frac{a}{3}\right]$$

$$= \frac{\pi bc}{4}\left(\frac{2a}{3}\right) \quad = \frac{\pi abc}{6}$$