UNIT-II

Eigen values and Eigen vectors

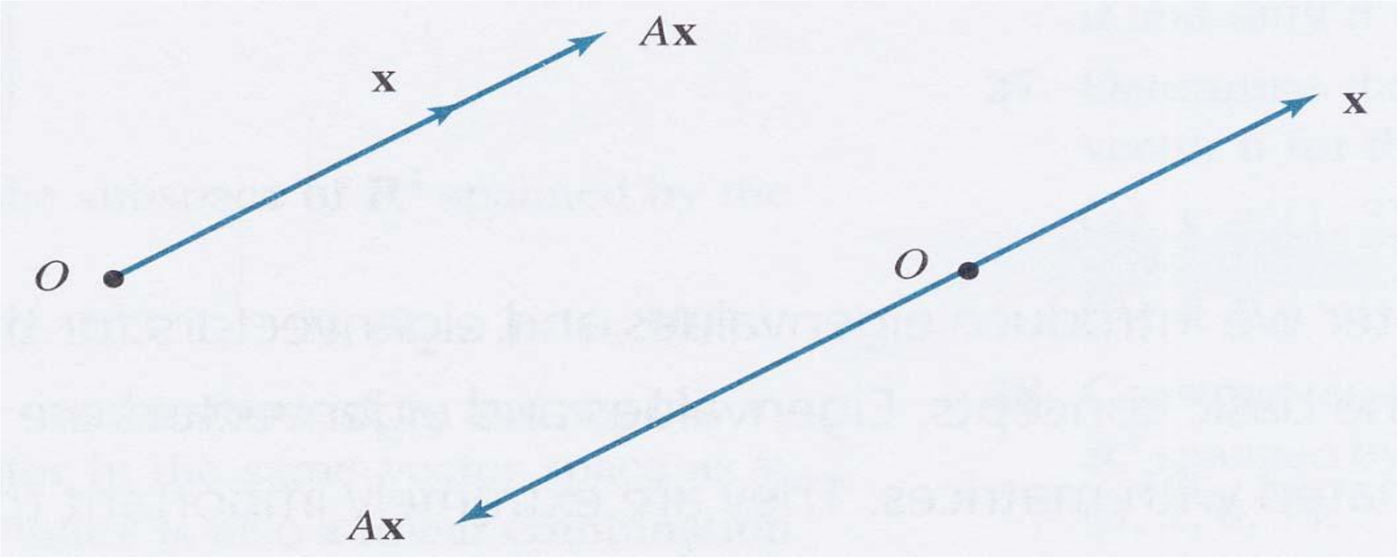
Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. A <u>*scalar*</u> λ is called an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ in $\mathbf{R}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector $\mathbf{x}$ is called an **eigenvector** corresponding to λ.



Computation of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Thus $A\mathbf{x} = \lambda\mathbf{x}$. This equation may be written

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

given

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

Solving the equation $|A - \lambda I_n| = 0$ for λ leads to all the eigenvalues of A.

On expending the determinant $|A - \lambda I_n|$, we get a polynomial in λ.

This polynomial is called the **characteristic polynomial** of A.

The equation $|A - \lambda I_n| = 0$ is called the **characteristic equation** of A.

Example:

Find the eigenvalues and eigenvectors of the matrix

Solution Let us first derive the characteristic polynomial of A. We get

$$A-\lambda I_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}$$

$$|A-\lambda I_2| = (-4-\lambda)(5-\lambda)+18 = \lambda^2-\lambda-2$$

We now solve the characteristic equation of A.

$$\lambda^2-\lambda-2=0 \Rightarrow (\lambda-2)(\lambda+1)=0 \Rightarrow \lambda = 2 \text{ or } -1$$

The eigenvalues of A are 2 and –1.

The corresponding eigenvectors are found by using these values of λ in the equation$(A-\lambda I_2)\mathbf{x} = \mathbf{0}$. There are many eigenvectors corresponding to each eigenvalue.

- For $\underline{\lambda = 2}$

  We solve the equation $(A - 2I_2)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

The matrix $(A - 2I_2)$ is obtained by subtracting 2 from the diagonal elements of A. We get

$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

This leads to the system of equations

$$-6x_1 - 6x_2 = 0$$
$$3x_1 + 3x_2 = 0$$

giving $x_1 = -x_2$. The solutions to this system of equations are $x_1 = -r$, $x_2 = r$, where r is a scalar. Thus the eigenvectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- For $\lambda = -1$

  We solve the equation $(A + 1I_2)x = 0$ for x.

The matrix $(A + 1I_2)$ is obtained by adding 1 to the diagonal elements of A. We get

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

This leads to the system of equations

$$-3x_1 - 6x_2 = 0$$

$$3x_1 + 6x_2 = 0$$

Thus $x_1 = -2x_2$. The solutions to this system of equations are $x_1 = -2s$ and $x_2 = s$, where s is a scalar. Thus the **eigenvectors** of A corresponding to $\lambda = -1$ are nonzero vectors of the form

$$v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example :

Find the eigenvalues and eigenvectors of the matrix

Solution The matrix $A - \lambda I_3$ is obtained by subtracting λ from the diagonal elements of A. Thus

$$A - \lambda I_3 = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I_3|$. Using row and column operations to simplify determinants, we get

$$|A - \lambda I_3| = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(9-\lambda)(2-\lambda)-8] = (1-\lambda)[\lambda^2 - 11\lambda + 10]$$

$$= (1-\lambda)(\lambda-10)(\lambda-1) = -(\lambda-10)(\lambda-1)^2$$

We now solving the characteristic equation of A:

$$-(\lambda-10)(\lambda-1)^2 = 0$$

$$\lambda = 10 \text{ or } 1$$

The eigenvalues of A are 10 and 1.

The corresponding eigenvectors are found by using three values of λ in the equation $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$.

- $\lambda_1 = 10$

  We get

$$(A - 10I_3)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

The solution to this system of equations are $x_1 = 2r$, $x_2 = 2r$, and $x_3 = r$, where r is a scalar.
Thus the eigenspace of $\lambda_1 = 10$ is the one-dimensional space of vectors of the form.

$$r \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

- $\lambda_2 = 1$

  Let $\lambda = 1$ in $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$. We get

$$(A - 1I_3)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

The solution to this system of equations can be shown to be $x_1 = -s - t$, $x_2 = s$, and $x_3 = 2t$, where s and t are scalars. Thus the eigenspace of $\lambda_2 = 1$ is the space of vectors of the form.

$$\begin{bmatrix} -s-t \\ s \\ 2t \end{bmatrix}$$

Separating the parameters *s* and *t*, we can write

$$\begin{bmatrix} -s-t \\ s \\ 2t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Thus the eigenspace of $\lambda = 1$ is a two-dimensional subspace of $\mathbf{R}^3$ with basis

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

If an eigenvalue occurs as a *k* times repeated root of the characteristic equation, we say that it is of **multiplicity** *k*. Thus λ=10 has multiplicity 1, while λ=1 has multiplicity 2 in this example.

- Exercise :
  1, 4, 9, 11, 13, 15, 24, 26, 32

Ex: Prove that if A is a diagonal matrix, then its eigenvalues are the diagonal elements.

Ex: Prove that if A and A^t have the same eigenvalues.

Ex: Prove that the constant term of the characteristic polynomial of a matrix A is $|A|$.

Diagonalization of Matrices

Definition

Let A and B be square matrices of the same size. B is said to be **similar** to A if there exists an invertible matrix C such that $B = C^{-1}AC$. The transformation of the matrix A into the matrix B in this manner is called a **similarity transformation**.

Example:

Consider the following matrices A and C with C is invertible. Use the similarity transformation $C^{-1}AC$ to transform A into a matrix B.

Solution

$$B = C^{-1}AC = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -10 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Similar matrices have the same eigenvalues.

Proof

Let A and B be similar matrices. Hence there exists a matrix C such that $B = C^{-1}AC$.

The characteristic polynomial of B is $|B - \lambda I_n|$. Substituting for B and using the multiplicative properties of determinants, we get

$$
\begin{aligned}
|B-\lambda I| &= |C^{-1}AC-\lambda I| = |C^{-1}(A-\lambda I)C| \\
&= |C^{-1}||A-\lambda I||C| = |A-\lambda I||C^{-1}||C| \\
&= |A-\lambda I||C^{-1}C| = |A-\lambda I||I| \\
&= |A-\lambda I|
\end{aligned}
$$

The characteristic polynomials of A and B are identical. This means that their eigenvalues are the same.

Example

(a) Show that the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable.

(b) Find a diagonal matrix *D* that is similar to *A*.

(c) Determine the similarity transformation that diagonalizes *A*.

Solution

(a) The eigenvalues and corresponding eigenvector of this matrix were found in Example 1 of Section 5.1. They are

$$\lambda_1 = 2, \mathbf{v}_1 = r\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1, \text{and} \mathbf{v}_2 = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Since *A*, a 2×2 matrix, has two linearly independent eigenvectors, it is diagonalizable.

(b) A is similar to the diagonal matrix D, which has diagonal elements $\lambda_1 = 2$ and $\lambda_2 = -1$. Thus

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \text{ is similar to } D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

(c) Select two convenient linearly independent eigenvectors, say

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let these vectors be the column vectors of the diagonalizing matrix C.

$$C = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

We get

$$C^{-1}AC = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D$$

Note

If A is similar to a diagonal matrix D under the transformation $C^{-1}AC$, then it can be shown that $A^k = CD^kC^{-1}$.
This result can be used to compute A^k. Let us derive this result and then apply it.

$$D^k = (C^{-1}AC)^k = \underbrace{(C^{-1}AC) \cdots (C^{-1}AC)}_{k \text{ times}} = C^{-1}A^kC$$

This leads to

$$\boxed{A^k = CD^kC^{-1}}$$

Example:

Compute A^9 for the following matrix A.

Solution

A is the matrix of the previous example. Use the values of C and D from that example. We get

$$D^9 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^9 = \begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix} = \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^9 = CD^9C^{-1}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -514 & -1026 \\ 513 & 1025 \end{bmatrix}$$

Example

Show that the following matrix A is not diagonalizable.

Solution

$$A-\lambda I_2 = \begin{bmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I_2| = 0 \Rightarrow (5-\lambda)(-1-\lambda)+9=0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)(\lambda-2)=0$$

There is a single eigenvalue, $\lambda = 2$. We find he corresponding eigenvectors. $(A - 2I)\mathbf{x} = \mathbf{0}$ gives $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad 3x_1 - 3x_2 = 0.$

Thus $x_1 = r$, $x_2 = r$. The eigenvectors are nonzero vectors of the form

$$r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace is a one-dimensional space. A is a 2×2 matrix, but it does not have two linearly independent eigenvectors. Thus A is not diagonalizable.

Let A be an $n \times n$ symmetric matrix.

(a) All the eigenvalues of A are real numbers.

(b) The dimension of an eigenspace of A is the multiplicity of the eigenvalues as a root of the characteristic equation.

(c) A has n linearly independent eigenvectors.

QUADRATIC FORMS

- A **quadratic form** is an homogeneous expression of every term degree two.

- Matrix form of Q.F=$X^T AX$ where A is symmetric matrix.

- The matrix A is called the **matrix of the quadratic form**.

QUADRATIC FORMS

Example : Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Compute $\mathbf{x}^T A\mathbf{x}$ for the following matrices.

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

a. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

QUADRATIC FORMS

- **Solution:**

a. $$x^TAx = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 . + 3x_2^2$$

a. There are two -2 entries in A.

$$x^TAx = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

CHANGE OF VARIBALE IN A QUADRATIC FORM

- If **x** represents a variable vector in , then a **change of variable** is an equation of the form

  $\mathbf{x} = P\mathbf{y}$, or equivalently, $\mathbf{y} = P^{-1}\mathbf{x}$ -----(1)

  where *P* is an invertible matrix and **y** is a new variable vector in Y

- Here **y** is the coordinate vector of **x** relative to the basis of determined by the columns of *P*.
- If the change of variable (1) is made in a quadratic form $\mathbf{x}^T A\mathbf{x}$, then

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T (P^T AP)\mathbf{y} \quad \text{-----(2)}$$

  and the new matrix of the quadratic form is $P^T AP$.

CHANGE OF VARIBALE IN A QUADRATIC FORM

- Since A is symmetric, Theorem guarantees that there is an orthogonal matrix P such that P^TAP is a diagonal matrix D, and the quadratic form in (2) becomes $\mathbf{Y}^T D\mathbf{Y}$.

- **Example :** Make a change of variable that transforms the quadratic form $Q(\mathrm{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$ into a quadratic form with no cross-product term.

- **Solution:** The matrix of the given quadratic form is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

CHANGE OF VARIBALE IN A QUADRATIC FORM

- The first step is to orthogonally diagonalize *A*.
- Its eigenvalues turn out to be $\lambda = 3$ and $\lambda = -7$.
- Associated unit eigenvectors are

$$\lambda = 3: \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}; \lambda = -7: \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

- These vectors are automatically orthogonal (because they correspond to distinct eigenvalues)

CHANGE OF VARIBALE IN A QUADRATIC FORM

- Let

$$P=\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, D=\begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

- Then $A=PDP^{-1}$ and $D=P^{-1}AP=P^{T}AP$

- A suitable change of variable is

$$x=Py \text{ ,where } x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

CHANGE OF VARIBALE IN A QUADRATIC FORM

- Then

$$x_1^2 - 8x_1x_2 - 5x_2^2 = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y})$$
$$= \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$
$$= 3y_1^2 - 7y_2^2$$

- To illustrate the meaning of the equality of quadratic forms in Example , we can compute *Q* (**x**) for $\mathbf{x} = (2, -2)$ using the new quadratic form.

CHANGE OF VARIBALE IN A QUADRATIC FORM

$$x = Py$$

- First, since
$$y = P^{-1}x = P^{T}x$$

so

$$y = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

- Hence

$$3y_1^2 - 7y_2^2 = 3(6/\sqrt{5})^2 - 7(-2/\sqrt{5})^2 = 3(36/5) - 7(4/5)$$
$$= 80/5 = 16$$

- This is the value of *Q* (**x**) when $x = (2, -2)$

CLASSIFYING QUADRATIC FORMS

- **Definition:** A quadratic form Q is:
  a. **positive definite** if $Q(\mathrm{x}) > 0$ for all $\mathrm{x} \neq 0$
  b. **negative definite** if $Q(\mathrm{x}) < 0$ for all $\mathrm{x} \neq 0$
  c. **indefinite** if Q (**x**) assumes both positive and negative values.

- Also, Q is said to be **positive semidefinite** for all **x**, and $Q(\mathrm{x}) \geq 0$ **negative semidefinite** if $Q(\mathrm{x}) \leq 0$ for all **x**.

QUADRATIC FORMS AND EIGENVALUES

- **Theorem :** Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A\mathbf{x}$ is:
  a. positive definite if and only if the eigenvalues of A are all positive,
  b. negative definite if and only if the eigenvalues of A are all negative, or
  c. indefinite if and only if A has both positive and negative eigenvalues.

QUADRATIC FORMS AND EIGENVALUES

- **Proof:** By the Principal Axes Theorem, there exists an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ such that

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A.

- Since P is invertible, there is a one-to-one correspondence between all nonzero **x** and all nonzero **y**.

QUADRATIC FORMS AND EIGENVALUES

- Thus the values of $Q(\mathbf{x})$ for $x \neq 0$ coincide with the values of the expression on the right side of (4), which is controlled by the signs of the eigenvalues $\lambda_1,\ldots,\lambda_n$, in three ways described in the theorem .