UNIT – II

Eigen Values and Eigen Vectors

Eigen Values:-

Let A = $[a_{ij}]_{n\times n}$ be a square matrix of order n &λ is the scalar quantity, is called the

1) The Matrix A - λI is called the characteristic Matrix is A where I is the unit matrix of order n.
2) The polynomial |A - λI| in λ of degree n is called characteristic polynomial of A.
3) The equation |A - λI| = 0

i.e., $\begin{bmatrix} a11-\lambda & a12 & . & a1n \\ a21 & a22-\lambda & . & a2n \\ . & . & . & . \\ an1 & an2 & . & ann-\lambda \end{bmatrix} = 0$ is called characteristic equation of A

Note:- The characteristic equation is of the form $(-1)^n\lambda^n + C_1\lambda^{n-1}+c_2\lambda^{n-2}+\ldots+c_n=0$

4) The roots of the characteristic equation |A - λI| = 0 are called characteristic roots (or) latent roots (or) Eigen values of the Matrix A.

Note: 1. The set of all eigen values of A is called the Spectrum of A.

2. The degree of the characteristic polynomial is equal to the order of the matrix.

Eigen Vectors:-

Let A = [aij]nxn, A non – zero vector x is said to be a characteristic vector of A if λ a scalar λ such that AX = λX.

If AX = λX, (x ≠ 0) we say that x is Eigen vector or characteristic vector of A corresponding to the Eigen value or characteristic value λ of A.

Solved Problems:

1) Find the Eigen values of A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol:- Step 1:- Given Matrix A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Step 2:- Characteristic equation |A - λI| = 0

$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$

(5-λ) (2-λ) – 4 = 0

$10-5\lambda-2\lambda+\lambda^2-4=0$

$\lambda^2-7\lambda+6=0$

Step 3:- The roots of characteristic equation is called eigen values or eigen roots or latent values.

$\lambda^2-7\lambda+6=0$

$\lambda^2-6\lambda-\lambda+6=0$

λ(λ-6) -1(λ-6) = 0

(λ-6) (λ-1) = 0

λ = 1, 6

∴ Eigen values are 1,6

2) Find the characteristic roots or eigen roots of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Sol:- Step1: Given matrix A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Step 2: Characteristic Equation

$|A-\lambda I| = 0$

$\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 0-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{bmatrix} = 0$

$\Rightarrow -\lambda^3+6\lambda-4 = 0$

$\lambda^3-6\lambda+4 = 0$

Step 3: roots of above egn are called eigen values.

$\lambda^3-6\lambda-4 = 0$

$(\lambda-2)(\lambda^2+2\lambda-2) = 0$

$\lambda = 2, \lambda = \frac{-2\pm\sqrt{4+8}}{2}$

$\lambda = 2, -1\pm\sqrt{3}$

Eigen roots are 2, $-1\pm\sqrt{3}$

Exercise problems:-

1) Find the eigen values A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

2) Find the eigen values A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

3) Find the eigen values A = $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

4) Find the eigen values A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Eigen vector problems

1) Find the Eigen values and Eigen vectors of the following matrix A = $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Sol: Step1:- given matrix A = $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Step2:- Characteristic equation $|A-\lambda I| = 0$

$\begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = 0$

$(5-\lambda)\{(6-x)(7-\lambda)-4\}+2\{-2(7-\lambda)-0\}+0=0$

$-\lambda^3+18\lambda^2-99\lambda+162=0$

$\lambda^3-18\lambda^2+99\lambda-162=0$

Step3:-$(\lambda-3)(\lambda-6)(\lambda-9)=0$

$\lambda=3, 6, 9$

$\therefore$ Eigen values are 3,6,9

Step3: Eigen vectors

1) Eigen vector corresponding to $\lambda=3$ $[A-\lambda I]x=0$; $[A-3I]x=0$

$$\begin{bmatrix} 5-3 & -2 & 0 \\ -2 & 6-3 & 2 \\ 0 & 2 & 7-3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Echelon form

$$R_2\to R_2+R_1\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ x2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3\to R_3+2R_1\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank = 2 = no. of non zero rows

N = no. of unknowns (or) no. of variables n = 3

$r<n \Rightarrow n-r=3-2=1$ we choose one variable to the one constant.

$2x_1-2x_2=0$

$x_1+2x_3=0$

let $x_3=k$

$2x_1=2x_2=2[-2k]=-4k$

$x1=\frac{-4}{2}k=-2k$

$$\text{Eigen vector } x_1=\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix}=\begin{bmatrix} -2k \\ -2k \\ \ \end{bmatrix}=k\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to 6 :- $[A-6I]x=0$

Using Echelon form

$$R_2\to R_2-2R_1\begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3\to 2R_3-R_2\begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

r = 2 , n = 3

we choose one variable to the one constant.

i.e., x3=k

$-x_1-2x_2 = 0$

$4x_2+2x_3 = 0$

$x_3 = k$

$4x_2= -2x_3 = -2k$

$X_2= -\frac{1}{2}= k$

$-x_1-2x_2 = 0 \Rightarrow -x_1 = 2x_2 = 2[\frac{-1}{2}]\ k$

$x_1 = k,\ x_2 = \frac{-1}{2}k,\ x_3 = k,$

Eigen vector $x_2 = \begin{bmatrix} 1 \\ x2 \\ 3 \end{bmatrix} = [-1/2k]$

$x_2 = \frac{k}{2}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

Eigenvector corresponding to 9 :- [A-9I]x = 0

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2\to 2R_2-R_1 \begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ 2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3\to 2R_3-R_2 \begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

r=2, n = 3

n-r= 3-2 = 1

Let $x_3 = k$

$-4x_1 - 2x_2 = 0$

$-4x_2 + 4x_3 = 0$

$-x_2 = -x_3$

$x_2 = x_3 = k$

$-4x_1-2x_2 = 0$

$-2x_1 = x_2$

$x_2 = -2x_1 = -2k$

$x_1 = \frac{-x2}{2} = \frac{-k}{2}$

$\therefore$ Eigen vector $x_3 = \begin{bmatrix} 1 \\ x2 \\ 3 \end{bmatrix} = \begin{bmatrix} -k/2 \\ k \\ \ \end{bmatrix}$

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$x_3 = \frac{}{2}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

Three eigen vectors are

$x_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

2) Find the characteristic roots and find the corresponding eigen vectors $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol :- Step1: Given Matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Step 2:- Characteristic Egn $|A-\lambda I| = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3-12\lambda^2+36\lambda-32 = 0$

$\Rightarrow (\lambda-2)(\lambda^2-10\lambda+16) = 0$

$(\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\lambda = 2, 2, 8$

Step 3:- Eigen values are 2,2,8

Eigen Vectors:- The eigen vector of A Corresponding to $\lambda = 2$

$[A-\lambda I]x = 0$, $[A-2I]x = 0$

$$\begin{bmatrix} -4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigen vector of A corresponding to $\lambda = 8$

$[A-8I]x = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-R_1$; $R_3 \to R_3-R_1$ $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 2 & -3 & -3 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3-R_1$ $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

r=2, n = 3, 1-r = 3-2 = 1 we have to select one variable to the one constant i.e, $x_3 = k$

$-2x_1-2x_2+2x_3 = 0$

$-3x_2+(-3)x_3 = 0$

$x_2 = -x_3 = -k$

$x_1 = 2k$

$$\Rightarrow x_3 = \begin{bmatrix} 1 \\ x2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -k \\ \end{bmatrix} = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \text{ Eigen vectors are } x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Exercise problems

I. Find the eigen values & Eigen vectors of the following matrixs.

1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Ans:- $\lambda = 0,0,3$ Eigen Vectors $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans:- $\lambda = 0,3,15$ Eigen Vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

3) $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Ans:- $\lambda = 2,3,6$ Eigen Vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Ans:- $\lambda = 1,3,6$ Eigen Vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

5) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Ans:- $\lambda = 1,2,-2$ Eigen Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/3 \\ 1 \\ -2 \end{bmatrix}$

6) $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Ans:- $\lambda = 1,2,3$ Eigen Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$

Diagonalization of a matrix

A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Also the matrix P is then said to diagonalize A or transform A to diagonal form.

Similarity of Matrix:- Let A & B be square matrices to A It Ǝ a non – singular matrix P of order n → B $P^{-1}AP$. It is denoted by A B. The transformation y = Px is called similarity transformation.

Thus a matrix is said to be diagonalizable if it is similar to a diagonal matrix.

Note:- A is nxn matrix. Then A is similar to a diagonal matrix D = diag $[\lambda_1, \lambda_2, \ldots\ldots \lambda_n]$

∴ An invertible matrix $P = [x_1, x_2 \ldots\ldots x_n] \rightarrow P^{-1}AP = D = \text{diag}([\lambda+, \lambda+, \ldots\ldots \lambda_n)$

Modal & Spectral Matrix:-

The matrix P in the above result which diagonalise the square matrix A is called the Modal matrix and the resulting diagonal D is called is known as spectral matrix.

Note:- If the eigen values of an nxn matrix are all district then it is always similar to a diagonal matrix.

Calculation of power of a matrix:-

Let A be the Square matrix. Then a non-singular matrix P can be found

$\rightarrow D = P^{-1}AP$

$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P$

$D^3 = P^{-1}A^3P$

$D^n = P^{-1}A^nP$........................... (1)

Premultiply (1) by P & Post multiply by P^{-1}

$PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})\,A^n\,(PP^{-1}) = A^n$

$\Rightarrow A^n = PD^nP^{-1}$

$$A^n = P\begin{bmatrix} \lambda^n & 0 & 0 & . & 0 \\ 0 & \lambda 2^n & 0 & . & 0 \\ 0 & 0 & \lambda 3^n & . & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & . & \lambda n^n \end{bmatrix} P^{-1}$$

1) Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find A^4 (or) find a matrix P which transform the matrix

$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ -4 & 4 & 3 \end{bmatrix}$ to diagonal form Hence calculate A^4 and find the eigen value A^{-1}

Sol:- $A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ -4 & 4 & 3 \end{bmatrix}$ Characteristic Equaltion $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(2-\lambda)(3-\lambda) = 0$

$\lambda = 1,2,3$

Characteristic vector corresponding to $\lambda = 1$

$[A-\lambda I] = 0$

$[A-I] = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} \\ \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Y+z = 0; $\Rightarrow$ y = -z

y+z = 0; let z = k

-4x+4y+2z = 0 $\quad$ y = - k

X = -k/2

Eigen vector $x_1 = \begin{bmatrix} x \\ \\ \end{bmatrix} = \begin{bmatrix} -2k/2 \\ \\ \end{bmatrix} = -k/2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\therefore x_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Characteristic vector corresponding to λ = 2

[A-λI]x = 0; [A-2I]x = 0

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 4R_1 \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z` \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2 \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

r=2, n = 3, n-r = 3-2 = 1 we have to give one variables to the one arbitrary constant.

-x+y+z=0; z = 0

Then we take x (or) y = y

∴ y = k

-x+k+0 = 0

x=k, y=k, z=0

$\Rightarrow x_2 = \begin{bmatrix} \\ y \\ \end{bmatrix} = \begin{bmatrix} \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

∴ Eigen value of A^{-1}

Characteristic vector corresponding to λ = 3

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 4R_1 \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \\ y \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

r=2, n=3, n-r=3-2 = 1

-2x+y+7 = 0

-y+z=0

Let z = k

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-y=-z=-k ⇒ y = k

-2z = -y-z = -k-k

-2x=-2k ⇒ x = k

Eigen vector x3 = $\begin{bmatrix} \\ \\ z \end{bmatrix} = \left[\begin{bmatrix} \\ \\ \end{bmatrix}\right] k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

P = [x_1 x_2 x_3]

Model matrix = p = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

P-1 = adjp/detp = $\begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ =D

2) P-1AP = D = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ = Diagonalization

Power of a matrix $A^n = PD^nP^{-1}$; $A^4=PD^4P^{-1}$

$A4 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$

Eigen value of $A^{-1} = 1/\lambda$ = 1/1,1/2,1/3

2. find the diagonal matrix that will diagonaize the real symmetric matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Also find the resulting diagonal matrix. (or) Diagonalize the matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Sol:- A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ Characteristic Equation $|A-\lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda(\lambda^2-14\lambda) = 0$

λ = 0, 0, 14 Eigen roots λ = 0, 0, 14

Eigen vector corresponding to λ = 14

[A-14I]x = 0

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 1, x_2 = 2, x_3 = 3$

Eigen vector $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

To the Eigen Vector corresponding to $\lambda = 0$

$[A-\lambda I]x =$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-2r_1;\ R_3 \to R_3-3R_1 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$r=1,\ n = 3,\ n-r = 3 – 1 = 2$

let $x2 = k_1,\ x_3 = k_2$

$x_1+2x_2+3x_3 = 0$

$x_1=-2k_1-3k_2$

$x_2=k_1$

$x_3=k_2$

Eigen vector $= \begin{bmatrix} -2k1-3k2 \\ 1 \\ 2 \end{bmatrix} = k_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$\therefore\ x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},\ x_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Normalised Model matrix $= p = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} \frac{1}{\|1\|} & \frac{2}{\|2\|} & \frac{3}{\|3\|} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & -3/\sqrt{10} \\ 2/\sqrt{14} & 1/\sqrt{5} & 0 \\ 3/\sqrt{14} & 0 & 1/\sqrt{10} \end{bmatrix}$

$\Rightarrow P^{-1} = P^{T} = \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -3/\sqrt{10} & 0 & 1/\sqrt{10} \end{bmatrix}$

$P^{-1}AP=P^{T}AP= \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -3/\sqrt{10} & 0 & 1/\sqrt{10} \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{5} & -3/\sqrt{10} \\ 2/\sqrt{14} & 1/\sqrt{5} & 0 \\ 3/\sqrt{14} & 0 & 1/\sqrt{10} \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix} \Rightarrow P^{-1}AP = P^{T}AP = D$$

\ A is reduced to diagonal form by orthogonal reduction.

Exercise problems:

1. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ by orthogonal reduction (or) Diagonalize the matrix.

2) Determine the diagonal matrix orthogonally similar to the following symmetric matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

3) Determine the diagonal matrix orthogonally similar to the following symmetrix matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

4) Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

5) Find a matrix P which transorm the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form.

Hence calculate A^4 (or) Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

6) Prove that the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

7) S.T. the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ cannot be diagonalized.

Quadratic forms

Quadratic form:-

A homogeneous expression of the second degree in any number of variables is called a quadratic form.

An expression of the form $Q = x^TAx = \sum_{i=1} . \sum_{j=1} .$ aij x i x j where aij's are constants is called quadratic form in n variables $x_1, x_2, \ldots\ldots x_n$. If the constants aij's are real numbers it is called a real quadratic form. $[x_1, x_2, \ldots\ldots x_n]$

$Q = x^TAx$ Ex-1) $3x^2+5xy-2y^2$ is a quadratic form in two variables x & y

2) $2x^2+3y^2-4z^2+2xy-3yz+5zx$ is a quadratic form of 3 variables x,y,& z

Symmetric Matrix :-

$Q = X^TAX$ is a quadratic form where A is known as real symmetric matrix.

A = symmetric Matrix = $\begin{bmatrix} coeff.\ of\ x1^2 & \frac{1}{2} coeff.\ of\ x1x2 & \frac{1}{2} coeff\ of\ x1x2 \\ \frac{1}{2} coeff\ of\ x1x2 & coeff.\ of\ x2^2 & \frac{1}{2} coeff\ of\ x2x3 \\ \frac{1}{2} coeff\ of\ x1x3 & \frac{1}{2} coef\ of\ x2x3 & coeff\ of\ x3^2 \end{bmatrix}$

Eg 1:- Write the symmetric matrix of the quadratic form $x_1^2 + 6x_1x_2 + 5x_2^2$

Sol:- Symmetric matrix of the quadratic form $x1^2+6x_1x_2+5x_2^2$

Sol:- A Symmetrix matrix $= \begin{bmatrix} 1 & \frac{6}{2} \\ \frac{6}{2} & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

2) Find the symmetric matrix of the quadratic form $x_1^2+2x_2^2+4x_2x_3+x_3x_4$

Sol:-

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 \\ 3 & 0 & 2 & 0 & \frac{1}{2} \\ 4 & 0 & 0 & \frac{1}{2} & 0 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

3) find the quadratic form of the given symmetric matrix A $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Sol:- Quadratic form = $X^TAX = [x\ \ y\ \ z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= ax^2+by^2+cz^2+2hxy+2gxz+2fyz$

Exercise Problems:-

Write the Symmetrix matrix of the following quadratic forms

1. $x_1^2+2x_2^2-7x3^2-4x_1x_2+8x_1x_3$
2. $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$
3. $2x_1x_2+6x_1x_3-4x_2x_3$
4. $x^2+2y^2+3z^2+4xy+5yz+6zx$
5. $x^2+y^2+z^2+2xt+2yz+3zt+4t^2$
6. Obtain the quadratic form of the following Matrices.

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 4 \\ 5 & 4 & 5 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 0 & 1 \\ 4 & 7 & 1 & 2 \end{bmatrix}$ 5) $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ 6) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 3 & 5 & 4 \end{bmatrix}$

Canonical form

The conanical form of a quadratic form x^TAx is y^TDy (or) $\lambda_1y_1^2+\lambda_2y_2^2+.....+\lambda_ny_n^2$

This form is also known as the sum of the squares form or principal axes form

Canonical form = $y^TDy = [y_1 y_2 y_3] \begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = \lambda_1y_1^2+\lambda_2y_2^2+\lambda_3y_3^2$

Reduction of Quadratic form to canonical form by Linear Transformation.

Consider a quadratic form in n variables

x^TAx and a non singular linear transformation $x = Py$ then $x^T= [Py]^T = y^TP^T$

$x^TAx = y^TP^TAPy = y^T(P^TAP)y = y^TDy$ where $D = P^TAP$

$\Rightarrow x^TAx = y^TDy$

Thus, the quadratic form x^TAx is reduced to the canonical form y^TDy. The diagonal Matrix D and matrix A and called Congruent matrices.

Reduction of Quadratic

Nature of the Quadratic form

The quadratic form x^TAx in n variables is said to be

1) Positive definite:-

If r = n & s = n (or) if all the eigen values are +ve.

2) Negative definite:-

If r = n & s = 0 (or) if all the eigen values are –ve.

3) Positive semidefinite (or) semipositive:-

If r<n & s=r (or) if all the eigen values of $A \geq 0$ & atleast one eigen value is zero.

4) semi negative:-

If r<n & s = 0 (or) if all the eigen values of $A \leq 0$ & atlease one eigen value is zero.

5) Indefinite:-

In all other cases (or) some are positive, -ve.

→**Index of a real quadratic form**

The number of positive terms in canonical form (or) normal form of a quadratic form is known as the index. It is denoted by 's'

Signature of a quadratic form

If r is the rank of a quadratic form & s is the number of positive terms in its normal form, then ∃ number of positive terms over the number of negative terms in a normal form of x^TAx . ∴ Signature = [+ve terms] – [-ve terms]

Note:- Signature = 2s-r

Where s→index

r→rank = no. of non zero rows.

Short Answer question:-

1) Find the nature, rank, Index of a quadratic form $2x^2+2y^2+2z^2+2yz$

Sol :- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} = 0$

$\lambda = 1, 2, 3$

Nature ;- all th eigen values are +ve

⇒ positive definite

Rank:- r = 3

Index : S = no. of positive terms = 3

Signature: - [+ve terms] – [-ve terms] = 3 – 0 = 3

Discuss the nature of the given quadratic form

1) $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$

2) $x^2+4xy+6xz-y^2+2yz+4z^2$

Reduction of Quadratic form to canonical form by orthogonal reduction:

1) Write the coefficient matrix A associated with the given quadratic form

2) A = symmetric Matrix = [ ]

3) Find the eigen values & eigen vectors.

4) Model Matrix $P = [x_1 \ x_2 \ x_3]$

5) Normalized model matrix $P = \left[\frac{1}{||\ 1||} \quad \frac{2}{||\ 2||} \quad \frac{3}{||\ 3||}\right]$

6) Find P^{-1}; $P^{-1} = P^T$

7) $P^{-1}AP = P^TAP = D = \begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix}$

8) Canoniclal form $= y^TDy = [y_1\ y_2\ y_3]\begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix}\begin{bmatrix} y1 \\ y2 \\ 3 \end{bmatrix}$

$= \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2$

9) Linear transformation is x = Py,

1. Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ to the normal form by orthogonal transformation . Also write the rank, Index, nature and signature.

Sol:- given quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characteristic equation is $|A-\lambda I| = 0$

$\begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} = 0$

$\lambda =$ 3, 1, 4; eigen values λ = 3, 1, 4

Eigen vectors $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$P = [x_1\ x_2\ x_3]\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

P = normalized model matrix $P=[e_1\ e_2\ e_3] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

P is orthogonal $P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

$P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$ & the quadratic form will be reduced to the normal form

<u>Canonical form = y^TDy</u>

$= [y_1\ y_2\ y_3]\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$

$= 3y_1^2+y_2^2+4y_3^2$

Index :- No.of positive terms = S = 3

Rank:- r = 3

Nature:- all eigen values are +ve = S = 3

Signature:- = [no of +ve terms] – [no. of –ve terms]

= 3-0 = 3

Orthogonal transformation is x = Py

$$x=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

$x= y_1/\sqrt{2}+ 1/\sqrt{6}y_2 + 1/\sqrt{3}\, y_3$

$y= 2/\sqrt{6}y2 - 1/\sqrt{3}\, y_3$

$z= -1/\sqrt{2}y1+ 1/\sqrt{6}y2 + 1/\sqrt{3}\, y_3$

Exercise:

Reduce the Quadratic form to canonical form by orthogonal Reduction. And write the transformation, nature index, rank, signature

1) $2x^2+2y^2+2z^2-2xy+2zx-2yz$

2) $x_1^2+3x_2^2+3x_3^2-2x_2x_3$

3) $3x^2+5y^2+3z^2-2yz+2zx-2xy$

4) $6x^2+3y^2+3z^2-2yz+4zx-4xy$

2) for the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2-6A+2I$

Sol:- $A=\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ characteristic egn is $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(3-\lambda)(-2-\lambda) = 0;\ \lambda=1,3,-2$

λ is the Eigen value of A & f(A) is a polynomial in A, then the eigen value of f(A) is $f(\lambda)$

$f(A) = 3A^3+5A^2-6A+2I$

Then the eigen value of f(A) are

$f(1) = 3(1)^3+5(1)^2-6(1)+2 = 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = 10$

Thus the Eigen value of $3A^3+5A^2-6A+2I$ are 4, 110, 10

→P.T. the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

Sol:- The characteristic equation is |A-λI| = 0

$$\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2=0$$

$$\lambda = 0,0$$

λ= 0, The characteristic vector. [A-λI]x = 0

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2=0$, $x_1=k$

The characteristic vector is $\begin{bmatrix} k \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The given matrix has only one i.j. charactestic vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to repeated characteristic value '0'

The matrix is not diagonalizable

Note: A is nilpolent matrix ⇒ A is not diagonalised.

$$\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

Sol:- $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ characteristic equation is

|A-λI| = 0

$$\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (8-\lambda)(2-\lambda)+8 = 0$$

$$16-8\lambda-2\lambda+\lambda^2+8=0$$

$$\lambda^2-10\lambda+24=0$$

$$\lambda^2-6\lambda-4\lambda+24=0$$

$$\lambda(\lambda-6) - 4(\lambda-6)=0$$

$$(\lambda-6)(\lambda-4)=0$$

$$\lambda= 6, 4$$

$B= 2A^2- ½ A+3I$

λ is the eigen value of A

Then the eigen value of B is

$B= 2(6)^2 - ½ (6)+3$, $B = 2(4)^2 - ½ (4)+3 = 72, 33$

Eigen value of B is 33,72

$$B = 2A^2 - ½ A + 3I= \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$$

Characteristic Equation $|B-\lambda I| = 0$

$$\begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2+105-2376 = 0$$

$$\lambda = 33, 72$$

Eigen value of B are 33 & 72

λ=33, the eigen vector of B is given by $[B-33I]x = 0$

$$\begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x = 1, x_2 = 1$

$\lambda=33$, $x1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

λ=72, the eigen vector of B is given by $[B-72I]x = 0$

$$\begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 = 1, x_1 = 2$

$\therefore$ The eigen vector for λ=72, $x2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

1) Find the inverse transformation of $y_1=2x_1+x_2+x_3$, $y_2 = x_1+x_2+2x_3$, $y_3 = x_1-2x_3$

Sol: The given transformation can be written as

$$\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

Y=Ax

$$|A| = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} = -1 \neq 0$$

Thus the matrix A is non-singular and hence the transformation is regular. The inverse transformation is given by $x=A^{-1}y$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

$x_1= 2y_1-2y_2-y_3$

$x_2=-4y_1+5y_2+3y_3$

$x_3= y_1-y_2-y_3$

2) S.T. the transformation $y_1=x_1\cos\theta = x_2\text{sing}\theta$, $y_2 = -x1\sin\theta+x_2\cos\theta$ is orthogonal.

Sol:- The given transformation can be written as Y=Ax

$Y = \begin{bmatrix} y1 \\ y2 \end{bmatrix}$ $A = \begin{bmatrix} c\ \theta & i\ \theta \\ -i\ \theta & c\ \theta \end{bmatrix}$, $x = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

Here the matrix of transformation is $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A^T$

the transformation is orthogonal.

Cayley – Hamilton Theorem

Theorem:- Every square matrix statisfies its own characteristic equation.

Applications of cayley – Hamilton Theorem

The important applications of Cayley – Hamilton theorem are

1) To find the inverse of a matrix

2) To find higher powers of a matrix.

1) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ verify cayley – Hamilton theorem

Find A^{-1}& A^4 using cayley – Hamilton theorem.

Sol: $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ Characteristic Equation $|A-\lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & -1-\lambda \end{bmatrix}$ $x^3-3\lambda^2-3\lambda+9=0$

By cayley – Hamilton theorem, matrix A should satisfy its characterstic Equation.

i.e., $A^3-3A^2-3A+9I = 0$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & +1 \end{bmatrix}$

$A^2=A.A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

$A^3=A^2.A = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$

$A^3 -3A^2-3A +9I =$

$\begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} -3 \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} -3 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} +9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3-3A^2-3A+9I = 0$

Hence cayley – Hamilton is verified.

To find A^{-1}:-

Multiplying equation (1) with A^{-1} on b/s

$A^{-1}[A^3-3A^2-3A+9]=0$

$A^2-3A-3AI+9A^{-1} = 0$

$9A^{-1} = 3A+3I-A^2$

$A^{-1} = \frac{1}{9}[3A+3I-A^2]$

$$A^{-1} = \frac{1}{9}[3A+3I-A^2] = \frac{1}{9}\left\{\begin{bmatrix}3 & 6 & -3\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix} + \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix} - \begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}\right\}$$

$$= \begin{bmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ \frac{2}{3} & \frac{-1}{3} & 0\\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3}\end{bmatrix}$$

Find A^4 :-

Multiplying with A

$A[A^3-3A^2-3A+9I] = 0$

$A^4 = 3A^3+3A^2-9A$

$$= 3\begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix} + 3\begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix} - 9\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} = \begin{bmatrix}9 & 72 & -72\\ 0 & 81 & -72\\ 0 & 0 & 9\end{bmatrix}$$

1) Show that the matrix satisfies its characteristic Equation Find A^{-1}& A^4 (or) verify cayley Hamilton Theorem. Find A^{-1}& A^4

1) $A = \begin{bmatrix}1 & -2 & 2\\ 1 & 2 & 3\\ 0 & -1 & 2\end{bmatrix}$

2) $A = \begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$

3) $A = \begin{bmatrix}1 & 0 & 3\\ 2 & -1 & -1\\ 1 & -1 & 1\end{bmatrix}$

4) $A = \begin{bmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{bmatrix}$

5) $A = \begin{bmatrix}3 & 4 & 1\\ 2 & 1 & 6\\ -1 & 4 & 7\end{bmatrix}$

1) using cayley – Hamilton theorm. Find A^{8}, If $A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

Sol:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Characteristic Equation

$|A-\lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$

$\lambda^2-5 = 0$

By cayley – Hamilton Theorem. Every square matrix satisfied its characteristic equation.

$A^2-5=0$

$A^2= 5I$

$A^8 = A^2.A^2.A^2 = [5I].[5I].[5I]$

$A^8 = 625I$

2) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find the value the matrix $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

Sol: The characteristic Equation $|A-\lambda I| = 0$

$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$

$x^3-5\lambda^2-7\lambda-3=0$ By Cayley Hamilton theorm

$A^3-5A^2+7A-3I=0$

We can rewrite the given expression as $A^5[A^3-5A^2+7A-3I] + A[A^3-5A^2+7A-3I]$

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$= A^5[A^3-5A^2+7A-3I] + A[A^3-5A^2+8A-2I]=I$

$= A^5(0) + A[A^3-5A^2+7A-3I] + A^2+A+I=I$

$A[A^3-5A^2+7A-3I] + (A+I)]+I$

$= A^2+A+I$

But $A^2+A+I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

Exercise:

1) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ write $2A^5-3A^4+A^2-4I$ as a linear polynomial in A

Sol:- $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ $|A-\lambda I| = 0$

$\begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda 2-5\lambda+7 = 0$

By cayley – Hamilton Theorm,

A must satisfy its characteristic equation.

$A^2-5A+7I = 0$

$A^2= 5A-7I$

$A^3 = 5A^2-7A$

$A^4=5A^3-7A^2$

$A^5=5A^4-7A^3$

$2A^5-3A^4+A^2-4I$

$=2[5A^4-7A^3]-3[5A^3-7A^2]+[5A-7I]-4I$

$= 7A^4-14A^3+A^2-4I$

$= 7[5A^3-7A^2]-14A^3+A^2-4I$

$= 21A^3-48A^2-4I$

$= 21(5A^2-7A) -48A^2-4I$

$= 57A^2-147A-4I$

$= 57(5A-7I) -147A-4I$

$= 138A-403I$ which is a linear poly in A

Unit – II(Important questions)

1. Find all the eigen values of $A^2+3A-2I$, If $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ 2 Marks
2. Find the nature, index, signature of the quadratic form $3x^2+5y^2+3z^2$. 3Marks
3. Find the Eigenvalues & Eigenvectors of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 5 Marks
4. Verify cayley – Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ Express $B= A^8-11A^7-4A^6+A^5+A^4-11A^3-3A^2+2A+I$ as a quadratic poly in A 5 Marks
5. Diagonalize the Matrix A = $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ hence find A^4 5 Marks
6. Reduce the Q.F. to C.F. C.F. Hence find its nature $x^2+y^2+z^2-2xy+4xz+4yz$ 5 Marks
7. Find the sum & product of the Eigen values of the matrix A $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{bmatrix}$ 2Marks
8. Write the quadratic form Corresponding to the matrix $A = \begin{bmatrix} 1 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 3 \end{bmatrix}$ 3 Marks
9. Find the eigen values $5A^2-2A^2+7A-3A^{-1}+I$ if $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ 5 Marks
10. Using cayley – Hamilton Then find A^{-1}& A^{-2} if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ 5 Marks

11. Reduce the Q.form $8x^2+7y^2+3z^2+12xy+4xz+8yz$ to canonical form and find rank, nature, index & signature 10 Marks

Properties of Eigen Values:

Theorm 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: Characterristic equation of A is . $|A-\lambda I|=0$

i.e, $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$ *expanding this we get*

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)-a_{12}(a\ polynomial\ of\ degree\ n-2)$

$+\ a_{13}\ (a\ polynomial\ of\ degree\ n+2)+\ldots+=0$

$\Longrightarrow (-1)[\lambda^n-(a_{11}+a_{22}+\ldots.+a_{nn})^{-1}+a\ polynomila\ of\ degree\ (n-2)]$

$(-1)^n\lambda^n+(-1)^{n+1}(Trave\ A)\lambda^{n-1}++a\ polynomial\ of\ degree\ (n-2) in\ \lambda=0$

$If\ \lambda_1,\lambda_2\ \ldots\ldots\lambda_n\ are\ the\ roots\ of\ this\ equation$

$$sum\ of\ the\ roots=\frac{(-1)^{n+1}(A)}{(-1)}=(A)$$

$urther\ |\ -\lambda\ |=(-1)^n\lambda^n+\ .+a_0$

$put\ \lambda=0\ then\ |A|=a_0$

$(-1)^n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+\ldots.+a_0=0$

$$Product\ of\ the\ roots=\frac{(-1)^n a_0}{(-1)^n}=a_0$$

$_0=|\ |=\det$

Hence the result

Theorm 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen value X, we have

$AX=\lambda X$------------(1)

Pre multiply (1) by A, $A(AX) = A(\lambda X)$

$(AA)X = \lambda(AX)$

$A^2X= \lambda(\lambda X)$

$A^2X= \lambda^2X$

λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector. Thus the theorm is true for n=2

let we assume it is true for n = k

i.e,, $A^K X = \lambda^K X$------------(2)

Premultiplying (2) by A, we get

$A(A^k X) = A(\lambda^K X)$

$(AA^K)X = \lambda^K(AX) = \lambda^K(\lambda X)$

$A^{K+1}X = \lambda^{K+1}X$

λ^{K+1} is eigen value of A^{K+1} with X itself as the corresponding eigen vector.

Thus, by M.I., λ^n is an eigen value of A^n

Theorm 3: A Square matrix A and its transpose A^T have the same eigen values.

Proof: We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A-\lambda I)^T| = |A^T - \lambda I|$ (or)

$|A-\lambda I| = |A^T - \lambda I|$

$|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

λ is eigen value of A if and only if λ is eigen value of A^T

Hence the theorm

Theorrm 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and $B A^{-1}$ have same eigen values.

Proof: Given A is invertile

i.e, A^{-1} exist

w e know that if A and P are the square matrices of order n such that P is non-singular then A and $P^{-1}AP$ hence the same eigen values.

Taking $A = B A^{-1}$ and P=A, we have

$B A^{-1}$ and $A^{-1}(B A^{-1})A$ have the same eigen value

$B A^{-1}$ and $(A^{-1}B)(A^{-1}A)$ have the same eigen values

$B A^{-1}$ and $(A^{-1}B)I$ have the same eigen values

$B A^{-1}$ and $A^{-1}B$ have the same eigen values

Theorm 5: If $\lambda_1, \lambda_2, \ldots.. \lambda_n$ are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, \ldots.. k\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

Proof: Let A be a square matrix of order n. Then $|KA - \lambda KI| = |K(A-\lambda I)| = K^n|A-\lambda I|$

Since $K\neq 0$, therefore $|KA-\lambda KI| = 0 \Rightarrow |A-\lambda I| = 0$

$i.e, K\lambda$ is an eigen value of KA $\Leftrightarrow$ if λ is an eigen value of A

Thus k λ_1, k λ_2 … k λ_n *are the eigen values of the matrix KA*

$\Leftrightarrow$ λ_1, λ_2 … λ_n are the eigen values of the matrix A

Theorm 6: If λ is an eigen value of the matrix. Then λ+K is an eigen value of the matrix A+KI

Proof: Let λ be an eigen value of A and X the corresponding eigen vector. Then by definition AX= λX

Now (A+KI)X = $(\lambda + KI)X$

$= \lambda X + KX$

$=(\lambda + K)X$

$\lambda + K$ is an eigen value of the matrix $A + KI$

Theorm 7: If λ_1, λ_2 … λ_n are the eigen values of A the $\lambda_1 - K$, $\lambda_2 - K$, … $\lambda_n - K$,

are the eigen values of the matrix $(A - KI)$ where K is a non − zero scalar

Proof: Since $\lambda_1, \lambda_2, \ldots.\lambda_n$ are the eigen values of A.

The characteristic polynomial of A is

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$-----------------------1

Thus the characteristic polynomial of A-KI is

$|(A - KI) - \lambda I| = |A - (k+\lambda)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \ldots..[\lambda_n - (\lambda + K)]$

$= [(\lambda_1 - K) - \lambda][(\lambda_2 - K) - \lambda] \ldots.[(\lambda_n - K) - \lambda]$

Which shows that the eigen values of A-KI are $\lambda_1 - K, \lambda_2 - K, \ldots \ldots.\lambda_n - K$

Theorm 8: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A find the eigen values of the matrix $(A - \lambda I)^2$

Sol: First we will find the eigen values of the matrix A- λI

Since $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A

The characteristics polynomial is

$|A- \lambda I| = (\lambda_1 - K)(\lambda_2 - K) \ldots..(\lambda_n - K)$ ------(1) *where K is scalar*

The characteristic polynomial of the matrix (A- λI) is

$|A- \lambda I-KI| = |A-(\lambda+K)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \ldots [\lambda_n - (\lambda+K)]$

$= [(\lambda_1 - \lambda) - K)]\,[(\lambda_2 - \lambda) - K] \ldots [(\lambda_n - \lambda) - K)]$

Which shows that eigen values of (A- λI) are $\lambda_1 - \lambda, (\lambda_2 - \lambda) \ldots.\lambda_n - \lambda$

We know that if the eigen values of A are $\lambda_1, \lambda_2 \ldots \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2 \ldots..\lambda_n^2$

Theorm 9: If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector then λ^{-1} *is an eigne vector of A^{-1} and corresponding eigen vectgor X itself.*

Proof: Since A is non-singular and product of the eigen values is equal to |A|. it follows that none of the eigen vectors of A is o.

If λ is an eigen vector of the non-singular matrix A and X is the corresponding eigen vector $\neq 0$ and AX= λX. Premultiplying this with A^{-1}, we get $A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$

X = $\lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X$

Hence λ^{-1} is an eigen value of A^{-1}

Theorm 10: If λ is an eigen value of a non – singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Proof: Since λ is an eigen value of a non-singular matrix, therfore $\lambda \neq 0$

Also λ is an eigen value of A implies that there exists a non-zero vector X such that AX = λX------- (1)

$\Longrightarrow (adj\ A)AX = (Adj\ A)(\lambda X)$

$\Longrightarrow [(adj\ A)A]X = \lambda(\text{adj A})X$

$\Longrightarrow |A|IX = \lambda\ (\text{adj A})X$

$\Rightarrow \frac{|A|}{\lambda}X = (adj\ A)X\ on\ (adj\ A)X = \frac{|A|}{\lambda}X$

$\Longrightarrow$ *Since X is a non – zero vector, therfore the relation (1)*

it is clear that $\frac{|A|}{X}$ is an eigen value of the matrix Adj A

Theorm 11: If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value

Proof: We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

Since A is an orthogonal matrix, therefore $A^{-1} = A^1$

$\frac{1}{\lambda}$ is an eigen value of A^1

But the matrices A and A^1 hence the same eigen values, since the determinants $|A - \lambda I|$ and $|A^1 - I\lambda|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

Theorm 12: If λ is eigen value of A then prove that the eigen value of B = $a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Proof: If X be the eigen vector corresponding to the eigen value λ, then AX = λX --- (1)

Premultiplying by A on both sides

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

This shows that λ^2 *is an eigen vector of* A^2

we have $B = a_0A^2+a_1A+a_2I$

$BX = (a_0A^2+a_1A+a_2I)X$

$= a_0A^2X+a_1AX+a_2X$

$= a_0A^2X+a_1\lambda X+a_2X \qquad = (a_0\lambda^2 X+a_1\lambda+a_2)X$

$(a_0\lambda^2 X+a_1\lambda+a_2)$ is an eigen value of B and the corresponding eigen vector of B is X.

<u>Theorm 14:</u> Suppose that A and P be square matrices of order n such that P is non singular then A and $P^{-1}AP$ have the same eigen values.

Proof: Consider the characteristic equation of $P^{-1}AP$

It is $|(P^{-1}AP)-\lambda I| = |P^{-1}AP-\lambda P^{-1}IP|$

$= |P^{-1}(A-\lambda I)P| = |P^{-1}|\,|A-\lambda I|\,|P|$

$= |A-\lambda I|$ since $|P^{-1}|\,|P| = 1$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same eigen values.

Proof: In the previous theorm take BA^{-1} in place of A and A in place of B.

We deduce that $A^{-1}(BA^{-1})A$ and (BA^{-1}) have the same eigen values.

i.e, $(A^{-1}B)(A^{-1}A)$ and BA^{-1} have the same eigen values.

i.e, $(A^{-1}B)I$ and BA^{-1} have the same eigen values

i.e, $A^{-1}B$ and BA^{-1} have the same eigen values

Corollary2: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

Proof: Notice that $AB=IAB = (B^{-1}B)(AB) = B^{-1}(BA)B$

Using the theorm above BA and $B^{-1}(BA)B$ have the same eigen values.

i.e, BA and AB have the same eigen values.

<u>Theorm 15:</u> The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A = \begin{vmatrix} a_{11} & a_{12}\,\dots\dots & a_{1n} \\ 0 & a_{22-\lambda}\,\dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0\,\dots\dots & a_{nn} \end{vmatrix}$ be a triangular matrix of order n

The characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} a_{11}-\lambda & a_{12} \cdots\cdots & a_{1n} \\ 0 & a_{22}-\lambda \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0 \cdots\cdots & a_{nn}-\lambda \end{vmatrix} = 0$

i.e, $(a_{11}-\lambda)(a_{22}-\lambda)\ldots\ldots(a_{nn}-\lambda)=0$

$\Longrightarrow \lambda = a_{11}, a_{22}, \ldots. a_{nn}$

Hence the eigen values of A are $a_{11}, a_{22}, \ldots. a_{nn}$ which are just the diagonal elements of A.

Note: lly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Theorm 16: The eigen values of a real symmetric matrix are always real.

Proof: Let λ be an eigen value of a real symmetric matrix A and Let X be the corresponding eigen vector then $AX=\lambda X ----(1)$

Take the conjugate $\bar{A}\bar{X} = \lambda\,\bar{X}$

Taking the transpose $\bar{X}^T(\bar{A})^T = \bar{\lambda}\,\bar{X}^T$

Since $\bar{A} = A \text{ and } A^T = A, we\ have\ \bar{X}^T A = \bar{\lambda}\,\bar{X}^T$

Post multiplying by X, we get $\bar{X}^T AX = \bar{\lambda}\,\bar{X}^T X$-------- (2)

Premultiplying (1) with $\bar{X}^T$, we get $\bar{X}^T AX = \lambda \bar{X}^T X$-------- (3)

(1) – (3) gives $(\lambda-\bar{\lambda})\bar{X}^T X = 0$ but $\bar{X}^T X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda - \bar{\lambda} \Rightarrow \lambda$ is real. Hence the result follows

Theorm 17: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Proof: Let λ_1, λ_2 be eigen values of a symmetric matrix A and let X_1, X_2 be the corresponding eigen vectors.

Let $\lambda_1 \neq \lambda_2$ we want to show that X_1 is orthogonal to X2 (i.e., $X_1^T X_2 = 0$)

Sice X_1, X_2 are eigen values of A corresponding to the eigen values λ_1, λ_2 we have

$AX_1= \lambda_1 X_1$ ----- (1) $\quad AX_2 = \lambda_2 X_2$--------- (2)

Premultiply (1) by X_2^T

$\Longrightarrow X_2^T AX_1 = \lambda_1 X_2^T X_1$

Taking transpose to above, we have

$\Longrightarrow X_1^T A^T (X_2^T)^T = \lambda_1 X_1^T A^T (X_2^T)^T$

$i.e, X_1^T AX_2 = \lambda_1 X_1^T X_2$............(3)

Premultiplying (2) by X_1^T, we get $X_1^T A X_2 = \lambda_2 X_1^T X_2$ − − − − − (4)
Hence from (3) and (4) we get
$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$
$\Rightarrow X_1^T X_2 = 0$
($\because \lambda_1 \neq \lambda_2$)
X_1 is orthogonal to X_2
Note: If λ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is f(λ)

Objective type questions

1. The Eigen values of $\begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$ are [ ]

   a) 1,2 b) 2,4 c) 3, 4 d) 1, 5

2. If the determinant of matrix of order 3 is 12. And two eigen values are 1 and 3, then the third eigen value is [ ]

   a) 2 b) 3 c) 1 d) 4

3. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ then the eigen values of A are [ ]

   a) 1, 1, 2 b) 1, 2, 3 c) 1, ½ , 1/3 d) 1, 2, ½

4. The sum of Eigen values of $A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}$ is [ ]

   a) 2 b) 3 c) 4 d) 5

5. If the Eigen values of A are (1,-1,2) then the Eogen values of Adj A are [ ]

   a) (-2,2,-1) b) (1,1,-2) c) (1,-1,1/2) d) (-1,1,4)

6. If the Eigen values of A are (2,3,4) then the Eigen values of 3A are [ ]

   a) 2,3,4 b) ½, 1/3, ¼ c) -2,3,2 d) 6,9,12

7. If the Eigen values of A are (2,3,-2) then the Eigen value of A-3I are [ ]

   a) -1,0,-5 b) 2,3,-2 c) -2,-3,2 d) 1,2,2

8. If A is a singular matrix then the product of the Eigen values of A is [ ]

   a) 1 b) -1 c) can't be decided d) 0

9. The Eigen vector corresponding to $\times = 2$ □□ $\begin{bmatrix}1 & 2 & -1\\ 0 & 2 & 2\\ 0 & 0 & -2\end{bmatrix}$ is [ ]

a) $\begin{bmatrix}2\\ 1\\ 0\end{bmatrix}$ b) $\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$ c) $\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ d) $\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$

10. If two Eigen vectors of a symmetric matrix of order 3 are $\begin{bmatrix}2\\ -1\\ 0\end{bmatrix}$ and $\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$ then the third eigen vector is [ ]

a) $\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$ b) $\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ c) $\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}$ d) $\begin{bmatrix}1\\ 2\\ 5\end{bmatrix}$

11. The Eigen values of $\begin{bmatrix}5 & 2\\ -1 & 2\end{bmatrix}$ are 3 and 4 then the eigen vectors are [ ]

a) $\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}$ b) $\begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}$ c) $\begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}-2\\ 1\end{bmatrix}$ d) $\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$

12. If the trace of A (2x2 matrix) is 5 and the determinant is 4, then the eigen values are [ ]

a) 2, 2 b) -2, 2 c) -1, -4 d) 1, 4

13. Sum of the eigen values of matrix A is equal to the [ ]

a) Principal diagonal elements of A b) Trace of matrix A c) A&B d) None

14. If A = $\begin{bmatrix}4 & 2\\ -3 & 3\end{bmatrix}$ then A^{-1} = [ ]

a) $\frac{1}{6}$ [7□ − □] b) $\frac{1}{4}$ [5□ − □] c) $\frac{1}{2}$ [7□ − □] d) $\frac{1}{18}$ [7□ − □]

15. If A = $\begin{bmatrix}6 & 2\\ 1 & -1\end{bmatrix}$ then $2A^2-8A-16I$ = [ ]

a) I b) 2A c) A-I d) 5I

16. Similar matrices have same [ ]

a) Characteristic Polynomial b) Characteristic equation

c) Eigen values d) All the above

17. If A = $\begin{bmatrix}1 & 2 & 5\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}$ then A^{-1} = [ ]

a) $\frac{1}{2}$ [□ + □ − □²] b) $\frac{1}{2}$ [□ + □ + □²]

c) $\frac{1}{2}$ [□ + 2□ − □²] d) $\frac{1}{2}$ [□ + 2□ − □²]

18. If A has eigen values (1,2) then the eigen values of $3A+4A^{-1}$ are [ ]

a) 3, 8 b) 7, 11 c) 7, 8 d) 3, 6

19. If A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then A^3 = [ ]

a) $2A^2+5A$ b) $4A^2+5A$ c) $2A^2+4A$ d) $5A^2+2A$

20. If $D = P^{-1}AP$ then A^2 = [ ]

a) PDP^{-1} b) $P^2D^2(P^{-1})^2$ c) $(P^{-1})^2D^2(p^2)$ d) PD^2P^{-1}

21. The product of Eigen values of A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is [ ]

a) 18 b) -18 c) 36 d) -36

22. If one of the eigen values of A is zero then A is [ ]

a) Singular b) Non-Singular c) Symmetric d) Non-Symmetric

23. If A is a square matrix, D is a diagonal matrix whose elements are eigen values of A and P is the matrix whose Columns are eigen vectors of A^4, then A^4 =________ [ ]

a) PDP^{-1} b) PD^4P^{-1} c) $P^{-1}D^2P$ d) $P^{-1}D^4P$

24. $\frac{|A|}{\lambda}$ is an eigen value of [ ]

a) Adj A b) A.adj A c) (adj A) A d) None

25. The characteristic equation of $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ is [ ]

a) $\lambda^2 - 3\lambda + 5 = 0$ b) $\lambda^2 + 3\lambda + 5 = 0$
c) $\lambda^2 + 3\lambda - 5 = 0$ d) $\lambda^2 - 3\lambda - 5 = 0$

26. If A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ the eigen values of A are 6 and 1 then the model matrix is [ ]

a) $\begin{bmatrix} [illegible] & [illegible] \\ 1 & -1 \end{bmatrix}$ b) $\begin{bmatrix} [illegible] & [illegible] \\ 1 & -1 \end{bmatrix}$ c) $\begin{bmatrix} [illegible] & [illegible] \\ 1 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

27. If A = $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ then the model matrix is [ ]

28. a) $\begin{bmatrix} [illegible] & [illegible] \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

29. If A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then the model matrix is [ ]

a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

30. If A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

30. If A = $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$

31. If the eigen values of A are 0, 3, 15 then the index and signature of X^TAX are [ ]

a) 2, 1 b) 2,2 c) 3,3 d) 1,1

32. If two eigen vectors of a symmetric matrix are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ □□□ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ then the third eigen vector is

i. a) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ c) $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

[ ]

33. Product of eigen values of matrix A is equal to [ ]

a) determinant of A b) Trace of A c) Principal diagonal of A d) None

34. If A and B are square matrices such that A is non-singular then $A^{-1}B$ and BA^{-1} have [ ]

a) different eigen values b) same eigen values

c) reciprocal eigen values d) None

35. The eigen values of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ □□□ [ ]

a) 2, 4, 5 b) -2,-4,-5 c) 1,2,3 d) 3,4,6

36. If A = $\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ then A^3-12A = [ ]

a) 12I b) 8I c) 10I d) 16I

37. If A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then $6A^2$-A^3+A = [ ]

a) 5I b) 10I c) 6I d) 8I

38. If A = $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ then A^3-$4A^2$+A+6I = [ ]

a) [0] b) I c) 3I d) 5I

39. If A = $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $\times = 2$ a□□ 3 then the modal matrix is [ ]

a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

40. If A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then D = [ ]

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$

41. If A = $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ then D = [ ]

a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

42. If λ is an eigen value of A then λ^m is eigen value of [ ]

a) A b) A^{-1} c) A^m d) A^{-m}

43. If A = $\begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ then the eigen values of A^2 are [ ]

a) -1, -9, -4 b) 1, -3, 2 c) 1, 3, -2 d) 1, 9, 4

44. If λ is the eigen value of A then the eigen values of A^{-1} are [ ]

a) $\frac{|A|}{\lambda}$ b) $\frac{1}{\lambda}$ c) $-\lambda$ d) λ

45. If the eigen values of A are 1, 3, 0 then $|A|$ = [ ]

a) 4 b) 1 c) 3 d) 0

46. The characteristic equation of $\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$ *is* [ ]

a) $\lambda^2 + 6\lambda + 1 = 0$ b) $\lambda^2 - 6\lambda - 1 = 0$

c) $\lambda^2 + 6\lambda - 1 = 0$ d) $\lambda^2 - 6\lambda + 1 = 0$

47. If A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $then\ p^{-1}A^2P =$ [ ]

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

48. If A = $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ the eigen values of A are (2, 2, -2) then $p^{-1}A^3P$ = [ ]

a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -21 \end{bmatrix}$ c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

49. If the eigen values of a matrix are (-2, 3, 6) and the corresponding eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ b) $\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 36 \end{bmatrix}$ d) $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$