UNIT – III
MAGNETOSTATICS

3 Electric potential

1. Introduction to Potential
2. Comments on Potential
3. Poisson's Equation and Laplace's Equation
4. The Potential of a Localized Charge Distribution
5. Electrostatic Boundary Conditions

3.1 introduction to potential

Any vector whose curl is zero is equal to the gradient of some scalar. We define a function:

$$V(p) = -\int_{\vartheta}^{p} \vec{E} \cdot d\vec{l}$$

Where is some standard reference point ; *V* depends only on the point *P*. *V* is called the *electric potential*.

$$V(b) - V(a) = -\int_{\vartheta}^{b} \vec{E} \cdot d\vec{l} - \left(-\int_{\vartheta}^{a} \vec{E} \cdot d\vec{l} \right) = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

The fundamental theorem for gradients

$$V(b) - V(a) = \int_{a}^{b} (\nabla V) \cdot d\vec{l}$$

so $\int_{a}^{b} (\nabla V) \cdot d\vec{l} = -\int_{a}^{b} \vec{E} \cdot d\vec{l} \Longrightarrow \boxed{\vec{E} = -\nabla V}$

3.2 Comments on potential

(1)The name Potential is not potential energy

$$\overleftharpoon{F} = q\overleftharpoon{E} = -q\nabla V \qquad \Delta U = \overleftharpoon{F} \cdot \overleftharpoon{X}$$

V: Joule/coulomb $\qquad U$: Joule

(2) Advantage of the potential formulation V is a scalar function, but E is a vector quantity

$$V(r) \qquad\qquad \vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

If you know *V*, you can easily get *E*: $\vec{E} = -\nabla V$.

E_x, E_y, E_z are not independent functions

$$\nabla \times \vec{E} = 0 \quad \text{so} \quad \frac{\partial E_x}{\partial y} \overset{①}{=} \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} \overset{②}{=} \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} \overset{③}{=} \frac{\partial E_z}{\partial x}$$

$$\partial_y \partial_z E_x = \partial_z \partial_y E_x \overset{①}{=} \partial_z \partial_x E_y = \partial_x \partial_z E_y \overset{②}{=} \partial_x \partial_y E_z = \partial_y \partial_x E_z \Rightarrow \partial_z E_x \overset{③}{=} \partial_x E_z$$

(3)The reference point ϑ

Changing the reference point amounts to adds a constant to the potential

$$V'(p) = -\int_{\vartheta'}^{p} \vec{E} \cdot d\vec{l} = -\int_{\vartheta'}^{\vartheta} \vec{E} \cdot d\vec{l} - \int_{\vartheta}^{p} \vec{E} \cdot d\vec{l} = K + V(p)$$

(Where K is a constant)

Adding a constant to *V* will not affect the potential difference between two point:

$$V'(b) - V'(a) = V(b) - V(a)$$

Since the derivative of a constant is zero: $\nabla V = \nabla V'$

For the different *V*, the field *E* remains the same.

Ordinarily we set $V(\infty) = 0$

(4) Potential obeys the superposition principle

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots \quad , \quad \vec{F} = Q\vec{E}$$

Dividing through by Q

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

Integrating from the common reference point to p ,

$$V = V_1 + V_2 + \cdots$$

(5) *Unit of potential*

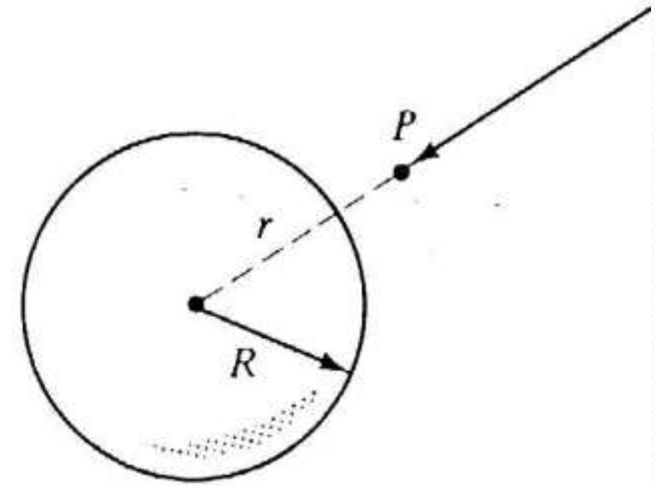
Volt=Joule/Coulomb

$$F : newton \qquad F \cdot x : Joule$$

$$F = qE = q\nabla V \rightarrow \frac{qV}{X}$$

$$V : \frac{F \cdot X}{q} \rightarrow$$ Joule/Coulomb

Example 2.6 Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge (the total charge is q).



solution:

$$\vec{E}_{in} = 0 \qquad \vec{E}_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

for r>R:

$$V(\vec{r}) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

for r≤R:

$$V(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad V(\vec{r}) \neq V(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

$$\uparrow$$

$$\vec{E}_{in} = 0$$

3.3 Poisson's Eq. & Laplace's Eq.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \nabla \cdot (-\nabla V) = -\nabla^2 V$$

$$\uparrow$$

$$\vec{E} = -\nabla V$$

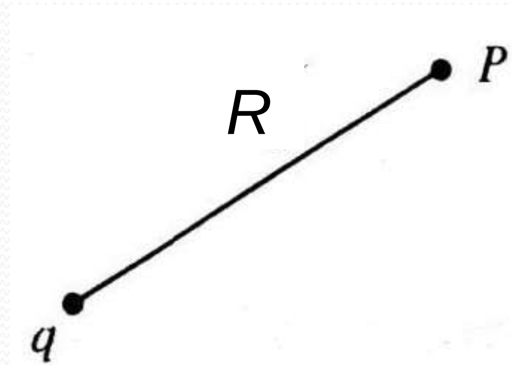
Poisson's Eq. $\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$

$\rho = 0$ $\boxed{\nabla^2 V = 0}$ Laplace's eq.

3.4 The Potential of a Localized Charge Distribution

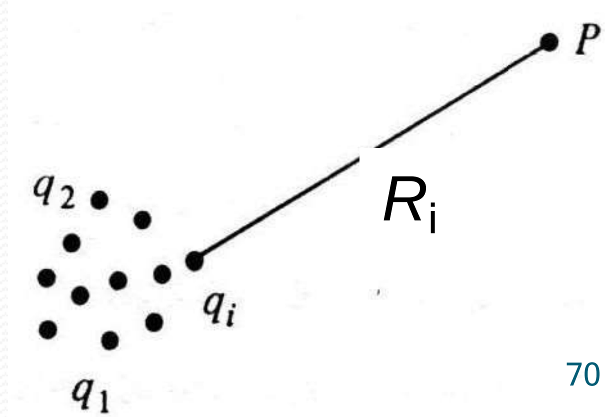
$$\vec{E} = -\nabla V \qquad V - V_\infty = -\int_\infty^r E\,dr' \qquad V_\infty = 0$$

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\bigg|_\infty^r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$



- Potential for a point charge

$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad R = \left|\vec{r} - \vec{r}_p\right|$$



- Potential for a collection of charge

$$V(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i} \qquad R_i = \left|\vec{r}_i - \vec{r}_p\right|$$

- Potential of a continuous distribution

for volume charge | for a line charge | for a surface charge

$$\delta q = \rho \, d\tau \qquad \delta q = \lambda \, dl \qquad \delta q = \sigma \, da$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{R} d\tau \qquad V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda}{R} dl \qquad V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{R} da$$

- **Corresponding electric field** $\left[\; -\nabla \frac{1}{R} = \frac{\hat{R}}{R^2} \; \right]$

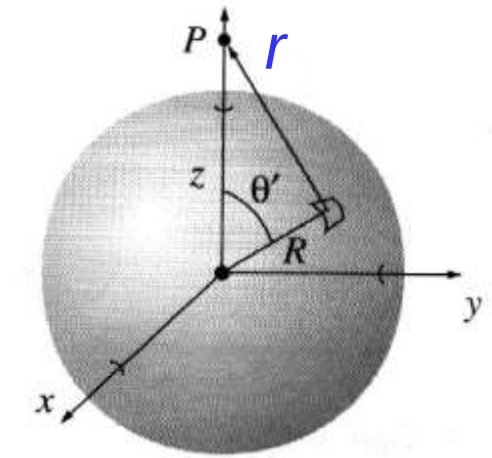
$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2}\rho \, d\tau \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2}\lambda \, dl \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2}\sigma \, da$$

3.4

Example 2.7 Find the potential of a uniformly charged spherical shell of radius *R*.

Solution:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{r}\, da', \qquad r^2 = R^2 + z^2 - 2Rz\cos\theta'$$



$$4\pi V(z) = \sigma \int \frac{R^2 \sin\theta' d\theta' d\varphi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 2\pi R^2 \sigma \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\, d\theta'$$

$$= 2\pi R^2 \sigma \left(\frac{1}{Rz}\sqrt{R^2 + z^2 - 2Rz\cos\theta'} \right)\Bigg|_0^{\pi}$$

$$= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz}\right)$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

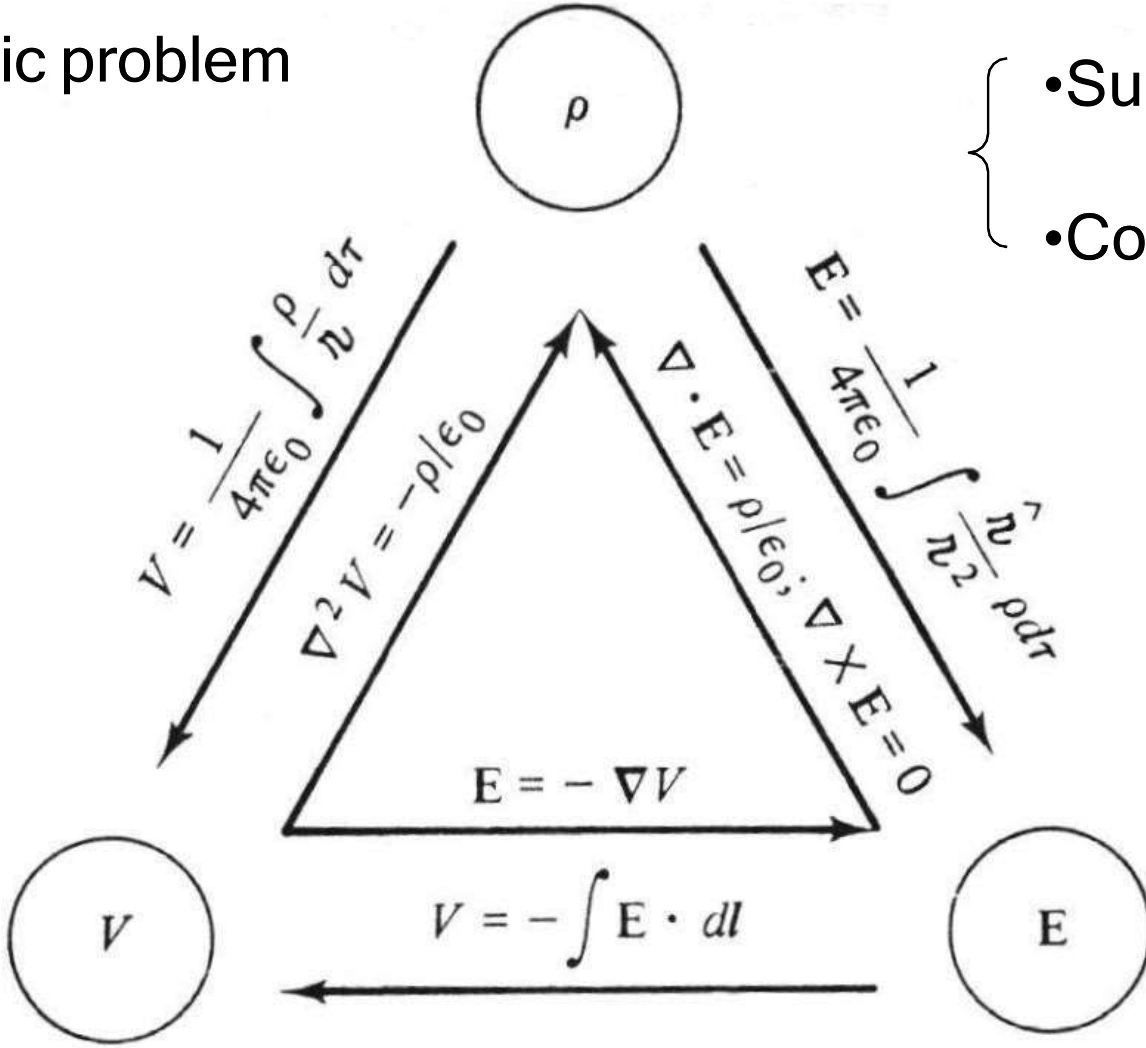
3.4

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z)-(z-R)] = \frac{R^2\sigma}{\varepsilon_0 z}, \quad outside$$

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z)-(R-z)] = \frac{R\sigma}{\varepsilon_0}, \quad inside$$

3.5 Electrostatic Boundary Condition

Electrostatic problem

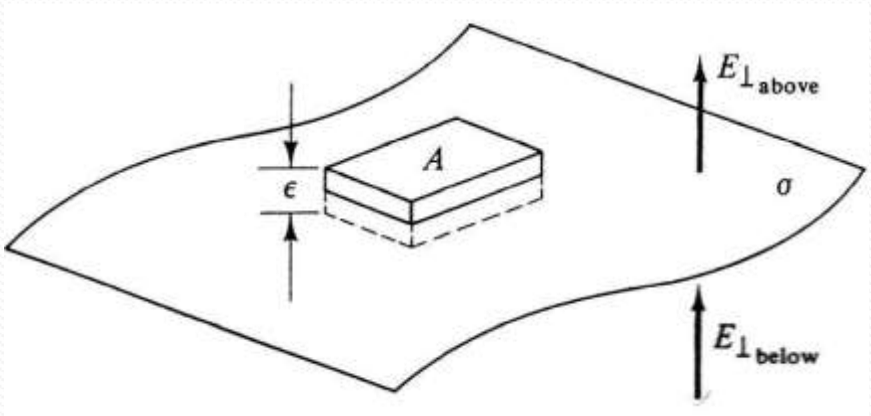


- •Superposition
- •Coulomb law

The above equations are differential or integral.
For a unique solution, we need boundary conditions. (e.q. , $V(\infty)=0$)
(boundary value problem. Dynamics: *initial value problem*.)

B.C. at surface with charge

$\perp$:



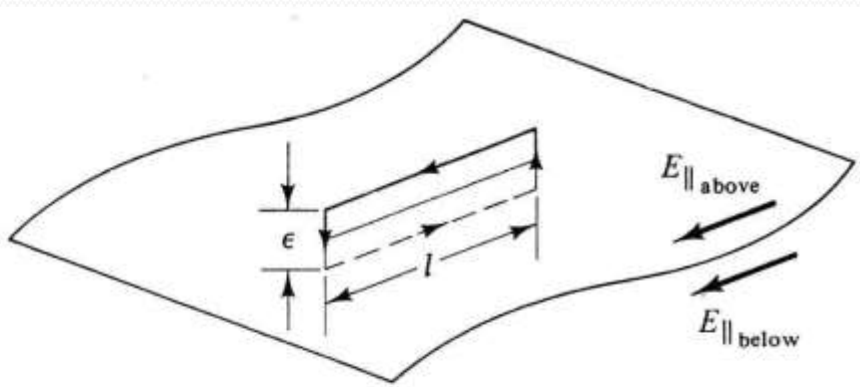
$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\parallel$$

$$E_{\perp above} \cdot A - E_{\perp below} \cdot A$$

$$E_{\perp above} - E_{\perp below} = \frac{\sigma}{\varepsilon_0}$$

$\parallel$:



$$\oint \vec{E} \cdot d\vec{l} = 0 \qquad (\because \nabla \times \vec{E} = 0)$$

$$E_{\parallel above} = E_{\parallel below}$$

$\perp + \parallel$

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

Potential B.C.

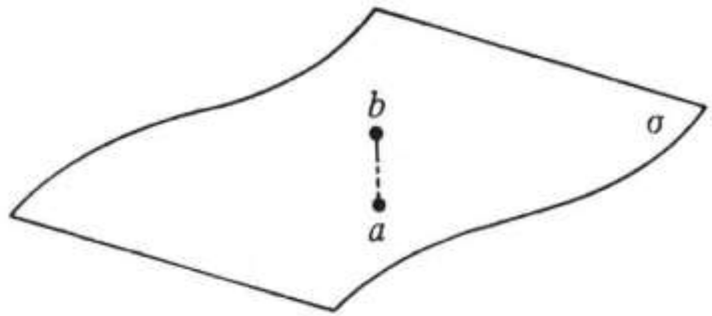


Figure 2.38

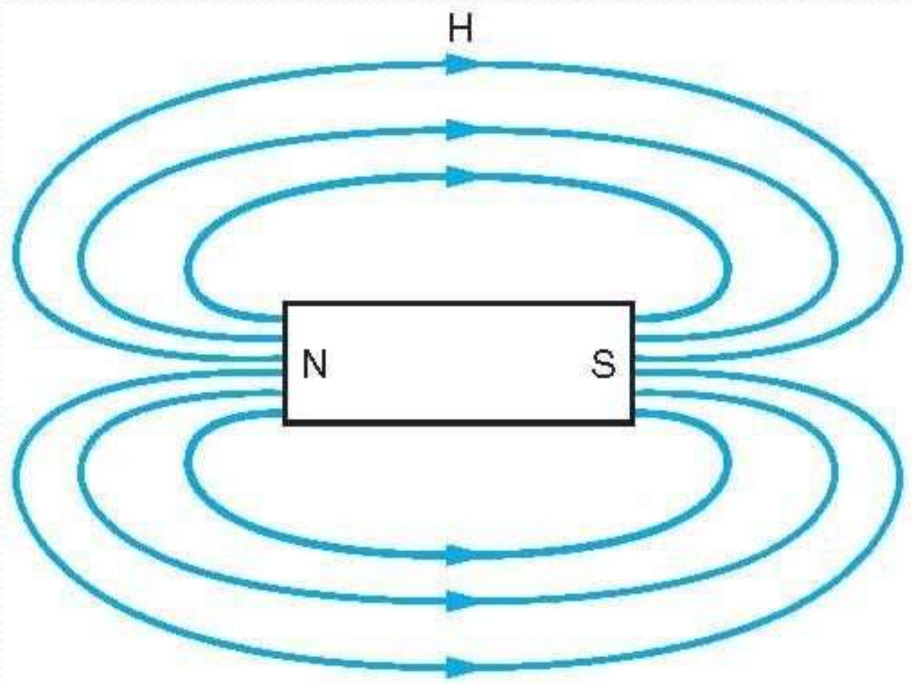
$$\because V_{above} - V_{below} = -\int_a^b \vec{E}\cdot d\vec{l} \xrightarrow{\overline{ab}\to 0} 0$$

$$\therefore \quad \boxed{V_{above} = V_{below}}$$

$$\because \vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \vec{E} = -\nabla V \qquad \frac{\partial}{\partial n} V = \nabla V \cdot \hat{n}$$

$$\therefore \quad \boxed{\frac{\partial}{\partial n} V_{above} - \frac{\partial}{\partial n} V_{below} = -\frac{\sigma}{\varepsilon_0}}$$

3.6 Magnetism and electricity have not been considered distinct phenomena until Hans Christian Oersted conducted an experiment that showed a compass deflecting in proximity to a current carrying wire

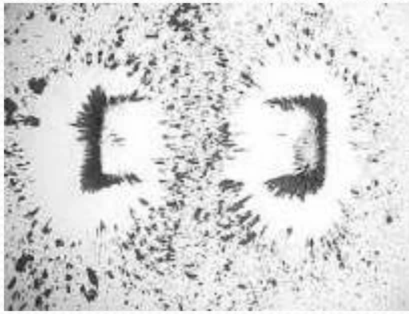


Produced by

-time varying electric fields

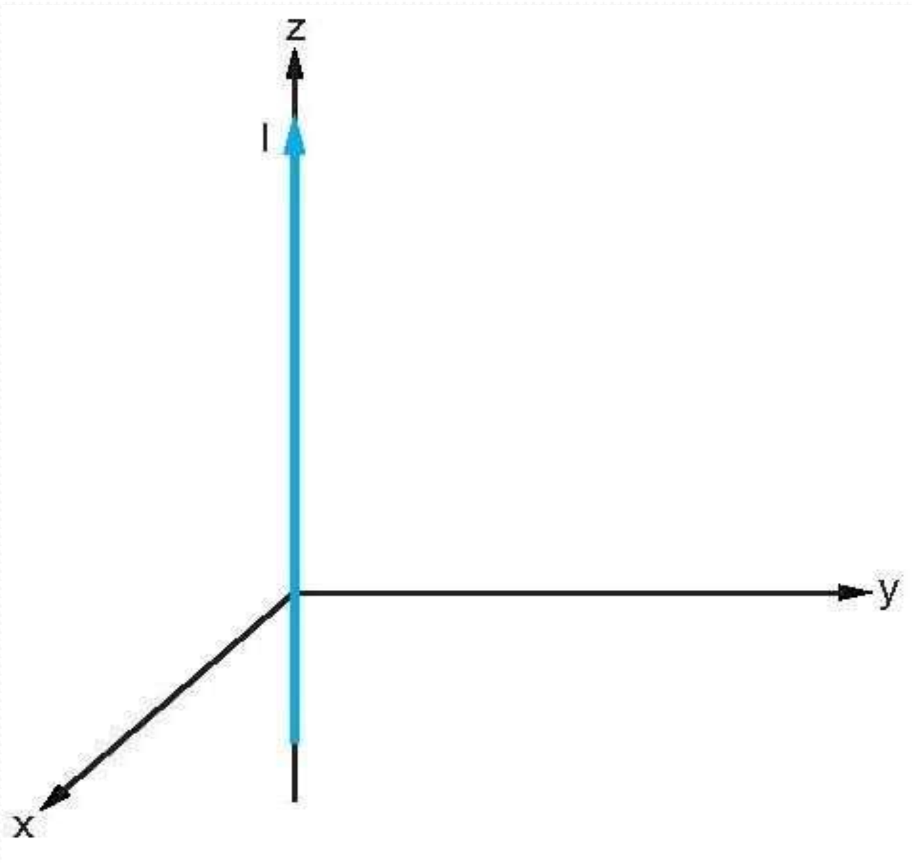
-permanent magnet (arises from quantum mechanical electron spin/ can be considered charge in motion=current)

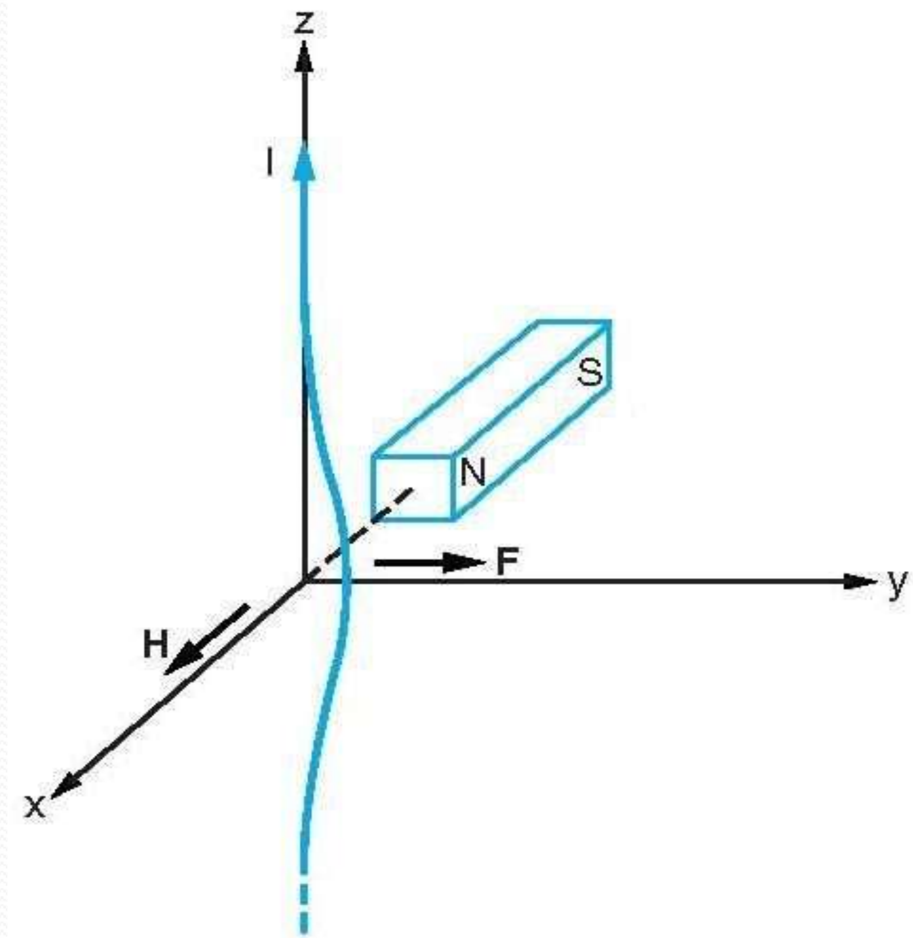
-steady electric currents



$$[H]_{SI} = \frac{A}{m}$$

If we place a wire with current I in the presence of a magnetic field, the charges in the conductor experience another force F_m





$F_m \sim q, u, B$ $\Rightarrow$ $\overline{H} = \dfrac{1}{\mu}\overline{B}$

q -charge
u -velocity vector
B -strength of the field (magnetic flux density)
μ_r -relative permeability
μ -absolute permeability
μ_0-permeability of the free space
$M=\chi_m H$- is the magnetization for linear and homogeneous medium

3.7 Relative permeabilities for a variety of materials

Material	$\mu/(H\ m^{-1})$	μ_r	Application
Ferrite U 60	1.00E-05	8	UHF chokes
Ferrite M33	9.42E-04	750	Resonant circuit RM cores
Nickel (99% pure)	7.54E-04	600	-
Ferrite N41	3.77E-03	3000	Power circuits
Iron (99.8% pure)	6.28E-03	5000	-
Ferrite T38	1.26E-02	10000	Broadband transformers
Silicon GO steel	5.03E-02	40000	Dynamos, mains transformers
supermalloy	1.26	1000000	Recording heads

3.8 Lorentz's Force equation

$$\overline{F}_e = q\,\overline{E}$$

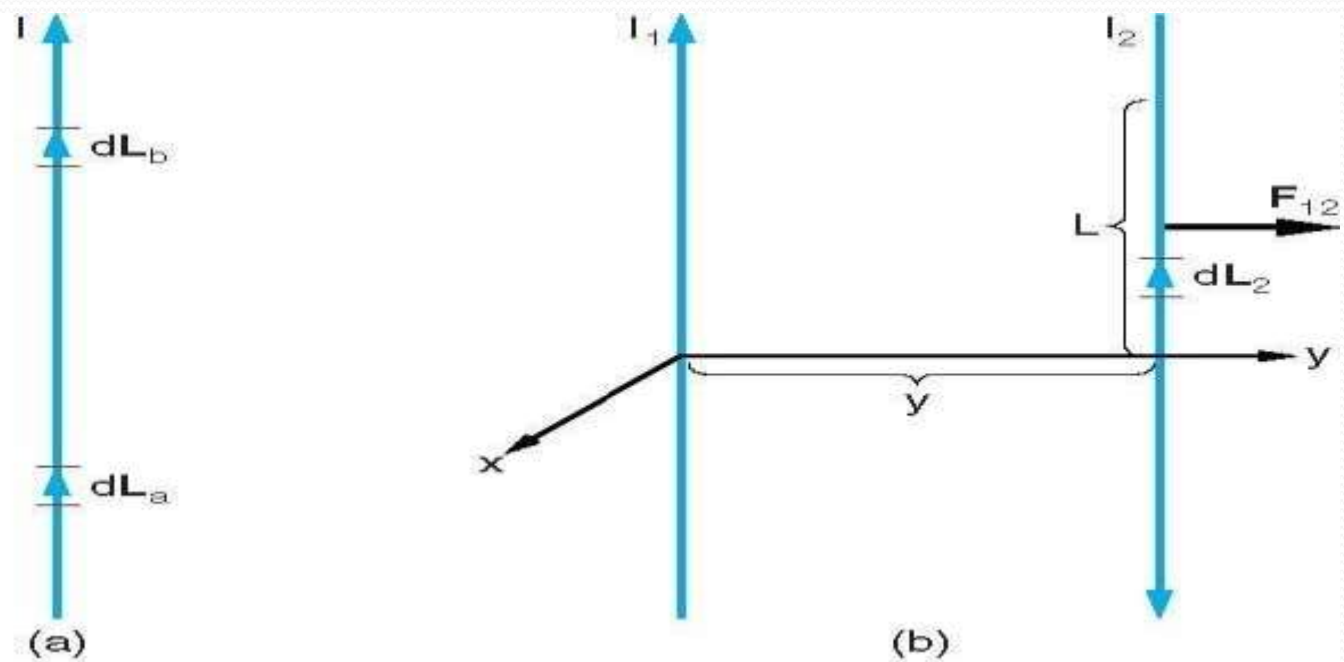
$$\overline{F}_m = q\,\overline{u} \times \overline{B}$$

$$\overline{F}_e + \overline{F}_m = F$$

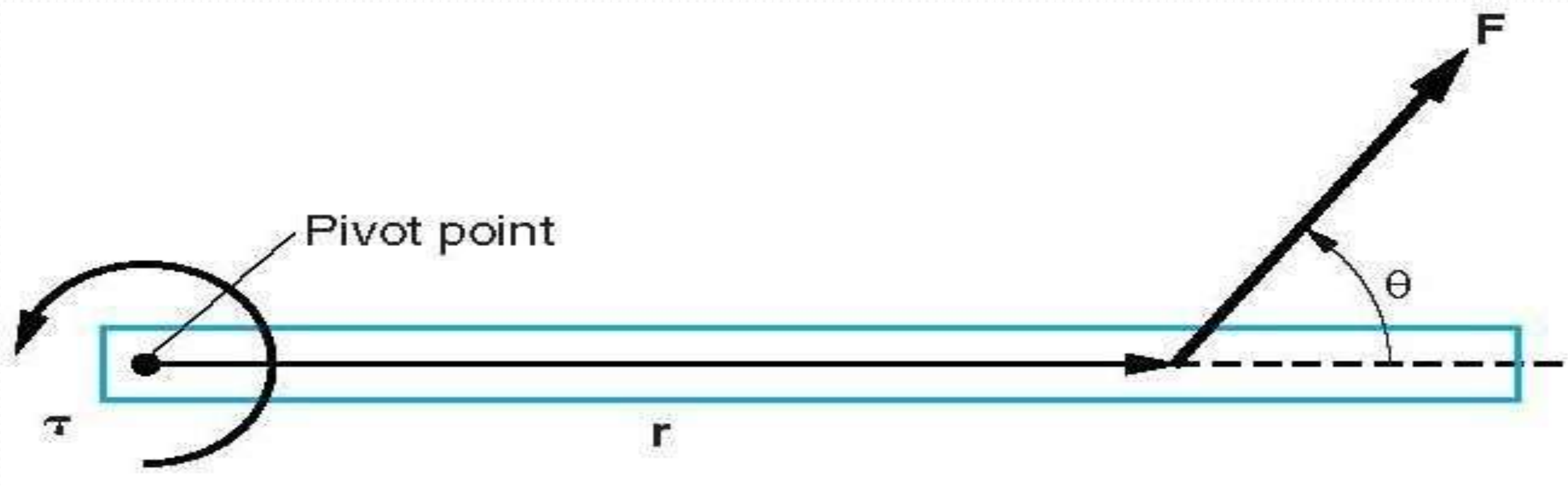
$$\overline{F} = q(\overline{E} + \overline{u} \times \overline{B})$$

Note: Magnetic force is zero for q moving in the direction of the magnetic field (sin0=0)

When electric current is passed through a magnetic field a force is exerted on the wire normal to both the magnetic field and the current direction. This force is actually acting on the individual charges moving in the conductor.

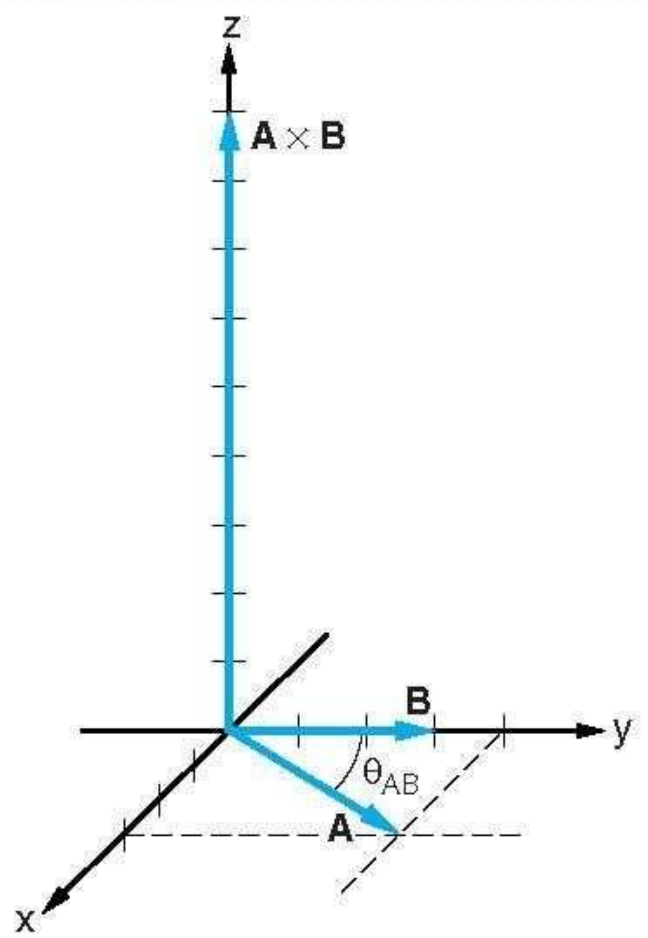


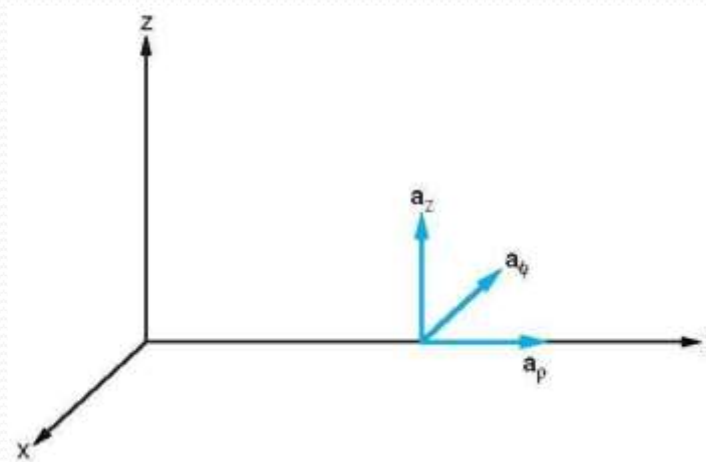
The magnetic force is exerting a torque on the current carrying coil

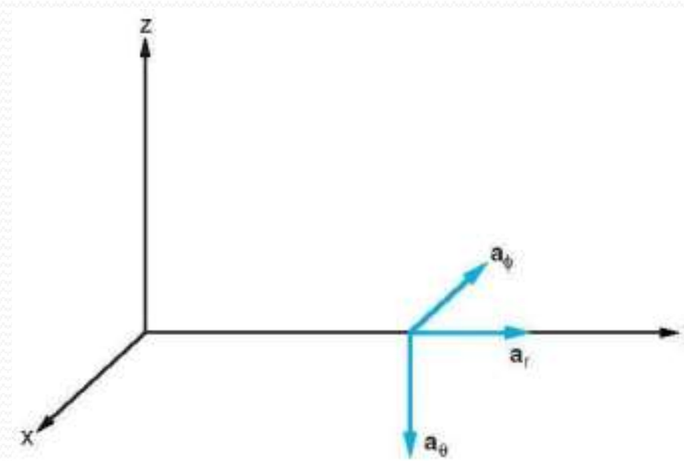


$$\tau = \overline{r} \times \overline{F_m} = |r| \times |F_m| \sin\theta \, \overline{a_n}$$

3.9 Cross product





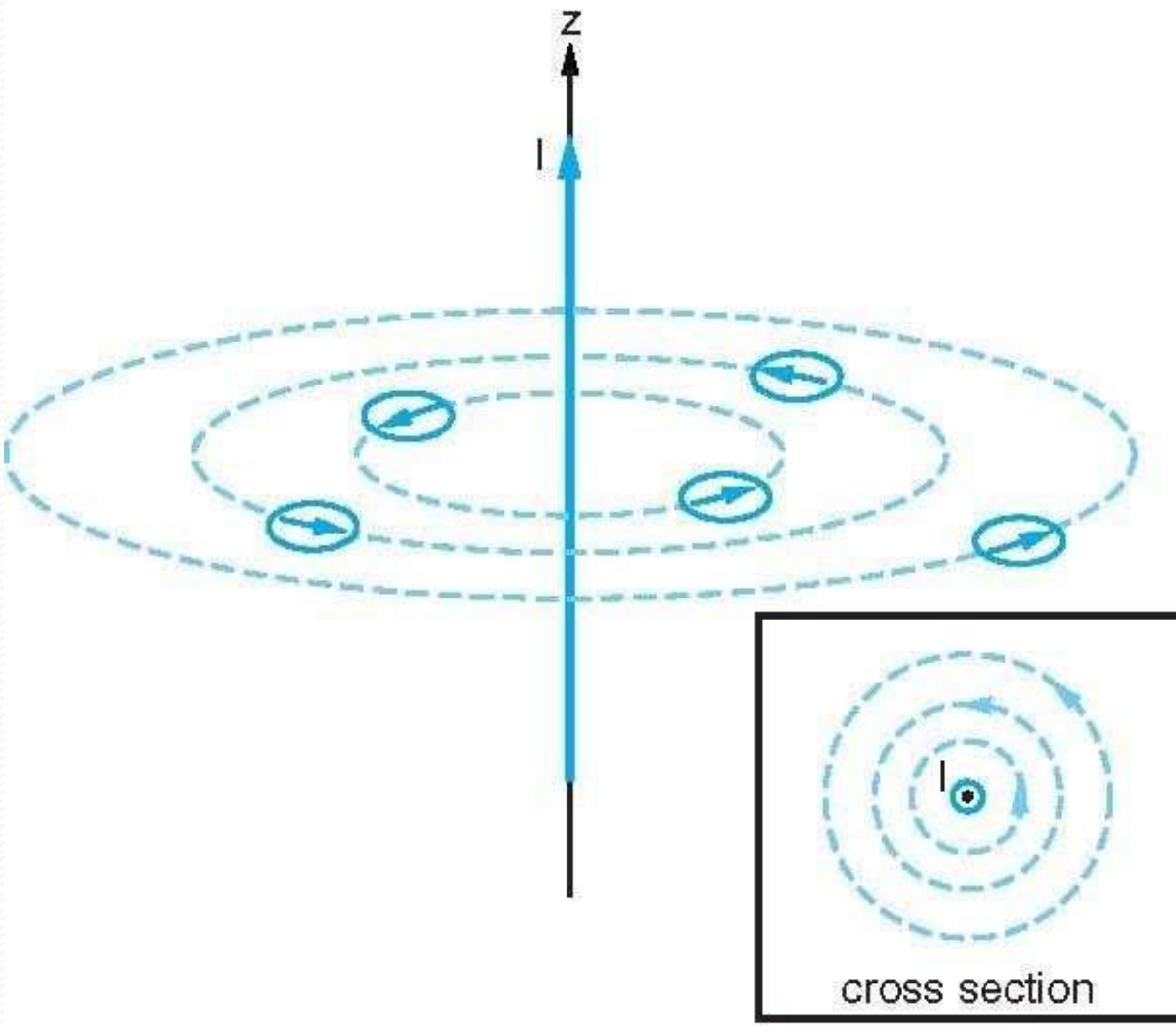


Fundamental Postulates of Magnetostatics in Free Space

$$\nabla \bullet \overline{B} = 0$$

$$\nabla \times \overline{B} = \mu_0 \overline{J}$$

$$\nabla \bullet \overline{J} = 0$$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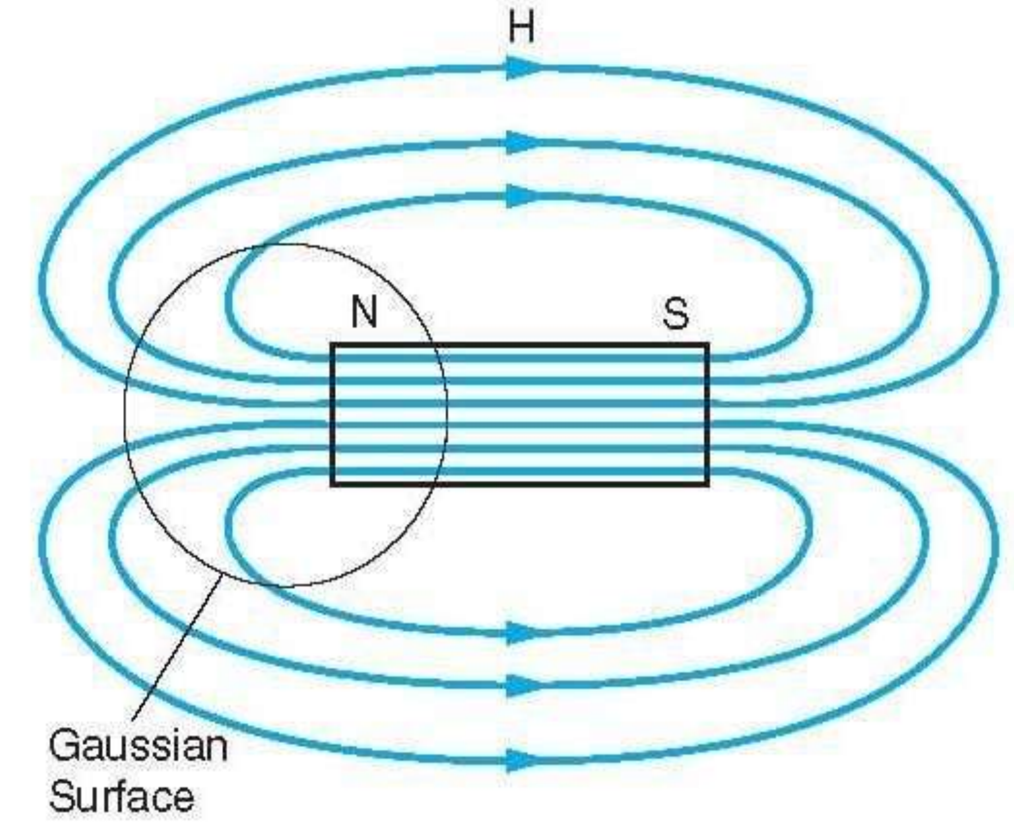
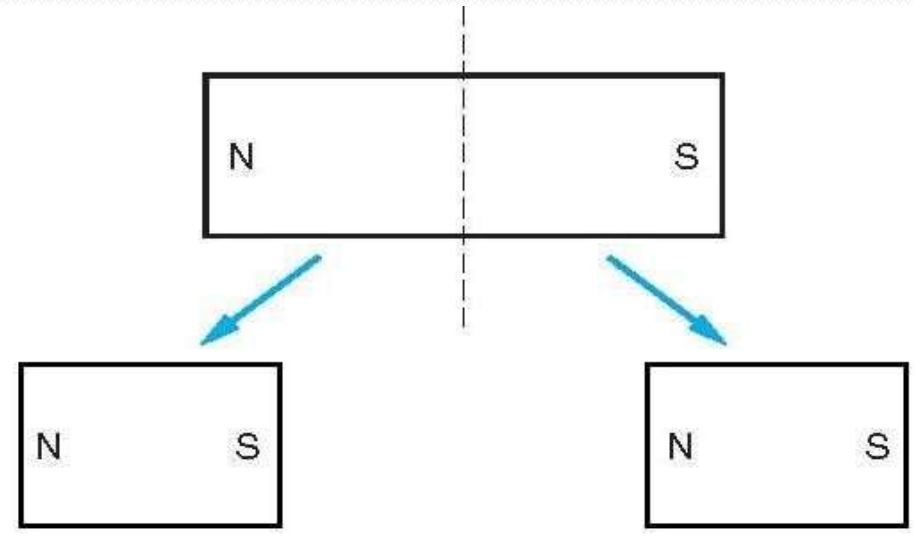
cross section

3.10 Law of conservation of magnetic flux



$$\oint_s \overline{B} \bullet d\overline{s} = 0$$

There are no magnetic flow sources, and the magnetic flux lines always close upon themselves



3.11 Ampere's circuital law

$$\int_s \left(\nabla \times \overline{B}\right) \bullet d\,\overline{s} = \mu_0 \int_s \overline{J} \bullet d\,\overline{s}$$

$$\oint_c \overline{B} \bullet d\,\overline{l} = \mu_0 I$$

The circulation of the magnetic flux density in free space around any closed path is equal to μ_0 times the total current flowing through the surface bounded by the path

Postulates of Magnetostatics in Free Space	
Differential Form	Integral Form
$\nabla \bullet \overline{B} = 0$	$\oint_s \overline{B} \bullet d\,\overline{s} = 0$
$\nabla \times \overline{B} = \mu_0 J$	$\oint_c \overline{B} \bullet d\,\overline{l} = \mu_0 I$