UNIT – V
TIME VARYING FIELDS AND FINITE ELEMENT METHOD

5.1 Time-Varying Fields

Stationary charges $\longrightarrow$ electrostatic fields

Steady currents $\longrightarrow$ magnetostatic fields

Time-varying currents $\longrightarrow$ electromagnetic fields

Only in a non-time-varying case can electric and magnetic fields be considered as independent of each other. In a time-varying (dynamic) case the two fields are interdependent. A changing magnetic field induces an electric field, and vice versa.

The Continuity Equation

Electric charges may not be created or destroyed (the principle of conservation of charge).

Consider an arbitrary volume *V* bounded by surface *S*. A net charge *Q* exists within this region. If a net current *I* flows across the surface out of this region, the charge in the volume must decrease at a rate that equals the current:

$$I = \int_S \bar{J} \cdot \overline{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_V \, dv$$

Divergence theorem

$$\int_V \nabla \cdot \bar{J} \, dv = -\int_V \frac{\partial \rho_V}{\partial t} dv$$

Partial derivative because ρ_V may be a function of both time and space

This equation must hold regardless of the choice of *V*, therefore the integrands must be equal:

$$\boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_V}{\partial t}} \quad (A/m^3)$$

the equation of continuity

For steady currents

$$\boxed{\nabla \cdot \bar{J} = 0}$$

Kirchhoff's current law follows from this

that is, steady electric currents are divergences or solenoidal.

Displacement Current

For magnetostatic field, we recall that $\nabla \times \overline{H} = \overline{J}$

Taking the divergence of this equation we have

$$\nabla \cdot \left(\nabla \times \overline{H}\right) = 0 = \nabla \cdot \overline{J}$$

However the continuity equation requires that

$$\nabla \cdot \overline{J} = -\frac{\partial \rho_V}{\partial t} \neq 0$$

Thus we must modify the magnetostatic curl equation to agree with the continuity equation. Let us add a term to the former so that it becomes

$$\nabla \times \overline{H} = \overline{J} + \overline{J}_d$$

where $\overline{J}$ is the conduction current density $\overline{J} = \sigma \overline{E}$, and $\overline{J}_d$ is to be determined and defined.

Displacement Current continued

Taking the divergence we have

$$\nabla \cdot \left(\nabla \times \overline{H}\right) = 0 = \nabla \cdot \overline{J} + \nabla \cdot \overline{J}_d \longrightarrow \nabla \cdot \overline{J}_d = -\nabla \cdot \overline{J}$$

In order for this equation to agree with the continuity equation,

Gauss' law

$$\nabla \cdot \overline{J}_d = -\nabla \cdot \overline{J} = \frac{\partial \rho_V}{\partial t} = \frac{\partial}{\partial t}\left(\nabla \cdot \overline{D}\right) = \nabla \cdot \frac{\partial \overline{D}}{\partial t}$$

$$\overline{J}_d = \frac{\partial \overline{D}}{\partial t} \longleftarrow$$ displacement current density

$$\boxed{\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}}$$

$$\int_S \nabla x \overline{H} \cdot \overline{ds} = \int_S \left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right) \cdot \overline{ds} \longrightarrow \boxed{\int_L \overline{H} \cdot \overline{dl} = I + \int_S \frac{\partial \overline{D}}{\partial t} \cdot \overline{ds}}$$

Stokes' theorem

Displacement Current continued

A typical example of displacement current is the current through a capacitor when an alternating voltage source is applied to its plates. The following example illustrates the need for the displacement current.

Using an unmodified form of Ampere's law

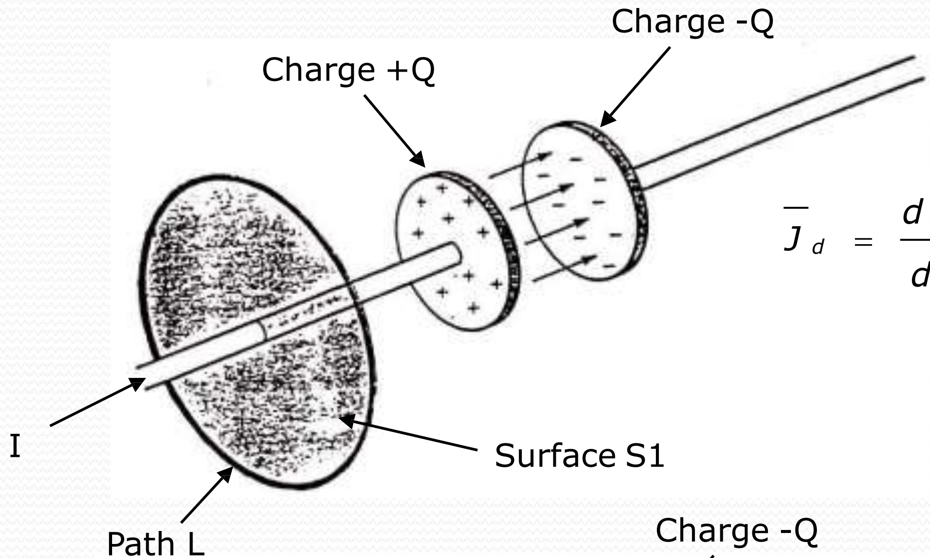
$$\begin{cases} \oint_L \overline{H} \cdot \overline{dl} = \int_{S_1} \overline{J} \cdot \overline{dS} = I_{enc} = I = \dfrac{dQ}{dt} \\ \oint_L \overline{H} \cdot \overline{dl} = \int_{S_2} \overline{J} \cdot \overline{dS} = I_{enc} = 0 \end{cases}$$

$\boxed{\overline{J}_d = 0}$

(no conduction current flows through S_2 ($\overline{J} = 0$))

To resolve the conflict we need to include $\overline{J}_d$ in Ampere's law.

Displacement Current continued



$$\bar{J}_d = \frac{d\bar{D}}{dt}$$

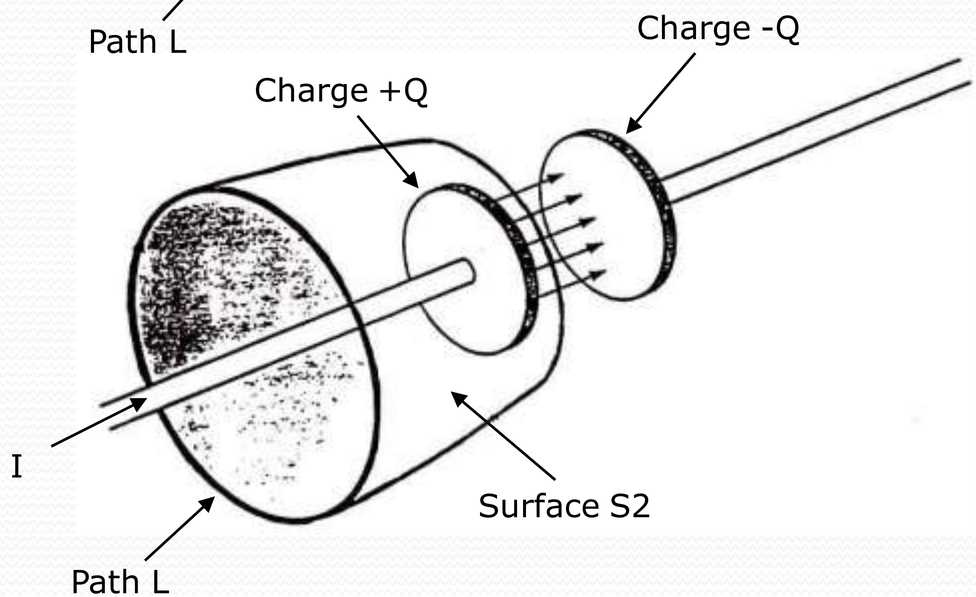
The total current density is $\bar{J} + \bar{J}_d$. In the first equation $\bar{J}_d = 0$ so it remains valid. In the second equation $\bar{J} = 0$ so that

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{S_2} \bar{J}_d \cdot d\bar{S} = \frac{d}{dt} \underbrace{\int_{S_2} \bar{D} \cdot d\bar{S}}_{Q} \quad \leftarrow \boxed{\bar{J} = 0}$$

$$= \frac{dQ}{dt} = I$$

$$\oint_{S_1 + S_2} \bar{D} \cdot d\bar{S} = Q$$

$$\int_{S_1} \bar{D} \cdot d\bar{S} = 0$$



So we obtain the same current for either surface though it is conduction current in S_1 and displacement current in S_2.

Faraday's Law

Faraday discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In differential (or point) form this experimental fact is described by the following equation

$$\boxed{\nabla \times \overline{E} = -\frac{\partial B}{\partial t}}$$

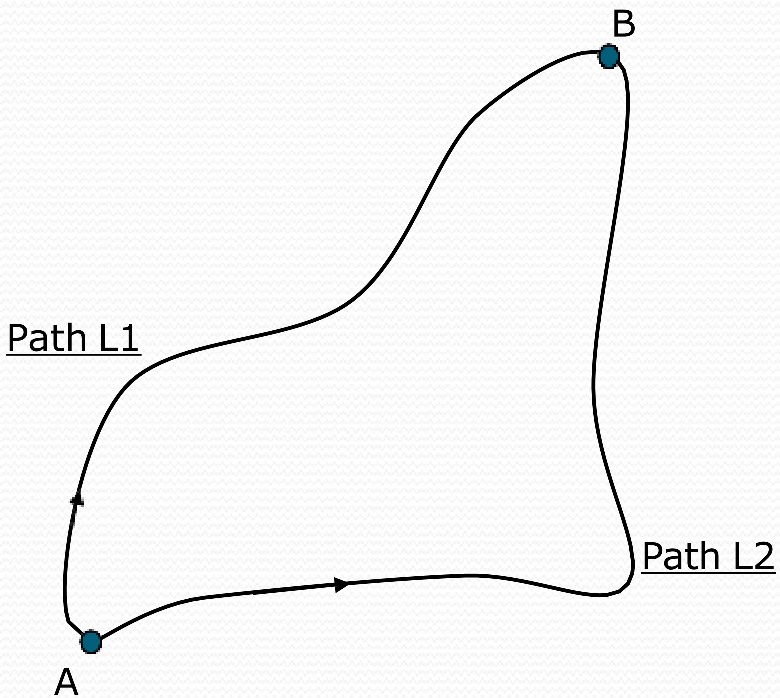
Taking the surface integral of both sides over an open surface and applying Stokes' theorem, we obtain

Integral form $\longrightarrow$ $\oint_L \overline{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \overline{B} \cdot \overline{dS} = -\frac{\partial \psi}{\partial t}$

where ψ is the magnetic flux through the surface S.

Faraday's Law continued

Time-varying electric field is not conservative.



The effect of electromagnetic induction. When time-varying magnetic fields are present, the value of the line integral of $\bar{E}$ from A to B may depend on the path one chooses.

Suppose that there is only one unique voltage $V_{AB} = V_A - V_B$. Then

$$V_{AB} = \oint_{L_1} \bar{E} \cdot d\bar{l} = \oint_{L_2} \bar{E} \cdot d\bar{l}$$

However,

$$\oint_L \bar{E} \cdot d\bar{l} = \underbrace{\int_{L_1} \bar{E} \cdot d\bar{l}}_{V_{AB}} - \underbrace{\int_{L_2} \bar{E} \cdot d\bar{l}}_{V_{AB}} = -\frac{\partial \psi}{\partial t}$$

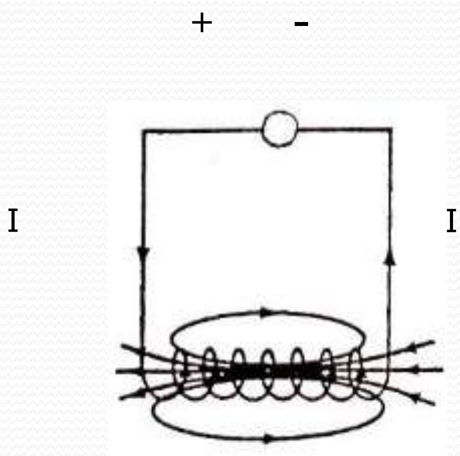
$$V_{AB} - V_{AB} \neq 0$$

$\partial \psi / \partial t = 0$

Thus V_{AB} can be unambiguously defined only if .(in practice, if than λ the $>$ dimensions of system in question)

Inductance

A circuit carrying current *I* produces a magnetic field $\bar{B}$ which causes a flux $\psi = \int B \cdot dS$ to pass through each turn of the circuit. If the medium surrounding the circuit is linear, the flux ψ is proportional to the current *I* producing it.



Magnetic field **B** produced by a circuit.

$$V = N \frac{\partial \psi}{\partial t}$$ (voltage induced across coil)

$$= Nk \frac{\partial I}{\partial t} = L \frac{\partial I}{\partial t}$$

$$L = Nk = \frac{N\psi}{I} = \frac{\lambda}{I}$$, H (henry)

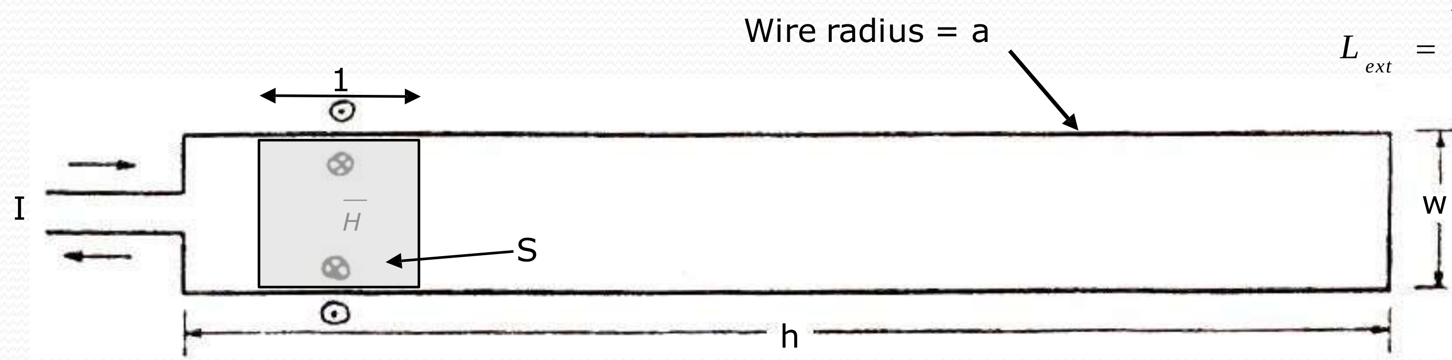
Self-inductance *L* is defined as the ratio of the magnetic flux linkage λ to the current *I*.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the internal inductance L_{in} while that produced by the flux external to it is called external inductance

$$L_{ext} \quad (L = L_{in} + L_{ext})$$

Let us find L_{ext} of a very long rectangular loop of wire for which $R = 0$, $a \ll w$ and $w \ll h$. This geometry represents a parallel-conductor transmission line. Transmission lines are usually characterized by per unit length parameters.

$$L_{ext} = \frac{\Psi}{I}$$



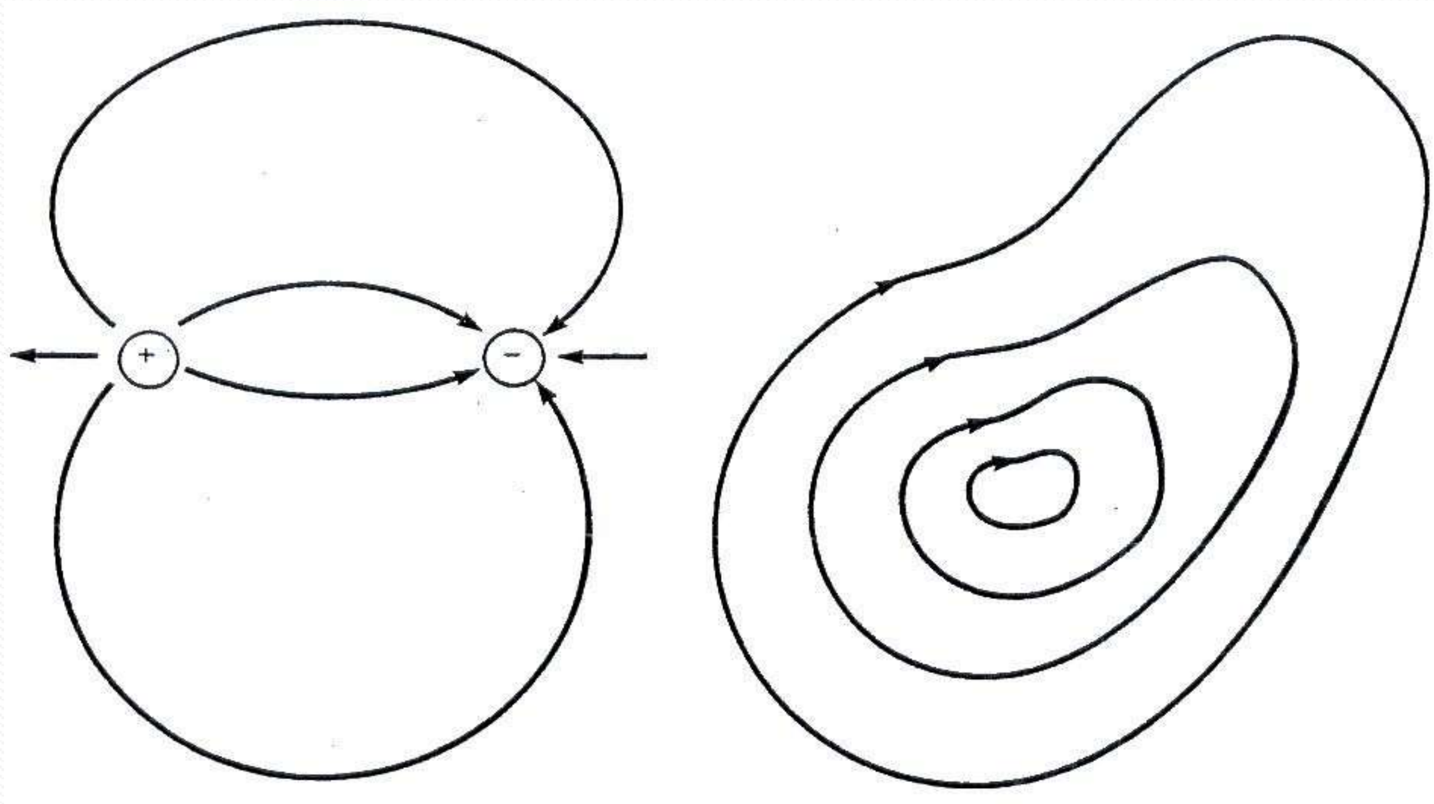
Finding the inductance per unit length of a parallel-conductor transmission line.

General Forms of Maxwell's Equations

	Differential	Integral	Remarks
1	$\nabla \cdot \overline{D} = \rho_V$	$\int_S \overline{D} \cdot d\overline{s} = \int_V \rho_V \, dv$	Gauss' Law
2	$\nabla \cdot \overline{B} = 0$	$\oint_S \overline{B} \cdot d\overline{s} = 0$	Nonexistence of isolated magnetic charge
3	$\nabla x \overline{E} = -\frac{\partial \overline{B}}{\partial t}$	$\int_L \overline{E} \cdot d\overline{l} = -\frac{\partial}{\partial t} \int_s \overline{B} \cdot d\overline{s}$	Faraday's Law
4	$\nabla x \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$	$\int_L \overline{H} \cdot d\overline{l} = \int_s \left(\overline{J} + \frac{\partial \overline{D}}{\partial t} \right) . d\overline{s}$	Ampere's circuital law

In 1 and 2, S is a closed surface enclosing the volume V

In 2 and 3, L is a closed path that bounds the surface S



Electric fields can originate on positive charges and can end on negative charges. But since nature has neglected to supply us with magnetic charges, magnetic fields cannot begin or end; they can only form closed loops.

Sinusoidal Fields

In electromagnetics, information is usually transmitted by imposing amplitude, frequency, or phase modulation on a sinusoidal carrier. Sinusoidal (or time-harmonic) analysis can be extended to most waveforms by Fourier and Laplace transform techniques.

Sinusoids are easily expressed in phasors, which are more convenient to work with. Let us consider the "curl H" equation.

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \overline{H} = f(x, y, z, t)$$

Its phasor representation is

$$\nabla \times \underline{\overline{H}} = \underline{\overline{J}} + j\omega \underline{\overline{D}} = \underline{\overline{J}} + j\omega\varepsilon \underline{\overline{E}}$$

$\underline{\overline{H}}$ is a vector function of position, but it is independent of time. The three scalar components of H are complex numbers; that is, if

$$\overline{H}(x, y, z, t) = \underbrace{f_1(x, y, z)\cos(\omega t + \phi_1)\overline{e}_x}_{} + \underbrace{f_2(x, y, z)\cos(\omega t + \phi_2)\overline{e}_y}_{}$$

then

$$\underline{\overline{H}}(x, y, z) = \overbrace{f_1(x, y, z)e^{j\phi_1}\overline{e}_x}^{} + \overbrace{f_2(x, y, z)e^{j\phi_2}\overline{e}_y}^{}$$

$e^{j\omega t}$

Point Form

$$\nabla \cdot \underline{\overline{D}} = \underline{\rho_V}$$

$$\nabla \cdot \underline{\overline{B}} = 0$$

$$\nabla \times \underline{\overline{E}} = -j\omega \underline{\overline{B}}$$

$$\nabla \times \underline{\overline{H}} = \underline{\overline{J}} + j\omega \underline{\overline{D}}$$

Integral Form

$$\int \underline{\overline{D}} \cdot d\overline{s} = \int \underline{\rho_V}\, dv$$

$$\int \underline{\overline{B}} \cdot d\overline{s} = 0$$

$$\int \underline{\overline{E}} \cdot d\overline{l} = -j\omega \int \underline{\overline{B}} \cdot d\overline{s}$$

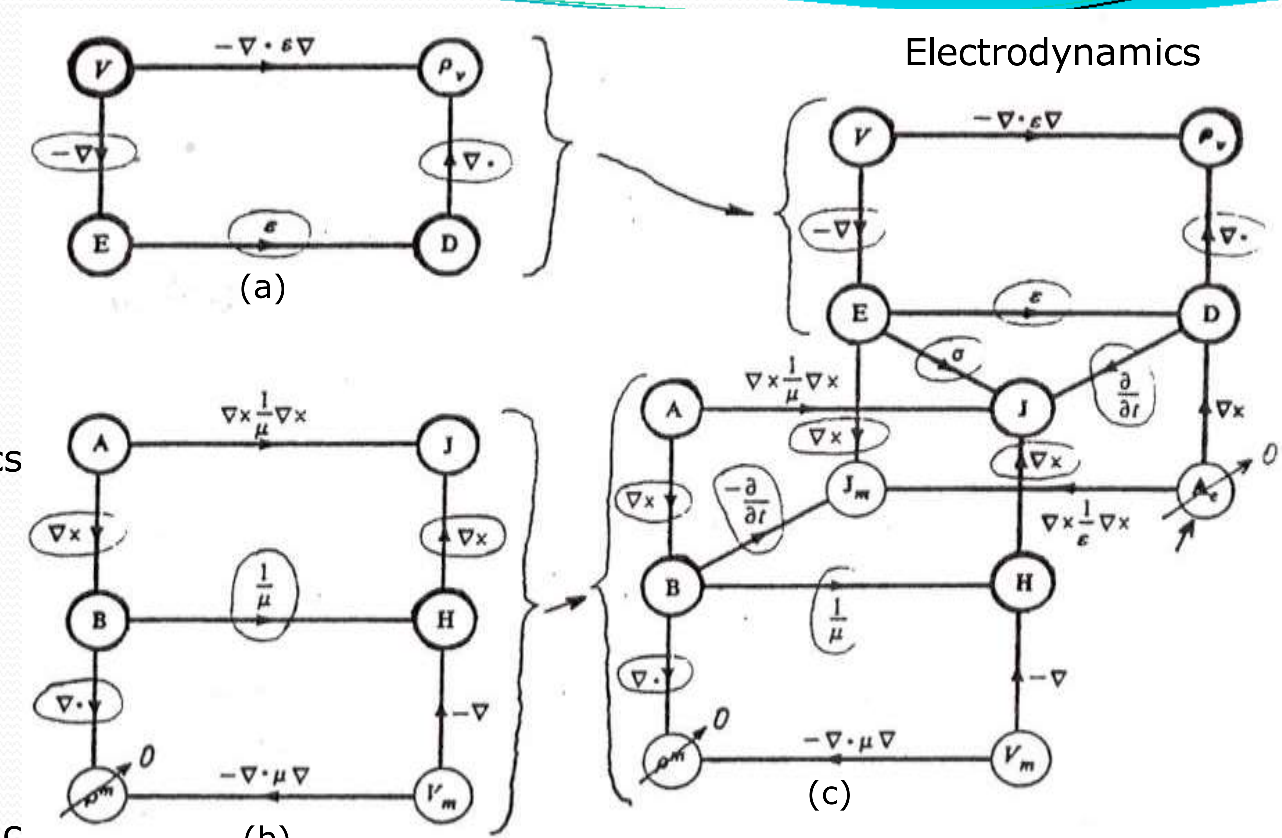
$$\int \underline{\overline{H}} \cdot d\overline{l} = \int \left(\underline{\overline{J}} + j\omega \underline{\overline{D}}\right) \cdot d\overline{s}$$

$$\overline{H}(x,y,z,t) = \mathrm{Re}\left[\underline{\overline{H}}(x,y,z) e^{j\omega t}\right.$$

Maxwell's Equations continued

Electrostatics

Magnetostatics



Free magnetic charge density (ρ^m) $= 0$

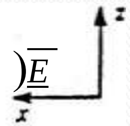
Electromagnetic flow diagram showing the relationship between the potentials and vector fields: (a) electrostatic system, (b) magnetostatic system, (c) electromagnetic system. [Adapted with permission from IEE Publishing Department]

The Skin Effect

When time-varying fields are present in a material that has high conductivity, the fields and currents tend to be confined to a region near the surface of the material. This is known as the "skin effect". Skin effect increases the effective resistance of conductors at high frequencies.

Let us consider the vector wave equation (Helmholtz's equation) for the electric field:

$$\nabla^2 \overline{E} = j\omega\mu\left(\sigma_E + j\omega\varepsilon\right)\overline{E}$$



$$\left(\nabla^2 \overline{E} = \nabla^2 E_x \overline{e_x} + \nabla^2 E_y \overline{e_y} + \nabla^2 E_z \overline{e_z}\right)$$

If the material in question is a very good conductor, so that $\sigma_E \gg \omega\varepsilon$ we can write

$$\boxed{\nabla^2 \overline{E} = j\omega\mu\sigma_E \overline{E}}$$

Since $\overline{E} = \overline{J}/\sigma_E$ we also have

$$\boxed{\nabla^2 \overline{J} = j\omega\mu\sigma_E \overline{J}}$$

Consider current flowing in the +*x* direction through a conductive material filling the half-space *z*<0. The current density is independent of *y* and *x*, so that we have

$$\frac{\partial^2 J_x}{\partial z^2} = \overbrace{j\omega\mu\sigma_E}^{\gamma^2} J_x \longrightarrow J_x = Ae^{-\frac{(1+j)z}{\delta}} + Be^{\frac{(1+j)z}{\delta}} = \cancel{Ae^{-\frac{(1+j)z}{\delta}}} \rightarrow 0 + Be^{\frac{(1+j)z}{\delta}}$$

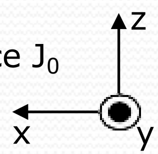
$$\gamma = \alpha + j\beta$$

$$\alpha = \beta = \frac{1}{\delta}$$

Current density at surface J_0

(Air) Z>0

(Metal) Z<0



Skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_E}} = \frac{1}{\sqrt{\pi\mu\sigma_E f}}$$

Magnitude of current density decreases exponentially with depth

$$\underline{J}_x = Ae^{-(1+j)z/\delta} + Be^{(1+j)z/\delta} = Ae^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} + Be^{\frac{z}{\delta}}e^{j\frac{z}{\delta}}$$

As $z \to -\infty$, the first term increases and would give rise to infinitely large currents. Since this is physically unreasonable, the constant *A* must vanish. From the condition when *z*=0 we have $\underline{J}_x = J_o$

$$J_o = Be^{(0)}e^{(0)} = B$$

and

$$\boxed{\underline{J}_x(z) = J_o e^{\frac{z}{\delta}} e^{j\frac{z}{\delta}}}$$

$$z \le 0$$

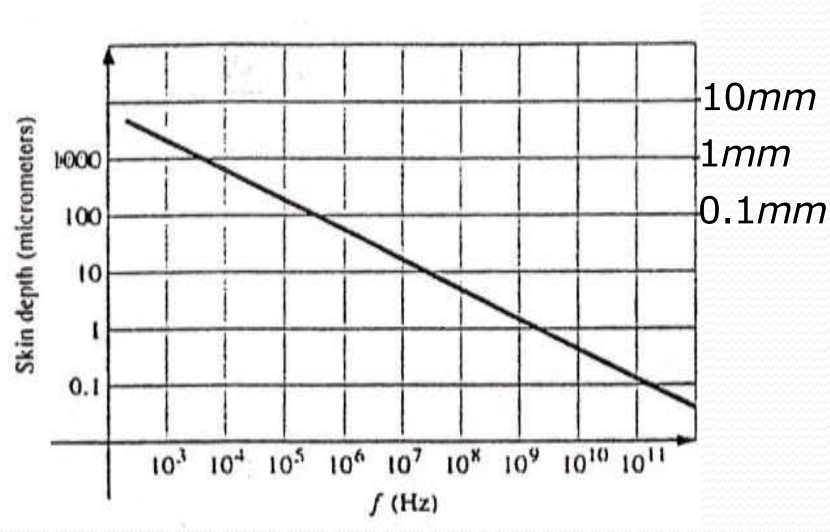
Current density magnitude decreases exponentially with depth. Its phase changes as well.

At $|z| = \delta$

$$|\underline{J}_x| = \frac{1}{e} J_o = 0.37 J_o$$

At $|z| = \pi\delta$

$$\underline{J}_x = -0.043 J_o$$



Skin depth δ versus frequency for copper

$$\begin{cases} 10\ kHz \to \delta = 0.6\,mm \\ 60\ Hz \to \delta = 8.6\,mm \end{cases}$$

Skin Depth and Surface Resistance for Metals

$$\delta = \frac{1}{\sqrt{\pi\mu\sigma_E f}} = \frac{A}{\sqrt{f}} \qquad R_S = \sqrt{\frac{\pi\mu f}{\sigma_E}} = B\sqrt{f}$$

Metal	$\sigma_E\,(\Omega m)^{-1}$	A	B
Silver	$6.81 \cdot 10^{7}$	$6.10 \cdot 10^{-2}$	$2.41 \cdot 10^{-7}$
Copper	$5.91 \cdot 10^{7}$	$6.55 \cdot 10^{-2}$	$2.59 \cdot 10^{-7}$
Gold	$4.10 \cdot 10^{7}$	$7.86 \cdot 10^{-2}$	$3.10 \cdot 10^{-7}$
Aluminum	$3.54 \cdot 10^{7}$	$8.46 \cdot 10^{-2}$	$3.34 \cdot 10^{-7}$
Iron	$1.02 \cdot 10^{7}$	$1.58 \cdot 10^{-1}$	$6.22 \cdot 10^{-7}$

Surface Impedance

If $\underline{J}_x(z) = J_o e^{\frac{z}{\delta}} e^{j\frac{z}{\delta}}$, the total current per unit width is

$$\frac{1}{1+j} = \frac{1}{\sqrt{2}} e^{-j45^\circ}$$

$$\underline{I}_w = \int_{-\infty}^{0} \underline{J}_x \, dz = \int_{-\infty}^{0} J_o e^{\frac{(1+j)z}{\delta}} \, dz = \frac{J_o \delta}{1+j}$$

45° out of phase with J_o

Since $J_o = \delta_E E_o$ we can write

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_E}}$$

$$\underline{I}_w = \frac{\sigma_E \delta}{1+j} E_o = \frac{1}{Z_S} E_o$$

Voltage per unit length

Surface impedance

$$\boxed{Z_S = \frac{\underline{E}_o}{\underline{I}_w} = \frac{1+j}{\sigma_E \delta} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma_E}} = R_S + jX_S}$$

$$(R_S = X_S)$$

$[Z_S] = \Omega$ or

"ohms per square"

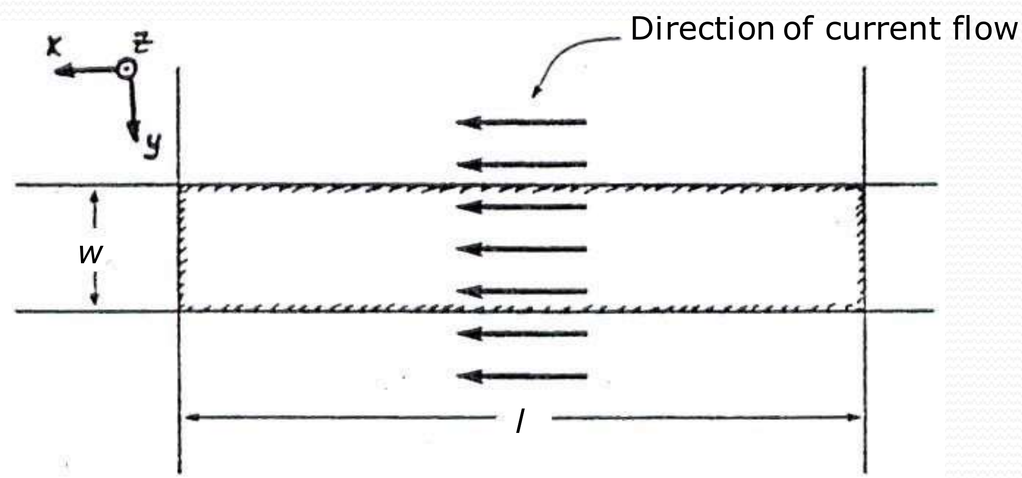
$$\underline{I} = \underline{I}_w w$$

$$\underline{V} = \underline{E}_o l$$

$$\frac{\underline{V}}{\underline{I}} = \frac{\underline{E}_o l}{\underline{I}_w w} = Z_S \left(\frac{l}{w}\right)$$

If $l = w$ (a square surface)

$$\frac{\underline{V}}{\underline{I}} = Z_S$$

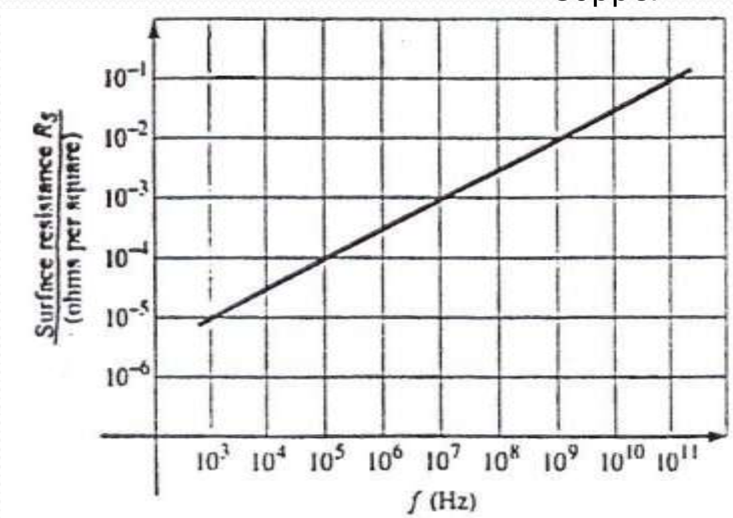


Illustrating the concept of "ohms per square"

$$\left(R_{dc} = \frac{l}{\sigma_E A} \right)$$

$$R_S = \frac{1}{\sigma_E \delta} = \sqrt{\frac{\pi f \mu}{\sigma_E}}$$

Copper



R_s is equivalent to the dc resistance per unit length of the conductor having cross-sectional area $1 \times \delta$

$$R_{ac} = \frac{l}{\underbrace{\sigma\delta \quad w}_{A}} = \frac{R_s l}{w}$$

For a wire of radius *a*,

$$w = 2\pi a$$

$$l = 1cm = 10^{4}\ \mu m$$

$$2r = d = 0.1mm = 100\ \mu m$$

$$f = 10\ GHz = 10^{10}\ Hz$$

$$\sigma_E = 5.91 \cdot 10^{7} (\Omega \cdot m)^{-1}$$

$$\mu = \mu_o = 4\pi \cdot 10^{-7}\ H/m$$

$$\delta = \frac{1}{\sqrt{\pi \mu \sigma_E f}} = 0.6\ \mu m << \frac{d}{2} = 50\ \mu m$$

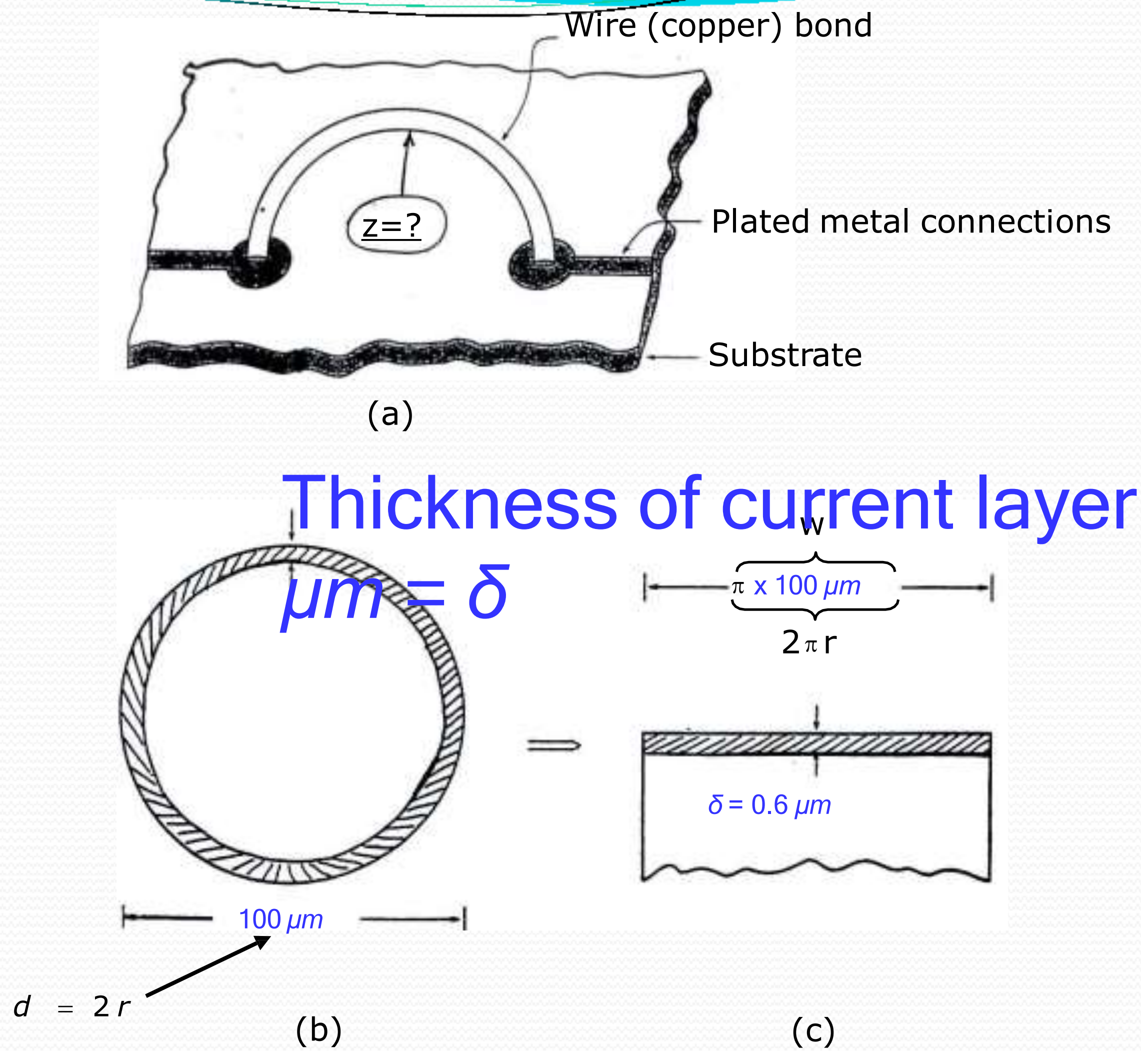
$$Z_s = \frac{(1 + j)}{\sigma_E \delta} = (1 + j) x\ 0.026\ \Omega$$

$$Z = Z_s \left(\frac{l}{w}\right) = Z_s \left(\frac{10^{4}\ \mu m}{\pi \cdot 100\ \mu m}\right)$$

$$= (0.86 + j\underbrace{0.86}_{X = \omega L_{int}})\Omega$$

$$Z = R + jX$$

$$L_{int} = X / \omega \quad \text{(Internal inductance)}$$



Finding the resistance and internal inductance of a wire bond. The bond, a length of wire connecting two pads on an IC, is shown in (a). (b) is a cross-sectional view showing skin depth. In (c) we imagine the conducting layer unfolded into a plane.

THANK YOU