UNIT – II
CONDUCTORS AND DIELECTRICS

2 Divergence and Curl of Electrostatic Fields

1. Field Lines and Gauss's Law

2. The Divergence of E

3. Application of Gauss's Law

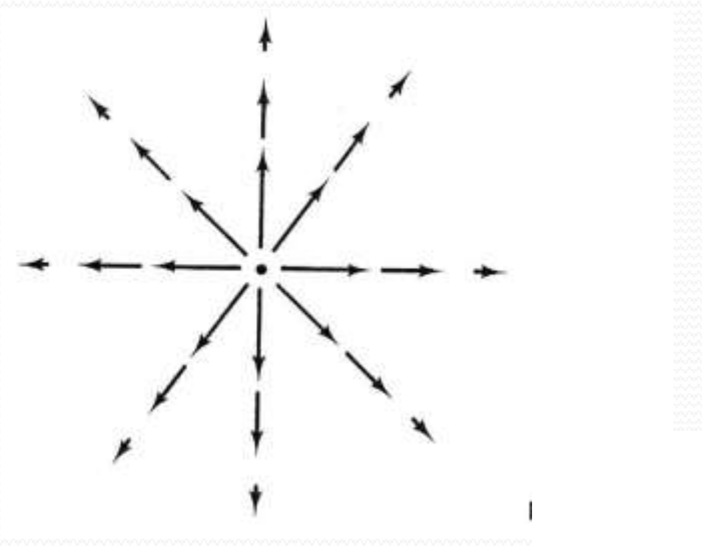
4. The Curl of E

2.1 Fields lines and Gauss's law

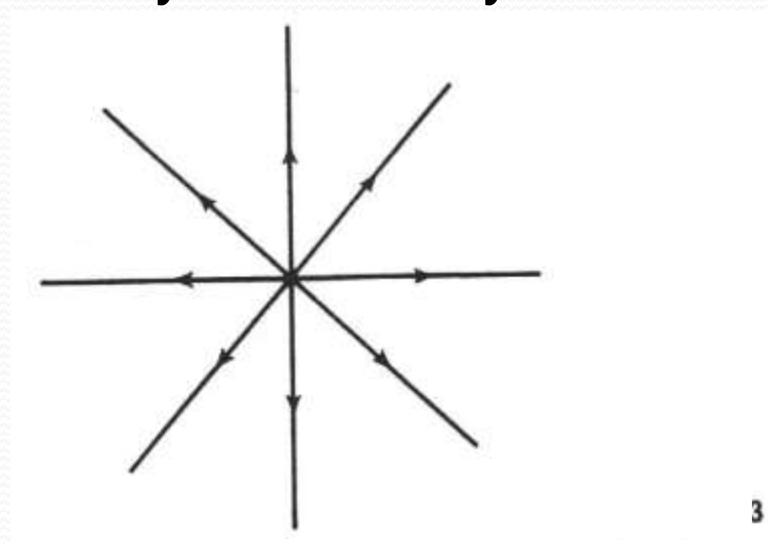
A single point charge q, situated at the origin

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Because the field falls off like $1/r^2$,the vectors get shorter as I go father away from the origin,and they always point radially outward. This vectors can be connect up the arrows to form the *field lines.* The magnitude of the field is indicated by the density of the lines.



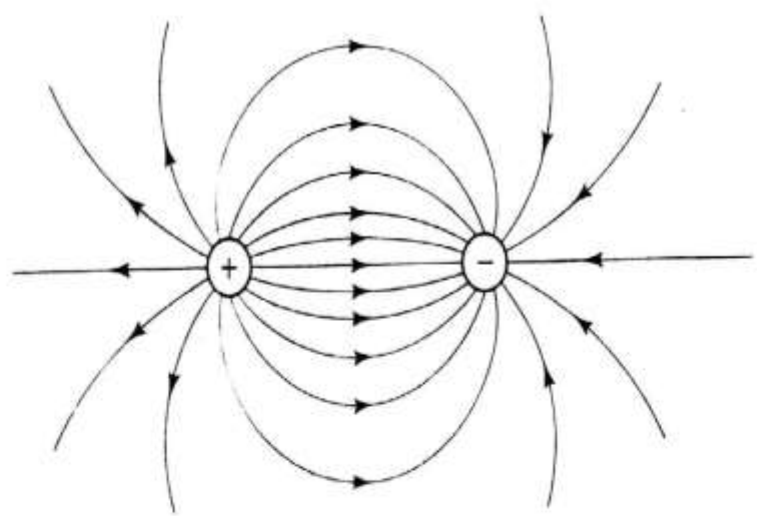
strength length of arrow



strength density of field line

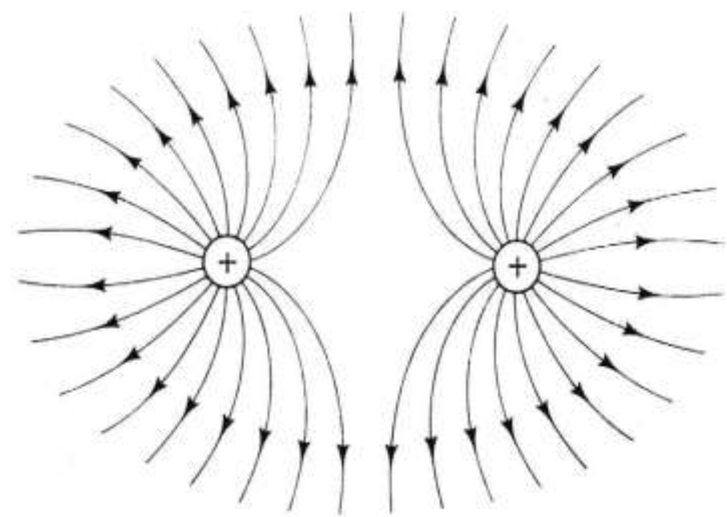
2.1

1. Field lines emanate from a point charge symmetrically in all directions.
2. Field lines originate on positive charges and terminate on negative ones.
3. They cannot simply stop in midair, though they may extend out to infinity.
4. Field lines can never cross.



Equal but opposite charges

Figure 2.14



Equal charges

Figure 2.15

2.1

Since in this model the fields strength is proportional to the number of lines per unit area, the *flux* of $\int(\vec{E}\cdot d\vec{a})$ is proportional to the the number of field lines passing through any surface .

The flux of E through a sphere of radius r is:

$$\int \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}\right) = \frac{1}{\varepsilon_0} q$$

The flux through any surface enclosing the charge is q/ε_0

According to the principle of superposition, the total field is the sum of all the individual fields:

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i \quad , \quad \int \vec{E}\cdot d\vec{a} = \sum_{i=1}^{n}\left(\int \vec{E}_i\cdot d\vec{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\varepsilon_0} q_i\right)$$

A charge outside the surface would contribute nothing to the total flux,since its field lines go in one side and out other.

2.1

Gauss's Law in integral form $\int \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

Turn integral form into a differential one , by applying the divergence theorem

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \int_{volume} (\nabla \cdot \vec{E}) d\tau$$

$$= \frac{1}{\varepsilon_0} Q_{enc} = \int_{volume} (\frac{1}{\varepsilon_0} \rho) d\tau$$

Gauss's law in differential form $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$

2.2 The Divergence of E

Calculate the divergence of E directly

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0}\int \nabla \cdot \left(\frac{\hat{R}}{R^2}\right)\rho(r')\,d\tau, \qquad \vec{R} = \vec{r} - \vec{r}'$$

The r-dependence is contained in R

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{all\,space} \frac{\hat{R}}{R^2}\rho(\vec{r}')\,d\tau'$$

From

$$\nabla \cdot \left(\frac{\hat{R}}{R^2}\right) = 4\pi\delta^3(R)$$

Thus

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0}\int 4\pi\delta^3(\vec{r} - \vec{r}')\rho(\vec{r}')\,d\tau' = \frac{1}{\varepsilon_0}\rho(\vec{r})$$

2.3 Application of Gauss's Law

Example 2.2 Find the field outside a uniformly charged sphere of radius a

Sol:

$$\int_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

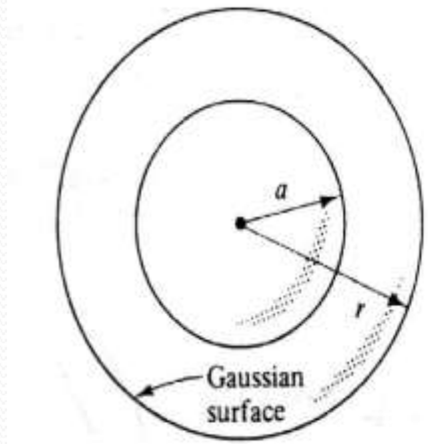


Figure 2.18

(a) E point radially outward ,as does $d\vec{a}$

$$\int_{surface} \vec{E} \cdot d\vec{a} = \int |E| d\vec{a}$$

(b)E is constant over the Gaussian surface

$$\oint_{surface} |E| d\vec{a} = |E| \oint_{surface} d\vec{a} = |E| 4\pi r^2$$

Thus $|E| 4\pi r^2 = \frac{1}{\varepsilon_0} q \quad \Longrightarrow \quad \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

2.3

1. Spherical symmetry. *Make your Gaussian surface a concentric sphere (Fig 2.18)*
2. Cylindrical symmetry. *Make your Gaussian surface a coaxial cylinder (Fig 2.19)*
3. Plane symmetry. *Use a Gaussian surface a coaxial the surface (Fig 2.20)*

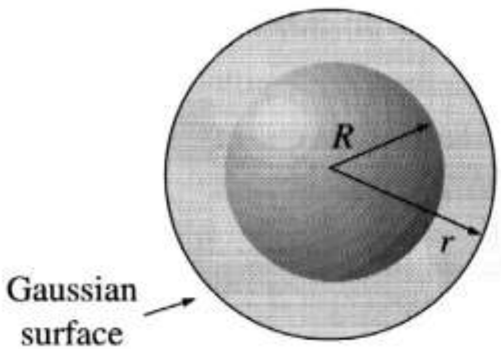


Figure 2.18

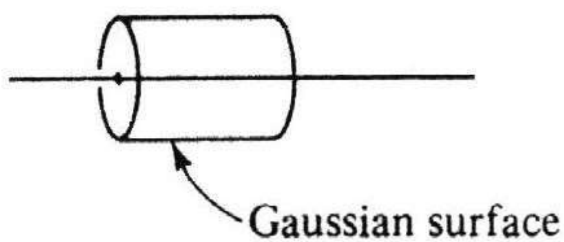


Figure 2.19

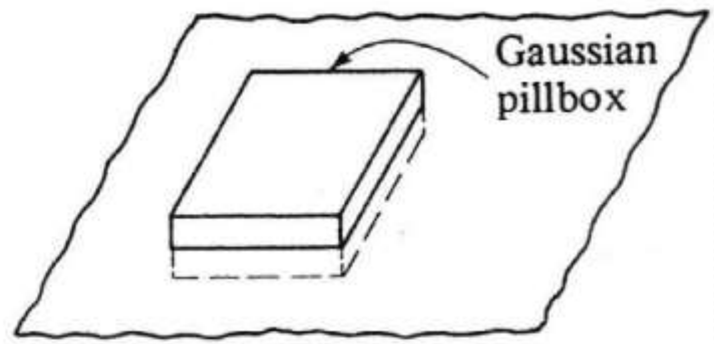


Figure 2.20

Example 2.3 Find the electric field inside the cylinder which contains charge density as $\rho = kr$

Solution:

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

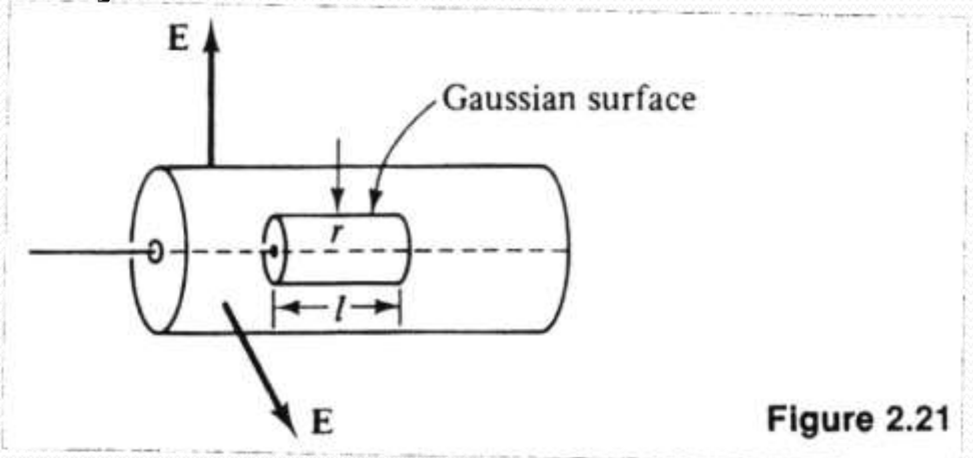


Figure 2.21

The enclosed charge is

$$Q_{enc} = \int \rho \, d\tau = \int (kr')(r'dr'd\varphi \, dz) = 2\pi kl \int_0^r r'^2 dr' = \frac{2}{3}\pi klr^3$$

$$\int \vec{E} \cdot d\vec{a} = \int |E| da = |E| \int da = |E| 2\pi rl$$ (by symmetry)

thus

$$|E| 2\pi rl = \frac{1}{\varepsilon_0} \frac{2}{3}\pi klr^3 \Rightarrow \boxed{\vec{E} = \frac{1}{3\varepsilon_0} kr^2 \hat{r}}$$

2.3

Example 2.4 An infinite plane carries a uniform surface charge σ. Find its electric field.

Solution: Draw a "Gaussian pillbox

Apply Gauss's law to this surface

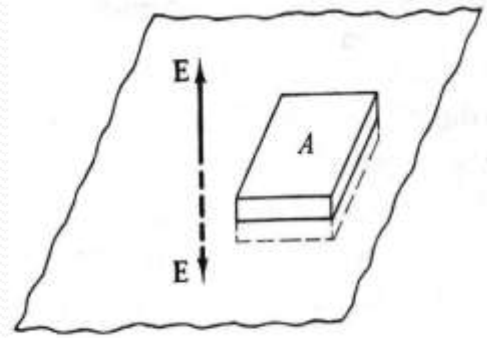


Figure 2.22

$$\int_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

By symmetry, E points away from the plane thus, the top and bottom surfaces yields

$$\int \vec{E} \cdot d\vec{a} = 2A\left|E\right|$$

(the sides contribute nothing)

$$2A\left|E\right| = \frac{1}{\varepsilon_0}\sigma A \Longrightarrow \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}}$$

2.3

Example 2.5 Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ .Find the field in each of the three regions.

Solution:

The field is (σ/ε_0), and points to the right, between the plane elsewhere it is zero.

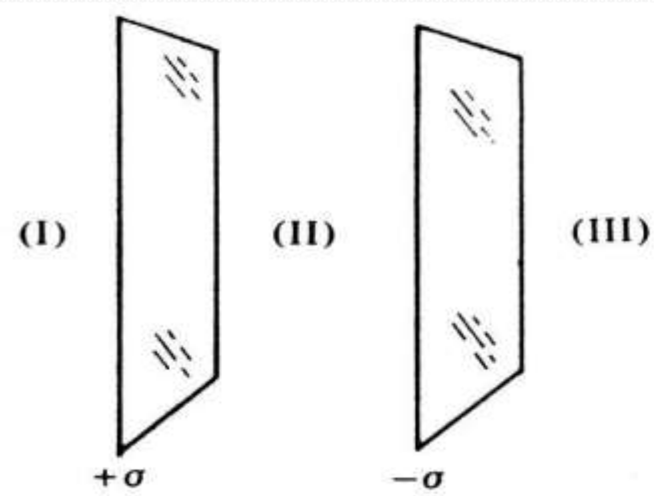


Figure 2.23

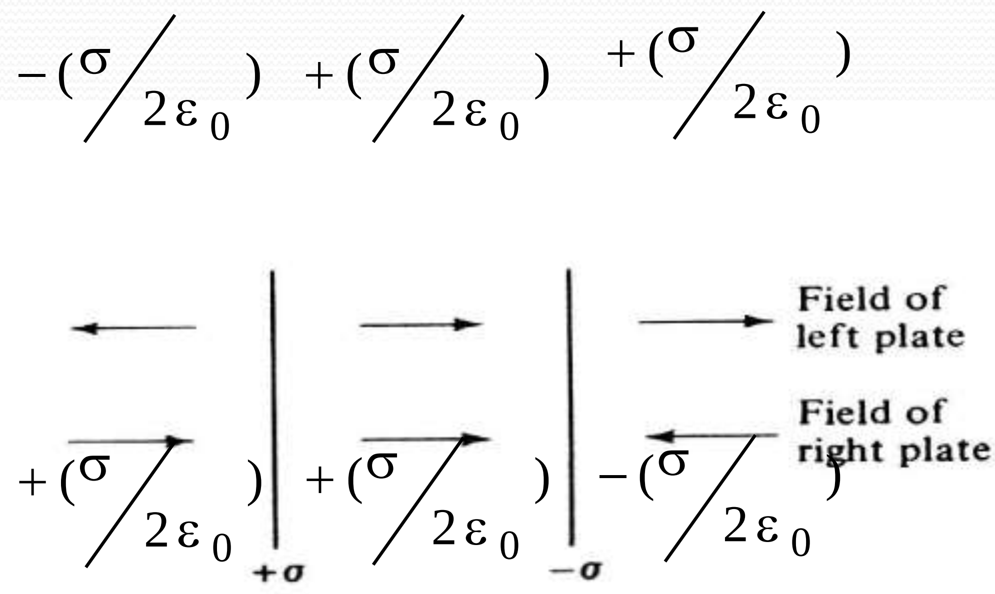


Figure 2.24

2.4 The Curl of E

A point charge at the origin. If we calculate the line integral of this field from some point a to some other point b :

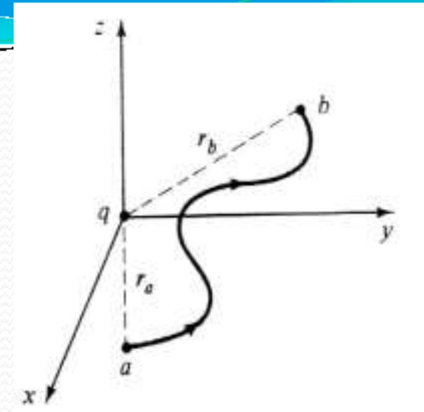


Figure 2.29

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical coordinate

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi} \qquad \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}d\vec{r}$$

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{-1}{4\pi\varepsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

This integral is independent of path. It depends on the two end points

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad (\text{if } r_a = r_b)$$

Apply by Stokes' theorem, $\nabla \times \vec{E} = 0$

The principle of superposition

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

so

$$\nabla \times \vec{E} = \nabla \times (\vec{E}_1 + \vec{E}_2 + \cdots) = (\nabla \times \vec{E}_1) + (\nabla \times \vec{E}_2) + \cdots = 0$$

$$\nabla \times \vec{E} = 0$$

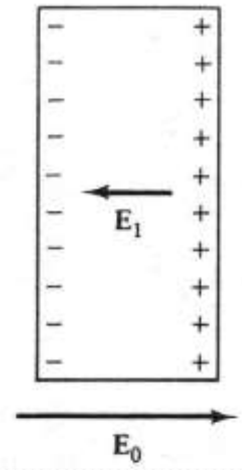
must hold for any static charge distribution whatever.

2.5 Basic Properties of Conductors

e^- are free to move in a conductor

(1) $\vec{E} = 0$ inside a conductor

otherwise, the free charges that produce $\vec{E}$ will move to make $\vec{E} = 0$ inside a conductor



(2) $\rho = 0$ inside a conductor

$$\because \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{E} = 0 \quad \Rightarrow \quad \rho = 0$$

(3) Any net charge resides on the surface

(4) V is constant, throughout a conductor.

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = 0$$

$$\therefore V(b) = V(a)$$

2.5

(5) $\vec{E}$ is perpendicular to the surface, just outside a conductor. Otherwise, $\vec{E}_{\parallel}$ will move the free charge to make $\vec{E}_{\parallel} = 0$ in terms of energy, free charges staying on the surface have a minimum energy.

σ

$$Energy = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

$$E_{in} = 0 \quad W_{in} = 0$$

ρ

$$Energy = \frac{3}{20\pi\varepsilon_0}\frac{q^2}{R}$$

$$E_{in} \neq 0 \quad W_{in} \neq 0$$

Example : A point charge q at the center of a spherical conducting shell. How much induced charge will accumulate there?

Solution :

$\because E_{in} = 0$ $\quad Q$ induced

$\therefore 4\pi a^2 \cdot \sigma_a = -q$

$$\sigma_a = -\frac{1}{4\pi}\frac{q}{a^2}$$

charge conservation

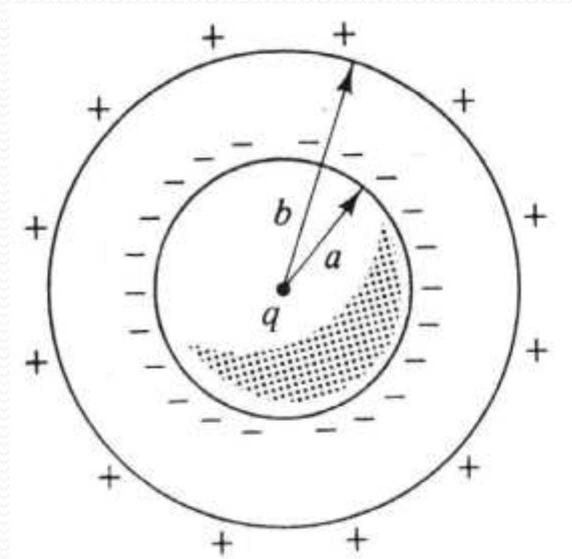
$4\pi b^2 \cdot \sigma_b = -4\pi a^2 \sigma_a$

$$\sigma_b = \frac{1}{4\pi}\frac{q}{b^2}$$

$E_{in} = 0$

$Q_{enc} = q + q_{induced} = 0$

$q_{induecd} = -q$



2.7 Induced Charge

Example 2.9 Within the cavity is a charge $+q$. What is the field outside the sphere?

$-q$ distributes to shield q and to make $E_{in} = 0$

i.e., $V_{surface} = constant$

from charge conservation and symmetry.

$+q$ uniformly distributes at the surface

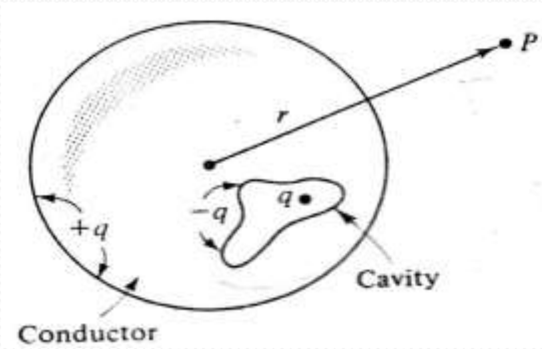
$$\therefore \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\int_a^b \vec{E} \cdot d\vec{l} = 0$$

a,b are arbitrary chosen

$$\vec{E}_{incavity} = 0$$



$\therefore$ a Faraday cage can shield out stray $\vec{E}$

2.8 The Surface Charge on a Conductor

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

⍰ $E_{in} = 0 \quad \text{or} \quad \vec{E}_{below} = 0$

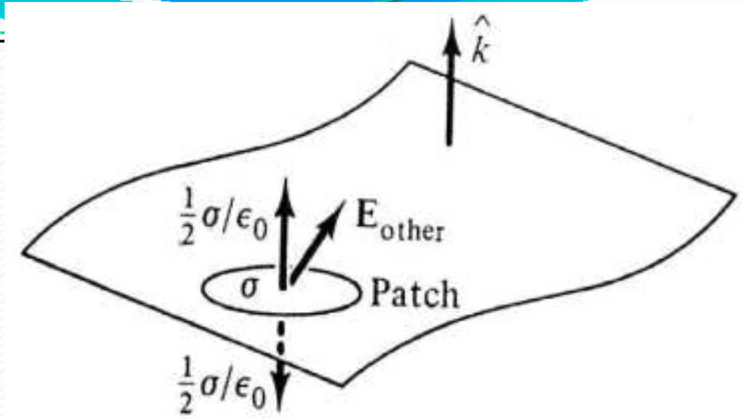
$$\therefore \; -\nabla V = \vec{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \text{or} \qquad \frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

if we know *V*, we can get σ.

2.8

Force on a surface charge,

$$\vec{f} = \sigma \vec{E}_{average} = \frac{1}{2}\sigma(\vec{E}_{above} + \vec{E}_{below})$$



why the average?

$$\vec{E}_{above} = \vec{E}_{other} + \frac{\sigma}{2\varepsilon_0}\hat{k} \qquad \vec{E}_{below} = \vec{E}_{other} - \frac{\sigma}{2\varepsilon_0}\hat{k}$$

$$\vec{E}_{other} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below}) \overset{\sigma \to \mathbf{0}}{=} \vec{E}_{average}$$

In case of a conductor

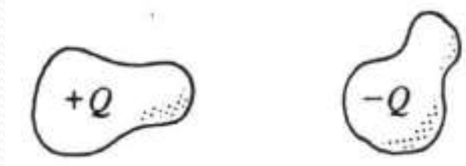
$$\vec{f} = \frac{1}{2}\sigma \vec{E}_{above} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

electrostatic pressure

$$P = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0}{2}E^2$$

2.9 Capacitors

Consider 2 conductors (Fig 2.53)



The potential difference

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$$

(V is constant.)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{r}}{r^2} \rho \, d\tau$$

double $\rho \rightarrow$ double $Q \rightarrow$ double $\vec{E} \rightarrow$ double V

Define the ratio between Q and V to be capacitance

$$C = \frac{Q}{V}$$

a geometrical quantity

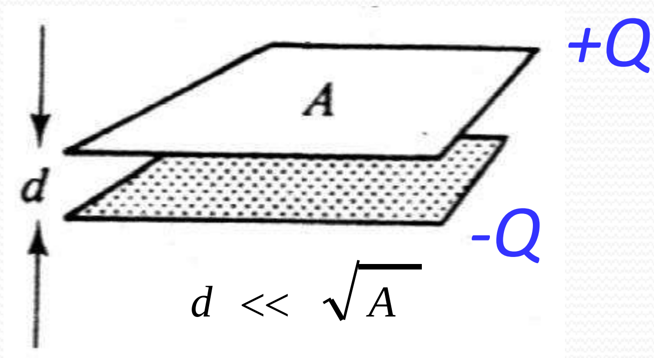
in mks 1 farad(F)= 1 Coulomb / volt

inconveniently large ;

$10^{-6} F$: microfarad

$10^{-12} F$: picofarad

Example 2.10 Find the capacitance of a "parallel-plate capacitor"?



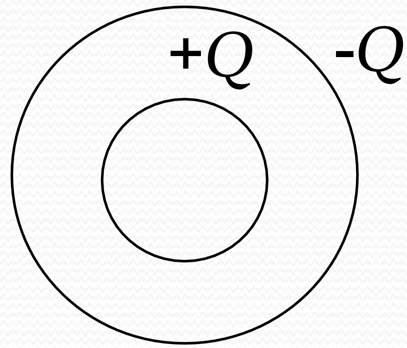
Solution:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

$$V = E \cdot d = \frac{Q}{A\varepsilon_0} d$$

$$C = \frac{A\varepsilon_0}{d}$$

Example 2.11 Find capacitance of two concentric spherical shells with radii a and b .



Solution:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

$$V = -\int_b^a \vec{E}\cdot d\vec{l} = -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$C = \frac{Q}{V} = 4\pi\varepsilon_0\frac{ab}{(b-a)}$$

The work to charge up a capacitor

$$dW = V\,dq = \left(\frac{q}{C}\right)dq$$

$$W = \int_0^Q \left(\frac{q}{C}\right)dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

2.10 Conductors and Dielectrics

- Conductors
  - Current, current density, drift velocity, continuity
  - Energy bands in materials
  - Mobility, micro/macro Ohm's Law
  - Boundary conditions on conductors
  - Methods of Images
- Dielectrics
  - Polarization, displacement, electric field
  - Permittivity, susceptibility, relative permittivity
  - Dielectrics research
  - Boundary conditions on dielectrics

Conductors and Dielectrics

- Polarization
  - Static alignment of charge in material
  - Charge aligns when voltage applied, moves no further
  - Charge proportional to voltage

- Conduction
  - Continuous motion of charge through material
  - Enters one side, exits another
  - Current proportional to voltage

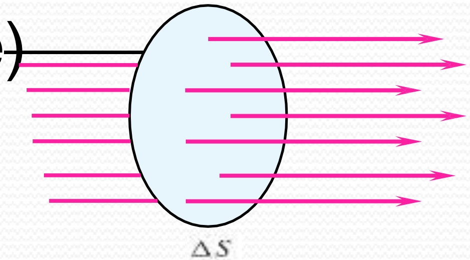
2.11 Current and current density

- Basic definition of current C/s = Amps

$$I = \frac{dQ}{dt}$$

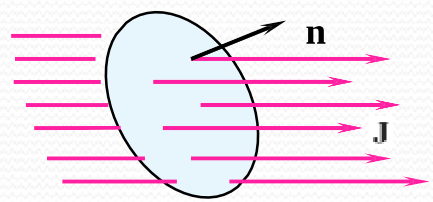
- Basic current density (J perp. surface)

$$\Delta I = J_N \Delta S$$



- Vector current density

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}}$$



Current density and charge velocity

- Basic definition of current

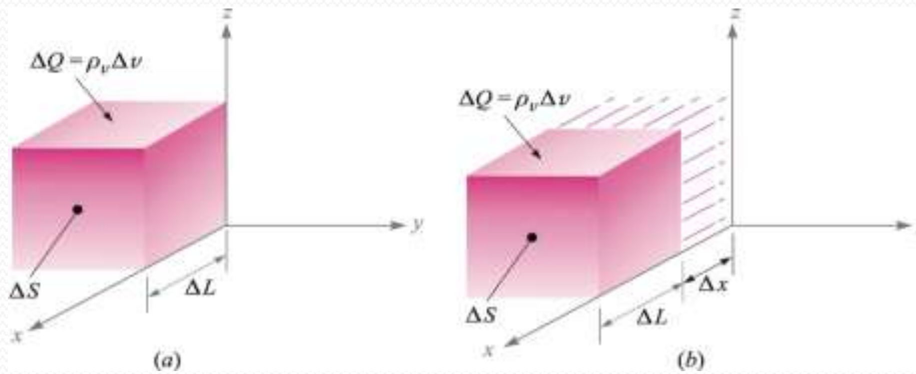
$$I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \, \Delta v}{\Delta t} = \frac{\rho_v \, dS \, \Delta x}{\Delta t} = \rho_v \, dS \, v$$

- Combining with earlier expression

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}}$$

- Gives current density

$$\boldsymbol{J} = \rho_v \, \boldsymbol{v}$$



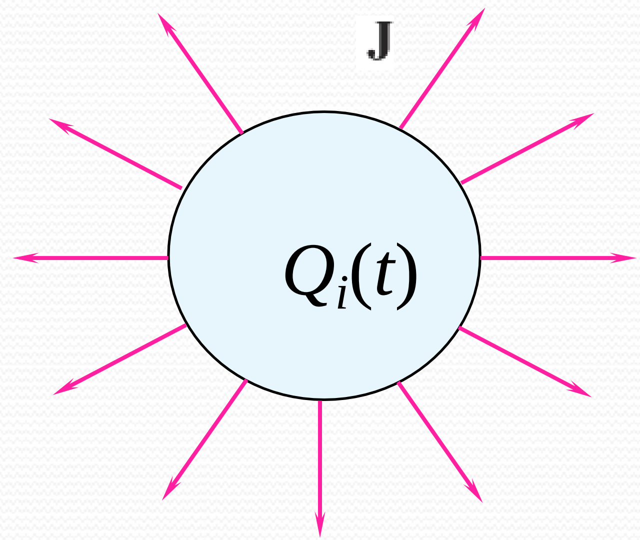
Charge and current continuity

- Current leaving any closed surface is time rate of change of charge within that surface

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} \quad = -\frac{dQ_i}{dt} \quad = -\frac{d}{dt}\int_{\text{vol}} \rho_v \, dv$$

- Using divergence theorem on left

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{\text{vol}} \rho_v \, dv$$



- Taking time derivative inside integral

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

- Equating integrands

$$\boxed{(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t}}$$

Example - charge and current continuity

- Given spherically symmetric current density

$$J = \frac{1}{r} e^{-t} \boldsymbol{a}_r$$

- Current increasing from r = 5m to r= 6m at t=1s

$$I = J_r S = \frac{1}{5} e^{-1} 4\pi 5^2 = 23.1 A \text{ @ } 5m, \qquad \frac{1}{6} e^{-1} 4\pi 6^2 = 27.7 \text{@ } 6m$$

- Current density from continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J = -\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r} e^{-t} \right) = -\frac{1}{r^2} e^{-t}$$

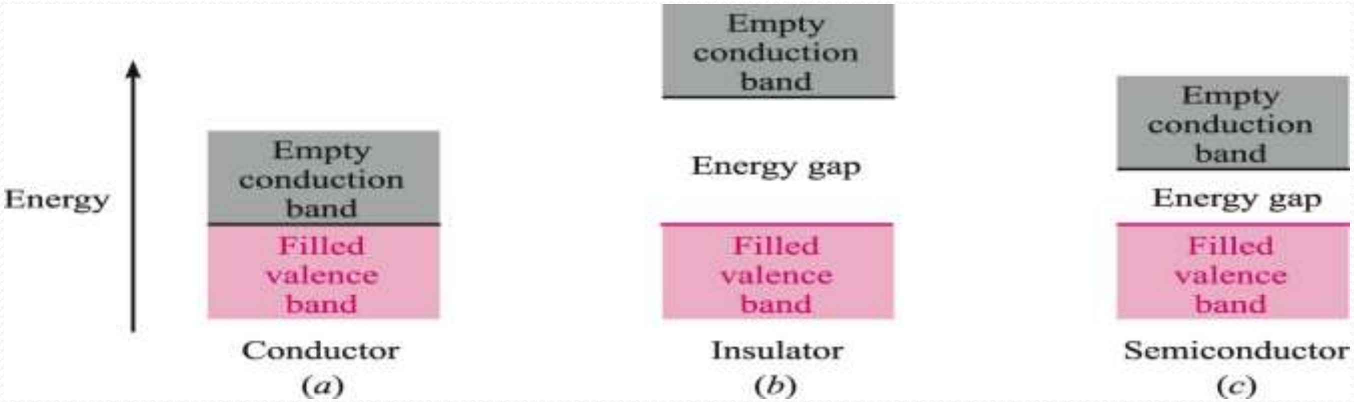
- Charge density ρ integral w.r.t. time

$$\rho = -\frac{1}{r^2} \int e^{-t} dt = \frac{1}{r^2} e^{-t} + const(r)$$

- Drift velocity is thus

$$v_r = \frac{J_r}{\rho_v} = \frac{\frac{1}{r} e^{-t}}{\frac{1}{r^2} e^{-t}} = r \quad m/s$$

2.12 Energy Band Structure in Three Material Types



Discrete quantum states broaden into energy bands in condensed materials with overlapping potentials

- Valence band - outermost filled band
- Conduction band - higher energy unfilled band

Band structure determines type of material

a) Insulators show large energy gaps, requiring large amounts of energy to lift electrons into the conduction band. When this occurs, the dielectric breaks down.

b) Conductors exhibit no energy gap between valence and conduction bands so electrons move freely

c) Semiconductors have a relatively small energy gap, so modest amounts of energy (applied through heat, light,or an electric field) may lift electrons from valence to conduction bands.

2.13 Ohm's Law (microscopic form)

Free electrons are accelerated by an electric field. The applied force on an electron of charge *Q = -e is*

$$\mathbf{F} = -e\mathbf{E}$$

But in reality the electrons are constantly bumping into things (like a terminal velocity) so they attain an equilibrium or *drift velocity*:

$$\boxed{\mathbf{v}_d = -\mu_e \mathbf{E}}$$

where μ_e is the electron *mobility*, expressed in units of $m^2/V\text{-}s$. The drift velocity is used in the current density through:

$$\mathbf{J} = \rho_v \mathbf{v}_d = -\rho_e \mu_e \mathbf{E} = \sigma \mathbf{E}$$

So Ohm's Law in point form (material property)

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

With the conductivity given as:

$$\boxed{\sigma = -\rho_e \mu_e}$$

$$\boxed{\sigma = -\rho_e \mu_e + \rho_h \mu_h}$$ S/m (electrons/holes)

Ohm's Law (macroscopic form)

- For constant electric field

$$|V_{ab}| = \int_b^a \boldsymbol{E} \cdot \boldsymbol{dL} = EL$$

- Ohm's Law becomes

$$\frac{I}{S} = J = \sigma E = \sigma \frac{V}{L}$$

- Rearranging gives

$$I = \frac{\sigma S}{L} V \qquad G = \frac{\sigma S}{L} \quad siemens \quad (conductance)$$

- Or

$$V = \frac{L}{\sigma S} I \qquad R = \frac{L}{\sigma S} = \frac{\rho L}{S} \quad ohms \quad (resistance)$$

- Variation with geometry
- Conductance vs. Resistance

2.14 Boundary conditions for conductors

- No electric field in interior
  - Otherwise charges repel to the surface
- No tangential electric field at surface
  - $E_t = 0$
  - Otherwise charges redistribute along surface
- Normal electric field at surface
  - $\varepsilon_o E_n = D_n = \rho_s$
  - Displacement Normal equals Charge Density (Gauss's Law)

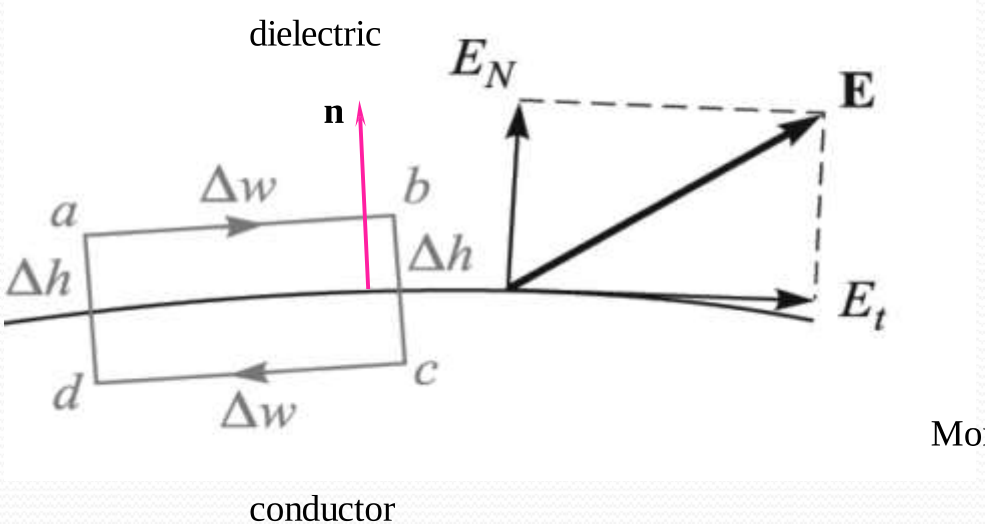
Boundary Condition for Tangential Electric Field E

Over the rectangular integration path, we use $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ or $\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$

To find:

$$E_t \Delta w \underbrace{- E_{N,\text{at}\,b} \tfrac{1}{2} \Delta h + E_{N,\text{at}\,a} \tfrac{1}{2} \Delta h}_{} = 0$$

These become negligible as Δh approaches zero.



Therefore

$$\boxed{E_t = 0}$$

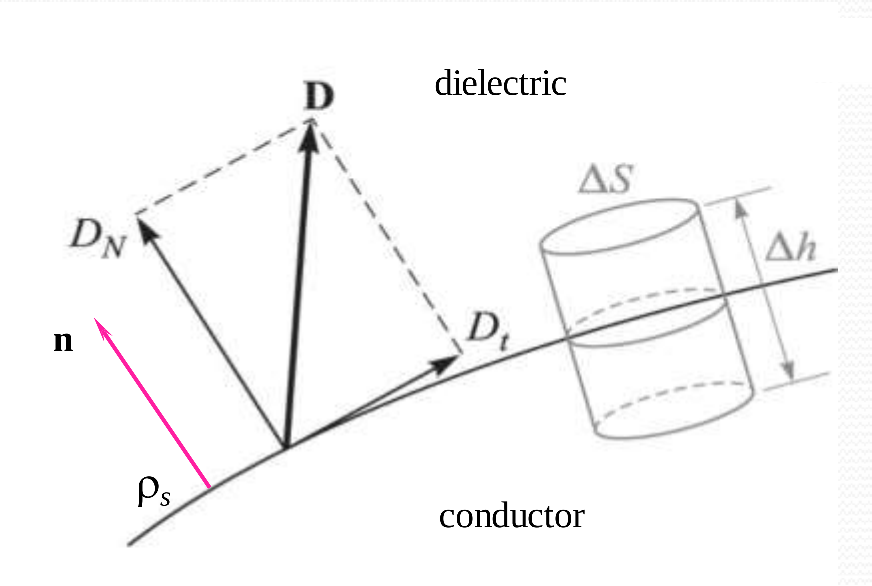
More formally: $\boxed{\mathbf{E} \times \mathbf{n}\big|_s = 0}$

Boundary Condition for the Normal Displacement D

Gauss' Law is applied to the cylindrical surface shown below:

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\text{top}} + \cancel{\int_{\text{bottom}}} + \cancel{\int_{\text{sides}}} = Q$$

This reduces to: $$D_N \Delta S = Q = \rho_S \Delta S$$ as Δh approaches zero



Therefore

$$\boxed{D_N = \rho_S}$$

More formally:

$$\boxed{\mathbf{D}\cdot\mathbf{n}\Big|_s = \rho_s}$$

2.15 Summary

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

At the surface:

$$\mathbf{E} \times \mathbf{n}\big|_s = 0$$

Tangential E is zero

$$\mathbf{D} \cdot \mathbf{n}\big|_s = \rho_s$$

Normal D is equal to the surface charge density

Example - Boundary Conditions for Conductors

- Potential given by

$$V = 100(x^2 - y^2)$$

- Potential at (2,-1,3) is 300 V. Also 300 V along entire surface where

$$300 = 100(x^2 - y^2)$$

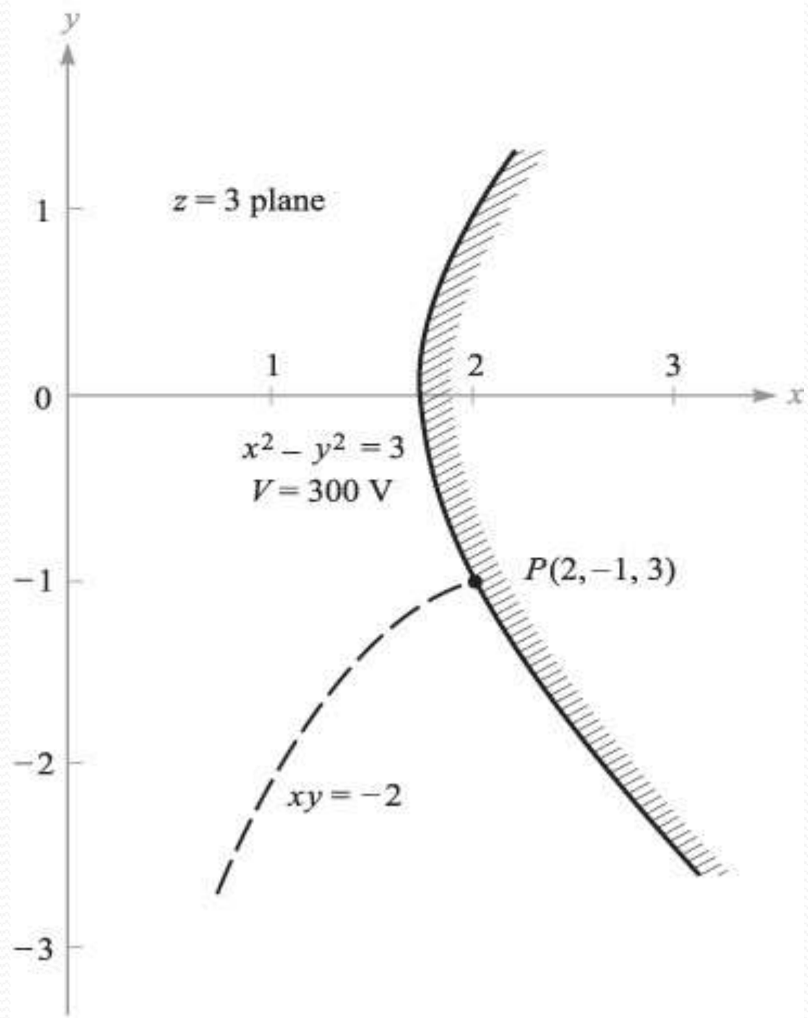
- Thus we can "insert" conductor in region provided the conductor follow hyperbola

$$3 = x^2 - y^2$$

- The Electric Field is at all times <u>normal</u> to conducting surface

$$E = -\nabla V = -200x\,\boldsymbol{a}_x + 200y\;\boldsymbol{a}_y$$

- Electric field at point 2,-1,3)
  - Ex = -400 V/m, Ey = -200 V/m
  - Down and to left



Example – Streamlines of Electric Field

- Slope of line equals electric field ratio

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{200\,y}{-200\,x} = -\frac{y}{x}$$

- Rearranging

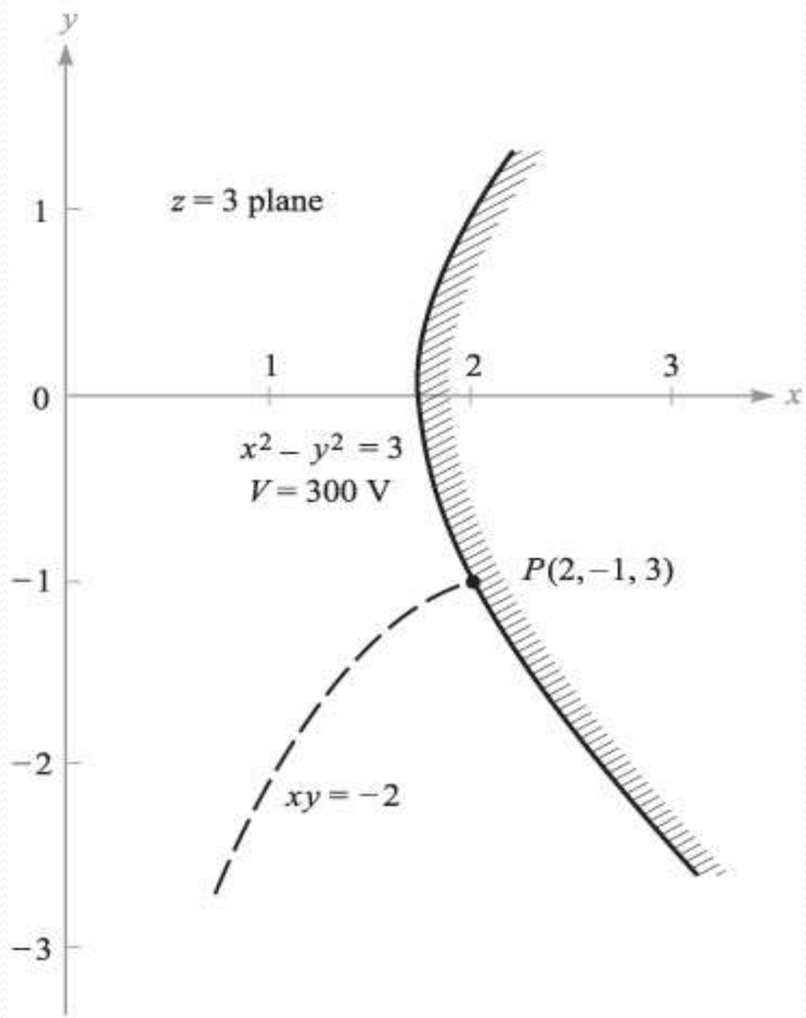
$$\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\ln x + \ln y = C_1$$

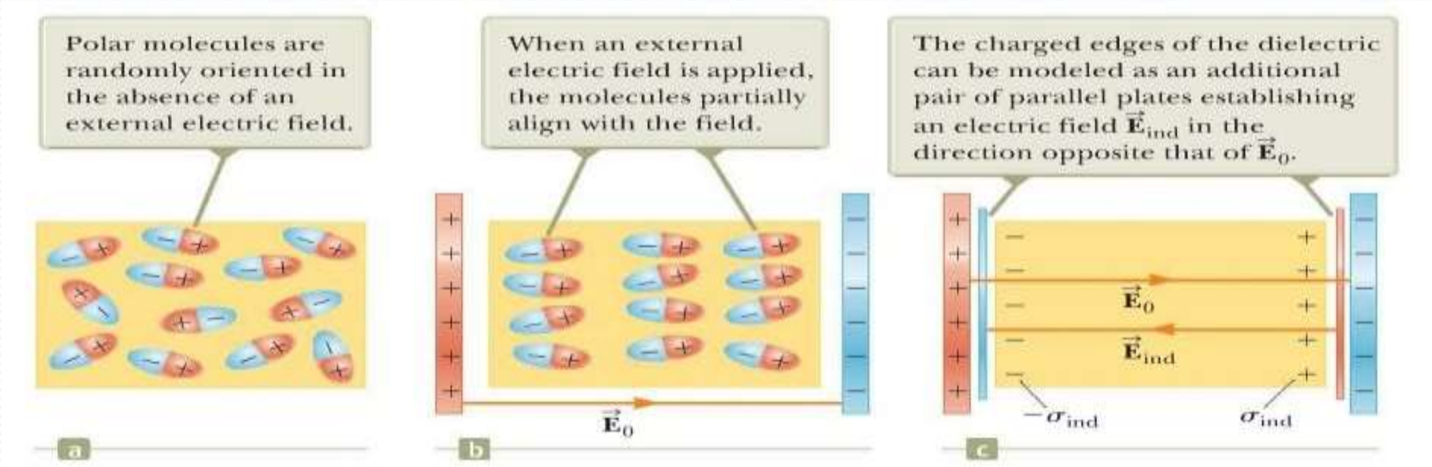
$$xy = C_2$$

- Evaluate at P(2,-1,3)

$$xy = -2$$



2.16 Dielectrics



- Material has random oriented dipoles
- Applied field aligns dipoles (negative at (+) terminal, positive at (-) terminal
- Effect is to cancel applied field, lower voltage
- OR, increase charge to maintain voltage
- Either increases capacitance C= Q/V

Review Dipole Moment

- Define dipole moment

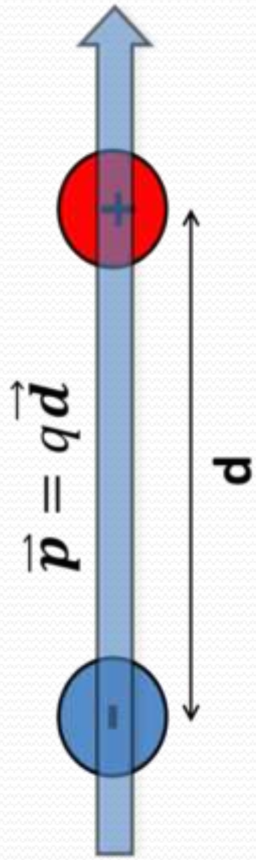
$$\boldsymbol{p} = Q\boldsymbol{d}$$

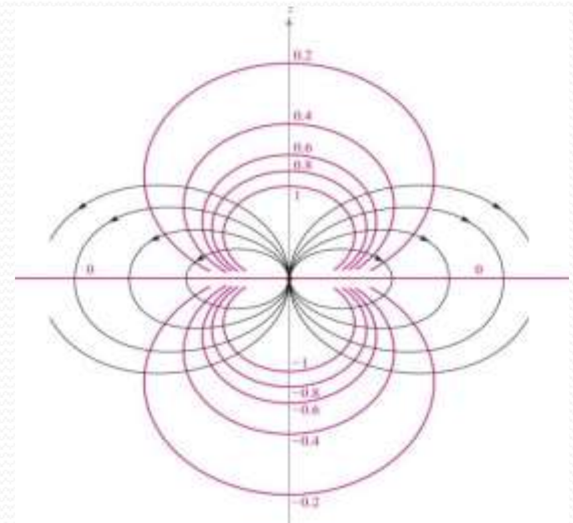
- Potential for dipole

$$\boxed{V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}}$$

- Written in terms of dipole moment and position

$$V = \frac{\mathbf{p}\cdot\mathbf{a}_r}{4\pi\epsilon_0 r^2}$$

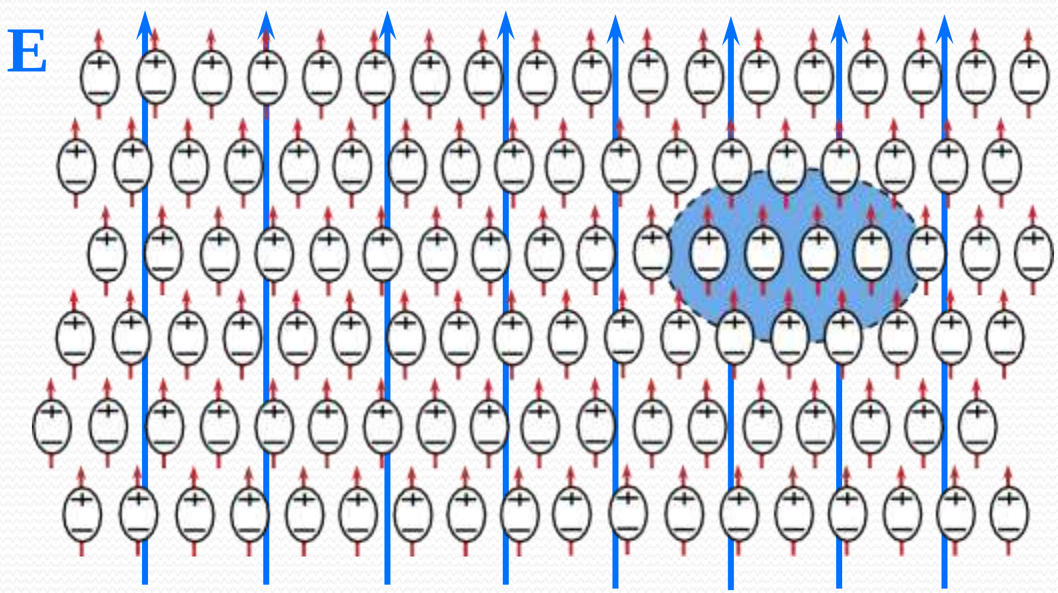




- Dipole moment determines “strength” of polar molecule amount of charge (Q) and offset (d) of charge

Polarization as sum of dipole moments (per volume)

Introducing an electric field may increase the charge separation in each dipole, and possibly re-orient dipoles so that there is some aggregate alignment, as shown here. The effect is small, and is greatly exaggerated here!

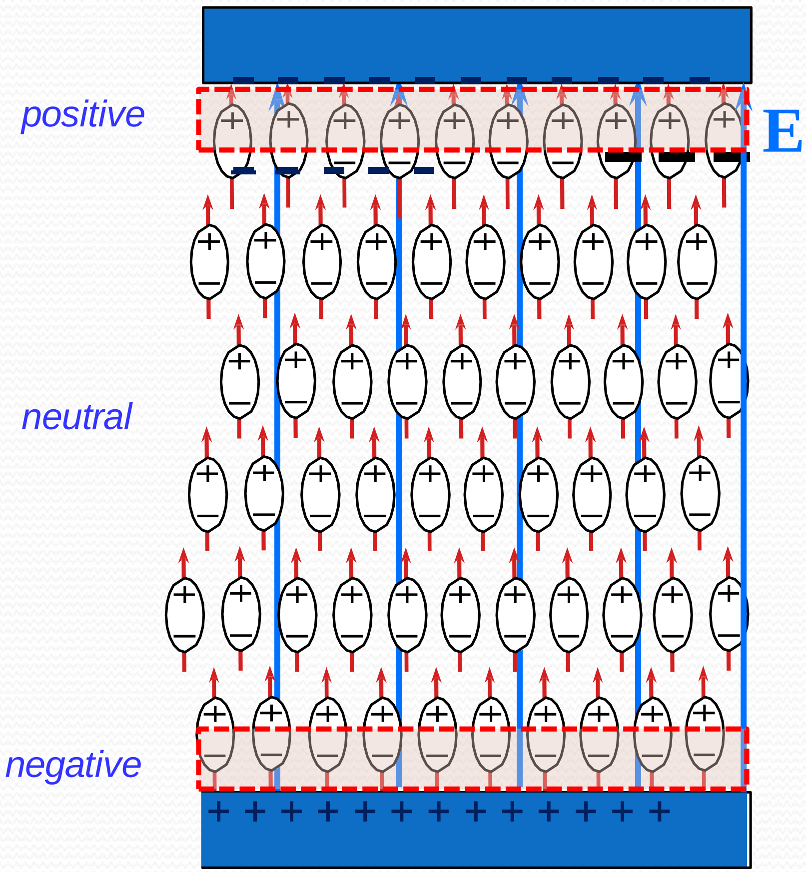


The effect is to increase **P**.

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i = n\boldsymbol{p} = nQ\boldsymbol{d}$$

n = charge/volume
p = polarization of individual dipole
P = polarization/volume

Polarization near electrodes



- From diagram
  - Excess positive bound charge near top negative electrode
  - Excess negative bound charge near bottom positive electrode
  - Rest of material neutral

- Excess charge in bound (red) volumes

$$\Delta Q = nQd\,\Delta S$$

- Writing in terms of polarization

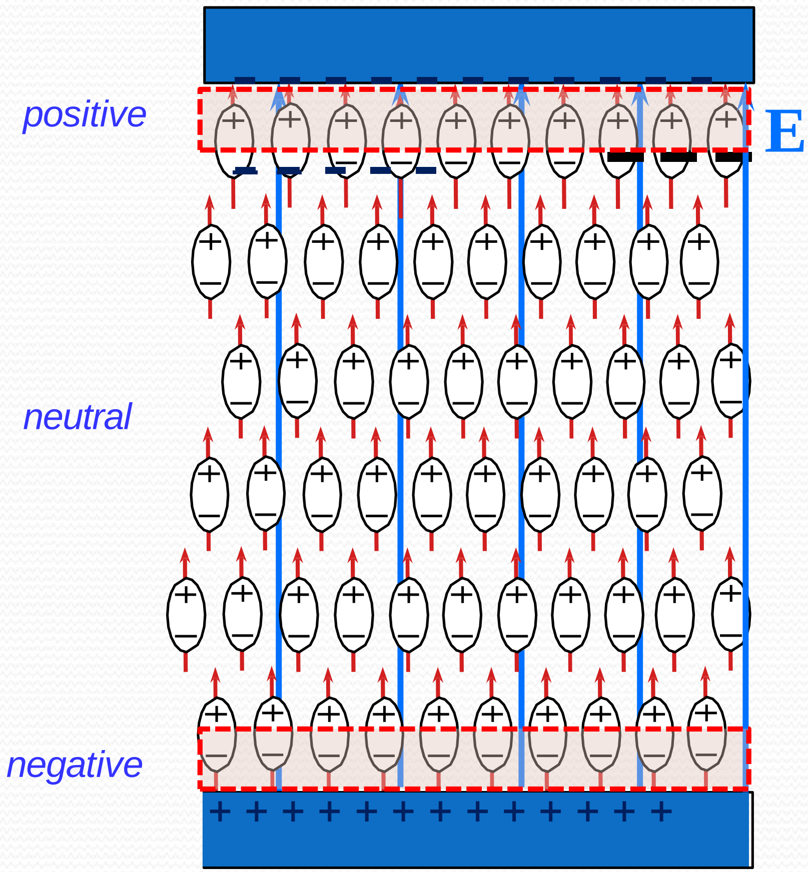
$$\Delta Q = \boldsymbol{P}\,\Delta S$$

- Writing similar to Gauss's law

$$Q_b = -\oint P \cdot ds$$

(Note dot product sign, outward normal leaves opposite charge enclosed)

Combining total, free, and bound charge



- Total, free, and bound charge

$$Q_t = Q_f + Q_b$$

- Total

$$Q_t = \oint \varepsilon_o E \cdot ds$$

- Free

$$Q_f = \oint D \cdot ds$$

- Bound

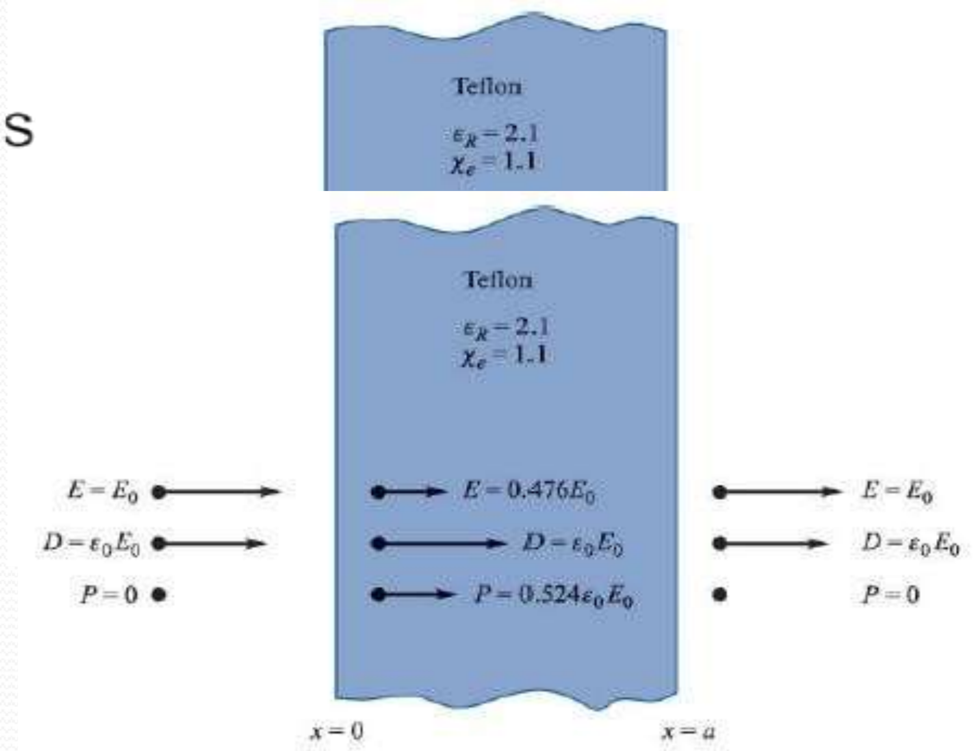
$$Q_b = -\oint P \cdot ds$$

- Combining

$$D = \varepsilon_o E + P$$

D, P, and E in Dielectric

- D continuous
- Polarization increases
- E decreases
- $\boldsymbol{D} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$
- C/m^2



Charge Densities

Taking the previous results and using the divergence theorem, we find the point form expressions:

Bound Charge: $$Q_b = \int_v \rho_b \, dv = -\oint_S \mathbf{P} \cdot d\mathbf{S} \longrightarrow \boxed{\nabla \cdot \mathbf{P} = -\rho_b}$$

Total Charge: $$Q_T = \int_v \rho_T \, dv = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \longrightarrow \boxed{\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T}$$

Free Charge: $$Q = \int_v \rho_v \, dv = \oint_S \mathbf{D} \cdot d\mathbf{S} \longrightarrow \boxed{\nabla \cdot \mathbf{D} = \rho_v}$$

2.17 Electric Susceptibility and the Dielectric Constant

A stronger electric field results in a larger polarization in the medium. In a *linear* medium, the relation between **P** and **E** is linear, and is given $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$

where χ_e is the electric *susceptibility* of the medium.

We may now write:
$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

wh~~ere the *dielectric constant*~~, or *relative permittivity* is defined as:

$$\epsilon_r = \chi_e + 1$$
$$\epsilon = \epsilon_r \epsilon_0$$

Leading to the overall permittivity of the medium: $\boxed{\mathbf{D} = \epsilon \mathbf{E}}$

Permittivity of Materials

- Typical permittivity for various solids and liquids.
  - Teflon - 2
  - Plastics - 3-6
  - Ceramics 8-10
  - Titanates>100
  - Acetone 2 1
  - Water 78
- Actual dielectric “constant” varies with:
  - Temperature
  - Direction
  - Field Strength
  - Frequency
  - Real & Imaginary components

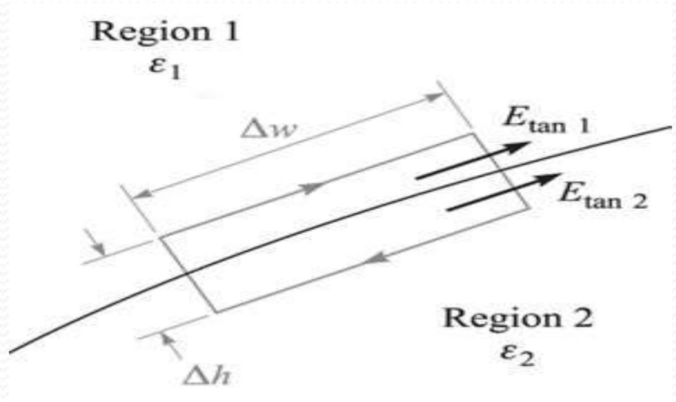
Other applications

- Other Applications
  - Bio
  - Liquid Crystal
  - Composite polymers
  - Titanates
  - Wireless characterization
  - MRI dyes
  - Ground water monitoring
  - Oil Drilling fluid characterization (GPR)

2.18 Boundary Condition for Tangential Electric Field **E**

Since **E** is conservative, we setup line integral straddling both dielectrics:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$



Left and right sides cancel, so

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

Leading to Continuity for tangential E

$$\boxed{E_{\tan 1} = E_{\tan 2}}$$

And **Discontinuity for tangential D**

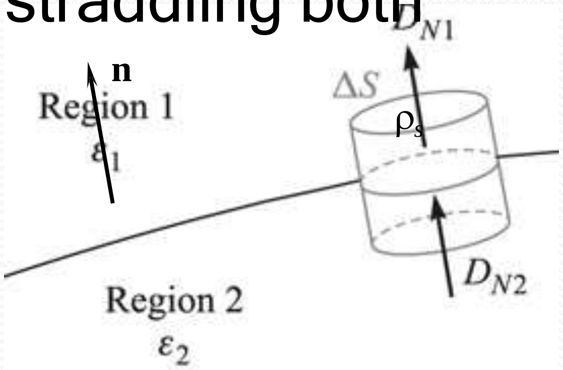
$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

E same, D higher in high permittivity material

Boundary Condition for Normal Displacement **D**

Apply Gauss' Law to the cylindrical volume straddling both dielectrics

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$



Flux enters and exits only through top and bottom surfaces, zero on sides

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

Leading to **Continuity for normal D (for ρ_S = 0)**

$$\boxed{D_{N1} - D_{N2} = \rho_S} \qquad \boxed{D_{N1} = D_{N2}}$$

And **Discontinuity for normal E**

$$\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$$

D same. E lower in high permittivity material

Bending of D at boundary

- Boundary conditions
  - D_N continuous
  - $\frac{D_{T1}}{D_{T2}} = \frac{\varepsilon_1}{\varepsilon_2}$
- Trigonometry
  - $\tan(\theta_1) = \frac{D_{T1}}{D_N}$
  - $\tan(\theta_2) = \frac{D_{T2}}{D_N}$
- Eliminating D_N

$$\frac{\tan(\theta_1)}{\tan(\theta_2)} = \frac{D_{T1}}{D_{T2}} = \frac{\varepsilon_1}{\varepsilon_2}$$