UNIT – IV
FORCE IN MAGNETIC FIELD AND MAGNETIC POTENTIAL

4 Work and Energy in Electrostatics

1. The Work Done in Moving a Charge
2. The Energy of a Point Charge Distribution
3. The Energy of a Continuous Charge Distribution
4. Comments on Electrostatic Energy

4.1 The Work Done in Moving a charge

A test charge Q feels a force Q $\vec{E}$ (from the stationary source charges).

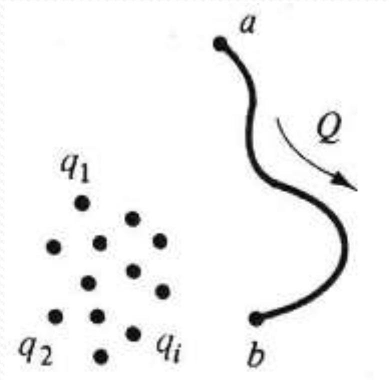
To move this test charge, we have to apply a force $\vec{F} = -Q\vec{E}$

conservative

The total work we do is

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q\left[V(b) - V(a)\right]$$

$$\boxed{V(b) - V(a) = \frac{W}{Q}}$$



So, bring a charge from ∞ to P, the work we do is

$$W = Q\left[V(P) - V(\infty)\right] = QV(P)$$

$$V(\infty) = 0$$

4.2 The Energy of a Point Charge Distribution

It takes no work to bring in first charges

$W_1 = 0$ for q_1

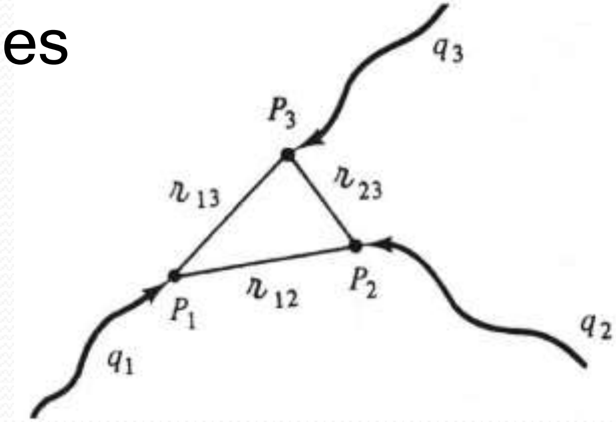


Figure 2.40

Work needed to bring in q_2 is :

$$W_2 = q_2 [\frac{1}{4\pi\varepsilon_0} \frac{q_1}{R_{12}}] = \frac{1}{4\pi\varepsilon_0} q_2 (\frac{q_1}{R_{12}})$$

Work needed to bring in q_3 is :

$$W_3 = q_3 [\frac{1}{4\pi\varepsilon_0} \frac{q_1}{R_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{R_{23}}] = \frac{1}{4\pi\varepsilon_0} q_3 (\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}})$$

Work needed to bring in q_4 is :

$$W_4 = \frac{1}{4\pi\varepsilon_0} q_4 [\frac{q_1}{R_{14}} + \frac{q_2}{R_{24}} + \frac{q_3}{R_{34}}]$$

4.2

Total work

$W=W_1+W_2+W_3+W_4$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} + \frac{q_1 q_4}{R_{14}} + \frac{q_2 q_4}{R_{24}} + \frac{q_3 q_4}{R_{34}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{R_{ij}}$$

$$\underline{\underline{R_{ij} = R_{ji}}} \qquad \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{R_{ij}} \qquad = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}\right)$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i)} \qquad V(P_i) = \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}$$

Dose not include the first charge

4.3 The Energy of a Continuous Charge Distribution

$$\delta q = \rho \, d\tau$$

Volume charge density

$$W = \frac{1}{2}\int \rho V d\tau$$

$$\rho = \varepsilon_0 \nabla \cdot \vec{E} \qquad \nabla \cdot (\vec{E} V) = (\nabla \cdot \vec{E})V + \vec{E} \cdot (\nabla V)$$

$$\frac{\varepsilon_0}{2}\int (\nabla \cdot \vec{E}) V d\tau = \frac{\varepsilon_0}{2}\left[\int \nabla \cdot (\vec{E} V) d\tau + \int \vec{E} \cdot (-\nabla V) d\tau\right]$$

∥ F.T.for $\nabla \cdot$ ∥

$$\oint_{surface} V\vec{E} \cdot d\vec{a} \qquad \vec{E}$$

$$= \frac{\varepsilon_0}{2}\left(\oint_{surface} V\vec{E} \cdot d\vec{a} + \oint_{volume} E^2 d\tau\right)$$

$$surface \to \infty \quad \Rightarrow \quad \boxed{W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau}$$

Example 2.8 Find the energy of a uniformly charged spherical shell of total charge q and radius R

Sol.1: $q = 4\pi R^2\sigma$ $\qquad V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{R}$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int \sigma V da = \frac{1}{2}\int \frac{q}{A}\frac{1}{4\pi\varepsilon_0}\frac{q}{R} da = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

Sol.2: $\overleftarrow{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r}$ $\qquad E^2 = \dfrac{q^2}{(4\pi\varepsilon_0)^2 r^4}$

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int_{space}^{outer} \frac{q^2}{(4\pi\varepsilon_o)^2}\frac{1}{r^4}\left(r^2 \sin\theta \, d\theta \, d\varphi \, dr\right)$$

$$= \frac{q^2}{32\pi^2\varepsilon_0}\left[2 \cdot 2\pi \int_R^\infty \frac{1}{r^2} dr\right] = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

4.4 Comments on Electrostatic Energy

(1) A perplexing inconsistent

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau \quad \Rightarrow \quad \text{Energy} > 0$$

or

$$\left.\begin{aligned} W &= \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i) \\ W &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_i q_j}{r_{ij}} \end{aligned}\right\} \Rightarrow \text{Energy} > 0 \quad \text{or} \quad < 0$$

Only for point charge

$$q_1 = q\ , \quad q_2 = -q \Rightarrow W = \frac{1}{4\pi\varepsilon_0}\frac{\left(-q^2\right)}{r_{12}} < 0$$

(∵ q_1 and q_2) are attractive

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$$ is more complete

the energy of a point charge itself,

$$W = \frac{\varepsilon_0}{2(4\pi\varepsilon_0)^2}\int \left(\frac{q^2}{r^4}\right)\left(r^2 \sin\theta d\theta d\varphi dr\right) = \frac{q^2}{8\pi\varepsilon_0}\int_0^\infty \frac{1}{r^2} dr = \infty$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i)$$, $V(P_i)$ does not include q_i

$$W = \frac{1}{2}\int \rho V d\tau$$, $V(P)$ is the full potential

There is no distinction for a continuous distribution, because

$$\rho(d\tau)_p \xrightarrow{d\tau \to 0} 0$$

(2)Where is the energy stored? In charge or in field ?

Both are fine in ES. But,it is useful to regard the energy as being stored in the field at a density

$$\varepsilon_0 \frac{E^2}{2} = \text{Energy per unit volume}$$

(3)The superposition principle,not for ES energy

$$W_1 = \frac{\varepsilon_0}{2}\int E_1^2 d\tau \qquad W_2 = \frac{\varepsilon_0}{2}\int E_2^2 d\tau$$

$$W_{tot} = \frac{\varepsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2) d\tau$$

$$= W_1 + W_2 + \varepsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) d\tau$$

4.5 Magnetic Fields in analogy with Electric Fields

Electric Field:

- Distribution of charge creates an electric field ***E*(*r*)** in the surrounding space.
- Field exerts a force ***F***=*q* ***E*(*r*)** on a charge *q* at ***r***

Magnetic Field:

- Moving charge or current creates a magnetic field ***B*(*r*)** in the surrounding space.
- Field exerts a force ***F*** on a charge moving *q* at ***r***
- (emphasis this chapter is on force law)

4.6 Magnetic Fields and Magnetic Forces

Magnetic Force on a moving charge

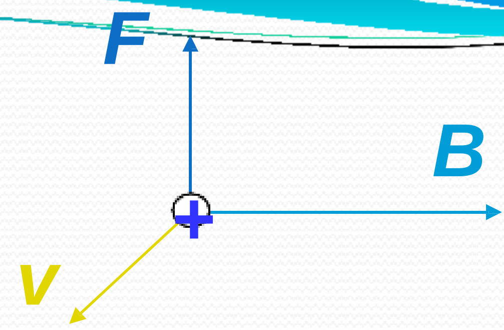
- proportional to electric charge
- perpendicular to velocity ***v***
- proportional to speed *v* (for a given geometry)
- perpendicular to Magnetic Field ***B***
- proportional to field strength *B* (for a given geometry)

$$\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$$

$\boldsymbol{F} = q\,\boldsymbol{v}$ ´ $\boldsymbol{B}$

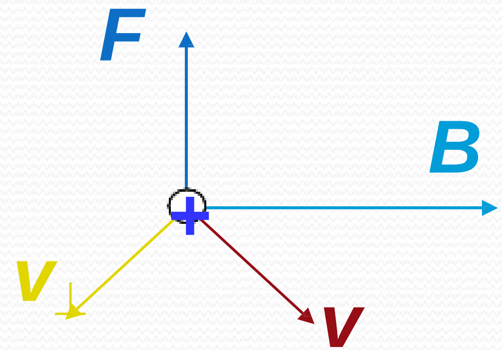
$F = |q|\, v\, B \sin\theta$

$\quad = |q|\, v\, B \qquad (\boldsymbol{v} \perp B)$



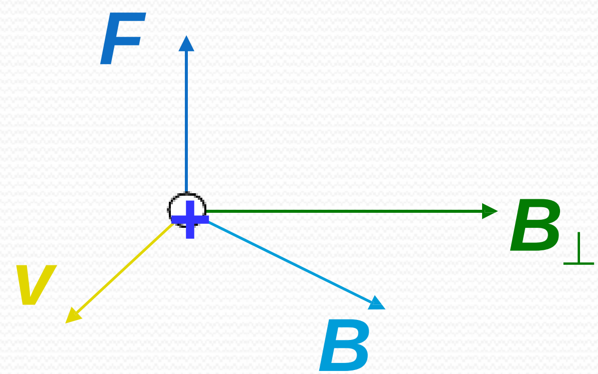
$\boldsymbol{F} = q\,\boldsymbol{v}$ ´ $\boldsymbol{B}$

$F = |q|\, v_{\perp}\, B$



$\boldsymbol{F} = q\,\boldsymbol{v}$ ´ $\boldsymbol{B}$

$F = |q|\, v\, B_{\perp}$



4.7 Magnetic Fields

Units of Magnetic Field Strength:

[B] = [F]/([q][v])

= N/(C m s^{-1})

= Tesla

Defined in terms of force on standard current

CGS Unit 1 Gauss = 10^{-4} Tesla

Earth's field strength ~ 1 Gauss

Direction = direction of velocity which generates <u>no</u> force

Electromagnetic Force:

$$\boldsymbol{F} = q\,(\,\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\,)$$

= *Lorentz Force Law*

4.8 Magnetic Field Lines and Magnetic Flux

Magnetic Field Lines

- Mapped out with compass
- Are not lines of force (***F*** is not parallel to ***B***)
- Field Lines never intersect

Magnetic Flux

$dF_B = \boldsymbol{B} \cdot d\boldsymbol{A}$

$$d\Phi_B = \vec{B} \cdot d\vec{A}$$

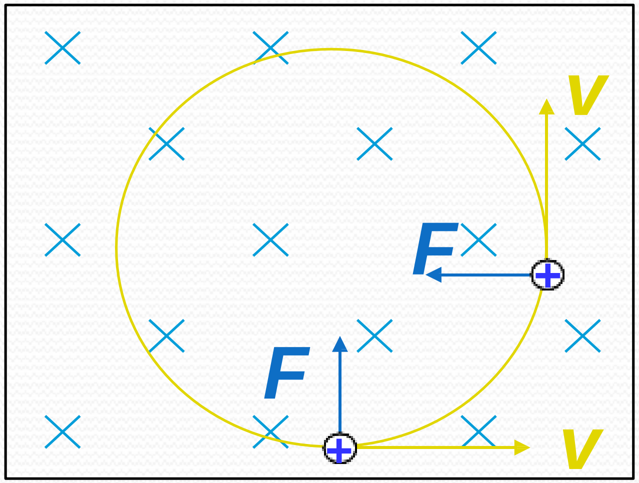
$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \quad \text{no magnetic charge! (no monopoles)}$$

4.9 Motion of Charged Particles in a Magnetic Field

Charged Particle moving perpendicular to the Magnetic Field

- Circular Motion!
- (simulations)



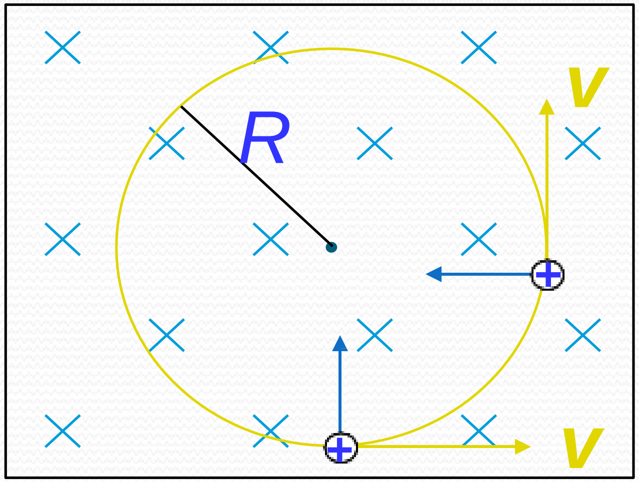
Charged Particle moving perpendicular to a uniform Magnetic Field

$$F = |q|vB = \frac{mv^2}{R}$$

$$R = \frac{mv}{|q|B}$$

$$\omega = \frac{v}{R} = \frac{|q|B}{m}$$

$$= \text{cyclotron frequency}$$



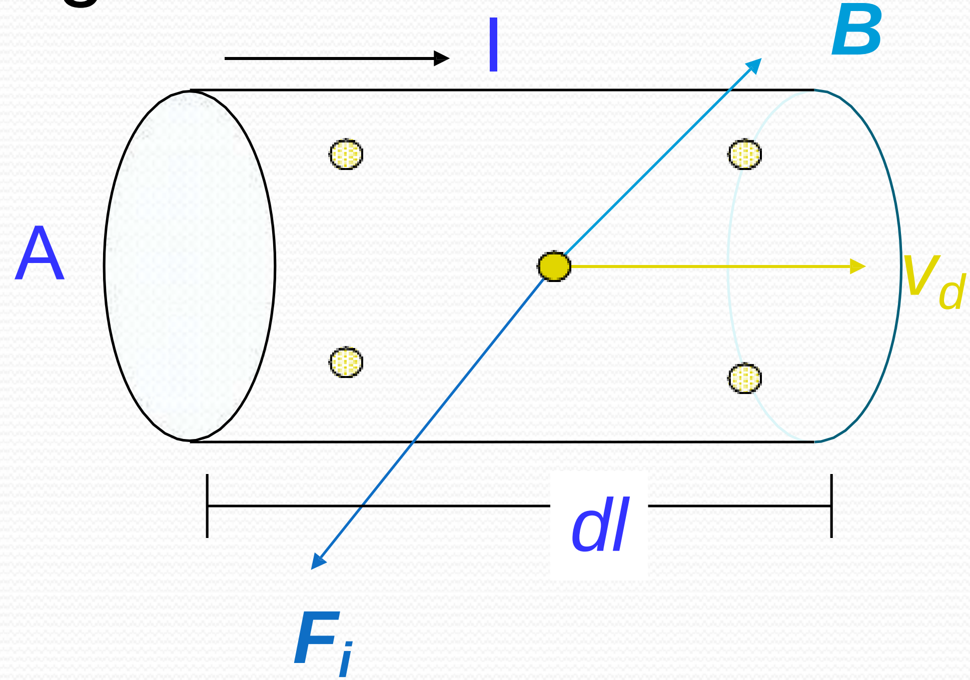
Work done by the Magnetic Field on a free particle:

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{x} \\ &= \left(q\vec{v} \times \vec{B}\right) \cdot \vec{v}\,dt \\ &= 0! \end{aligned}$$

=> no change in Kinetic Energy!

Motion of a free charged particle in **<u>any</u>** magnetic field has constant speed.

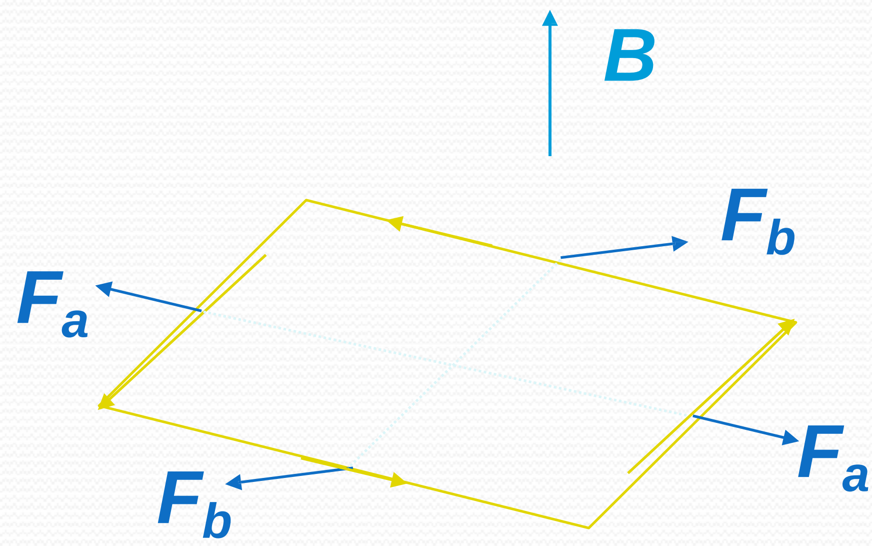
4.11 Magnetic Force on a Current Carrying Wire



$$
\begin{aligned}
F &= \sum F_i = \sum q_i \vec{v}_i \times \vec{B} \\
&= Nq\,\vec{v}_d \times \vec{B} = n \cdot volume \cdot q\vec{v}_d \times \vec{B} \\
&= nAdlq\,\vec{v}_d \times \vec{B} = \vec{J}Adl \times \vec{B} \\
&= Id\vec{l} \times \vec{B} \quad (RHR)
\end{aligned}
$$

4.12 Torque on a Current Loop (from **F** = *I* ***l*** x **B**)

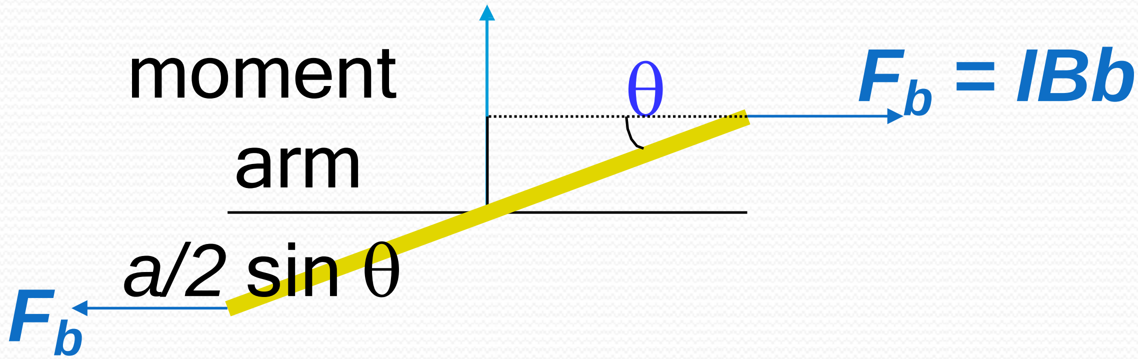
Rectangular loop in a magnetic field (directed along z axis) short side length a, long side length b, tilted with short sides at an angle with respect to ***B***, long sides still perpendicular to ***B***.



Forces on short sides cancel: no net force or torque.

Forces on long sides cancel for no net force but there is a net torque.

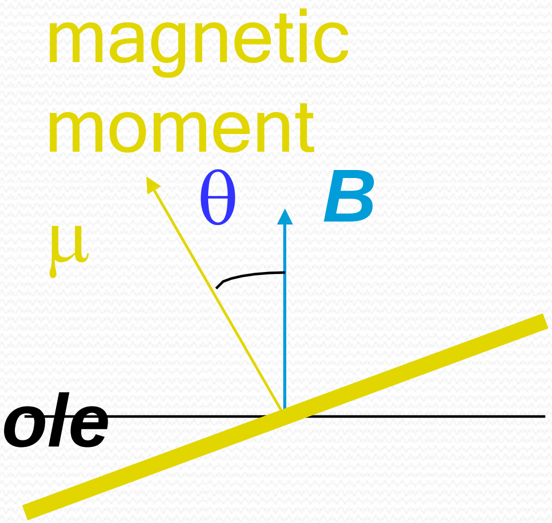
Torque calculation: Side view



$$\tau = F_b\, a/2 \sin\theta + F_b\, a/2 \sin\theta$$

$$= Iab\, B \sin\theta \quad = I\, A\, B \sin\theta$$

$$= \mathrm{I}\, \mathrm{A} \times \mathrm{B} = \mu \times \mathrm{B}$$

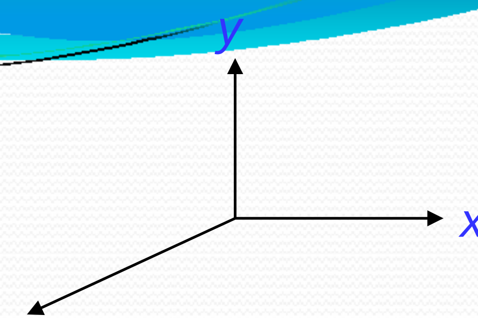


Magnetic Dipole ~ Electric Dipole

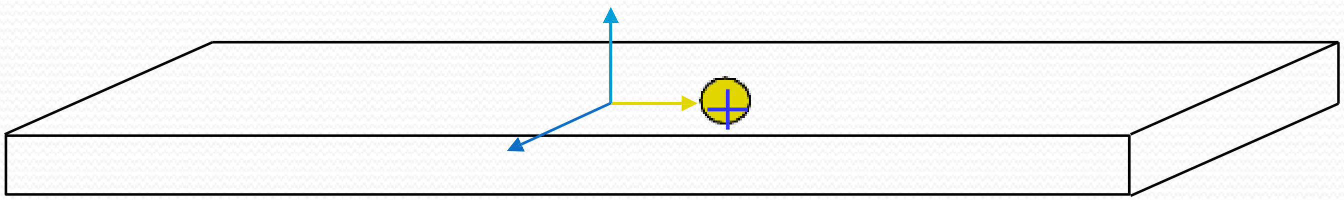
$\boldsymbol{U} = -\mu \cdot \mathrm{B}$

Switch current *direction* every 1/2 rotation => DC motor

4.13 Hall Effect

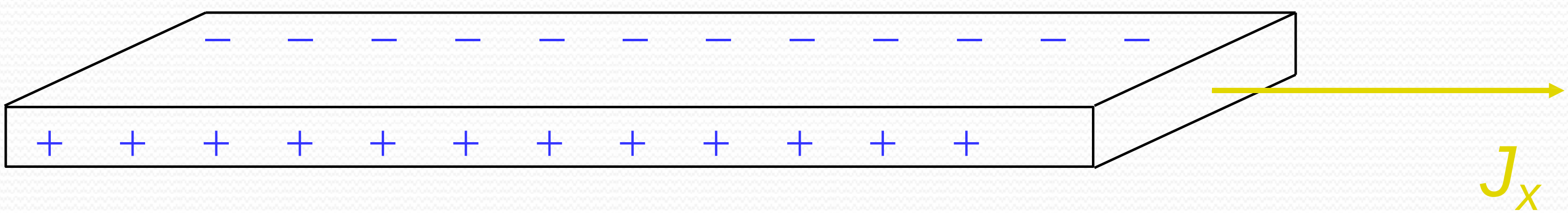


Conductor in a uniform magnetic field

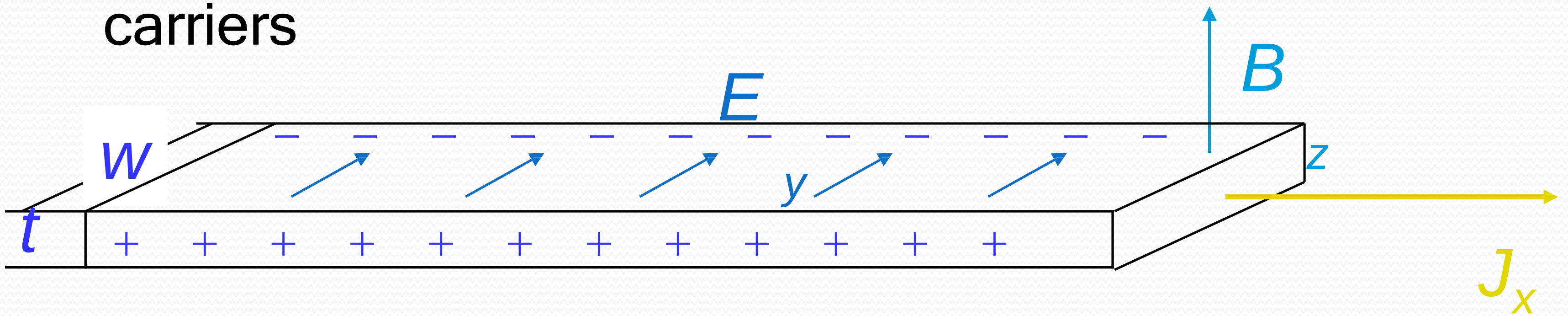


Magnetic force on charge carriers $\boldsymbol{F} = q\,\boldsymbol{v_d} \times \boldsymbol{B}$

$F_z = qv_dB$ Charge accumulates on edges



Equilibrium: Magnetic Force = Electric Force on bulk charge carriers



Charge accumulates on edges $F_z = 0 = qv_dB_y + q\,E_z$

$$v_d = -\frac{E_z}{B_y}$$

$$J_x = nqv_d = -nq\frac{E_z}{B_y}$$

$$nq = \frac{-J_x B_y}{E_z}$$

Hall EMF $\quad V_H = E_z w$

$$I = J_x tw$$

$$nq = \frac{-IB_y}{V_H t}$$