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ENGINEERING COLLEGE
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Estd.2007

UNIT – V

MULTI VARIABLE CALCULUS

(INTEGRATION)

Multiple Integrals:

Definite Integrals: Let y =f(x) be a function of one variable define and bounded on [a,b] consider the sum $\sum_{i=1}^{n} f(x_i)\,\delta x_i$ of this sum tends to a finite limit as n => ∞ such that length of δx_i tends to O for arbitrary choice of the t_i's. The limit is define to be the definite integral $\int_a^b f(x)dx$.

The generalization of this definition to two dimensions is called a double integral and to three dimensions is called a triple integral.

Double Integral: An expression of the form $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)dy\,dx$ or $\int_a^b \int_{x_1(y)}^{x_2(y)} f(x,y)dx\,dy$ is called an iterated integral or double integral.

4) Evaluate $\int_0^1 \int_0^x e^{x+y}$dy dx

Ans: = $\frac{1}{2}(e-1)^2$

5) **Evaluate** $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)$ **dxdy**

$= \int_{x=0}^{1} dx \int_{y=x}^{\sqrt{x}} (x^2+y^2)$dy

$= \int_{x=0}^{1} dx\ [x^2y + \frac{y^3}{3}]_x^{\sqrt{x}}$ dx

$= \int_{x=0}^{1} [x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3}] - [x^3 + \frac{x^3}{3}]]$ dx

$= \int_{x=0}^{1} [x^{\frac{5}{2}} + \frac{(x)^{\frac{3}{2}}}{3} - \frac{4x^3}{3}]$ dx

$= [\frac{(x)^{\frac{7}{2}}}{\frac{7}{2}} + \frac{(x)^{\frac{5}{2}}}{3.\frac{5}{2}} - \frac{4x^4}{3\ .4}]_0^1$

$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30+14-35}{105} = \frac{9}{105} = \frac{3}{35}$

P.T $\int_1^2 \int_3^4 (xy + e^y)$ dx dy

$= \int_3^4 \int_1^2 (xy + e^y)$ dy dx

L.H.S = $\int_1^2 [\int_3^4 (xy + e^y)$ dx]dy

$= \int_1^2 [y . \frac{x^2}{2} + e^y . x]_3^4 dy$

$= \int_1^2 \{[y . \frac{16}{2} + 4e^y] - [y . \frac{9}{2} + 3e^y]\} dy$

$= \int_1^2 [y . \frac{7}{2} + e^y] dy$

$= [y . \frac{7}{2} + e^y]_1^2$

$= (\frac{7}{4} * 4 + e^2) - (\frac{7}{4} + e)$

$= 7 - \frac{7}{4} + e^2 - e$

$= \frac{21}{4} + e^2 - e$

R.H.S $= \int_3^4 [\int_1^2 (xy + e^y) dx] dy$

$= \int_3^4 [x . \frac{y^2}{2} + e^y]_1^2 dx$

$= \int_3^4 \{[x . \frac{4}{2} + e^2] - [\frac{x}{2} + e]\} dx$

$= \int_3^4 [\frac{3x}{2} + e^2 - e] dx$

$= [\frac{3}{2}\frac{x^2}{2} + e^2 x - ex]_3^4$

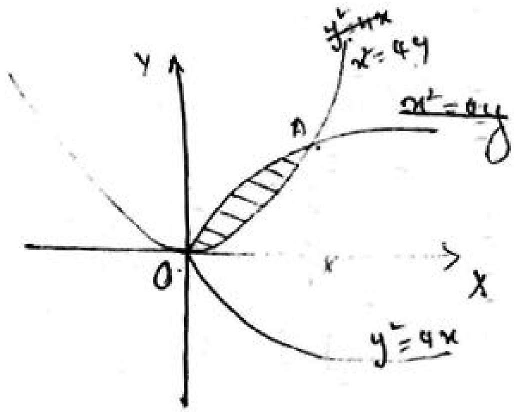
$= (\frac{3}{4} * 9 + 3e^2 - 3e) - (\frac{3}{4} * 16 + 4e^2 - 4e)$

$= \frac{21}{4} + e^2 - e$

∴ L.H.S =R.H.S

3. Evaluate $\int\int y \, dx \, dy$ where R is the region bounded by the parabolas y^2 = 4x and x^2 = 4y

The co-ordinates of points O & A are (0,0) and (4,4).



$$\int\int y\,dx\,dy = \int_{x=0}^{4} dx \int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y\,dy$$

$$= \int_{x=0}^{4} dx \left(\frac{y^2}{4}\right)_{\frac{x^2}{4}}^{2\sqrt{x}}$$

$$= \int_{x=0}^{4}\left[\frac{4x}{2} - \frac{x^4}{2.16}\right] dx$$

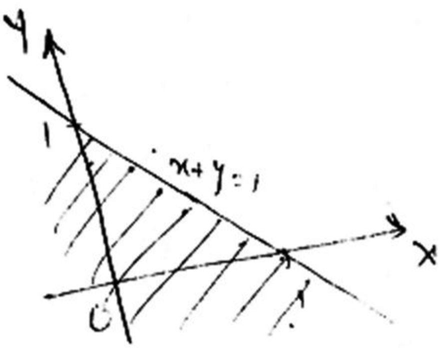
$$= \frac{1}{2}\left[4 . \frac{x^2}{2} - \frac{x^2}{16.5}\right]_0^4$$

$$= \frac{1}{2}\left[32 - \frac{64}{16}\right]$$

$$= \frac{1}{2}\left[\frac{160-64}{5}\right] = \frac{1}{2}\left[\frac{96}{5}\right] = \frac{48}{5}$$

$$\therefore \int\int y\,dx\,dy = \frac{48}{5}$$

4. Evaluate $\int\int x^2 + y^2$ dx dy in positive quadrant for which x+y≤1.



$\int\int x^2 + y^2$dxdy =

$$= \int_{x=0}^{1} dx \int_{y=0}^{1-x} (x^2 + y^2)\, dy$$

$$= \int_{x=0}^{1} \left(x^2 y + \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1 - 3x + 3x^2 - x^3)\right] dx$$

$$= \left(\frac{2x^3}{3} - \frac{4}{3}.\frac{x^4}{4} + \frac{1}{3}x - \frac{x^2}{2}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

$$\therefore \int\int x^2 + y^2 dxdy = \frac{1}{6}$$

Change of order of integration:

5. Evaluate the following integrals by changing the order of integration.

Sol:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy$$

The area of integration lies between y=0 which is x-axis and

$y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$

Which is a circle. Also limits of x are 0 to 1.

Hence the region of integration is OAB and is divided into vertical strip for changing the order of integration, we shall divide the region of integration into horizontal strips.

The new limits of integration become x = 0 to x= $\sqrt{1-y^2}$ and those for 'y' will be y=0 to y = 1.

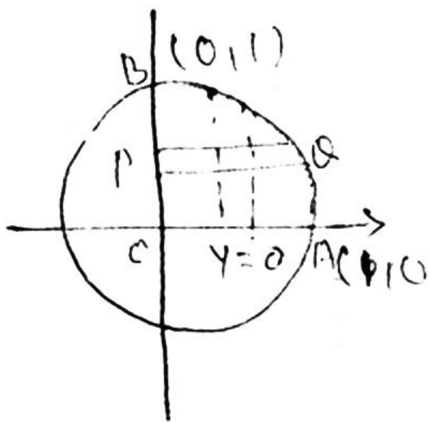
Hence

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy = \int_{y=0}^{1} dy \int_{x=0}^{\sqrt{1-y^2}} y^2\, dx$$

$$= \int_{y=0}^{1} y^2 [\,(x)_0^{\sqrt{1-y^2}}\,] dy$$

$$= \int_{y=0}^{1} y^2 \sqrt{1-y^2}\, dy$$

Put y = sinθ dy=cos$\theta\, d\theta$



y=0 => $\theta = 0$,

y=1 => $\theta = \frac{\pi}{2}$

hence I= $\int_{\theta=0}^{\frac{\pi}{2}} sin^2\theta cos^2\theta d\theta$

$= \frac{1}{4} \cdot \frac{\pi}{2} \cdot \frac{1}{2} \qquad = \frac{\pi}{16}$

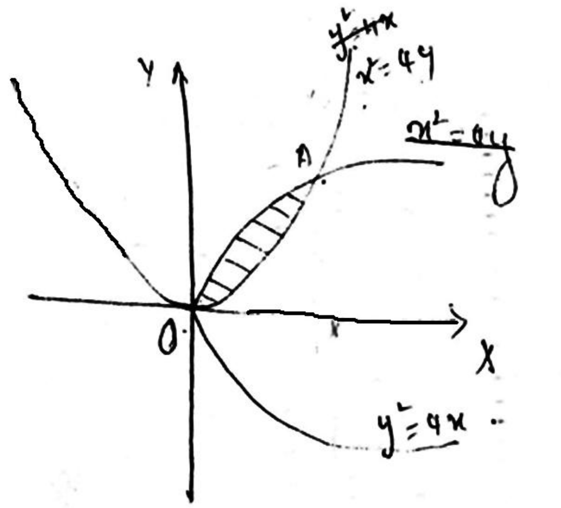
6. $\int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx

Sol. The region of integration lies between x=y a straight line and passing through the origin x=a and y=0. Also the limits for y are 0 to a, which is ΔOAB and the region is divided by horizontal strips.

By changing the order of integration take a vertical strip PQ so that the new limits become y=0 to y=x and x varies from 0 to a.

Hence I = $\int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx

$= \int_{x=0}^{a} dx \int_{y=0}^{x} \frac{x}{x^2+y^2}$ dy



$= \int_{x=0}^{a} x. \ (\frac{1}{x} tan^{-1}\frac{y}{x})_0^x dx$

$= \int_{x=0}^{a} tan^{-1}(1)dx$

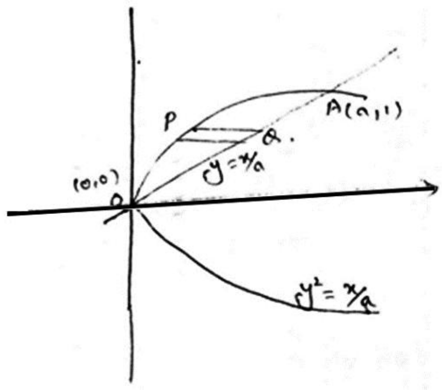
$= \frac{\pi}{4}(x)_0^a \ = \frac{\pi a}{4}$

$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx = $\frac{\pi a}{4}$

7. $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$

Sol. The region of integration is defined by y= $\sqrt{\frac{x}{a}}$ => $y^2 = \frac{x}{a}$ which is a parabola and y= $\frac{x}{a}$ => x=ay is a straight line passing through the origin. The points of intersection are O(0,0) and A(a,1). The limits for x are 0 to a.

Integration is done by taking strip parallel to y-axis. By changing the order of integration take a strip PQ parallel to x-axis. The limits for x in this case will be x=a y^2 to x=ay and that for y will be y=0 to y=1.



$$\therefore \text{I} = \int_0^a \int_{y=\frac{x}{a}}^{y=\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=ay^2}^{x=ay} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=0}^{1} \left(\frac{x^3}{3} + y^2 x\right)_{ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3 y^3}{3} + ay^3 - \frac{a^3 y^6}{3} - ay^4\right) dy$$

$$= \left(\frac{a^3}{3}\frac{y^4}{4} + a\frac{y^4}{4} - \frac{a^3}{3}\frac{y^7}{7} - a\frac{y^5}{5}\right)_0^1$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \qquad = \frac{a^3}{28} + \frac{a}{20}$$

$$\therefore \int_0^a \int_{y=\frac{x}{a}}^{y=\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy = \frac{a^3}{28} + \frac{a}{20}$$

Change of variables:

Let x and y be functions of u and v and let x = $\emptyset(u, v)$ and y = x(u,v) then $\int_R \int f(x,y)dx\,dy$ is transformed into $\int_{R^1} \int f\{\emptyset(u,v), \text{x(u,v)}\}\, |\mathcal{J}| du\, dv$

Where $\mathcal{J} = \frac{\partial(x,y)}{\partial(u,v)}$ is the jacobian of transformation from (x,y) to(u,v) co-ordinates and $R^1$is the region in the uv plane corresponding to R in the xy plane.

In polar co-ordinates x=rcosθ , y= rsinθ

$$\mathcal{J} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = \text{r}$$

$$\therefore \int_R \int f(x,y)dx\,dy = \int_{R^1} \int f\{\text{rcos}\theta, \text{rsin}\theta\}\, r d\theta$$

8. Evaluate the following integrals by changing to polar co-ordinates.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \text{dxdy}$$

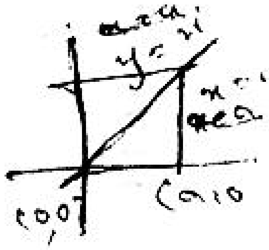
Since both x and y vary from 0 to ∞, the region of integration is the xoy plane, change to polar co-ordinates, x= $r\cos\theta$, y = $r\sin\theta$ dx dy = r dr dθ and $(x^2 + y^2) = r^2$. In the region of integration 'r' varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

Put t $= r^2$

$\therefore$dt = 2r dr

r =0 => t=0



r=∞ => t = ∞

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_0^\infty e^{-t} \frac{dt}{2} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [-e^{-t}]_0^\infty d\theta$$

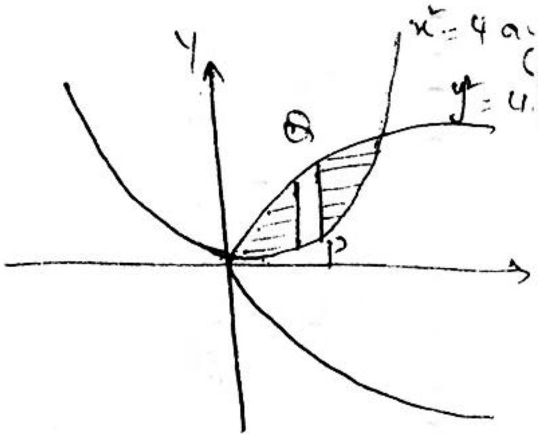
$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0-1) d\theta$$

$$= \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

9. Show by double integration, the area between the parabolas y^2 = 4ax and x^2 = 4ay is $\frac{16}{3}a^2$

Sol:

The P OI of given curves is A(0,0) and B(4a,4a). by taking a vertical strip parallel to y-axis. We get the area between the two parabolas as:



$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx$$

$$= \int_{x=0}^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} \text{dy}$$

$$= \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) \text{dx}$$

$$= \left[2\sqrt{a}.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

$$\therefore \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx = \frac{16}{3}a^2$$

Triple integrals:

Let f(x,y,z) be a function which is defined at all points in a finite region v in space. Let δx, δy, δz be an elementary volume v enclosing of the point (x,y,z) thus the triple summation.

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \sum\sum\sum f(x,y,z)\,\delta x, \delta y, \delta z$$

If it exists is written as $\iiint f(x,y,z)$ dx dydz which is called the triple integral of f(x,y,z) over the region v.

If the region v is bounded by the surfaces $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, $z=z_1$, $z=z_2$ then

$$\iiint f(x,y,z)\text{dxdydz} = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$$

Note:

(i) If x_1, x_2; y_1, y_2; z_1, z_2 are all constants then the order of integration is immaterial provide the limits of integration are changed accordingly.

i.e.

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)dx\,dy\,dz$$

$$= \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x,y,z)dx\,dz\,dy$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dz\,dy\,dx$$

(ii) If, however z_1, z_2 are functions of x and y and y_1, y_2 are functions of x while $x_1$and x_2 are constants then the integration must be performed first w.r.to 'z' then w.r.to 'y' and finally w.r.to 'x'.

i.e.

$\iiint f(x,y,z)$dxdydz =

$$= \int_{x=a}^{x=b}\int_{y=\emptyset_1(x)}^{y=\emptyset_2(x)}\int_{z=\delta_1(x,y)}^{z=(x,y)} f(x,y,z)dz\,dy\,dx$$

10. Evaluate the following integrals:

(i) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\,dy\,dx$

Sol

$$\int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\ dz\right] dy\right\} dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[xy\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}\right\} dy\,dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[\frac{xy\,(1-x^2-y^2)}{2}\right]\right\} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}[\,xy - x^3y - xy^3]\right\} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\,dx$$

$=\frac{1}{2}\int_0^1 \frac{x-x^3-x^3+x^5)}{2} - \frac{x(1-2x^2+x^4)}{4}\,dx$

$=\frac{1}{2}\int_0^1 \frac{x-2x^3+x^5)}{2} - \frac{2x^3-x-x^5)}{4}\,dx$

$=\frac{1}{8}\int_0^1 x - 2x^3 + x^5 dx$

$=\frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} - \frac{x^6}{6}\right]_0^1$

$=\frac{1}{8}[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}]=\frac{1}{48}$

$\therefore \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\,dy\,dx = \frac{1}{48}$

(ii) $\int_1^e\int_1^{logy}\int_1^{e^x} \log y\ dz\,dx\,dy$

Sol. $I = \int_{y=1}^{e}\int_{x=1}^{logy}\int_{z=1}^{e^x} \log z\ dz\,dx\,dy$

Consider $\int_{z=1}^{e^x} \log z\ dz = [\text{zlogz} - \text{z}]_1^{e^x}$

$= e^x \log_e e^x - e^x + 1$

$= xe^x - e^x + 1$

$= e^x(x-1) + 1$

$I = \int_{y=1}^{e}\int_{x=1}^{logy}\{(x-1)\,e^x + 1\}\,dx$

Consider $\int_{x=1}^{x=logy}\{(x)\,e^x - e^x + 1\}\,dx$

$= [xe^x - e^x - e^x + 1]_{x=1}^{logy}$

$= [xe^x - 2e^x + 1]_1^{logy}$

$= [\text{ylogy} - 2\text{y} + \text{logy}] - [e-2e+1]$

$= (\text{y}+1)\text{logy} - 2\text{y} + (\text{e}-1)$

$\therefore I = \int_{y=1}^{e} ylogy + logy - 2y + (e-1)\,dy$

$= \left[\frac{y^2}{2}logy - \frac{y^2}{4} + ylogy - y - y^2 + (e-1)y\right]_1^e$

$= [\frac{e^2}{2}loge - \frac{e^2}{4} + eloge - e - e^2 + (e-1)e] -$

- $[\frac{1}{2}log1 \;\; -\frac{1}{4}+ \; log1 - 1 - 1 + e - 1]$

$=(\frac{e^2}{2} \;\; - \frac{e^2}{4} + e - 2e) - (-\frac{1}{4}-3-e)$

$= \frac{2e^2 - e^2 - 8e + 1 + 12}{4} = \frac{1}{4}[e^2 \; -8e+13]$

$\int_1^e \int_1^{logy} \int_1^{e^x} \log y \; dz \; dx \; dy = \frac{1}{4}[e^2 \; -8e+13]$

APPLICATIONS OF INTEGRATION

Length of curves (Rectification):

The process of finding the length of an arc of the curve is called rectification .

Equation of curve	**Length of arc**

Cartesion form:
i) y=f(x) & x= a, x=b;
ii) x=f(y) & y=a, y=b;
	$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}$ dx.
	$S = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2}$ dy.

Parametric equation: x=x( ) , y=y( ) & = $_1$, $\theta =$ $_2$	$S = \int_{\theta 1}^{\theta 2} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$.
Polar form : i) r =f(θ) & $\theta = \alpha, \theta = \beta$ ii) $\theta = \emptyset$(r) & r =r_1 , r =r_2	$S = \int_\alpha^\beta \sqrt{(r)^2 + (\frac{dr}{d\theta})^2}$ d . $S = \int_{r1}^{r2} \sqrt{1 + (r)^2 (\frac{d\theta}{dr})^2}$ dr.

1. **Find the length of the arc of the curve y = log $(\frac{e^x - 1}{e^x + 1})$ from x =1 to x =2Sol: Given** y =log($e^x - 1$) – log($e^x + 1$)

$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = e^x [\frac{e^x - 1 + e^x + 1}{e^{2x} - 1}] = \frac{2e^x}{e^{2x} - 1}$

Hence required length = $\int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}$ dx.

$$= \int_1^2 \sqrt{1 + \frac{4e^{2x}}{(e^{2x}-1)^2}}\, dx.$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx. \quad = \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx.$$

$$= \int_1^2 \sqrt{\frac{\frac{(e^{2x}+1)}{e^x}}{\frac{(e^{2x}-1)}{e^x}}}\, dx. \quad = \int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}\, dx.$$

$$= \log\left[(e^x - e^{-x})\right]_1^2$$

$$= \log\left(\frac{e^2-e^{-2}}{e^1-e^{-1}}\right)$$

$$= \log\left(\frac{(e^1+e^{-1})(e^1-e^{-1})}{e^1-e^{-1}}\right)$$

$$= \log\left(e + \frac{1}{e}\right)$$

2. Find length of the loop of the curve $x = t^2$, $y = t - t^3/2$

Sol : Here 'x' is even function of 't'. & 'y' is an odd func of t , so the curve is Symmetrical about x- axis.

Intersection with the co-ordinate axis:

Put y=0 => $t - \frac{t^3}{3} = 0 \Rightarrow t\left[1 - \frac{t^2}{3}\right] = 0$

⇨ t= 0 or $3 = t^2 \Rightarrow t = \sqrt{3}$

⇨ ∴ x = 0 or $x = \sqrt{3} - \frac{3\sqrt{3}}{3} = 0$

or x =3

Thus the curve cuts x-axis at A(0,3) and O(0,0) due to symmetry a loop is formed in between these two points.

∴ Required length $= 2\int_{t=0}^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= 2\int_{t=0}^{\sqrt{3}} \sqrt{4t^2 + (1-t^2)^2}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(1+t^2)^2}\, dt$$

$= 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\left[\sqrt{3} + \frac{3\sqrt{3}}{3}\right] = 2\sqrt{3}\,[2] = 4\sqrt{3}$

3. Find the perimeter of the cardiod r=a(1-cosθ).

Sol: This cardiod is symmetrical

about the initial line and the maximum value of r is 2a when $\theta = \pi$ and minimum value of r is 0 when $\theta = 0$

∴ It lies with in the circle r=2a

∴ perimeter = 2 × length of curve

$= 2\int_{\pi}^{0} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= -2\int_{0}^{\pi} \sqrt{a^2(1-cos\theta)^2 + a^2 sin^2\theta}\, d\theta$

$= -2a\int_{0}^{\pi} \sqrt{(2-2cos\theta)^2}\, d\theta$

$= -2a\int_{0}^{\pi} \sqrt{2[1-(1-2sin^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= -2a\,(2)\int_{0}^{\pi} sin\left(\frac{\theta}{2}\right) d\theta$

$= -4a\left[\frac{-cos\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$

$= -8a\,[-cos\frac{\pi}{2} + cos0]$

$= -8a$

∴ perimeter = 8a

Volumes of solids of Revolution:

Region	Volume of solid generated
Cartesian form:	
(i) y = f(x) the x-axis and the lines x =a, x=b	$V = \pi\int_a^b y^2 dx$
(ii) x = g(y) the y-axis and the lines y =c, y=d	$V = \pi\int_c^d x^2 dy$
(iii) $y=y_1(x)$, $y = y_2(x)$ the x-axis and ordinates x=a,x=b	$V = \pi\int_a^b (y_2^2 - y_1^2) dx$

(iv) $x=x_1(y)$, $x = x_2(y)$ the y-axis and ordinates y=a,y=b	$V = \pi \int_a^b (x_2^2 - x_1^2)dy$
Parametric form:	
(i) $x = \emptyset(t)$, $y = \varphi(t)$ the x-axis and ordinates $t=t_1$, $t=t_2$	$V = \pi \int_{t1}^{t2} y^2 \frac{dx}{dt} dt$
(ii) $x = \emptyset(t)$, $y = \varphi(t)$ the y-axis and abscissae $t=t_1$, $t=t_2$	$V = \pi \int_{t1}^{t2} x^2 \frac{dy}{dt} dt$
Polar form:	
(i) $r = f(\theta)$ the initial line $\theta = 0$ and the radii vectors $\theta = \alpha$, $\theta = \beta$.	$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin\theta \, d\theta$
(ii) $r = f(\theta)$ the initial line $\theta = \frac{\pi}{2}$ perpendicular to the initial line and the radii vectors $\theta = \alpha$, $\theta = \beta$.	$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \cos\theta \, d\theta$
(iii) $r = f(\theta)$ the initial line $\theta = r$ and the radii vectors $\theta = \alpha$, $\theta = \beta$.	$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin(\theta - r) \, d\theta$

1) Find the volume of a sphere of radius 'a'.

Sol: Sphere is formed by the revolution of the area enclosed by a semi-circle about its diameter.

Equation to circle of radius 'a' is $x^2 + y^2 = a^2$-------(1)

In Semi-circle 'x' varies from –a to a.

Required volume = $\int_{-a}^{a} \pi y^2$ dx

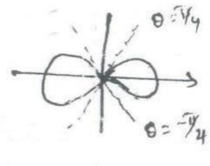
$= \pi \int_{-a}^{a} (a^2 - x^2)$ dx

$= \pi [a^2 x - \frac{x^3}{3}]_{-a}^{a}$

$$= \pi\ [a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}]$$

$$= \pi\ [2a^3 - \frac{2a^3}{3}] = \frac{4\pi a^3}{3}$$

2) Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$



Sol: Given $r^2 = a^2\cos 2\theta$ the upper half of the loop. 'θ' varies from 0 to $\frac{\pi}{4}$

Required volume obtained by revolution of the loop about the line $\theta = \frac{\pi}{2}$

$$\text{Volume} = 2 * \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} r^2 \cos\theta \, d\theta$$

$$= \frac{4\pi}{3} \int_0^{\frac{\pi}{4}} (a^2\cos\theta)^{\frac{3}{2}} \cos\theta \, d\theta$$

$$= \frac{4\pi}{3} \int_0^{\frac{\pi}{4}} a^3 (1 - 2\sin^2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$$

Let $\sqrt{2}\sin\theta = \sin\emptyset$

$\sqrt{2}\cos\theta \, d\theta = \cos\emptyset \, d\emptyset$

When $\theta = 0 \Rightarrow \sin\emptyset = 0 \Rightarrow \emptyset = 0$

$\theta = \frac{\pi}{4} \Rightarrow \sin\emptyset = \sqrt{2}.\frac{1}{\sqrt{2}} \Rightarrow \emptyset = \frac{\pi}{2}$

$$= \frac{4\pi}{3} a^3 \int_0^{\frac{\pi}{2}} (1 - 2\sin^2\emptyset)^{\frac{3}{2}} \frac{1}{\sqrt{2}} \cos\emptyset \, d\emptyset$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\frac{\pi}{2}} (\cos^4\emptyset) \, d\emptyset$$

since $\int \cos^4\theta \, d\theta = \frac{(n-1)}{n} . \frac{(n-3)}{(n-2)}$ --------- $\frac{1}{2} * \frac{\pi}{2}$ (if n is even)

$$= \frac{2\sqrt{2}\pi a^3}{3} . \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2} = \frac{\sqrt{2}\pi^2 a^3}{8}$$

Surface areas of Revolution:

Equation of curve	Axis of revolution	Surface Area
Cartesian Form:		

(i) $y = f(x)$ (ii) $x = \emptyset(y)$	X – axis Y – axis	$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$
Parametric Form:		
(i) $x = x(t)$, $y = y(t)$ (ii) $x = x(t)$, $y = y(t)$	X – axis Y – axis	$S = 2\pi \int_{t1}^{t2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $S = 2\pi \int_{t1}^{t2} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
Polar Form:		
(i) $r = f(\theta)$ (ii) $r = f(\theta)$	$\theta = 0$ (x –axis) $\theta = \frac{\pi}{2}$ (y –axis)	$S = 2\pi \int_{\theta 1}^{\theta 2} r\sin\theta \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ $S = 2\pi \int_{\theta 1}^{\theta 2} r\cos\theta \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

1) Find the surface area of the solid generated by the revolution of the cycloid $x=a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

Sol : Given $x=a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = -a\sin\theta$$

For cycloid θ varies from 0 to 2π

$$\text{Surface area} = 2\pi \int_0^{2\pi} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi \int_0^{2\pi} a(1 + \cos\theta)\ 2a.\cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 2\pi\ 2a^2 \int_0^{2\pi} 2\cos^3\left(\frac{\theta}{2}\right) d\theta$$

$$= 8\pi a^2 \int_0^{2\pi} \cos^3\left(\frac{\theta}{2}\right) d\theta$$

Let $t = \frac{\theta}{2}$ $\quad 2dt = d\theta$

$\theta = 0 \Rightarrow t = 0$

$\theta = 2\pi$ => t = π

$= 8\pi a^2 \int_0^{\pi} cos^3 t\ 2\ dt$

$= 16\,\pi a^2 2 \int_0^{\frac{\pi}{2}} cos^3 t\ \ dt$

$= 32\ \pi\ a^2 . \frac{2}{3} . 1$

$= \frac{64\pi a^2}{3}$

2) Find the area of the surface of the revolution generated by revolving about the x-axis of the arc of the parabola y^2 = 12x from x = 0 to x = 3

Sol: $y = 2\sqrt{3}\sqrt{x} \Rightarrow \frac{dy}{dx} = 2\sqrt{3} . \frac{1}{2\sqrt{x}}$

surface area $= 2\pi \int_0^3 y \sqrt{1 + (\frac{dy}{dx})^2}\ dx$

$= 2\pi \int_0^3 2\sqrt{3}\sqrt{x} \sqrt{1 + \frac{3}{x}}\ dx$

$= 4\pi\sqrt{3} \int_0^3 \sqrt{x} \sqrt{\frac{x+3}{x}}\ dx$

$= 4\pi\sqrt{3} [\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}]_0^3$

$= \frac{8\pi}{\sqrt{3}} [(6)^{\frac{3}{2}} - (3)^{\frac{3}{2}}]$

$= \frac{8\pi}{\sqrt{3}} (3)^{\frac{3}{2}} [(2)^{\frac{3}{2}} - 1]$

$= 24\pi\ [2\sqrt{2}\ -1]$

3) Find the length of arc of the curve x = 2$\theta - sin2\theta$, y= 2 $sin^2\theta$ as θ varies from 0 to π

Sol: $\frac{dx}{d\theta} = 2 - 2cos2\theta = 2(1 - cos2\theta)$

$\frac{dy}{d\theta} = 4 sin\theta\ cos\theta = 2\ sin2\theta$

$\frac{dx}{d\theta} = 2(1 - cos2\theta)$

$$\frac{ds}{d\theta} = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}$$

$$= \sqrt{4\,(1 - cos2\theta)^2 + 4\,sin^2 2\theta}$$

$$= 2\sqrt{(1 - 2cos2\theta)^2 + cos^2 2\theta + sin^2 2\theta}$$

$$= 2\sqrt{(1 - 2cos2\theta)+1}$$

$$= 2\sqrt{2(1 - cos2\theta)}$$

$$= 2\sqrt{2.2\ sin^2\theta} = 2.2\, sin\theta = 4\, sin\theta$$

Length of arc = $\int_0^{\pi} \frac{ds}{d\theta} d\theta = \int_0^{\pi} 4\ sin\theta\, d\theta$

$= -4\,[cos\theta]_0^{\pi}$ $\quad = -4\,[cos\pi - cos0]$

$= -4[-1-1]$

$=-4[-2] = 8$

4) Find the length of the curve $9x^2 = 4\,(1 + y^2)^3$ from the point $(\frac{2}{3}, 0)$ to the point $(\frac{10\sqrt{5}}{3}, 2)$

Sol : Given curve is $9x^2 = 4\,(1 + y^2)^3$

$$x^2 = \frac{4}{9}(1 + y^2)^3$$

$$x = \frac{2}{3}(1 + y^2)^{\frac{3}{2}}$$

$$\frac{dx}{dy} = \frac{2}{3} * \frac{3}{2}(1 + y^2)^{\frac{1}{2}}.2y$$

$$= 2y\,(1 + y^2)^{\frac{1}{2}}$$

$$\frac{ds}{dy} = \sqrt{1 + (\frac{dx}{dy})^2} = \sqrt{1 + 4y^2(1 + y^2)}$$

$$= \sqrt{1 + 4y^2 + 4y^4)}$$

$$= \sqrt{(2y^2)^2 + 2.2\,y^2.1 + 1^2)}$$

$$= (2y^2 + 1)$$

Length of curve $\quad = \int_0^2 (2y^2 + 1)dy$

$= (2.\frac{y^3}{3} + y)_0^2$

$= (2.\frac{8}{3} + 2) - 0$

$= \frac{16+6}{3} = \frac{22}{3}$

PROBLEMS

1). Show that the volume of the solid generated by the revolution of the loop of the curve $y^2(a-x) = x^2(a+x)$ about the x-axis is 2π [log2 - $\frac{2}{3}$] a^3

Sol: Given curve is $y^2(a-x) = x^2(a+x)$

It contains only even powers of y hence it is symmetric about x-axis for x=0 => y = 0 and for y = 0 => x =0 or x=-a

∴it passes through (0,0) and (-a,0)

$\therefore V = \pi\int_{-a}^{0} y^2 dx$

$V = \pi\int_{-a}^{0} \frac{x^2(a+x)}{(a-x)} dx$

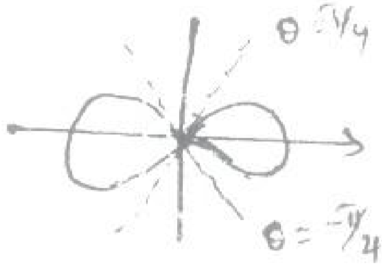
$V = \pi\int_{-a}^{0} \frac{(ax^2+x^3)}{(a-x)} dx$

$V = \pi\int_{-a}^{0} -x^2 - 2ax - 2a^2 + \frac{2a^3}{(a-x)}\ dx$

$V = \pi[\frac{x^3}{3} - \frac{2ax^2}{2} - 2a^2x - 2a^3\log|a-x|]_{-a}^{0}$

$V = 2\pi a^3$ [log2 - $\frac{2}{3}$]

2). Find the volume of the solid generated by revolving the lemniscates $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$



Given

curve is $r^2 = a^2 \cos 2\theta$ the upper half of the loop θ varies from θ to $\frac{\pi}{4}$

Required volume obtained by revolution of the loop about the line OY i.e. $\theta = \frac{\pi}{2}$ Volume = $2. \int_0^{\frac{\pi}{4}} \frac{2\pi}{3} r^3 \cos\theta \, d\theta$

$= 2. \int_0^{\frac{\pi}{4}} \frac{2\pi}{3} (a^2 \cos 2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$

$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$

$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} (1 - 2\sin^2(\theta))^{\frac{3}{2}} \cos\theta \, d\theta$

Let $\sqrt{2}\sin\theta = \sin\emptyset$

$\Rightarrow \sqrt{2} \cos\theta \, d\theta = \cos\emptyset \, d\emptyset$

When $\theta = 0 \Rightarrow \sin\emptyset = 0 \Rightarrow \emptyset = 0$

$\theta = \frac{\pi}{4} \Rightarrow \sin\emptyset = \sqrt{2} \cdot \frac{1}{\sqrt{2}} \Rightarrow \emptyset = \frac{\pi}{2}$

$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}} (1 - 2\sin^2(\emptyset))^{\frac{3}{2}} \frac{1}{\sqrt{2}} \cos\emptyset \, d\emptyset$

$= \frac{2\sqrt{2}\pi a^3}{3} \int_0^{\frac{\pi}{2}} \cos^2(\emptyset) d\emptyset$

$= \frac{2\sqrt{2}\pi a^3}{3} * \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2} = = \frac{\sqrt{2}\pi^2 a^3}{8}$ V= $\frac{\sqrt{2}\pi^2 a^3}{8}$

3). The part of the parabola cut off by the Latusrectum is rotated

(i) about the Latus rectum (ii) about the axis.

Show that the volumes generated are in the ratio 16:15

Sol: Equation of the parabola is $y^2 = 4ax$

Let v_1 be the volume generated when rotated about the Latus rectum and v_2 be the volume generated when rotated about the axis equation to Latus rectum is x=a.

When the area of the parabola cut off by the latus rectum revolves about the latus rectum any point(x,y)

On the parabola describes a circle radius a-x

$$\therefore v_1 = \pi \int_{-2a}^{2a} (a-x)^2 dy$$

$$= \pi \int_{-2a}^{2a} (a - \frac{y^2}{4a})^2 dy$$

$$= 2\pi \int_0^{2a} a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2} dy$$

$$v_1 = 2\pi [a^2 y - \frac{y^3}{6} + \frac{y^5}{80a^2}]_0^{2a}$$

$$= 2\pi [2a^3 - \frac{8a^3}{6} + \frac{32a^5}{80a^2}]$$

$$= 2\pi [2a^3 - \frac{4a^3}{3} + \frac{2a^3}{5}]$$

$$= 2\pi a^3 [\frac{30-20+6}{15}] = 2\pi a^3 \frac{16}{15} = \frac{32\pi a^3}{15}$$

$$V_2 = \pi \int_0^a (y)^2 \, dx$$

$$= \pi \int_0^a 4ax \, dx$$

$$= \pi [\frac{4ax^3}{2}]_0^a = 2\pi [a^3 - 0] = 2\pi a^3$$

$$\therefore \frac{v1}{V2} = \frac{32\pi a^3}{15 * 2\pi a^3} = \frac{16}{15}$$

4) Find the surface area of the solid generated by the revolution of the cycloid x=a(θ +sinθ), y = a(1+cosθ) about its base is $\frac{64}{3}\pi a^2$.

Sol: Given equation of cycloid is x=a(θ +asinθ), y = a(1+cosθ)

$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\frac{dy}{d\theta} = -a\sin\theta$$

$\frac{ds}{d\theta} = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}$

$= \sqrt{a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)}$

$= a\sqrt{2(1 + \cos\theta)}$

$= a\sqrt{2.2.\cos^2(\frac{\theta}{2})}$

$= 2.a\cos(\frac{\theta}{2})$

For the arc of cycloid θ varies from $\theta = 0$ to $\theta = 2\pi$

$\therefore$surface area = $\int_0^{2\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$

$= 2\pi \int_0^{2\pi} a(1 + \cos\theta)\, 2a\cos(\frac{\theta}{2})\, d\theta$

$= 4\pi a^2 \int_0^{2\pi} 2.\cos^2\left(\frac{\theta}{2}\right).\cos(\frac{\theta}{2})\, d\theta$

$= 4\pi a^3 \int_0^{2\pi} 2.\cos^3\left(\frac{\theta}{2}\right) d\theta$

$= 8\pi a^2 \int_0^{\pi} \cos^3 t\,(2dt)$

Let $t = \frac{\theta}{2}$ | $\theta = 0$ $t = 0$

$d\theta = 2dt$ | $\theta = 2\pi$ $t = \pi$

$= 16\pi a^2 * 2\int_0^{\frac{\pi}{2}} \cos^3 t\,(dt)$ [since $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$]

$= 32\pi a^2(\frac{2}{3}).1 = (\frac{64\pi a^2}{3})$

5). Find the area of the surface of the revolution generated by revolving about the x-axis of the arc of the parabola y^2 = 12x from x=0 to x=3

Sol: Given parabola $y^2 = 12x$

⇨ $y = 2\sqrt{3}.\sqrt{x}$

⇨ $\frac{dy}{dx} = 2\sqrt{3}.\frac{1}{2\sqrt{x}} = \frac{\sqrt{3}}{\sqrt{x}}$

⇨ $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$

$= \sqrt{1 + \frac{3}{x}} \quad = \sqrt{\frac{x+3}{x}}$

Surface area = $\int_0^3 2\pi y \frac{ds}{dx}\, dx$

$= 2\pi \int_0^3 2\sqrt{3}\sqrt{x} \sqrt{\frac{x+3}{x}}\, dx$

$= 4\pi\sqrt{3} \int_0^3 (x+3)^{\frac{1}{2}} dx$

$= 4\pi\sqrt{3} \left[\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

$= \frac{8\pi}{\sqrt{3}} [6^{\frac{3}{2}} - 3^{\frac{3}{2}}]$

$= \frac{8\pi}{\sqrt{3}} 3^{\frac{3}{2}} [2^{\frac{3}{2}} - 1]$

$= 8\pi \,.\, 3\, [2^{\frac{3}{2}} - 1]$

$= 24\pi\, [2^{\frac{3}{2}} - 1]$

6) The lemniscate $r^2 = a^2 \cos 2\theta$ revolves about the initial line find the surfacearea of the solid generated.

Sol: Given curve is $r^2 = a^2 \quad \theta$

cos2Differentiating w.r.to

'θ' $2r \frac{dr}{d\theta} = -a^2\, 2 \sin 2\theta$

$\frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}$

$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}}$

$= \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}}$

$= a\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}$

$= \frac{a}{\sqrt{\cos 2\theta}}$

If r = 0 => $\cos 2\theta = 0$ i.e $2\theta = \pm\frac{\pi}{2}$

⇨ $\theta = \frac{-\pi}{4}, \frac{\pi}{4}$

The curve consists of two equal loops.

∴ Required surface area = $2\int_0^{\frac{\pi}{4}} 2\pi y \frac{dr}{d\theta}\, d\theta$

$= 4\pi \int_0^{\frac{\pi}{4}} r \sin\theta \frac{a}{\sqrt{cos2\theta}}\, d\theta$

$= 4\pi \int_0^{\frac{\pi}{4}} a\sqrt{cos2\theta} \sin\theta \frac{a}{\sqrt{cos2\theta}}\, d\theta$

$= 4\pi a^2 \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta$

$= 4\pi a^2 [-\cos\theta]_0^{\frac{\pi}{4}}$

$= 4\pi a^2[\frac{-1}{\sqrt{2}} + 1] = 4\pi a^2[1 - \frac{1}{\sqrt{2}}]$

MULTIVARIABLE CALCULUS(INTEGRATION)

1) The length of the curve $y = \frac{2}{3} x^{\frac{3}{2}}$ from x =1 to x = 4 is

(a) $\frac{2}{3} 5^{\frac{3}{2}}$ (b) $\frac{2}{3} 2^{\frac{3}{2}}$ (c) $\frac{2}{3}(5^{\frac{3}{2}} - 2^{\frac{3}{2}})$ (d) $\frac{2}{3}(5^{\frac{3}{2}} + 2^{\frac{3}{2}})$

2) The length of the curve $y = \frac{4}{3} x^{\frac{3}{2}}$ from x =0 to x = 20 is

(a) |2| (b) $|2|\frac{1}{3}$ (c) $|2|\frac{2}{3}$ (d) None

3) The length of the curve $x=t^2 - 3t$, $y=3t^2$ from t=0 to t=1 is

(a) 4 (b) 8 (c) 6 (d) 2

4) The length of the curve $x=e^t \sin t$, $y=e^t \cos t$ from t=0 to $t=\frac{\Pi}{2}$ is

(a) $2e^{\frac{\pi}{2}}$ (b) $e^{\frac{\pi}{2}}$-1 (c) 2($e^{\frac{\pi}{2}}$-1) (d) $\sqrt{2}\,e^{\frac{\pi}{2}}$-1

5) The perimeter of the asteroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is

(a) 4a (b) 6a (c) 8a (d) 12a

6) The perimeter of the loop of curve $9y^2 = (x-2)(x-5)^2$ is

(a) $2\sqrt{3}$ (b) $\frac{\sqrt{3}}{2}$ (c) $4\sqrt{3}$ (d) $\frac{\sqrt{3}}{4}$

7) The perimeter of the cardioids r=a(1-cos θ) is

(a)4a (b)2a (c) 6a (d) 8a

8) The upper half of the cardioids r=a(1+cos θ) is bisected by the line

(a)$\theta=\frac{\pi}{3}$ (b) $\theta=\frac{\pi}{4}$ (c) $\theta=\frac{\pi}{6}$ (d) None

9. The volume generated by revolution of r=2acos θ between

$\theta = 0 \ to\ \theta = \frac{\pi}{2}$ is

(a) $\frac{1\pi a^3}{3}$ (b) $\frac{2\pi a^3}{3}$ (c) $\frac{4\pi a^3}{3}$ (d) $\frac{5\pi a^3}{3}$

10. $\int_0^2 \int_0^x y\,dy\,dx$

(a) $\frac{4}{3}$ (b) $\frac{4}{5}$ (c) $\frac{2}{3}$ (d) $\frac{2}{5}$

11. $\int_{x=0}^{a} \int_{y=0}^{b} (x^2+y^2)$ dx dy

(a) $\frac{1}{3}(a^2+b^2)$ (b) $\frac{a}{3}(a^2+b^2)$ (c) $\frac{b}{3}(a^2+b^2)$ (d) $\frac{ab}{3}(a^2+b^2)$

12. $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi}{4}$

13. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}$dxdy

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

14. $\int_0^\pi \int_0^{a\sin\theta} r\,dr\,d\theta$

(a) $\frac{\pi a^2}{4}$ (b) $\frac{\pi a}{4}$ (c) $\frac{\pi a^2}{2}$ (d) $\frac{\pi a}{2}$

15. The iterated integral for $\int_{-1}^{1}\int_{0}^{1-x^2} f(x,y)dx\,dy$ after changing order of integration is-------

Ans: $\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\,dy$

FILL IN THE BLANKS;

16. $\int_{0}^{a}\int_{0}^{\sqrt{x^2+y^2}} dx\,dy$ after changing to polar co-ordinates is…………..

17. $\int_{0}^{a}\int_{0}^{\frac{a\sqrt{b^2-y^2}}{b}} xy\,dx\,dy$ after changing the order of integration is…………..

18. $\int_{0}^{1}\int_{1}^{2}\int_{2}^{3} x\,y\,z\,dx\,dy\,dz$…………

19. The area enclosed by the parabolas x^2 = y and y^2=x is……………..

20. The area of the region bounded by y^2 = 4ax and x^2 = 4ay is…………………

21. the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is…………….

22. $\int_{0}^{1}\int_{0}^{1}\int_{0}^{1} x\,y\,x\,dx\,dy\,dz$…………………..

23. $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz$

(a) 12 (b) 24 (c) 48 (d) 36

24. The volume of tetrahedron formed by the surfaces x=0, y=0,z=0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ = 1 is……………..

25. The volume of tetrahedron bounded by the co-ordinate planes and the plane x+y+z=1 is………………..

INTEGRATION AND ITS APPLICATIONS

(Assignment Problems)

1) (i) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

(i) Evaluate $\int_0^5 \int_0^{x^2} x(x^2 + y^2)$ dx dy

2) (i) Evaluate $\iint (x^2 + y^2)$ dx dy in the positive quadratic for which $x + y \leq 1$

(ii) Evaluate $\iint (x^2 + y^2)$ dx dy over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3) Evaluate $\int_0^{\frac{\pi}{4}} \int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2 - b^2}}$

4) Evaluate $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}$ dx dy by changing into polar co-ordinates.

5) By changing the order of integration, evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

6) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} x\ xyz\,dz\,dy\,dx$