UNIT-III

UNIT – III MAGNETOSTATICS

Static Magnetic Fields – Biot-Savart Law – Oerstead's experiment – Magnetic Field Intensity (MFI) due to a Straight, Circular &Solenoid Current Carrying Wire – Maxwell's Second Equation. Ampere's Circuital Law and its Applications Viz., MFI Due to an Infinite Sheet of Current and a Long Current Carrying Filament – Point Form of Ampere's Circuital Law – Maxwell's Third Equation – Numerical Problems.

Magnetic Force — Lorentz Force Equation – Force on Current Element in a Magnetic Field - Force on a Straight and Long Current Carrying Conductor in a Magnetic Field - Force Between two Straight and Parallel Current Carrying Conductors – Magnetic Dipole and Dipole moment – A Differential Current Loop as a Magnetic Dipole – Torque on a Current Loop Placed in a Magnetic Field – Numerical Problems.

7.3 Biot-Savart Law

Consider a conductor carrying a direct current I and a steady magnetic field produced around it. The Biot-Savart law allows us to obtain the **differential magnetic field intensity** $d\overline{H}$, produced at a point P, due to a differential current element IdL. The current carrying conductor is shown in the Fig. 7.6.

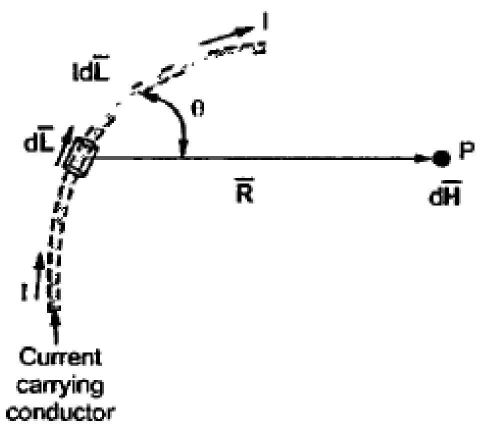


Fig. 7.6

Consider a differential length dL hence the differential current element is IdL. This is very small part of the current carrying conductor. The point P is at a distance R from the differential current element. The θ is the angle between the differential current element and the line joining point P to the differential current element.

The **Biot-Savart law** states that,

The magnetic field intensity $d\overline{H}$ produced at a point P due to a differential current element IdL is,

1. Proportional to the product of the current I and differential length dL.
2. The sine of the angle between the element and the line joining point P to the element.
3. And inversely proportional to the square of the distance R between point P and the element.

Mathematically, the Biot-Savart law can be stated as,

$$d\bar{H} \propto \frac{I\, dL \sin\theta}{R^2} \quad \text{... (1)}$$

$$\therefore \quad \boxed{d\bar{H} = \frac{k\, I\, dL \sin\theta}{R^2}} \quad \text{... (2)}$$

where k = Constant of proportionality

In SI units, $k = \frac{1}{4\pi}$

$$\therefore \quad d\bar{H} = \frac{I\, dL \sin\theta}{4\pi R^2} \quad \text{... (3)}$$

Let us express this equation in vector form.

Let dL = Magnitude of vector length $d\bar{L}$ and

$\bar{a}_R$ = Unit vector in the direction **from differential current element to point P**

Then from rule of cross product,

$$d\bar{L} \times \bar{a}_R = dL\, |\bar{a}_R| \sin\theta = dL \sin\theta \quad \text{... } |\bar{a}_R| = 1$$

Replacing in equation (3),

$$d\bar{H} = \frac{I\, d\bar{L} \times \bar{a}_R}{4\pi R^2} \text{ A/m} \quad \text{... (4)}$$

But $\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\bar{R}}{R}$

Hence,

$$\boxed{d\bar{H} = \frac{I\, d\bar{L} \times \bar{R}}{4\pi R^3} \text{ A/m}} \quad \text{... (5)}$$

The equations (4) and (5) is the mathematical form of Biot-Savart law.

According to the direction of cross product, the direction of $d\bar{H}$ is normal to the plane containing two vectors and in that normal direction which is along the progress of right handed screw, turned from $d\bar{L}$ through the smaller angle θ towards the line joining element to the point P. Thus the direction of $d\bar{H}$ is normal to the plane of paper. For the case considered, according to right handed screw rule, the direction of $d\bar{H}$ is going into the plane of the paper.

The entire conductor is made up of all such differential elements. Hence to obtain total magnetic field intensity $\bar{H}$, the above equation (4) takes the integral form as,

$$\boxed{\bar{H} = \oint \frac{I\, d\bar{L} \times \bar{a}_R}{4\pi R^2}} \quad \text{... (6)}$$

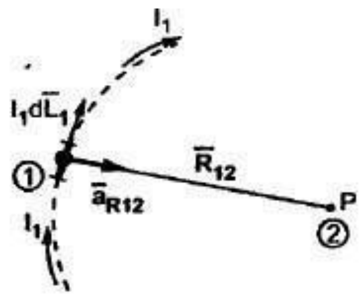


Fig. 7.7

The closed line integral is required to ensure that all the current elements are considered. This is because current can flow only in the closed path, provided by the closed circuit. If the current element is considered at point 1 and point P at point 2, as shown in the Fig. 7.7 then,

$$d\bar{H}_2 = \frac{I_1\, d\bar{L}_1 \times \bar{a}_{R12}}{4\pi R_{12}^2} \text{ A/m} \qquad \text{... (7)}$$

where I_1 = Current flowing through dL_1 at point 1

$d\bar{L}_1$ = Differential vector length at point 1

$\bar{a}_{R12}$ = Unit vector in the direction from element at point 1 to the point P at point 2

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{R}_{12}}{R_{12}}$$

$$\therefore \quad \boxed{\bar{H}_2 = \oint \frac{I_1\, d\bar{L}_1 \times \bar{a}_{R12}}{4\pi R_{12}^2} \text{ A/m}} \qquad \text{... (8)}$$

This is called **integral form of Biot-Savart law.**

7.3.1 Biot-Savart Law Interms of Distributed Sources

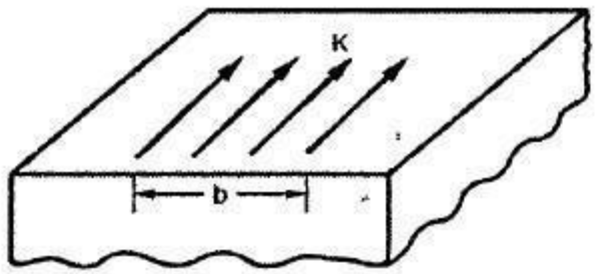


Fig. 7.8 Surface current density

Consider a surface carrying a uniform current over its surface as shown in the Fig. 7.8. Then the **surface current density** is denoted as $\bar{K}$ and measured in amperes per metre (A/m). Thus for uniform current density, the current I in any width b is given by $I = Kb$, where width b is perpendicular to the direction of current flow.

Thus if dS is the differential surface area considered of a surface having current density $\bar{K}$ then,

$$I\, d\bar{L} = \bar{K}\, dS \qquad \text{... (9)}$$

If the current density in a volume of a given conductor is $\bar{J}$ measured in A/m^2 then for a differential volume dv we can write,

$$I\, d\bar{L} = \bar{J}\, dv \qquad \text{... (10)}$$

Hence the Biot-Savart law can be expressed for surface current considering $\bar{K}$ dS while for volume current considering $\bar{J}$ dv.

$$\therefore \qquad \boxed{\bar{H} = \int_S \frac{\bar{K} \times \bar{a}_R\, dS}{4\pi R^2} \text{ A/m}} \qquad \text{... (11)}$$

and

$$\boxed{\bar{H} = \int_{vol} \frac{\bar{J} \times a_R\, dv}{4\pi R^2} \text{ A/m}} \qquad \text{... (12)}$$

The Biot-Savart law is also called **Ampere's law for the current element.** Let us study now the various applications of Biot-Savart law.

7.4 $\bar{H}$ due to Infinitely Long Straight Conductor

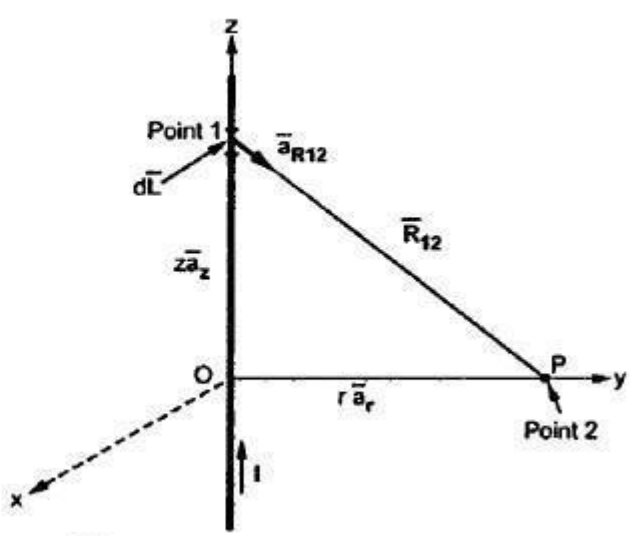


Fig. 7.10 $\bar{H}$ due to infinitely long straight conductor

Consider an infinitely long straight conductor, along z-axis. The current passing through the conductor is a direct current of I Amp. The field intensity $\bar{H}$ at a point P is to be calculated, which is at a distance 'r' from the z-axis. This is shown in the Fig. 7.10.

Consider small differential element at point 1, along the z-axis, at a distance z from origin.

$$\therefore \quad I\, d\bar{L} = I\, dz\, \bar{a}_z \qquad \text{... (1)}$$

The distance vector joining point 1 to point 2 is $\bar{R}_{12}$ and can be written as,

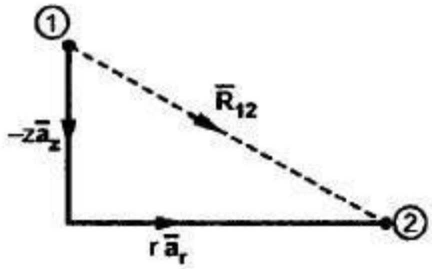


Fig. 7.11

$$\bar{R}_{12} = -z\,\bar{a}_z + r\,\bar{a}_r \quad \text{... (2)}$$

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}} \quad \text{... (3)}$$

$$\therefore\ d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix} = r\,dz\,\bar{a}_\phi$$

While obtaining cross product, $|\bar{R}_{12}|$ is neglected for convenience and must be considered for further calculations.

$$\therefore \quad I\,d\bar{L} \times \bar{a}_{R12} = \frac{I\,r\,dz\,\bar{a}_\phi}{\sqrt{r^2+z^2}} \quad \text{... (4)}$$

According to Biot-Savart law, $d\bar{H}$ at point 2 is,

$$d\bar{H} = \frac{I\,dL \times a_{R12}}{4\pi R_{12}^2} = \frac{I\,r\,dz\,\bar{a}_\phi}{4\pi\sqrt{r^2+z^2}\,(\sqrt{r^2+z^2})^2}$$

$$= \frac{I\,r\,dz\,\bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \quad \text{... (5)}$$

Thus total field intensity $\bar{H}$ can be obtained by integrating $d\bar{H}$ over the entire length of the conductor.

$$\therefore \quad \bar{H} = \int_{z=-\infty}^{\infty} d\bar{H} = \int_{z=-\infty}^{\infty} \frac{I\,r\,dz\,\bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \quad \text{... (6)}$$

Put $z = r\tan\theta$, $z^2 = r^2\tan^2\theta$

and $dz = r\sec^2\theta\,d\theta$, $z = -\infty$, $\theta = -\frac{\pi}{2}$ and $z = +\infty$, $\theta = +\frac{\pi}{2}$

$$\therefore \quad \bar{H} = \int_{\theta=-\pi/2}^{\pi/2} \frac{I\,r\,r\sec^2\theta\,d\theta\,\bar{a}_\phi}{4\pi\left(r^2+r^2\tan^2\theta\right)^{3/2}}$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \frac{I\,r^2\sec^2\theta\,d\theta\,\bar{a}_\phi}{4\pi r^3\sec^3\theta} \quad \text{... } 1+\tan^2\theta = \sec^2\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \frac{I}{4\pi r}\,\frac{1}{\sec\theta}\,d\theta\,a_\phi = \frac{I}{4\pi r}\int_{\theta=-\pi/2}^{\pi/2} \cos\theta\,d\theta\,\bar{a}_\phi$$

$$= \frac{I}{4\pi r}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} \bar{a}_\phi = \frac{I}{4\pi r}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]\bar{a}_\phi$$

$$= \frac{I}{4\pi r}\left[1-(-1)\right] a_\phi = \frac{2I}{4\pi r}\bar{a}_\phi$$

$$\therefore \quad \boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \quad A/m} \quad \text{... (7)}$$

$$\boxed{\bar{B} = \mu \bar{H} = \frac{\mu I}{2\pi r}\bar{a}_\phi \ \ Wb/m^2} \quad \text{... (8)}$$

The following observations are important about $\bar{H}$:

1. The magnitude of magnetic field intensity $\bar{H}$ is not a function of ϕ or z. It is inversely proportional to r which is the perpendicular distance of the point from the conductor.

2. The direction of $\bar{H}$ is tangential i.e. circumferential along $\bar{a}_\phi$. This direction is going into the plane of the paper at point P.

3. The streamlines i.e. magnetic flux lines are in the form of concentric circles around the conductor. Thus if conductor is viewed from the top with I coming out of the paper towards observer, then the streamlines are anticlockwise.

7.6 $\bar{H}$ at the Centre of a Circular Conductor

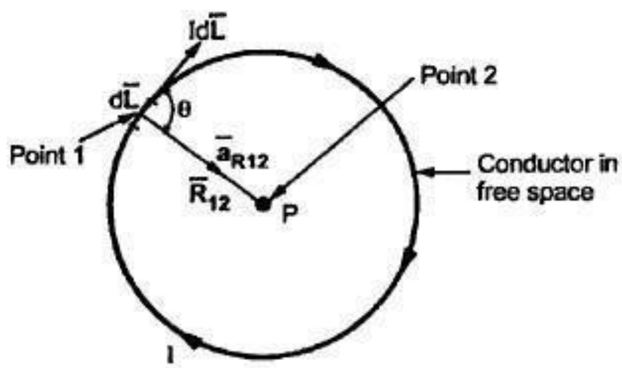


Fig. 7.15

Consider the current carrying conductor arranged in a circular form as shown in the Fig. 7.15.

The $\bar{H}$ at the centre of the circular loop is to be obtained. The conductor carries the direct current I.

Consider the differential length $d\bar{L}$ at a point 1.

The direction of $d\bar{L}$ at a point 1 is tangential to the circular conductor at point 1.

Let θ = Angle between I $d\bar{L}$ and $\bar{a}_{R12}$

$\bar{a}_{R12}$ = Unit vector in the direction of $\bar{R}_{12}$

$\bar{R}_{12}$ = Distance vector joining differential current element at point 1 to point P at point 2 which is centre of circle.

Using the definition of cross product,

$$\therefore \quad I\,d\bar{L} \times \bar{a}_{R12} = I\left|d\bar{L}\right|\left|\bar{a}_{R12}\right| \sin\theta\,\bar{a}_N = I\,dL\,\sin\theta\,\bar{a}_N \quad \text{... (1)}$$

$\bar{a}_N$ = Unit vector normal to the plane containing $d\bar{L}$ and $\bar{a}_{R12}$ i.e. normal to the plane in which the circular conductor is lying

According to Biot-Savart law, the differential magnetic field intensity $d\bar{H}$ at point P is,

$$d\bar{H} = \frac{I\,d\bar{L}\times\bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\,dL\sin\theta\,\bar{a}_N}{4\pi R^2} \quad \text{... } R = R_{12} = \text{Radius}$$

Hence total magnetic field intensity $\bar{H}$ at point P can be obtained by integrating $d\bar{H}$ around the circular closed path.

$$\therefore \quad \bar{H} = \oint d\bar{H} = \oint \frac{I\,dL\sin\theta\,\bar{a}_N}{4\pi R^2} = \frac{I\sin\theta\,\bar{a}_N}{4\pi R^2}\oint dL \quad \text{... (2)}$$

$$\text{But} \quad \oint dL = \text{Circumference of the circle} = 2\pi R \quad \text{... (3)}$$

$$\therefore \quad \bar{H} = \frac{I\sin\theta\,2\pi R\,\bar{a}_N}{4\pi R^2} = \frac{I\sin\theta}{2R}\,\bar{a}_N \quad \text{... (4)}$$

As $Id\bar{L}$ is tangential to the circle and R_{12} is the radius, angle θ must be 90°.

$$\therefore \quad \boxed{\bar{H} = \frac{I\sin 90^\circ}{2R}\bar{a}_N = \frac{I}{2R}\bar{a}_N \quad \text{A/m}} \quad \text{... (5)}$$

$\bar{a}_N = \bar{a}_z$ if the circular loop is placed in xy plane

Now $\bar{B} = \mu_0\bar{H}$ ... for free space

The flux density $\bar{B}$ at centre of the circular conductor carrying direct current I, placed in a free space is given by,

$$\boxed{\bar{B} = \frac{\mu_0 I}{2R}\bar{a}_N \quad \text{Wb/m}^2} \quad \text{... (6)}$$

7.7 $\bar{H}$ on the Axis of a Circular Loop

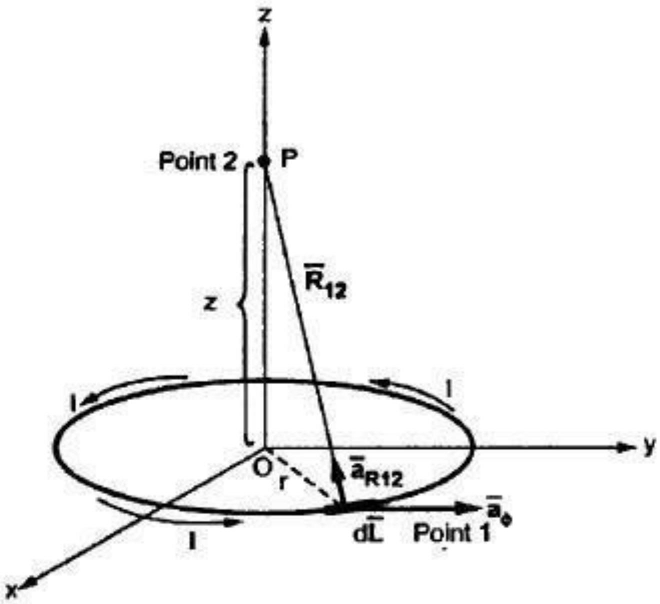


Fig. 7.16

Consider a circular loop carrying a direct current I, placed in xy plane, with z-axis as its axis as shown in the Fig. 7.16. The magnetic field intensity $\bar{H}$ at point P is to be obtained. The point P is at a distance z from the plane of the circular loop, along it's axis.

The radius of the circular loop is r. Consider the differential length $d\bar{L}$ of the circular loop as shown in the Fig. 7.16.

In the cylindrical co-ordinate system,

$$d\bar{L} = dr\,\bar{a}_r + r\,d\phi\bar{a}_\phi + dz\bar{a}_z$$

But $d\bar{L}$ is in the plane for which r is constant and z = 0 = constant plane. The $I\,d\bar{L}$ is tangential at point 1 in $\bar{a}_\phi$ direction.

$$\therefore \qquad I\,d\bar{L} = I\,r\,d\phi\,\bar{a}_\phi \qquad \text{... (1)}$$

The unit vector $\bar{a}_{R12}$ is in the direction along the line joining differential current element to the point P.

$$\therefore \qquad \bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} \qquad \text{... (2)}$$

From the Fig. 7.17, it can be observed that,

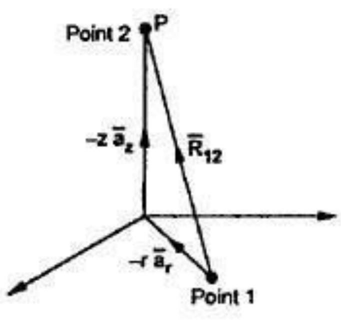


Fig. 7.17

$$\bar{R}_{12} = -r\,\bar{a}_r + z\,\bar{a}_z \qquad \text{... from 1 to 2}$$

$$\therefore \qquad |\bar{R}_{12}| = \sqrt{(-r)^2 + (z)^2} = \sqrt{r^2 + z^2}$$

$$\therefore \qquad \bar{a}_{R12} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}} \qquad \text{... (3)}$$

$$\text{Now} \quad d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & rd\phi & 0 \\ -r & 0 & z \end{vmatrix} = zr\,d\phi\,\bar{a}_r + r^2\,d\phi\bar{a}_z$$

Note that while calculating cross product $|R_{12}|$ is neglected for convenience, which must be considered in further calculations.

According to Biot-Savart law, the differential field strength $d\overline{H}$ at point P is given by,

$$d\overline{H} = \frac{I\, d\overline{L}\times\overline{a}_{R12}}{4\pi R_{12}^2} = \frac{I\left[z\, r\, d\phi\, \overline{a}_r + r^2 d\phi\, \overline{a}_z\right]}{4\pi\sqrt{r^2+z^2}\,(\sqrt{r^2+z^2})^2} \qquad \text{... (4)}$$

Note that $\left|\overline{R}_{12}\right|$ neglected while obtaining the cross product is considered in $d\overline{H}$.

The total $\overline{H}$ is to be obtained by integrating $d\overline{H}$ over the circular loop i.e. for $\phi=0$ to 2π.

Note : It can be observed that though $d\overline{H}$ consists of two components $\overline{a}_r$ and $\overline{a}_z$, due to radial symmetry all $\overline{a}_r$ components are going to cancel each other. So $\overline{H}$ exists only along the axis in $\overline{a}_z$ direction. Let us prove this mathematically.

$$\overline{H} = \int_{\phi=0}^{2\pi} \frac{I\left[z\, r\, \overline{a}_r + r^2\, \overline{a}_z\right] d\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \qquad \text{... (5)}$$

$$= \frac{I}{4\pi}\left\{\int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}}\overline{a}_r + \int_{\phi=0}^{2\pi} \frac{r^2\, \overline{a}_z\, d\phi}{\left(r^2+z^2\right)^{3/2}}\right\} \qquad \text{... (6)}$$

Consider first integral to prove that its value is zero due to radial symmetry.

$$\int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}}\overline{a}_r = \int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}}\left[\cos\phi\, \overline{a}_x + \sin\phi\, \overline{a}_y\right]$$

The unit vector $\overline{a}_r$ is expressed in rectangular co-ordinate system as

$\cos\phi\overline{a}_x + \sin\phi\overline{a}_y$.

Now $\quad \displaystyle\int_{\phi=0}^{2\pi} \cos\phi\, d\phi = [\sin\phi]_{\phi=0}^{2\pi} = \sin 2\pi - \sin 0 = 0$

And $\quad \displaystyle\int_{\phi=0}^{2\pi} \sin\phi\, d\phi = [-\cos\phi]_0^{2\pi} = -\cos 2\pi - [-\cos 0] = -1+1 = 0$

$$\therefore \quad \int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}}\, \overline{a}_r = 0$$

This proves that $\overline{H}$ at P can not have any radial component.

$$\therefore \qquad \overline{H} = \frac{I}{4\pi}\int_{\phi=0}^{2\pi} \frac{r^2\, d\phi}{\left(r^2+z^2\right)^{3/2}}\overline{a}_z = \frac{I\, r^2\, \overline{a}_z}{4\pi\left(r^2+z^2\right)^{3/2}}\int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{I\, r^2\, \overline{a}_z\, [\phi]_0^{2\pi}}{4\pi\left(r^2+z^2\right)^{3/2}} = \frac{I\, r^2\, 2\pi\overline{a}_z}{4\pi\left(r^2+z^2\right)^{3/2}}$$

$$\therefore \qquad \bar{H} = \frac{I r^2}{2(r^2+z^2)^{3/2}} \bar{a}_z \ \text{A/m} \qquad \text{... (7)}$$

where $\quad r$ = Radius of the circular loop

$\quad z$ = Distance of point P along the axis

Note : If point P is shifted at the centre of the circular loop i.e. $z = 0$, we get the result obtained in earlier section.

$$\boxed{\bar{H} = \frac{I r^2}{2(r^2)^{3/2}} \bar{a}_z = \frac{I}{2r} \bar{a}_z \ \text{A/m}}$$

where a_z is the unit vector normal to xy plane in which the circular loop is lying.

7.8 Ampere's Circuital Law

In electrostatics, the Gauss's law is useful to obtain the $\bar{E}$ in case of complex problems. Similarly in the magnetostatics, the complex problems can be solved using a law called **Ampere's circuital law** or **Ampere's work law.**

The Ampere's circuital law states that,

The line integral of magnetic field intensity $\bar{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

The mathematical representation of Ampere's circuital law is,

$$\boxed{\oint \bar{H} \cdot d\bar{L} = I} \qquad \text{... (1)}$$

The law is very helpful to determine $\bar{H}$ when the current distribution is symmetrical.

7.8.1 Proof of Ampere's Circuital Law

Consider a long straight conductor carrying direct current I placed along z axis as shown in the Fig. 7.26. Consider a closed circular path of radius r which encloses the straight conductor carrying direct current I. The point P is at a perpendicular distance r from the conductor. Consider $d\bar{L}$ at point P which is in $\bar{a}_\phi$ direction, tangential to circular path at point P.

$$\therefore \qquad d\bar{L} = r \, d\phi \, \bar{a}_\phi \qquad \text{... (2)}$$

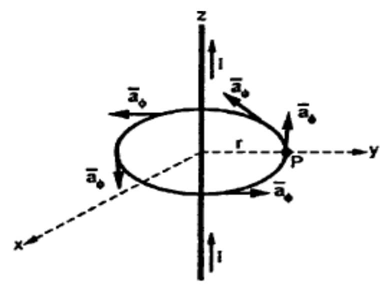


Fig. 7.26

While $\overline{H}$ obtained at point P, from Biot-Savart law due to infinitely long conductor is,

$$\overline{H} = \frac{I}{2\pi r}\bar{a}_\phi \qquad \text{... (3)}$$

$$\therefore \qquad \overline{H}\cdot d\overline{L} = \frac{I}{2\pi r}\bar{a}_\phi \cdot r\, d\phi\, \bar{a}_\phi$$

$$= \frac{I}{2\pi r} r\, d\phi = \frac{I}{2\pi} d\phi \qquad \text{.....} \ \bar{a}_\phi \cdot \bar{a}_\phi = 1$$

Integrating $\overline{H}\cdot d\overline{L}$ over the entire closed path,

$$\oint \overline{H}\cdot d\overline{L} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi}[\phi]_0^{2\pi} = \frac{I\, 2\pi}{2\pi}$$

$$= I = \text{Current carried by conductor}$$

This proves that the integral $\overline{H}\cdot d\overline{L}$ along the closed path gives the direct current enclosed by that closed path.

7.9 Applications of Ampere's Circuital Law

Let us steady the various cases and the application of Ampere's circuital law to obtain $\bar{H}$.

7.9.1 $\overline{H}$ due to Infinitely Long Straight Conductor

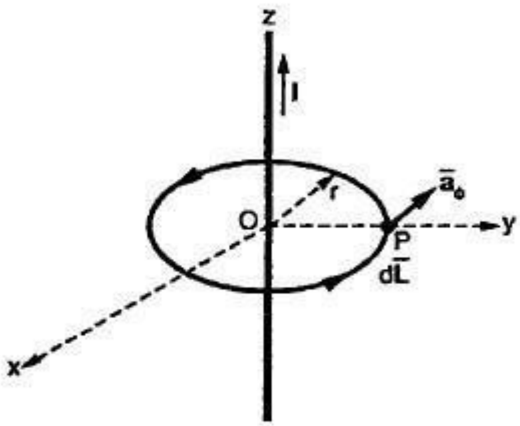


Fig. 7.27

Consider an infinitely long straight conductor placed along z-axis, carrying a direct current I as shown in the Fig. 7.27. Consider the Amperian closed path, enclosing the conductor as shown in the Fig. 7.27. Consider point P on the closed path at which $\overline{H}$ is to be obtained. The radius of the path is r and hence P is at a perpendicular distance r from the conductor.

The magnitude of $\overline{H}$ depends on r and the direction is always tangential to the closed path i.e. $\bar{a}_\phi$. So $\overline{H}$ has only component in $\bar{a}_\phi$ direction say H_ϕ.

Consider elementary length $d\overline{L}$ at point P and in cylindrical co-ordinates it is $r\,d\phi$ in $\bar{a}_\phi$ direction.

$$\therefore \quad \overline{H} = H_\phi \bar{a}_\phi \quad \text{and} \quad d\overline{L} = r\,d\phi\bar{a}_\phi$$

$$\therefore \quad \overline{H}\cdot d\overline{L} = H_\phi \bar{a}_\phi \cdot r\,d\phi\bar{a}_\phi = H_\phi\, r\,d\phi$$

According to Ampere's circuital law,

$$\oint \overline{H}\cdot d\overline{L} = I$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\,d\phi = I$$

$$\therefore \quad H_\phi\, r \int_{\phi=0}^{2\pi} d\phi = I$$

$$\therefore \quad H_\phi\, r(2\pi) = I$$

$$\therefore \quad H_\phi = \frac{I}{2\pi r}$$

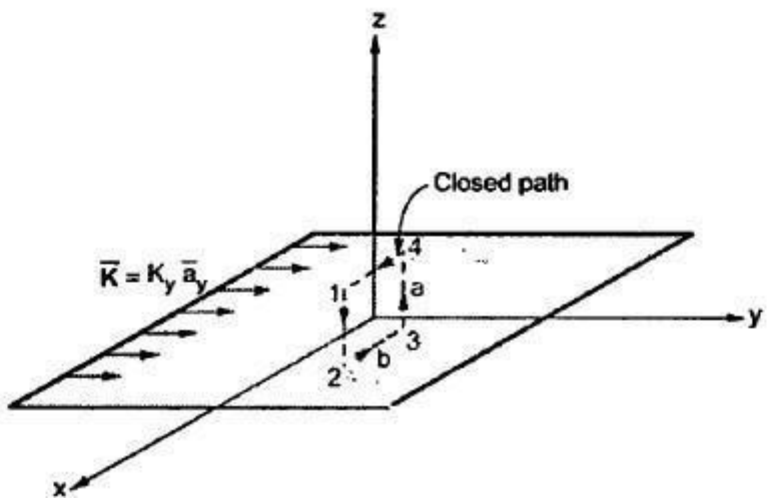
Hence $\overline{H}$ at point P is given by,

$$\boxed{\overline{H} = H_\phi \bar{a}_\phi = \frac{I}{2\pi r}\bar{a}_\phi \quad \text{A/m}}$$

7.9.3 $\bar{H}$ due to Infinite Sheet of Current

Consider an infinite sheet of current in the z = 0 plane. The surface current density is $\bar{K}$. The current is flowing in positive y direction hence $\bar{K} = K_y \bar{a}_y$. This is shown in the Fig. 7.32.

Consider a closed path 1-2-3-4 as shown in the Fig. 7.32. The width of the path is b while the height is a. It is perpendicular to the direction of current hence in xz plane.



The current flowing across the distance b is given by $K_y b$.

$$\therefore \qquad I_{enc} = K_y b \qquad \text{... (6)}$$

Consider the magnetic lines of force due to the current in $\bar{a}_y$ direction, according to right hand thumb rule. These are shown in the Fig. 7.33.

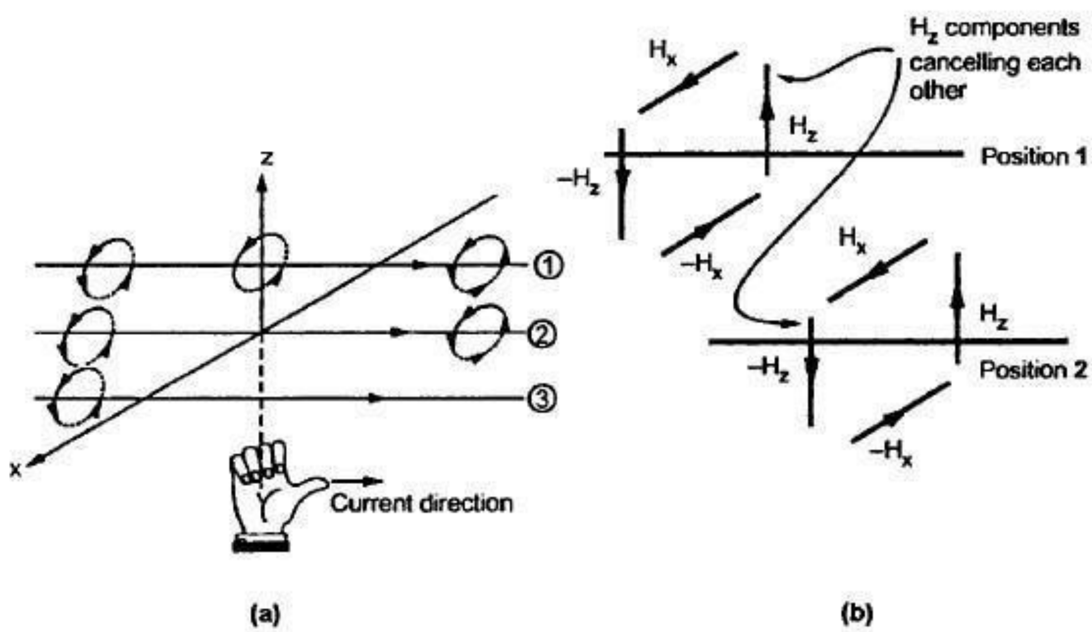


Fig. 7.33

In Fig. 7.33 (b), it is clear that in between two very closely spaced conductors, the components of $\overline{H}$ in z direction are oppositely directed ($-H_z$ for position 1 and $+H_z$ for position 2 between the two positions). All such components cancel each other and hence $\overline{H}$ can not have any component in z direction.

As current is flowing in y direction, $\overline{H}$ can not have component in y direction.

So $\overline{H}$ has only component in x direction.

$$\therefore \qquad \overline{H} = H_x \, \bar{a}_x \qquad \text{... for } z > 0 \qquad \text{... (7 (a))}$$

$$= -H_x \, \bar{a}_x \qquad \text{... for } z < 0 \qquad \text{... (7 (b))}$$

Applying Ampere's circuit law,

$$\oint \overline{H} \cdot d\overline{L} = I_{enc} \qquad \text{... (8)}$$

Evaluate the integral along the path 1-2-3-4-1.

For path 1-2, $d\overline{L} = dz \, \bar{a}_z$,

For path 3-4, $d\overline{L} = dz \, \bar{a}_z$

But $\overline{H}$ is in x direction while $\bar{a}_x \cdot \bar{a}_z = 0$.

Hence **along the paths 1-2 and 3-4,** the integral $\oint \overline{H} \cdot dL = 0$.

Consider path 2-3 along which $d\overline{L} = dx \, \bar{a}_x$.

$$\therefore \qquad \int_2^3 H \cdot dL = \int_2^3 (-H_x \, \bar{a}_x) \cdot (dx \bar{a}_x) = H_x \int_2^3 dx = b \, H_x$$

The path 2-3 is lying in $z < 0$ region for which $\overline{H}$ is $-H_x \, \bar{a}_x$. **And limits from 2 to 3, positive x to negative x hence effective sign of the integral is positive.**

Consider path 4-1 along which $d\overline{L} = dx \, \bar{a}_x$ and it is in the region $z > 0$ hence $\overline{H} = H_x \, \bar{a}_x$.

$$\therefore \qquad \int_4^1 H \cdot dL = \int_1^1 (H_x \, \bar{a}_x) \cdot (dx \bar{a}_x) = H_x \int_1^1 dx = b \, H_x$$

$$\therefore \qquad \oint \overline{H} \cdot d\overline{L} = b \, H_x + b \, H_x = 2 \, b \, H_x \qquad \text{... (9)}$$

Equating this to I_{enc} in equation (6),

$$2 \, b \, H_x = K_y \, b$$

$$\therefore \qquad H_x = \frac{1}{2} K_y \qquad \text{... (10)}$$

$$\text{Hence,} \qquad \overline{H} = \frac{1}{2} K_y \, \bar{a}_x \quad \text{for} \quad z > 0 \qquad \text{... (11 (a))}$$

$$= -\frac{1}{2} K_y \, \bar{a}_x \quad \text{for} \quad z < 0 \qquad \text{... (11 (b))}$$

In general, for an infinite sheet of current density $\bar{K}$ A/m we can write,

$$\boxed{\bar{H} = \frac{1}{2}\bar{K}\times\bar{a}_N} \quad \text{... (12)}$$

where $\bar{a}_N$ = Unit vector normal from the current sheet to the point at which $\bar{H}$ is to be obtained.

8.2 Force on a Moving Point Charge

According to the discussion in the previous chapters, a static electric field $\bar{E}$ exerts a force on a static or moving charge Q. Thus according to Coulomb's law, the force $\bar{F}_e$ exerted on an electric charge can be obtained. The force is related to the electric field intensity $\bar{E}$ as,

$$\boxed{\bar{F}_e = Q\,\bar{E}\ \ N} \quad \text{... (1)}$$

For a positive charge, the force exerted on it is in the direction of $\bar{E}$. This force is also refered as **electric force** $(\bar{F}_e)$.

Now consider that a charge is placed in a steady magnetic field. It experiences a force only if it is moving. Then a **magnetic force** $(\bar{F}_m)$ exerted on a charge Q, moving with a velocity $\bar{v}$ in a steady magnetic field $\bar{B}$ is given by,

$$\boxed{\bar{F}_m = Q\,\bar{v}\times\bar{B}\ \ N} \quad \text{... (2)}$$

The magnitude of the magnetic force $\bar{F}_m$ is directly proportional to the magnitudes of Q, $\bar{v}$ and $\bar{B}$ and also the sine of the angle between $\bar{v}$ and $\bar{B}$. The direction of $\bar{F}_m$ is perpendicular to the plane containing $\bar{v}$ and $\bar{B}$ both, as shown in the Fig. 8.1.

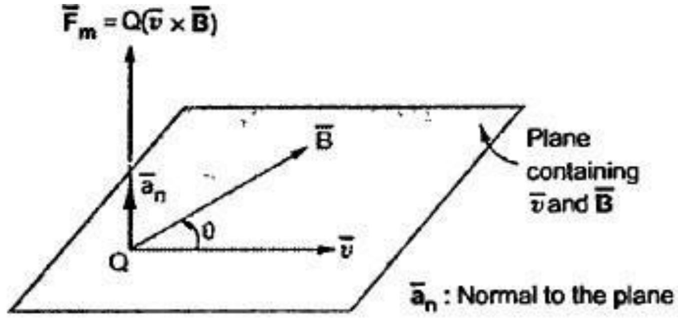


Fig. 8.1 Magnetic force on a moving charge in magnetic field

From equation (1) it is clear that the electric force $\bar{F}_e$ is independent of the velocity of the moving charge. In other words, the electric force exerted on the moving charge by the electric field is independent of the direction in which the charge is moving. Thus the electric force performs work on the charge. On the other hand, the magnetic force $\bar{F}_m$ is dependent on the velocity of the moving charge. But $\bar{F}_m$ cannot perform work on a moving charge as it is at right angle to the direction of motion of charge. ($\bar{F} \cdot d\bar{L} = 0$).

The total force on a moving charge in the presence of both electric and magnetic fields is given by,

$$\boxed{F = \bar{F}_e + \bar{F}_m = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) \quad N} \qquad \text{... (3)}$$

Above equation is called **Lorentz Force Equation** which relates mechanical force to the electrical force. If the mass of the charge is m, then we can write,

$$\boxed{\bar{F} = m\,\bar{a} = m\,\frac{d\bar{v}}{dt} = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) \quad N} \qquad \text{... (4)}$$

8.3 Force on a Differential Current Element

The force exerted on a differential element of charge dQ moving in a steady magnetic field is given by,

$$d\bar{F} = dQ\,\bar{v} \times \bar{B} \quad N \qquad \text{... (1)}$$

The current density $\bar{J}$ can be expressed interms of velocity of a volume charge density as,

$$\bar{J} = \rho_v\,\bar{v} \qquad \text{... (2)}$$

But the differential element of charge can be expressed in terms of the volume charge density as,

$$dQ = \rho_v \; dv \qquad \text{... (3)}$$

Substituting value of dQ in equation (1),

$$\therefore \qquad d\bar{F} = \rho_v \, dv \, \bar{v} \times \bar{B}$$

Expressing $d\bar{F}$ interms of $\bar{J}$ using equation (2), we can write,

$$d\bar{F} = \bar{J} \times \bar{B} \; dv \qquad \text{... (4)}$$

But we have already studied in previous chapters, the relationship between current element as,

$$\bar{J} \, dv = \bar{K} \, dS = I \, d\bar{L}$$

Then the force exerted on a surface current density is given by,

$$d\bar{F} = \bar{K} \times \bar{B} \; dS \qquad \text{... (5)}$$

Similarly the force exerted on a differential current element is given by,

$$dF = (I \, dL \times B) \qquad \text{... (6)}$$

Integrating equation (4) over a volume, the force is given by,

$$\bar{F} = \int_{vol} \bar{J} \times \bar{B} \, dv \qquad \text{... (7)}$$

Integrating equation (5) over either open or closed surface, we get,

$$\bar{F} = \int_{S} \bar{K} \times \bar{B} \; dS \qquad \text{... (8)}$$

Similarly integrating equation (6) over a closed path, we get,

$$\boxed{\bar{F} = \oint I \, d\bar{L} \times \bar{B}} \qquad \text{... (9)}$$

If a conductor is straight and the field $\bar{B}$ is uniform along it, then integrating equation (6) we get simple expression for the force as,

$$\boxed{\bar{F} = I \, \bar{L} \times \bar{B}} \qquad \text{... (10)}$$

The magnitude of the force is given by,

$$\boxed{F = I \, L \, B \sin \theta} \qquad \text{... (11)}$$

Actually the magnetic field exerts a magnetic force on the electrons which constitutes the current I. But these electrons are part of the conductor, this magnetic force gets transfered to the conductor lattice. Now this transfered force can perform work on a conductor as a whole.

8.4 Force between Differential Current Elements

While discussing the electrostatic fields, we have studied that a point charge exert force on another point charge, separated by distance R. If these charges are of same type (i.e. both positive or negative), then they repel each other. But when two charges are of different type (i.e. one positive and other negative), then they attract each other.

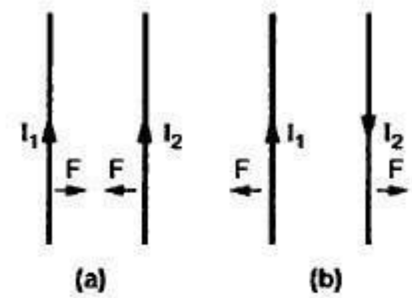


Fig. 8.2 Force between two parallel current carrying conductors

Now consider that two current carrying conductors are placed parallel to each other. Each of this conductor produces its own flux around it. So when such two conductors are placed closed to each other, there exists a force due to the interaction of two fluxes. The force between such parallel current carrying conductors depends on the directions of the two currents. If the directions of both the currents are same, then the conductors experience a force of attraction as shown in the Fig. 8.2 (a). And if the directions of two currents are opposite to each other, then the conductors experience a force of repulsion as shown in the Fig. 8.2 (b).

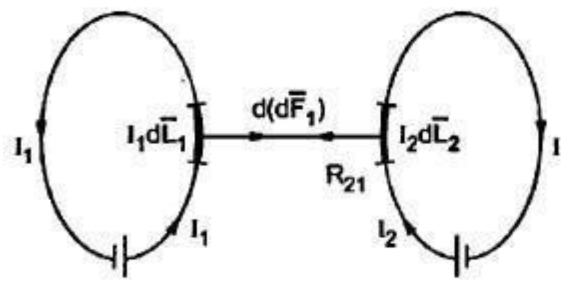


Fig. 8.3 Force between two current elements

Let us now consider two current elements $I_1\, d\bar{L}_1$ and $I_2 d\bar{L}_2$ as shown in the Fig. 8.3. Note that the directions of I_1 and I_2 are same.

Both the current elements produce their own magnetic fields. As the currents are flowing in the same direction through the elements, the force $d(d\bar{F}_1)$ exerted on element $I_1 d\bar{L}_1$ due to the magnetic field $d\bar{B}_2$ produced by other element $I_2 d\bar{L}_2$ is the force of attraction.

From the equation of force the force exerted on a differential current element is given by,

$$d\left(d\bar{F}_1\right) = I_1 d\bar{L}_1 \times d\bar{B}_2 \qquad \text{... (1)}$$

According to Biot-Savart's law, the magnetic field produced by current element $I_2 d\bar{L}_2$ is given by, for free space,

$$d\bar{B}_2 = \mu_0 d\bar{H}_2 = \mu_0 \left[\frac{I_2 d\bar{L}_2 \times \bar{a}_{R21}}{4\pi R_{21}^2}\right] \qquad \text{... (2)}$$

Substituting value of $d\bar{B}_2$ in equation (1), we can write,

$$d(d\bar{F}_1) = \mu_0 \frac{I_1 d\bar{L}_1 \times \left(I_2 d\bar{L}_2 \times \bar{a}_{R21}\right)}{4\pi R_{21}^2} \qquad \text{... (3)}$$

The equation (3) represents force between two current elements. It is very much similar to Coulomb's law. By integrating $d(d\,F_2)$ twice, the total force $\bar{F}_1$ on current element 1 due to current element 2 is given by,

$$\boxed{\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times \left(d\bar{L}_2 \times \bar{a}_{R21}\right)}{R_{21}^2}} \qquad \text{... (4)}$$

Exactly following same steps, we can calculate the force $\bar{F}_2$ exerted on the current element 2 due to the magnetic field $\bar{B}_1$ produced by the current element 1. Thus,

$$\boxed{\bar{F}_2 = \frac{\mu_0 I_2 I_1}{4\pi} \oint_{L_2} \oint_{L_1} \frac{d\bar{L}_2 \times \left(d\bar{L}_1 \times \bar{a}_{R12}\right)}{R_{12}^2}} \qquad \text{... (5)}$$

Actually equation (5) is obtained from equation (4) by interchanging the subscripts 1 and 2. By using back-cab rule for expanding vector triple product, we can show that

$$\bar{F}_2 = -\bar{F}_1 \qquad \text{... (6)}$$

Thus, above condition indicates that both the forces $\bar{F}_1$ and $\bar{F}_2$ obey Newton's third law that for every action there is equal and opposite reaction.

For the two current carrying conductors of length l each, the force exerted is given by

$$\boxed{F = \frac{\mu I_1 I_2 l}{2\pi d}} \qquad \text{... (7)}$$

where I_1 and I_2 are the currents flowing through conductor 1 and conductor 2 and d is the distance of separation between two conductors.

If the two currents flow in same directions, the current carrying conductors attract each other. While if, the two currents flow in opposite direction to each other, the current carrying conductors repel each other.

8.5.2 Magnetic Dipole Moment

The **magnetic dipole moment** of a current loop is defined as the product of current through the loop and the area of the loop, directed normal to the current loop. From the definition it is clear that, the magnetic dipole moment is a vector quantity. It is denoted by $\overline{m}$. The direction of the magnetic dipole moment $\overline{m}$ is given by the right hand thumb rule. The right hand thumb indicates the direction of the unit vector in which $\overline{m}$ is directed and the figures represents the current direction. The **magnetic dipole moment** is given by

$$\boxed{\overline{m} = (IS)\overline{a}_n \;\; A \cdot m^2} \qquad \dots (12)$$

In the previous section we have obtained the expression for the torque along the axis of rotation of a planar coil as,

$$\overline{T} = BIS\left(-\overline{a}_y\right)$$

Using definition for the magnetic dipole moment, the torque can be expressed as,

$$\boxed{\overline{T} = \overline{m} \times \overline{B} \quad Nm} \qquad \dots (13)$$

Above expression is in general applicable in calculating the overall torque on a planar loop of any arbitrary shape. But the basic requirement is that the magnetic field must be uniform. The torque is always in the direction of axis of rotation. When the planar loop or coil is normal to the mangetic field, the sum of the forces on the planar loop as well as the torque will be zero.

8.8 Magnetic Boundary Conditions

The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other are called **boundary conditions** for magnetic fields or simply **magnetic boundary conditions**. When we consider magnetic boundary conditions, the conditions of $\overline{B}$ and $\overline{H}$ are studied at the boundary. The boundary between the two different magnetic materials is considered. To study conditions of $\overline{B}$ and $\overline{H}$ at the boundary, both the vectors are resolved into two components ;

a) Tangential to boundary and

b) Normal (perpendicular) to boundary.

Consider a boundary between two isotropic, homogeneous linear materials with different permeabilities μ_1 and μ_2 as shown in the Fig. 8.14. To determine the boundary conditions, let us use the closed path and the Gaussian surface.

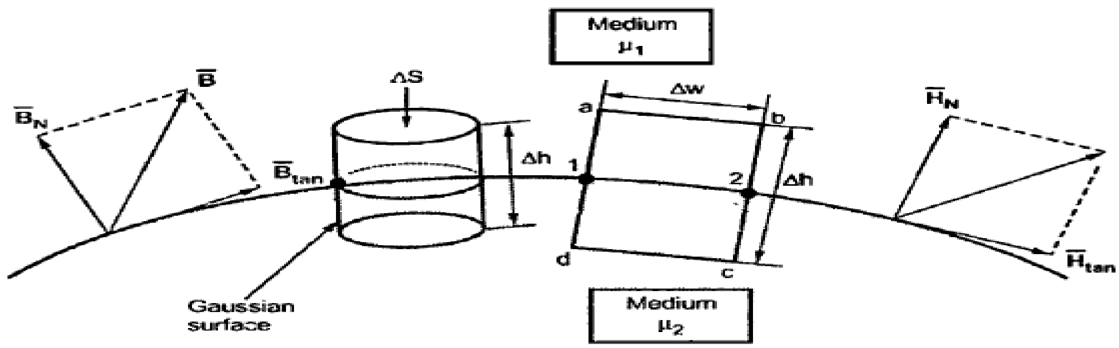


Fig. 8.14 Boundary between two magnetic materials of different permeabilities

8.8.1 Boundary Conditions for Normal Component

To find the normal component of $\overline{B}$, select a closed Gaussian surface in the form of a right circular cylinder as shown in the Fig. 8.14. Let the height of the cylinder be Δh and be placed in such a way that $\Delta h/2$ is in medium 1 and remaining $\Delta h/2$ is in medium 2. Also the axis of the cylinder is in the normal direction to the surface.

According to the Gauss's law for the magnetic field,

$$\oint_S \overline{B} \cdot d\overline{S} = 0 \quad \text{... (1)}$$

The surface integral must be evaluated over three surfaces, (i) Top, (ii) Bottom and (iii) Lateral.

Let the area of the top and bottom is same, equal to ΔS.

$$\therefore \oint_{top} \overline{B} \cdot d\overline{S} + \oint_{bottom} \overline{B} \cdot d\overline{S} + \oint_{lateral} \overline{B} \cdot d\overline{S} = 0 \quad \text{... (2)}$$

As we are very much intetrested in the boundary conditions, reduce Δh to zero. As $\Delta h \to 0$, the cylinder tends to boundary and only top and bottom surfaces contribute in the surface integral. Thus surface integrals are calculated for top and bottom surfaces only. These surfaces are very small. Let the magnitude of normal component of $\overline{B}$ be B_{N_1} and B_{N_2} in medium 1 and medium 2 respectively. As both the surfaces are very small, we can assume B_{N_1} and B_{N_2} constant over their surfaces. Hence we can write,

For top surfaces :

$$\oint_{Top} \overline{B} \cdot d\overline{S} = B_{N_1} \oint_{Top} d\overline{S} = B_{N_1} \Delta S \quad \text{... (3)}$$

For bottom surface :

$$\oint_{Bottom} \overline{B} \cdot d\overline{S} = B_{N_2} \oint_{Bottom} d\overline{S} = B_{N_2} \Delta S \quad \text{... (4)}$$

For lateral surface

$$\oint_{\text{Lateral}} \overline{B} \cdot d\overline{S} = 0 \qquad \text{... (5)}$$

Putting values of surface integrals in equation (2), we get

$$B_{N_1} \Delta S - B_{N_2} \Delta S = 0 \qquad \text{... (6)}$$

Note that the negative sign is used for one of the surface integrals because normal component in medium 2 is entering the surface while in medium 1 the component is leaving the surface. Hence B_{N_1} and B_{N_2} are in opposite direction.

From equation (6), we can write,

$$B_{N_1} \Delta S = B_{N_2} \Delta S$$

i.e.

$$\boxed{B_{N_1} = B_{N_2}} \qquad \text{... (7)}$$

Thus the normal component of $\overline{B}$ is continuous at the boundary.

As the magnetic flux density and the magnetic field intensity are related by

$$\overline{B} = \mu \overline{H}$$

Thus, equation (7) can be written as,

$$\mu_1 H_{N_1} = \mu_2 H_{N_2}$$

$$\therefore \quad \boxed{\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}}} \qquad \text{... (8)}$$

Hence the normal component of $\overline{H}$ is not continuous at the boundary. The field strengths in two media are inversely proportional to their relative permeabilities.

8.8.2 Boundary Conditions for Tangential Component

According to Ampere's circuital law,

$$\oint \overline{H} \cdot d\overline{L} = I \qquad \text{... (9)}$$

Consider a rectangular closed path abcda as shown in the Fig. 8.13. It is traced in clockwise direction as a-b-c-d-a. This closed path is placed in a plane normal to the boundary surface. Hence $\oint \overline{H} \cdot d\overline{L}$ can be divided into 6 parts.

$$\oint \overline{H} \cdot d\overline{L} = \int_a^b \overline{H} \cdot d\overline{L} + \int_b^1 \overline{H} \cdot d\overline{L} + \int_1^c \overline{H} \cdot d\overline{L} + \int_c^d \overline{H} \cdot d\overline{L} + \int_d^2 \overline{H} \cdot d\overline{L} + \int_2^a \overline{H} \cdot d\overline{L} = I \qquad \text{... (10)}$$

From the Fig. 8.14 it is clear that, the closed path is placed in such a way that its two sides a-b and e-d are parallel to the tangential direction to the surface while the other two sides are normal to the surface at the boundary. This closed path is placed in such a way that half of its portion is in medium 1 and the remaining is in medium 2. The rectangular path is an elementary rectangular path with elementary height Δh and elementary width Δw. Thus over small width Δw, $\overline{H}$ can be assume constant say H_{tan1} in medium 1 and H_{tan2} in medium 2. Similarly over a small height $\frac{\Delta h}{2}$, $\overline{H}$ can be assumed constant say H_{N1} in medium 1 and H_{N2} in medium 2. Now assume that $\overline{K}$ is the surface current normal to the path. Also from the Fig. 8.14 it is clear that the normal and tangential components in medium 1 and medium 2 are in opposite direction. Thus equation (10) can be written as,

$$K \cdot dw = H_{tan1}(\Delta w) + H_{N1}\left(\frac{\Delta h}{2}\right) + H_{N2}\left(\frac{\Delta h}{2}\right) - H_{tan2}(\Delta w)$$

$$- H_{N2}\left(\frac{\Delta h}{2}\right) - H_{N1}\left(\frac{\Delta h}{2}\right) \qquad \text{... (11)}$$

To get conditions at boundary, $\Delta h \to 0$. Thus,

$$K \cdot dw = H_{tan1}(\Delta w) - H_{tan2}(\Delta w)$$

$$H_{tan1} - H_{tan2} = K \qquad \text{... (12)}$$

In vector form, we can express above relation by a cross product as

$$\boxed{\bar{H}_{tan1} - \bar{H}_{tan2} = \bar{a}_{N_{12}} \times \bar{K}} \quad \text{... (13)}$$

where $\bar{a}_{N_{12}}$ is the unit vector in the direction normal at the boundary from medium 1 to medium 2.

For $\bar{B}$, the tangential components can be related with permeabilities of two media using equation (12),

$$\therefore \quad \frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = K \quad \text{... (14)}$$

Consider a special case that the boundary is free of current. In other words, media are not conductors; so K = 0. Then equation (12) becomes

$$H_{tan1} - H_{tan2} = 0$$

$$\text{or} \quad H_{tan1} = H_{tan2} \quad \text{... (15)}$$

For tangential components of $\bar{B}$ we can write,

$$\frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = 0$$

$$\therefore \quad \frac{B_{tan1}}{\mu_1} = \frac{B_{tan2}}{\mu_2}$$

$$\therefore \quad \boxed{\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}} \quad \text{... (16)}$$

From equations (15) and (16) it is clear that tangential component of $\bar{H}$ are continuous, while tangential component of $\bar{B}$ are discontinuous at the boundary, with the condition that the boundary is current free.

Let the fields make angles α_1 and α_2 with the normal to the interface as shown in the Fig. 8.15.

Interms of angle α_1 and α_2, we can write relationship between normal components and tangential components of $\bar{B}$.

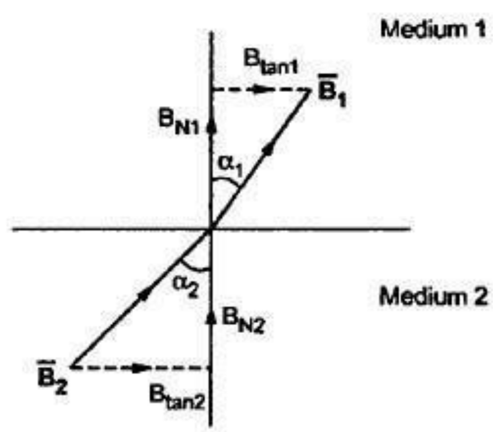


Fig. 8.15 Component of $\bar{B}$ at boundary

In medium 1,

$$\tan \alpha_1 = \frac{B_{tan1}}{B_{N1}} \quad \text{... (17)}$$

Similarly in medium 2,

$$\tan \alpha_2 = \frac{B_{tan2}}{B_{N2}} \quad \text{... (18)}$$

Dividing equation (17) by equation (18)

$$\therefore \qquad \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{B_{tan1}}{B_{N1}} \cdot \frac{B_{N2}}{B_{tan2}}$$

As we know, $B_{N1} = B_{N2}$,

$$\boxed{\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{B_{tan1}}{B_{tan1}} = \frac{\mu_{r1}}{\mu_{r2}}} \qquad \text{... (19)}$$

Consider an interface between air (medium 1) and soft iron (medium 2) For air, $\mu_{r1} = 1$. For soft iron, let $\mu_{r2} = 7000$. Then

$$\frac{B_{tan1}}{B_{tan2}} = \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{1}{7000}$$

If $\alpha_2 = 85°$, then $\alpha_1 = 0.093°$ and $B_{tan1} = 0$.

Thus practically when fields cross a medium of **high** μ_r to **low** μ_r, then the magnetic fields $\overline{B}$ and $\overline{H}$ are **always perpendicular to the boundary.**