UNIT-IV

UNIT - IV MAGNETIC POTENTIAL

Scalar **Magnetic Potential and Vector Magnetic Potential and its Properties - Vector MagneticPotential due to Simple Configuration – Vector Poisson's Equations.**

Self and Mutual Inductances – Neumann's Formulae – Determination of Self Inductance of a Solenoid and Toroid and Mutual Inductance Between a Straight, Long Wire and a Square Loop Wire in the Same Plane – Energy Stored and Intensity in a Magnetic Field – Numerical Problems.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a **scalar magnetic potential** and write:

$\vec{H} = -\nabla V_m$ .... (4.21)

From Ampere's law , we know that

$\nabla \times \vec{H} = \vec{J}$ . ... (4.22)

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$ .... (4.23)

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$. Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, V_m in general is not a single valued function of position.

This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 4.8.

In the region $a < \rho < b$, $\vec{J} = 0$ and $\vec{H} = \dfrac{I}{2\pi\rho}\hat{a}_\phi$

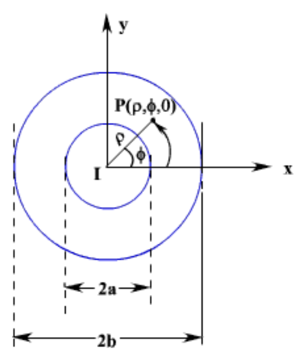

Fig. 4.8: Cross Section of a Coaxial Line

If V_m is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

$$\therefore V_m = -\frac{I}{2\pi}\phi + c$$

If we set $V_m = 0$ at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$\therefore$ At $\phi = \phi_0$ $\quad V_m = -\frac{I}{2\pi}\phi_0$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but V_m this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of V_m keeps changing as we complete additional laps to pass through the same point. We introduced V_m analogous to electostatic potential V. But for static electric fields, $\nabla \times \vec{E} = 0$ and $\oint \vec{E} \cdot d\vec{l} = 0$, whereas for steady magnetic field $\nabla \times \vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H} \cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the **vector magnetic potential** which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla . \vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla . \left(\nabla \times \vec{A}\right) = 0$, we can write $\vec{B} = \nabla \times \vec{A}$.

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation.

We have introduced the vector function $\vec{A}$ and related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla \cdot \vec{A}$ is made as follows.

$$\nabla\times\nabla\times\vec{A} = \mu\nabla\times\vec{H} = \mu\vec{J}$$ ....(4.24)

By using vector identity, $\nabla\times\nabla\times\vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$ ....(4.25)

$$\nabla(\nabla.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J}$$ ....(4.26)

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$. Putting $\nabla.\vec{A} = 0$, we get $\nabla^2\vec{A} = -\mu\vec{J}$ which is vector poisson equation. In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x = -\mu J_x$$ ....(4.27a)

$$\nabla^2 A_y = -\mu J_y$$ ....(4.27b)

$$\nabla^2 A_z = -\mu J_z$$ ....(4.27c)

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$ ....(4.28)

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho}{R}dv', \qquad R = \left|\vec{r} - \vec{r}'\right|$$ ....(4.29)

In case of time varying fields we shall see that $\nabla.\vec{A} = \mu\varepsilon\frac{\partial V}{\partial t}$, which is known as Lorentz condition, *V* being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi}\int_{V'}\frac{J_x}{R}dv'$$ ....(4.30)

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}}{R}dv'$$ ....(4.31)

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$. Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{S}\frac{\vec{K}}{R}ds' \qquad \text{.............................. (4.33)}$$

The magnetic flux ψ through a given area S is given by

$$\psi = \int_{S}\vec{B}.d\vec{s} \qquad \text{. .... (4.34)}$$

Substituting

$$\vec{B} = \nabla \times \vec{A}$$

$$\psi = \int_{S}\nabla \times \vec{A}.d\vec{s} = \oint_{C}\vec{A}.d\vec{l} \qquad \text{. .... (4.35)}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Inductance and Inductor:

Resistance, capacitance and inductance are the three familiar parameters from circuit theory. We have already discussed about the parameters resistance and capacitance in the earlier chapters. In this section, we discuss about the parameter inductance. Before we start our discussion, let us first introduce the concept of flux linkage. If in a coil with N closely wound turns around where a current I produces a flux ϕ and this flux links or encircles each of the N turns, the flux linkage Λ is defined as $\Lambda = N\phi$. In a linear medium, where the flux is proportional to the current, we define the self inductance L as the ratio of the total flux linkage to the current which they link.

i.e., $$L = \frac{\Lambda}{I} = \frac{N\phi}{I} \qquad \text{...(4.47)}$$

To further illustrate the concept of inductance, let us consider two closed loops C_1 and C_2 as shown in the figure 4.10, S_1 and S_2 are respectively the areas of C_1 and C_2 .

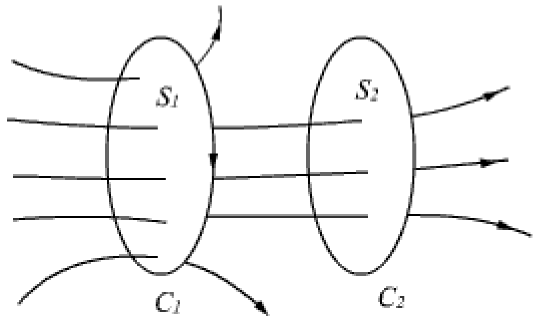

Fig 4.10

if a current I_1 flows in C_1 , the magnetic flux B_1 will be created part of which will be linked to C_2 as shown in Figure 4.10.

$$\phi_{12} = \int_{S_2} \vec{B}_1 . d\vec{S}_2 \quad \text{.................................(4.48)}$$

In a linear medium, ϕ_{12} is proportional to I_1. Therefore, we can write

$$\phi_{12} = L_{12} I_1 \quad \text{.................................(4.49)}$$

where L_{12} is the mutual inductance. For a more general case, if C_2 has N_2 turns then

$$\Lambda_{12} = N_2 \phi_{12} \quad \text{................................. (4.50)}$$

and $\Lambda_{12} = L_{12} I_1$

or $L_{12} = \dfrac{\Lambda_{12}}{I_1}$ .... (4.51)

i.e., the mutual inductance can be defined as the ratio of the total flux linkage of the second circuit to the current flowing in the first circuit.

As we have already stated, the magnetic flux produced in C_1 gets linked to itself and if C_1 has N_1 turns then $\Lambda_{11} = N_1 \phi_{11}$, where ϕ_{11} is the flux linkage per turn.

Therefore, self inductance

$$L_{11} (\text{or } L \text{ as defined earlier}) = \frac{\Lambda_{11}}{I_1} \quad \text{..(4.52)}$$

As some of the flux produced by I_1 links only to C_1 & not C_2.

$$\Lambda_{11} = N_1\phi_{11} > N_2\phi_{12} = \Lambda_{12} \qquad \text{.................................}(4.53)$$

Further in general, in a linear medium, $L_{12} = \frac{d\Lambda_{12}}{dI_1}$ and $L_{11} = \frac{d\Lambda_{11}}{dI_1}$

Example 1: Inductance per unit length of a very long solenoid:

Let us consider a solenoid having n turns/unit length and carrying a current *I*. The solenoid is air cored.

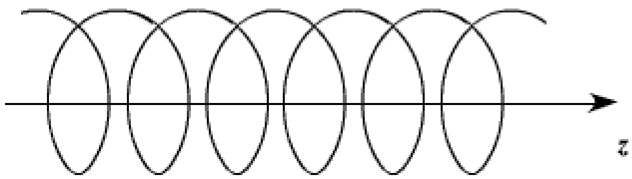

Fig 4.11: A long current carrying solenoid

The magnetic flux density inside such a long solenoid can be calculated as

$$\vec{B} = \mu_0 n I \hat{a}_3 \qquad \text{.................................} (4.54)$$

where the magnetic field is along the axis of the solenoid.

If S is the area of cross section of the solenoid then

$$\phi = BS = \mu_0 n I S \qquad \text{.................................}(4.55)$$

The flux linkage per unit length of the solenoid

$$\Lambda = n\phi = \mu_0 n^2 I S \qquad \text{.................................}(4.56)$$

$\therefore$ The inductance per unit length of the solenoid

$$L = \frac{\Lambda}{I} = \mu_0 n^2 S \qquad \text{.................................}(4.57)$$

Example 2: Self inductance per unit length of a coaxial cable of inner radius 'a' and outer radius 'b'. Assume a current I flows through the inner conductor.

Solution:

Let us assume that the current is uniformly distributed in the inner conductor so that inside the inner conductor.

i.e., $0 \le \rho \le a$

$$\vec{B}_i = \hat{a}_\phi \frac{\mu_0 I}{2\pi\rho} \frac{\pi\rho^2}{\pi a^2}$$

$$= \frac{I \mu_0 \rho}{2\pi a^2} \hat{a}_\phi$$ . ...(4.58)

and in the region , $a \le \rho \le b$

$$\vec{B}_e = \hat{a}_\phi \frac{\mu_0 I}{2\pi\rho}$$ .... (4.59)

Let us consider the flux linkage per unit length in the inner conductor. Flux enclosed between the region ρ and $\rho + d\rho$ (and unit length in the axial direction).

$$d\phi_i = \frac{\mu_0 I}{2\pi a^2} d\rho$$ . ...(4.60)

Fraction of the total current it links is $\frac{\rho^2}{a^2}$

$$\therefore \quad d\Lambda_i = \frac{\mu_0 I}{2\pi} \frac{\rho^3}{a^4} d\rho$$

$$\Lambda_i = \int_0^a \frac{\mu_0 I}{2\pi} \frac{\rho^3}{a^4} d\rho = \frac{\mu_0 I}{8\pi}$$ . ...(4.61)

Similarly for the region $a \le \rho \le b$

$$d\phi_e = \frac{\mu_0 I}{2\pi\rho} d\rho = d\lambda e$$ . ...(4.62)

$$\lambda_e = \frac{\mu_0 I}{2\pi} \int_a^b \frac{d\rho}{\rho} = \frac{\mu_0 I}{2\pi} \ln\frac{b}{a}$$

& ....(4.63)

Total linkage $\Lambda = \Lambda_i + \Lambda_e$

$$= \frac{\mu_0 I}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]$$ . ...(4.64)

The self inductance, $L = \frac{\Lambda_i + \Lambda_e}{I} = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]$ ....(4.65)

Here, the first term $\frac{\mu_0}{8\pi}$ arises from the flux linkage internal to the solid inner conductor and is the internal inductance per unit length.

In high frequency application and assuming the conductivity to be very high, the current in the internal conductor instead of being distributed throughout remain essentially concentrated on the surface of the inner conductor (as we shall see later) and the internal inductance becomes negligibly small.

Example 3: Inductance of an *N* turn toroid carrying a filamentary current *I*.

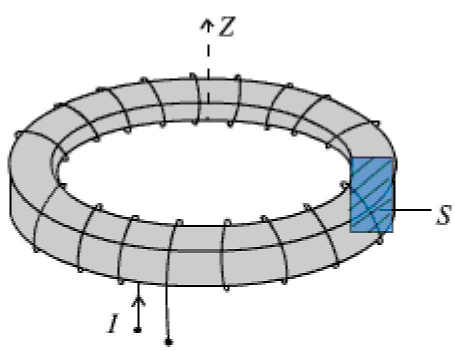

Fig 4.12: ***N*** **turn toroid carrying filamentary current *I*.**

Solution: Magnetic flux density inside the toroid is given by

$$\vec{B} = \frac{\mu I}{2\pi\rho}\hat{a}_\phi \quad \text{.................................(4.66)}$$

Let the inner radius is 'a' and outer radius is 'b'. Let the cross section area 'S' is small compared to the mean radius of the toroid $\rho_0\left[=\frac{a+b}{2}\right]$

Then total flux

$$\phi = N\frac{\mu I}{2\pi\rho_0}S \quad \text{.................................(4.67)}$$

and flux linkage

$$\Lambda = \frac{\mu N^2 IS}{2\pi\rho_0} \qquad \text{.................................(4.68)}$$

The inductance

$$L = \frac{\Lambda}{I} = \frac{\mu N^2 S}{2\pi\rho_0} \qquad \text{.................................(4.69)}$$

Energy stored in Magnetic Field:

So far we have discussed the inductance in static forms. In earlier chapter we discussed the fact that work is required to be expended to assemble a group of charges and this work is stated as electric energy. In the same manner energy needs to be expended in sending currents through coils and it is stored as magnetic energy. Let us consider a scenario where we consider a coil in which the current is increased from 0 to a value *I*. As mentioned earlier, the self inductance of a coil in general can be written as

$$L = \frac{d\Lambda}{di} = N\frac{d\phi}{di} \qquad \text{.................................(4.70a)}$$

or $L\,di = N\,d\phi$ ....(4.70b)

If we consider a time varying scenario,

$$L\frac{di}{dt} = N\frac{d\phi}{dt} \qquad \text{................................ (4.71)}$$

We will later see that $N\frac{d\phi}{dt}$ is an induced voltage.

$\therefore v = L\frac{di}{dt}$ is the voltage drop that appears across the coil and thus voltage opposes the change of current.

Therefore in order to maintain the increase of current, the electric source must do an work against this induced voltage.

$$\begin{aligned} dW &= vi\,dt \\ &= Li\,di \end{aligned} \qquad \text{..(4.72)}$$

& $$W = \int_0^I Li\,di = \frac{1}{2}LI^2$$ (Joule) .... (4.73)

which is the energy stored in the magnetic circuit.

We can also express the energy stored in the coil in term of field quantities.

For linear magnetic circuit

$$W = \frac{1}{2}\frac{N\phi}{I}I^2 = \frac{1}{2}N\phi I$$ .... (4.74)

Now, $$\phi = \int_s \vec{B}.d\vec{S} = BA$$ ....(4.75)

where A is the area of cross section of the coil. If l is the length of the coil

$$NI = Hl$$

$$\therefore W = \frac{1}{2}HBAl$$ .... (4.76)

Al is the volume of the coil. Therefore the magnetic energy density i.e., magnetic energy/unit volume is given by

$$W_m = \frac{W}{Al} = \frac{1}{2}BH$$ .... (4.77)

In vector form

$$W_m = \frac{1}{2}\vec{B}.\vec{H}$$ J/mt^3.. (4.78)

is the energy density in the magnetic field.