NARSIMHA REDDY ENGINEERING COLLEGE
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UNIT – IV

MULTI VARIABLE CALCULUS

(PARTIAL DIFFERENTIATION AND APPLICATION)

FUNCTION OF SEVERAL VARIABLES

Functions of Several Variable

A Symbol 'Z' which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write Z = f(x,y).

Limit of a Function f(x,y):-

The function f(x,y) defined in a Region R, is said to tend to the limit 'ι' as x→a and y→b iff corresponding to a positive number ∈, There exists another positive number δ such that $| f(x,y) – ι | < \in$ for $0 < (x-a)^2 + (y-b)^2 < \delta^2$ for every point (x,y) in R.

Continuity:-

A function f(x,y) is said to be continuous at the point (a,b) if

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b).$$

Homogeneous Function:-

An expression of the form,

$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \; - - - \; - + a_n y^n$ in which every term is of n^{th} degree, is called a homogeneous function of order 'n'.

Euler's Theorem:-

If z = f(x,y) be a homogeneous function of order 'n' in x and y, then $x \frac{6z}{6} + Y\frac{6Z}{6F} = nz$

Total Derivatives:-

if u = f(x,y)

where x = ϕ(t) , y = ψ(t)

then $\frac{du}{dydt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y\, dt}$

1) if f(x,y) = c

then

$$\frac{dy}{dx} = -\frac{(\partial u/\partial x)}{(\partial u/\partial y)}$$

2) if u = f(x,y) where x = ϕ(s,t), y = ψ(s,t)

then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Eulers theroms problems;

1.Verify Eulers therom for the function xy+yz+zx

Sol; Let f(x,y,z)=xy+yz+zx

f(kx,ky,kz)= ^{2}f(x,y,z)

This is homogeneous fuction of second degree

We have $\frac{6f}{6x}$=y+z $\frac{6f}{6y}$=x+z $\frac{6f}{6z}$=x+y

$x\frac{6f}{6}+y\frac{6f}{6}+z\frac{6f}{6}=x(y+z)+y(x+z)+z(x+y)$

$=xy+xz+yx+yz+zx+zy$

$=2(xy+yz+zx)$

$=2f(x,y,z)$

PROLEMS;

1.Verify the Eulers therom for z= $\frac{1}{2+\ \ +\ 2}$

2.Verify the Eulers therom for u= $\sin^{-1}\frac{x}{\ }+\tan^{-1}\frac{y}{\ }$

3.Verify the Eulers therom for u= $x^2\tan^{-1}\frac{y}{\ } - y^2\tan^{-1}\frac{x}{\ }$ **and also prove that**

$\frac{6^2}{6\ 6} = \frac{2\_\ 2}{2+\ 2}$

Jacobian (J) : Let U = u (x , y) , V = v(x , y) are two functions of the independent variables x , y. The jacobian of (u , v) w.r.t (x , y) is given by

$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ Note : $J\left(\frac{u,v}{x,y}\right) \times J\left(\frac{x,y}{u,v}\right) = 1$

Similarly of U = u(x, y , z) , V = v (x, y , z) , W = w(x, y , z)

Then the Jacobian of u , v , w w.r.to x , y , z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Solved Problems:

1. If x + y² = u , y + z² = v , z + x² = w find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

$= 1(1-0) - 2y(0 - 4xz) + 0$

$= 1 - 2y(-4xz)$

$= 1 + 8xyz$

$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]}$ $\frac{1}{1+8xyz}$

2. S.T the functions $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related. ('07 S-1)

Sol: Given $u = x + y + z$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

$w = x^3 + y^3 + z^3 - 3xyz$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \Rightarrow c_1 - c_2$

$c_2 \Rightarrow c_2 - c_3$

$$=6 \begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$=6[2(x - y) (y^2 + xy - xz - z^2) - 2(y - z)(x^2 + xz - yz - y^2)]$

$=6[2(x - y)(y - z)(x + y + z) - 2(y - z)(x - y)(x + y + z)]$

$=0$

3. If $x + y + z = u$, $y + z = uv$, $z = uvw$ then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ ('06 S-1)

Sol: $x + y + z = u$

$y + z = uv$

$z = uvw$

$y = uv - uvw = uv(1 - w)$

$x = u - uv = u(1 - v)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R2 \Rightarrow R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$= uv[\, u - uv + uv]$

$= u^2v$

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$ ('07 S-2)

Sol: Given $u = x^2 - y^2$, $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\quad$ $= 2r\cos\theta \, r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\quad$ $= r^2\sin 2\theta$

$= r^2\cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

$= 4r^2\,[r\cos^2 2\theta + r\sin^2 2\theta]$

$= 4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$

$= 4r^3$

5. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ ('08 S-4)

Sol: Given $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz(-1/x^2) = \frac{-yz}{x^2}$, $u_y = \frac{z}{x}$, $u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$, $v_y = xz(-1/y^2) = \frac{-xz}{y^2}$, $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $w_y = \frac{x}{z}$, $w_z = xy(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

= 1[-1(1-1) -1(-1-1) + (1+1)]

= 0 -1(-2)+(2)

=2 + 2

=4

Assignment

Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x=\sqrt{vw}$, $y=\sqrt{wu}$, $z=\sqrt{uv}$ and $u = r\sin\theta\cos\emptyset$, $v = r\sin\theta\sin\emptyset$, $w = r\cos\theta$

6. If $x = e^r\sec\theta$, $y = e^r\tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)}\cdot\frac{\partial(r,\theta)}{\partial(x,y)} = 1$ ('08 S-2)

Sol: Given $x = e^r\sec\theta$, $y = e^r\tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r\sec\theta = x$, $x_\theta = e^r\sec\theta\tan\theta$

$y_r = e^r\tan\theta = y$, $y_\theta = e^r\sec^2\theta$

$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log(x^2 - y^2)$

⇨ $r = \frac{1}{2}\log(x^2 - y^2)$

$r_x = \frac{1}{2}\frac{1}{(x2 - y2)}(2x) = \frac{x}{(x2 - y2)}$

$r_y = \frac{1}{2}\frac{1}{(x2 - y2)}(-2y) = \frac{-y}{(x2 - y2)}$

$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$

⇨ $\sin\theta = \frac{y}{x}$, $\theta = \sin^{-1}(\frac{y}{x})$

$$\theta_x = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}(1/x) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta & \tan\theta \\ & e^r\sec2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\,e^r\sec\theta\tan\theta$$

$$= e^{2r}\sec\theta[\sec^2\theta - \tan^2\theta] = e^{2r}\sec\theta$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x2 - y2)} & \frac{-y}{(x2 - y2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2 - y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} \quad = \frac{1}{x\sqrt{x^2-y^2}} \quad = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Functional Dependence

Two functions u and v are functionally dependent if their Jacobian

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

If the Jacobian of u, v is not equal to zero then those functions u, v are functionally independent.

** **Maximum & Minimum for function of a single Variable:**

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find $f'(x)$ and equate it to zero

(ii) solve the above equation we get x_0, x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x = x0)} > 0$, then f(x) is minimum at x_0

If $f^{11}(x)_{(x = x0)}, < 0$, f(x) is maximum at x_0. Similarly we do this for other stationary points.

PROBLEMS:

1. **Find the max & min of the function $f(x) = x^5 - 3x^4 + 5$ ('08 S-1)**

Sol : Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 - 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 - 12x^3 = 0$

$X = 0$, $x = 12/5$

$f^{11}(x) = 20x^3 - 36x^2$

At $x = 0 \Rightarrow f^{11}(x) = 0$. So f is neither maximum nor minimum at s = 0

At $x = (12/5)$ $f^{11}(x) = 20(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

** **Maxima & Minima for functions of two Variables**:

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a_1,b_1) (a_2,b_2) (a_3,b_3) ………………

2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$

i) IF $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$.

ii) IF $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

iii) IF $ln - m^2 < 0$ and at (a_1,b_1) then f(x ,y) is neither maximum nor minimum at (a_1,b_1). In this case (a_1,b_1) is saddle point.

iv) IF $ln - m^2 = 0$ and at (a_1,b_1) , no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

PROBLEMS:

1. Locate the stationary points & examine their nature of the following functions.

('07 S -2)

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2, (x > 0, y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$---------------------- > (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$---------------------- > (2)

Adding (1) & (2) ,

$x^3 + y^3 = 0$

$\Rightarrow = x = -y$-------------------> (3)

(1) $\Rightarrow x^2 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial 2u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4$ & $n = \frac{\partial 2u}{\partial y2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0) , $ln - m^2 = (0-4)(0-4) - 16 = 0$

(0,0) is not a extrem value.

2. Investigate the maxima & minima if any of the function $f(x) = x^3y^2(1-x-y)$.

('08 S – 4)

Sol: Given $f(x) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$---------------- > (2)

From (1) & (2) $4x + 3y - 3 = 0$--------------------- 2

$2x + 3y - 2 = 0$---------------------- 3

$2x = 1 \Rightarrow x = ½$

$4(½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2$, $y = (1/3)$

$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3) - 12(1/2)(1/3) - 6(1/2)(1/3) = 1/3 - 1/3 - 1/9 = -1/9$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2,1/3)} = 6(1/2)(1/3) - 8(1/2)(1/3) - 9(1/2)(1/3) = \frac{6-4-3}{12} = \frac{-1}{12}$

$n = \frac{\partial 2f}{\partial y2} = 2x^3 - 2x^4 - 6x^3y$

$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = (1/8) - 2(1/2) - 6(1/2)(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$ln - m^2 = (-1/9)(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

The function has a maximum value at (1/2 , 1/3)

3. Find three positive numbers whose sum is 100 and whose product is maximum.

('08 S-1)

Sol: Let x ,y ,z be three +ve numbers.

Given $x + y + z = 100$

⇨ $Z = 100 - x - y$

Let $f(x,y) = xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0 \Rightarrow y(100 - 2x - y) = 0$-----------------> (1)

$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0 \Rightarrow x(100 - x - 2y) = 0$-------------------> (2)

From (1) & (2)

$100 - 2x - y = 0$

$200 - 2x - 4y = 0$

$-100 + 3y = 0 \Rightarrow 3y = 100 \Rightarrow y = 100/3$

$100 - x - (200/3) = 0 \Rightarrow x = 100/3$

$l = \frac{\partial^2 f}{\partial x^2} = -2y$

$\left(\frac{\partial^2 f}{\partial x^2}\right)(100/3 , 100/3) = -200/3$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 100 - 2x - 2y$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)(100/3 , 100/3) = 100 - (200/3) - (200/3) = -(100/3)$

$n = \frac{\partial^2 f}{\partial y^2} = -2x$

$\left(\frac{\partial^2 f}{\partial y^2}\right)(100/3 , 100/3) = -200/3$

$ln - m^2 = (-200/3)(-200/3) - (-100/3)^2 = (100)^2/3$

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\quad \therefore z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required no. are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$ ('08 S-3)

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0 , x = \pm 1$

$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0 \quad \Rightarrow -4y(1-y^2) = 0 \Rightarrow y = 0, y = \pm 1$

$l = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$

$n = \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$

we have $ln - m^2 = (4-12x^2)(-4+12y^2) - 0$

$= -16 + 48x^2 + 48y^2 - 144x^2y^2$

$= 48x^2 + 48y^2 - 144x^2y^2 - 16$

i) At $(0, \pm 1)$

$ln - m^2 = 0 + 48 - 0 - 16 = 32 > 0$

$l = 4 - 0 = 4 > 0$

f has minimum value at $(0, \pm 1)$

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$

The minimum value is '-1 '.

ii) At $(\pm 1, 0)$

$ln - m^2 = 48 + 0 - 0 - 16 = 32 > 0$

$l = 4 - 12 = -8 < 0$

f has maximum value at $(\pm 1, 0)$

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

The maximum value is '1 '.

iii) At $(0,0)$, $(\pm 1, \pm 1)$

$ln - m^2 < 0$

$l = 4 - 12x^2$

$(0, 0)$ & $(\pm 1, \pm 1)$ are saddle points.

F has no max & min values at $(0, 0)$, $(\pm 1, \pm 1)$.

Assignment

1. Find the maximum value of x,y,z when x + y + z = a .

   [Ans: $\frac{m^m n^n p^p (a^{m+n+p})}{(m+n+p)^{m+n+p}}$]

***Extremum** : A function which have a maximum or minimum or both is called 'extremum'

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points:** - To get stationary points we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ i.e the pairs (a_1, b_1), (a_2, b_2)....................Are called Stationary.

*Maxima & Minima for a function with constant condition :Lagrangian Method

Suppose $f(x , y , z) = 0$------------ (1)

$\emptyset(x , y , z) = 0$----------- ---(2)

$F(x , y , z) = f(x , y , z) + \gamma \emptyset(x , y , z)$ where γ is called Lagrange's constant.

1. $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0$---------------- (3)

   $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0$---------------- (4)

   $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0$---------------- (5)

2. Solving the equations (2) (3) (4) & (5) we get the stationary point (x, y, z).
3. Substitute the value of x , y , z in equation (1) we get the extremum.

Problem:

1. **Find the minimum value of $x^2 + y^2 + z^2$ given $x + y + z = 3a$ ('08 S-2)**

Sol: $u = x^2 + y^2 + z^2$

$\emptyset = x + y + z - 3a = 0$

Using Lagrange's function

$F(x , y , z) = u(x , y , z) + \gamma \emptyset(x , y , z)$

For maxima or minima

$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 2x + \gamma = 0$------------- (1)

$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 2y + \gamma = 0$------------- (2)

$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 2z + \gamma = 0$------------- (3)

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

OBJECTIVE TYPE QUESTIONS

13. If u=x^2 - 2y ,v=x+y then $\frac{\partial(u,v)}{\partial(x,y)}$ = ________________

    (a) $(x+1)^2$ (b) 2(x+1) (c) 3(x+1) (d) None

16. If u(1-v)=x, uv=y then $J\left(\frac{u,v}{x,y}\right)$. $J\left(\frac{x,y}{u,v}\right)$ =

    (a) 0 (b) 1 (c) xy (d) None

17. If u= $\frac{x+y}{1-xy}$, v = tan^{-1} x + tan^{-1} y then $J\left(\frac{u,v}{x,y}\right)$. $J\left(\frac{x,y}{u,v}\right)$ =

    (a) 0 (b) 1 (c) xy (d) None

18. Are u=x$\sqrt{1-x^2}$, v=2x functionally dependent? If so what is $\left(\frac{u,v}{x,y}\right)$?

    (a) yes,1 (b) yes,0 (c) No,0 (d) None

19. If u=x^2y ,v=xy^2 then $\frac{\partial(u,v)}{\partial(x,y)}$ is

    (a)5x^2y^2 (b)4 x^2y^2 (c) $2x^2y^2$ (d) $3x^2y^2$

(Assignment Questions)

{ Functions of Several Variables}

1. If x+y^2=u , y+z^2=v ,z+ x^2=w find $\frac{\partial(x.y.z)}{\partial(u.v.w)}$.
2. If x+y+z=u, y+z= uv, z=uvw then evaluate $\frac{\partial(x.y.z)}{\partial(u.v.w)}$.
3. S.T the functions u=x+y+z, v=$x^2+y^2+z^2$-2xy-2zx and w=$x^3+y^3+z^3$-3xyz are functionally related.
4. Find the max & min values of the function f(x)=x^5-3x^4+5.
5. Find three positive numbers whose sum is 100 and whose product is maximum.
6. Locate the stationary points & examine their nature of the following functions u=$x^4+y^4-2x^2+4xy-2y^2$ (x>0,y>0).
7. If u= $\frac{yz}{x}$,v=$\frac{xz}{y}$, w=$\frac{xy}{z}$, find $\frac{\partial(u.v.w)}{\partial(x.y.z)}$.