UNIT-II

UNIT – II CONDUCTORS AND DIELECTRICS

Behavior of Conductors in an Electric Field-Conductors and Insulators – Electric Field inside a Dielectric Material – Polarization – Dielectric Conductors and Dielectric Boundary Conditions – Capacitance-Capacitance of Parallel Plate, Spherical & Co-axial capacitors – Energy Stored and Energy Density in a Static Electric Field – Current Density – Conduction and Convection Current Densities – Ohm's Law in Point Form – Equation of Continuity – Numerical Problems.

5.2 Current and Current Density

The current is defined as the rate of flow of charge and is measured in amperes.

Key Point : *A current of 1 ampere is said to be flowing across the surface when a charge of one coulomb is passing across the surface in one second.*

The current is considered to be the motion of the positive charges. The conventional current is due to the flow of electrons, which are negatively charged. Hence the direction of conventional current is assumed to be opposite to the direction of flow of the electrons.

The current which exists in the conductors, due to the drifting of electrons, under the influence of the applied voltage is called **drift current.**

While in dielectrics, there can be flow of charges, under the influence of the electric field intensity. Such a current is called the **displacement current or convection current.** The current flowing across the capacitor, through the dielectric separating its plates is an example of the convection current.

The analysis of such currents, in the field theory is based on defining a current density at a point in the field.

The current density is a vector quantity associated with the current and denoted as $\bar{J}$.

5.2.1 Relation between I and $\bar{J}$

Consider a surface S and I is the current passing through the surface. The direction of current is normal to the surface S and hence direction of $\bar{J}$ is also normal to the surface S.

Consider an incremental surface area dS as shown in the Fig. 5.1 (a) and $\bar{a}_n$ is the unit vector normal to the incremental surface dS.

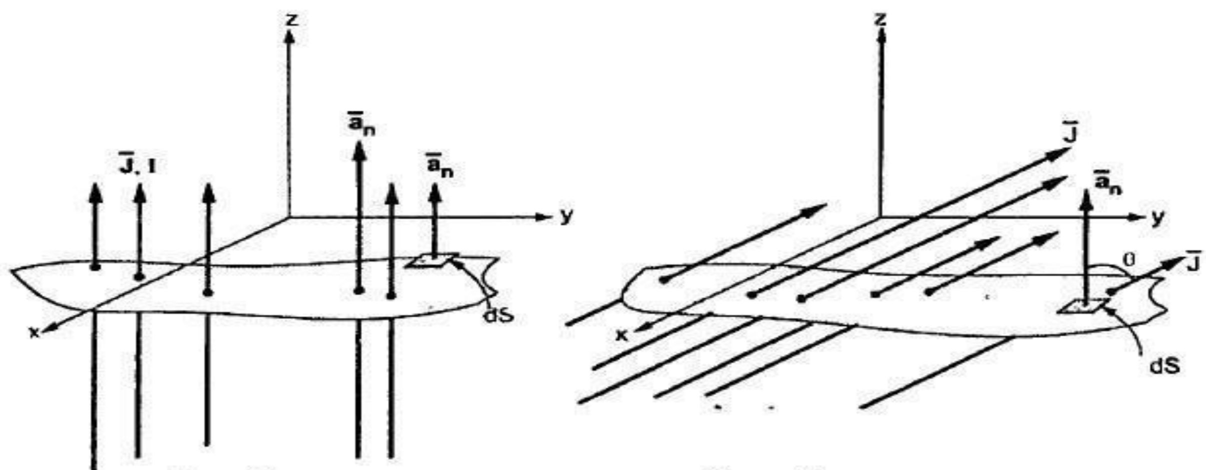


Fig. 5.1

$$\therefore \quad d\bar{S} = dS\,\bar{a}_n \quad \text{while} \quad \bar{J} = J\,\bar{a}_n \qquad \text{... (1)}$$

Then the differential current dI passing through the differential surface dS is given by the dot product of the current density vector $\bar{J}$ and $d\bar{S}$.

$$\therefore \quad dI = \bar{J} \cdot d\bar{S} \quad \text{(dot product)} \qquad \text{... (2)}$$

When $\bar{J}$ and $d\bar{S}$ are at right angles ($\theta = 90°$) then

$$dI = J\,\bar{a}_n \cdot dS\,\bar{a}_n = J\,dS \qquad \text{... (3)}$$

and

$$I = \int_S J\,dS \qquad \text{... (4)}$$

where J = Current density in A/m^2.

But if $\bar{J}$ is **not normal** to the differential area $d\bar{S}$ then the total current is obtained by integrating the incremental current which is dot product of $\bar{J}$ and $d\bar{S}$, over the surface S. This is shown in the Fig. 5.1 (b). Thus in general,

$$\boxed{I = \int_S \bar{J} \cdot d\bar{S} \quad \text{(Dot product)}} \qquad \text{... (5)}$$

Thus if $\bar{J}$ is in A/m^2 and $d\bar{S}$ is in m^2 then the current obtained is in amperes (A). It may be noted that $\bar{J}$ need not be uniform over S and S need not be a plane surface.

5.2.2 Relation between $\bar{J}$ and ρ_v

The set of charged particles give rise to a charge density ρ_v in a volume v. The current density $\bar{J}$ can be related to the velocity with which the volume charge density i.e. charged particles in volume v crosses the surface S at a point. This is shown in the Fig. 5.2. The velocity with which the charge is getting transferred is $\bar{v}$ m/s. It is a vector quantity.

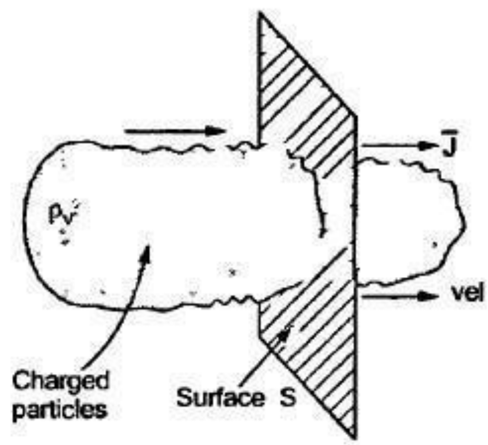


Fig. 5.2

To derive the relation between $\bar{J}$ and ρ_v, consider differential volume Δv having charge density ρ_v as shown in the Fig. 5.3. The elementary charge that volume carries is,

$$\Delta Q = \rho_v \Delta v \quad \text{... (6)}$$

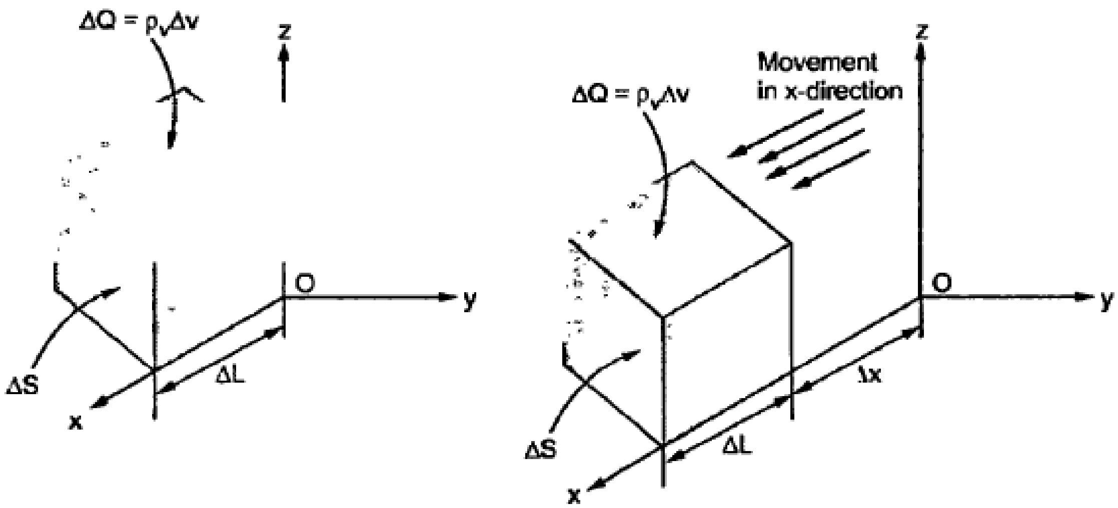


Fig. 5.3 Incremental charge moving in x-direction

Let ΔL is the incremental length while ΔS is the incremental surface area hence incremental volume is,

$$\Delta v = \Delta S \, \Delta L \qquad \text{... (7)}$$

$$\therefore \qquad \Delta Q = \rho_v \, \Delta S \, \Delta L \qquad \text{... (8)}$$

Let the charge is moving in x-direction with velocity $\bar{v}$ and thus velocity has only x component v_x.

[Note : Velocity is denoted by small italic letter while the volume is denoted by small normal letter.]

In the time interval Δt the element of charge has moved through distance Δx, in x-direction as shown in the Fig. 5.3. The direction is normal to the surface ΔS and hence resultant current can be expressed as,

$$\Delta I = \frac{\Delta Q}{\Delta t} \qquad \text{... (9)}$$

But now, $\Delta Q = \rho_v \, \Delta S \Delta x$ as the charge corresponding the length Δx is moved and responsible for the current.

$$\therefore \qquad \Delta I = \rho_v \, \Delta S \frac{\Delta x}{\Delta t} \qquad \text{.. (10)}$$

But $\qquad \dfrac{\Delta x}{\Delta t}$ = Velocity in x-direction i.e. v_x

$$\therefore \qquad \Delta I = \rho_v \, \Delta S v_x \qquad \text{... (11)}$$

Note that $\qquad v_x$ = x component of velocity $\bar{v}$

But $\qquad \Delta I = \bar{J} \, \Delta S$ when $\bar{J}$ and ΔS are normal

Here $\bar{J}$ and ΔS are normal to each other hence comparing the two equations,

$$J_x = \rho_v v_x = \text{x component of } \bar{J} \qquad \text{... (12)}$$

In this case $\bar{J}$ has only x component.

In general, the relation between J and ρ_v can be expressed as,

$$\boxed{\bar{J} = \rho_v \, \bar{v}} \qquad \text{... (13)}$$

where $\qquad \bar{v}$ = Velocity vector

Such a current is called convection current and the current density is called convection current density.

Key Point : *The convection current density is linearly proportional to the charge density and the velocity with which the charge is transferred.*

5.3 Continuity Equation

The continuity equation of the current is based on the **principle of conservation of charge**. The principle states that, The charges can neither be created nor be destroyed.

Consider a closed surface S with a current density $\bar{J}$, then the total current I crossing the surface S is given by,

$$\boxed{I = \oint_S \bar{J} \cdot d\bar{S}} \quad \text{... (1)}$$

The current flows outwards from the closed surface. It has been mentioned earlier that **the current means the flow of positive charges.** Hence the current I is constituted due to outward flow of positive charges from the closed surface S. According to principle of conservation of charge, there must be decrease of an equal amount of positive charge inside the closed surface. Hence the outward rate of flow of positive charge gets balanced by the rate of decrease of charge inside the closed surface.

Let $\quad Q_i$ = Charge within the closed surface

$-\frac{dQ_i}{dt}$ = Rate of **decrease** of charge inside the closed surface

The negative sign indicates decrease in charge.

Due to principle of conservation of charge, this rate of decrease is same as rate of outward flow of charge, which is a current.

$$\therefore \quad I = \oint_S \bar{J} \cdot d\bar{S} = -\frac{dQ_i}{dt} \quad \text{... (2)}$$

This is the **integral form of the continuity equation of the current.**

The negative sign in the equation indicates outward flow of current from the closed surface. So the equation (2) is indicating **outward flowing current I.**

If the current is entering the volume then

$$\oint_S \bar{J} \cdot d\bar{S} = -I = +\frac{dQ_i}{dt}$$

The point form of the continuity equation can be obtained from the integral form. Using the **divergence theorem,** convert the surface integral in integral form to the volume integral.

$$\therefore \quad \boxed{\oint_S \bar{J} \cdot d\bar{S} = \int_{vol} (\nabla \cdot \bar{J})\, dv} \quad \text{...(3)}$$

$$\therefore \quad -\frac{dQ_i}{dt} = \int_{vol} (\nabla \cdot \bar{J})\, dv \quad \text{... (4)}$$

But $$Q_i = \int_{vol} \rho_v \, dv \quad \text{... (5)}$$

where $\quad \rho_v$ = Volume charge density

$$\therefore \quad \int_{vol} (\nabla \cdot \bar{J})\, dv = -\frac{d}{dt}\left[\int_{vol} \rho_v\, dv\right] = -\int_{vol} \frac{\partial \rho_v}{\partial t}\, dv \qquad \text{... (6)}$$

For a constant surface, the derivative becomes the partial derivative.

$$\therefore \quad \int_{vol} (\nabla \cdot \bar{J})\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t}\, dv \qquad \text{... (7)}$$

If the relation is true for any volume, it must be true even for incremental volume Δv.

$$\therefore \quad (\nabla \cdot \bar{J})\, \Delta v = -\frac{\partial \rho_v}{\partial t}\, \Delta v \qquad \text{... (8)}$$

$$\therefore \quad \boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}} \qquad \text{... (9)}$$

This is the **point form or differential form of the continuity equation of the current.**

The equation states that the current or the charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

5.3.1 Steady Current

For steady currents which are not the functions of time, $\partial \rho_v / \partial t = 0$ hence,

$$\boxed{\nabla \cdot \bar{J} = 0 \qquad \text{(Steady current)}} \qquad \text{... (10)}$$

For such currents, the rate of flow of charge remains constant with time. The steady currents have no sources or sinks, as it is constant.

5.4 Conductors

Let us study the behaviour and properties of the conductors. Under the effect of applied electric field, the available free electrons start moving. The moving electrons strike the adjacent atoms and rebound in the random directions. This is called drifting of the electrons. After some time, the electrons attain the constant average velocity called **drift** velocity (v_d). The current constituted due to the drifting of such electrons in metallic conductors is called **drift current**. The drift velocity is directly proportional to the applied electric field.

$$\therefore \qquad \bar{v}_d \propto \bar{E} \qquad \text{... (1)}$$

The constant of proportionality is called **mobility** of the electrons in a given material and denoted as μ_e. It is positive for the electrons.

$$\therefore \qquad \boxed{\bar{v}_d = -\mu_e \bar{E}} \qquad \text{... (2)}$$

The **negative sign** indicates that the velocity of the electrons is against the direction of field $\bar{E}$.

$$\text{Now} \quad \mu \text{ (Mobility)} = \frac{\text{Velocity}}{\text{Field}} = \frac{\text{m/s}}{\text{V/m}} = \frac{\text{m}^2}{\text{V-s}}$$

Thus mobility is measured in square metres per volt-second (m^2/V-s). The typical values of mobility are 0.0012 for aluminium, 0.0032 for copper etc.

According to relation between $\bar{J}$ and $\bar{v}$ we can write,

$$\bar{J} = \rho_v \bar{v} \qquad \text{... (3)}$$

But in the material, the number of protons and electrons is same and it is always electrically neutral. Hence $\rho_v = 0$ for the neutral materials. The drift velocity is the velocity of free electrons hence the above relation can be expressed as,

$$\boxed{\bar{J} = \rho_e \bar{v}_d} \qquad \text{... (4)}$$

where $\qquad \rho_e$ = Charge density due to free electrons

The charge density ρ_e can be obtained as the product of number of free electrons/m^3 and the charge 'e' on one electron. Thus ρ_e = ne where n is number of free electrons per m^3.

Substituting equation (2) in equation (4) we get,

$$\boxed{\bar{J} = -\rho_e \mu_e \bar{E}} \qquad \text{... (5)}$$

5.4.1 Point Form of Ohm's Law

The relationship between $\bar{J}$ and $\bar{E}$ can also be expressed in terms of conductivity of the material.

Thus for a metallic conductor,

$$\boxed{\bar{J} = \sigma \bar{E}} \qquad \text{... (6)}$$

where $\qquad \sigma$ = Conductivity of the material

The conductivity is measured in mhos per metre (℧/m). The equation (6) is called **point form of Ohm's law**. The unit of conductivity is also called Siemens per metre (S/m).

The typical values of conductivity are 3.82×10^7 for aluminium, 5.8×10^7 for copper etc. expressed in mho/m. For the metallic conductors the conductivity is constant over wide ranges of current density and electric field intensity. In all directions, metallic conductors have same properties hence called **isotropic** in nature. Such materials obey the Ohm's law very faithfully.

Comparing the equation (5) and equation (6) we can write,

$$\sigma = -\rho_e \mu_e \qquad \dots (7)$$

This is conductivity interms of mobility of the charge density of the electrons.

The resistivity is the reciprocal of the conductivity. The conductivity depends on the temperature. As the temperature increases, the vibrations of crystalline structure of atoms increases. Due to increased vibrations of electrons, drift velocity decreases, hence the mobility and conductivity decreases. So **as temperature increases, the conductivity decreases and resistivity increases.**

5.4.2 Resistance of a Conductor

Consider that the voltage V is applied to a conductor of length L having uniform cross-section S, as shown in the Fig. 5.5.

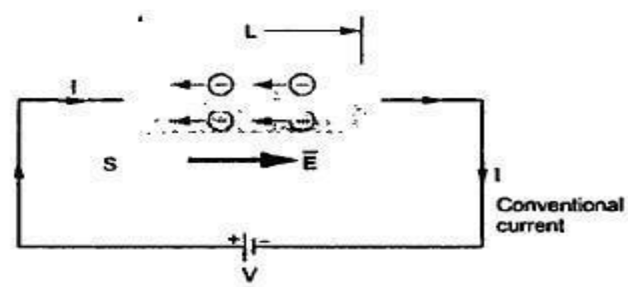


Fig. 5.5 Conductor subjected to voltage V

The direction of $\bar{E}$ is same as the direction of conventional current, which is opposite to the flow of electrons. The electric field applied is uniform and its magnitude is given by,

$$E = \frac{V}{L} \qquad \dots (8)$$

The conductor has uniform cross-section S and hence we can write,

$$I = \int_S \bar{J} \cdot d\bar{S} = JS \qquad \dots (9)$$

The current direction is normal to the surface S.

Thus, $$J = \frac{I}{S} = \sigma E \qquad \dots (10)$$

And using equation (8) in equation (10) we get,

$$J = \frac{\sigma V}{L} \quad \text{where } \sigma = \text{Conductivity of the material} \qquad \dots (11)$$

$$\therefore \qquad V = \frac{JL}{\sigma} = \frac{IL}{\sigma S} = \left(\frac{L}{\sigma S}\right) I \qquad \text{... (12)}$$

$$\therefore \qquad \boxed{R = \frac{V}{I} = \frac{L}{\sigma S}} \qquad \text{... (13)}$$

Thus the ratio of potential difference between the two ends of the conductors to the current flowing through it is **resistance** of the conductor.

The equation (12) is nothing but the Ohm's law in its normal form given by V = IR. The equation is true for the uniform fields and resistance is measured is ohms (Ω).

For **nonuniform fields**, the resistance R is defined as the ratio V to I where V is the potential difference betweem two specified equipotential surfaces in the material and I is the current crossing the more positive surface of the two, into the material. Mathematically the resistance for nonuniform fields is given by,

$$\boxed{R = \frac{V_{ab}}{I} = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_S \bar{J} \cdot d\bar{S}} = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_S \sigma \bar{E} \cdot d\bar{S}}} \qquad \text{... (14)}$$

The numerator is a line integration giving potential difference across two ends while the denominator is a surface integration giving current flowing through the material.

The resistance can also be expressed as,

$$\boxed{R = \frac{L}{\sigma S} = \frac{\rho_c L}{S}\ \Omega} \qquad \text{... (15)}$$

where $\rho_c = \frac{1}{\sigma}$ = Resistivity of the conductor in Ω-m

5.4.3 Properties of Conductor

Consider that the charge distribution is suddenly unbalanced inside the conductor. There are number of electrons trying to reside inside the conductor. All the electrons are negatively charged and they start repelling each other due to their own electric fields. Such electrons get accelerated away from each other, till all the electrons causing interior imbalance, reach at the surface of the conductor. The conductor is surrounded by the insulating medium and hence electrons just driven from the interior of the conductor, reside over the surface. Thus,

1. Under static conditions, no **charge** and **no electric field** can exist at any point **within the conducting material.**
2. The charge can exist on the surface of the conductor giving rise to surface charge density.
3. Within a conductor, the charge density is always zero.
4. The charge distribution on the surface depends on the shape of the surface.
5. The conductivity of an ideal conductor is infinite.
6. The conductor surface is an equipotential surface.

5.6 Dielectric Materials

It is seen that the conductors have large number of free electrons while insulators and dielectric materials do not have free charges. The charges in dielectrics are bound by the finite forces and hence called **bound charges.** As they are bound and not free, they cannot contribute to the conduction process. But if subjected to an electric field $\bar{E}$, they shift their relative positions, against the normal molecular and atomic forces. This shift in the relative positions of bound charges, allows the dielectric to store the energy.

The shifts in positive and negative charges are in opposite directions and under the influence of an applied electric field $\bar{E}$ such charges act like small electric dipoles.

Key Point : ***When the dipole results from the displacement of the bound charges, the dielectric is said to be polarized.***

And these electric dipoles produce an electric field which opposes the externally applied electric field. This process, due to which separation of bound charges results to produce electric dipoles, under the influence of electric field $\bar{E}$, is called ***polarization.***

5.6.1 Polarization

To understand the polarization, consider an atom of a dielectric. This consists of a nucleus with positive charge and negative charges in the form of revolving electrons in the orbits. The negative charge is thus considered to be in the form of cloud of electrons. This is shown in the Fig. 5.6.

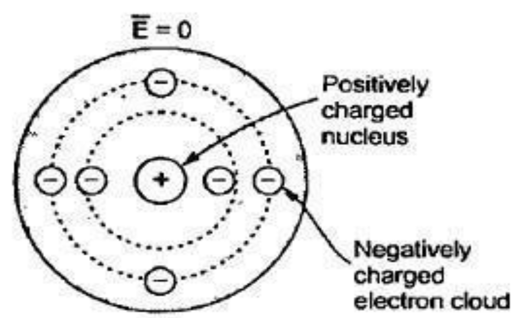
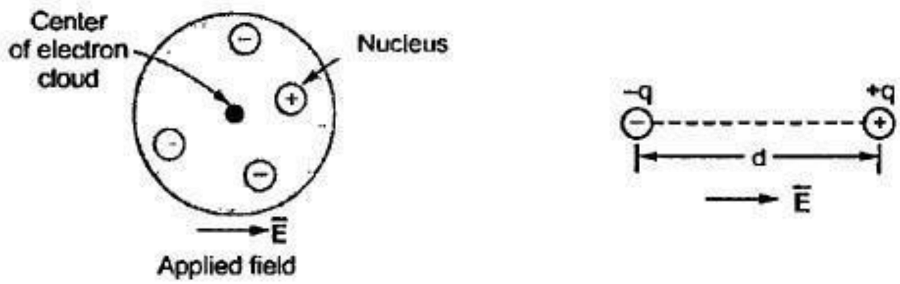


Fig. 5.6 Unpolarized atom of a dielectric

Note that $\bar{E}$ applied is zero. The number of positive charges is same as negative charges and hence atom is electrically neutral. Due to symmetry, both positive and negative charges can be assumed to be point charges of equal amount, coinciding at the centre. Hence there cannot exist an electric dipole. This is called **unpolarized atom.**

When electric field $\bar{E}$ is applied, the symmetrical distribution of charges gets disturbed. The positive charges experience a force $\bar{F} = Q\bar{E}$ while the negative charges experience a force $\bar{F} = -Q\bar{E}$ in the opposite direction.

Now there is separation between the nucleus and the centre of the electron cloud as shown in the Fig. 5.7 (a). Such an atom is called **polarized atom.**



(a) Polarized atom **(b) Equivalent dipole**

Fig. 5.7

It can be seen that an electron cloud has a centre separated from the nucleus. This forms an electric dipole. The equivalent dipole formed is shown in the Fig. 5.7 (b). The dipole gets aligned with the applied field. This process is called **polarization** of dielectrics.

There are two types of dielectrics,

1. Nonpolar and 2. Polar.

In **nonpolar** molecules, the dipole arrangement is totally absent, in absence of electric field E. It results only when an externally field $\bar{E}$ is applied to it. In **polar** molecules, the permanent displacements between centres of positive and negative charges exist. Thus dipole arrangements exist without application of $\bar{E}$. But such dipoles are randomly oriented. Under the application of $\bar{E}$, the dipoles experience torque and they align with the direction of the applied field $\bar{E}$. This is called polarization of polar molecules.

The examples of nonpolar molecules are hydrogen, oxygen and the rare gases. The examples of polar molecules are water, sulphur dioxide, hydrochloric acid etc.

5.7 Boundary Conditions

When an electric field passes from one medium to other medium, it is important to study the conditions at the boundary between the two media. The conditions existing at the boundary of the two media when field passes from one medium to other are called **boundary conditions.** Depending upon the nature of the media, there are two situations of the boundary conditions,

1. Boundary between conductor and free space.

2. Boundary between two dielectrics with different properties.

The free space is nothing but a dielectric hence first case is nothing but the boundary between conductor and a dielectric. For studying the boundary conditions, the Maxwell's equations for electrostatics are required.

$$\oint \bar{E} \cdot \overline{dL} = 0 \quad \text{and} \quad \oint \bar{D} \cdot d\bar{S} = Q$$

Similarly the field intensity $\bar{E}$ is required to be decomposed into two components namely tangential to the boundary $\left(E_{tan}\right)$ and normal to the boundary $\left(\bar{E}_N\right)$.

$$\therefore \qquad \bar{E} = \bar{E}_{tan} + \bar{E}_N$$

Similar decomposition is required for flux density $\bar{D}$ as well.

Let us study the various cases of boundary conditions in detail.

5.8 Boundary Conditions between Conductor and Free Space

Consider a boundary between conductor and free space. The conductor is ideal having infinite conductivity. Such conductors are copper, silver etc. having conductivity of the order of 10^6 S/m and can be treated ideal. For ideal conductors it is known that,

1. The field intensity inside a conductor is zero and the flux density inside a conductor is zero.

2. No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.

3. The charge density within the conductor is zero.

Thus $\bar{E}, \bar{D}$ and ρ_v **within the conductor are zero.** While ρ_S is the surface charge density on the surface of the conductor.

To determine the boundary conditions let us use the closed path and the Gaussian surface.

Consider the conductor free space boundary as shown in the Fig. 5.8.

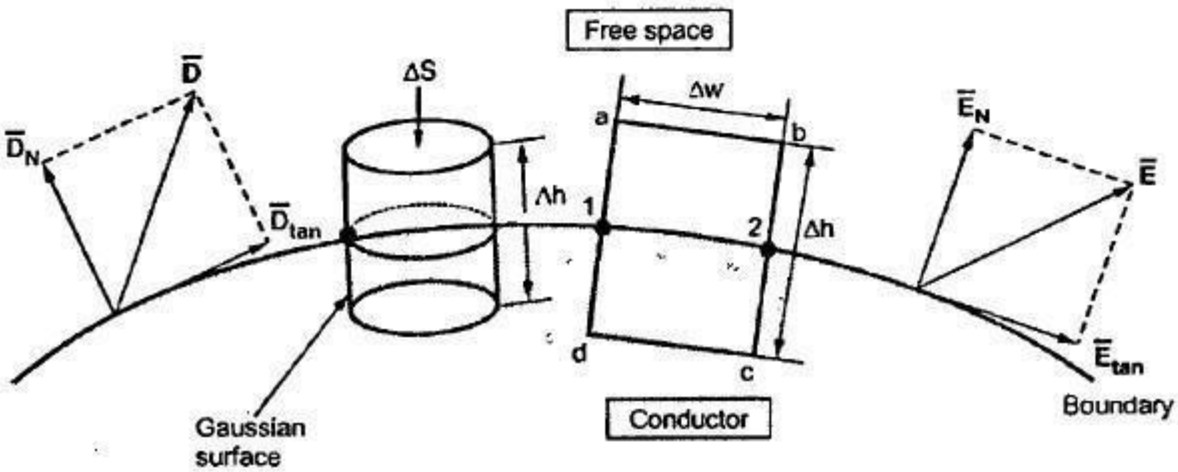


Fig. 5.8 Boundary between conductor and free space

5.8.1 $\bar{E}$ at the Boundary

Let $\bar{E}$ be the electric field intensity, in the direction shown in the Fig. 5.11, making some angle with the boundary. This $\bar{E}$ can be resolved into two components :

1. The component tangential to the surface ($\bar{E}_{tan}$).

2. The component normal to the surface ($\bar{E}_N$).

It is known that,

$$\oint \bar{E} \cdot d\bar{L} = 0 \qquad \text{... (1)}$$

The integral of $\bar{E} \cdot d\bar{L}$ carried over a closed contour is zero i.e. work done in carrying unit positive charge along a closed path is zero.

Consider a rectangular closed path abcda as shown in the Fig. 5.8. It is traced in clockwise direction as a-b-c-d-a and hence $\oint \bar{E} \cdot d\bar{L}$ can be divided into four parts.

$$\oint \bar{E} \cdot d\bar{L} = \int_a^b \bar{E} \cdot d\bar{L} + \int_b^c \bar{E} \cdot d\bar{L} + \int_c^d \bar{E} \cdot d\bar{L} + \int_d^a \bar{E} \cdot d\bar{L} = 0 \qquad \text{... (2)}$$

The closed contour is placed in such a way that its two sides a-b and c-d are parallel to tangential direction to the surface while the other two are normal to the surface, at the boundary.

The rectangle is an elementary rectangle with elementary height Δh and elementary width Δw. The rectangle is placed in such a way that half of it is in the conductor and remaining half is in the free space. Thus $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space.

Now the portion c-d is in the conductor where $\bar{E} = 0$ hence the corresponding integral is zero.

$$\therefore \quad \int_a^b \bar{E}\cdot d\bar{L} + \int_b^c \bar{E}\cdot d\bar{L} + \int_d^a \bar{E}\cdot d\bar{L} = 0 \qquad \text{... (3)}$$

As the width Δw is very small, E over it can be assumed constant and hence can be taken out of integration.

$$\therefore \quad \int_a^b \bar{E}\cdot d\bar{L} = \bar{E}\int_a^b d\bar{L} = \bar{E}(\Delta w) \qquad \text{... (4)}$$

But Δw is along tangential direction to the boundary in which direction $\bar{E} = \bar{E}_{tan}$.

$$\therefore \quad \int_a^b E\cdot d\bar{L} = E_{tan}(\Delta w) \quad \text{where} \quad E_{tan} = |\bar{E}_{tan}| \qquad \text{... (5)}$$

Now b-c is parallel to the normal component so we have $\bar{E} = \bar{E}_N$ along this direction. Let $E_N = |\bar{E}_N|$.

Over the small height Δh, E_N can be assumed constant and can be taken out of integration.

$$\therefore \quad \int_b^c E\cdot d\bar{L} = \bar{E}\int_b^c d\bar{L} = E_N\int_b^c d\bar{L} \qquad \text{... (6)}$$

But out of b-c, b-2 is in free space and 2-c is in the conductor where $\bar{E} = 0$.

$$\therefore \quad \int_b^c d\bar{L} = \int_b^2 d\bar{L} + \int_2^c d\bar{L} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2} \qquad \text{... (7)}$$

$$\therefore \quad \int_b^c E\cdot dL = E_N\left(\frac{\Delta h}{2}\right) \qquad \text{... (8)}$$

Similarly for path d-a, the condition is same as for the path b-c, only direction is opposite.

$$\therefore \quad \int_d^a \bar{E}\cdot d\bar{L} = -E_N\left(\frac{\Delta h}{2}\right) \qquad \text{... (9)}$$

Substituting equations (4), (8) and equation (9) in (3) we get,

$$\therefore \quad E_{tan}\,\Delta w + E_N\left(\frac{\Delta h}{2}\right) - E_N\left(\frac{\Delta h}{2}\right) = 0 \qquad \text{... (10)}$$

$$\therefore \qquad E_{tan}\,\Delta w = 0 \qquad \text{But } \Delta w \neq 0 \text{ as finite}$$

$$\therefore \qquad \boxed{E_{tan} = 0} \qquad \text{... (11)}$$

Thus the **tangential component of the electric field intensity is zero at the boundary between conductor and free space.**

Key Point : ***Thus the $\overline{E}$ at the boundary between conductor and free space is always in the direction perpendicular to the boundary.***

Now $\qquad \overline{D} = \varepsilon_0 \overline{E}$ for free space

$$\therefore \qquad \boxed{D_{tan} = \varepsilon_0 E_{tan} = 0} \qquad \text{... (12)}$$

Thus the **tangential component of electric flux density is zero at the boundary between conductor and free space.**

Key Point : *Hence electric flux density $\overline{D}$ is also only in the normal direction at the boundary between the conductor and the free space.*

5.8.2 D_N at the Boundary

To find normal component of $\overline{D}$, select a closed Gaussian surface in the form of right circular cylinder as shown in the Fig. 5.8. Its height is Δh and is placed in such a way that $\Delta h/2$ is in the conductor and remaining $\Delta h/2$ is in the free space. Its axis is in the normal direction to the surface.

According to Gauss's law, $\oint_S \bar{D} \cdot d\bar{S} = Q$

The surface integral must be evaluated over three surfaces,

i) Top, ii) Bottom and iii) Lateral.

Let the area of top and bottom is same equal to ΔS.

$$\therefore \quad \int_{top} D \cdot d\bar{S} + \int_{bottom} \bar{D} \cdot d\bar{S} + \int_{lateral} \bar{D} \cdot d\bar{S} = Q \qquad \text{... (13)}$$

The bottom surface is in the conductor where $\bar{D} = 0$ hence corresponding integral is zero.

The top surface is in the free space and we are interested in the boundary condition hence top surface can be shifted at the boundary with $\Delta h \to 0$.

$$\therefore \quad \int_{top} \bar{D} \cdot d\bar{S} + \int_{lateral} \bar{D} \cdot d\bar{S} = Q \qquad \text{... (14)}$$

The lateral surface area is $2\pi r\, \Delta h$ where r is the radius of the cylinder. But as $\Delta h \to 0$, this area reduces to zero and corresponding integral is zero.

While only component of $\bar{D}$ present is the normal component having magnitude D_N. The top surface is very small over which D_N can be assumed constant and can be taken out of integration.

$$\therefore \quad \int_{top} \bar{D} \cdot d\bar{S} = D_N \int_{top} d\bar{S} = D_N\, \Delta S \qquad \text{... (15)}$$

From Gauss's law,

$$\therefore \quad D_N\, \Delta S = Q \qquad \text{... (16)}$$

But at the boundary, the charge exists in the form of surface charge density $\rho_S\ C/m^2$.

$$\therefore \quad Q = \rho_S\, \Delta S \qquad \text{... (17)}$$

Equating equation (16) and (17),

$$\therefore \quad D_N\, \Delta S = \rho_S\, \Delta S$$

$$\therefore \quad \boxed{D_N = \rho_S} \qquad \text{... (18)}$$

Thus the flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.

$$\therefore \quad D_N = \varepsilon_0 E_N = \rho_S \qquad \text{... (19)}$$

$$\therefore \quad \boxed{E_N = \frac{\rho_S}{\varepsilon_0}} \qquad \text{... (20)}$$

5.10 Concept of Capacitance

Consider two conducting materials M_1 and M_2 which are placed in a dielectric medium having permittivity ε. The material M_1 carries a positive charge Q while the material M_2 carries a negative charge, equal in magnitude as Q. There are no other charges present and total charge of the system is zero. In conductors, charge cannot reside within the conductor and it resides only on the surface. Thus for M_1 and M_2, charges + Q and – Q reside on the surfaces of M_1 and M_2 respectively. This is shown in the Fig. 5.13.

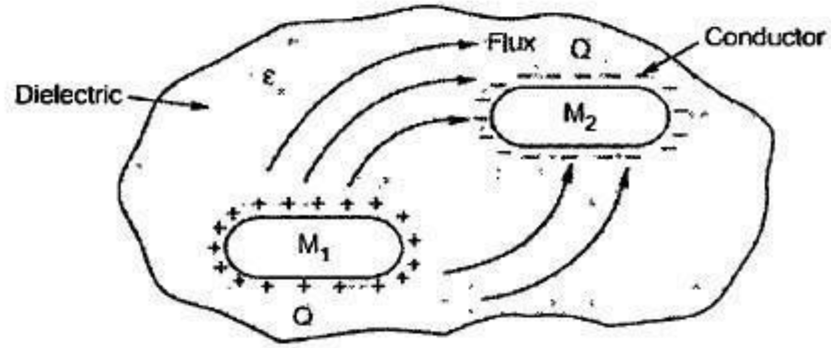


Fig. 5.13 Concept of capacitance

Such a system which has two conducting surfaces carrying equal and opposite charges, separated by a dielectric is called **capacitive system** giving rise to a **capacitance.**

The electric field is normal to the conductor surface and the electric flux is directed from M_1 towards M_2 in such a system. There exists a potential difference between the two surfaces of M_1 and M_2. Let this potential is V_{12}. The ratio of the magnitudes of the total charge on any one of the two conductors and potential difference between the conductors is called the **capacitance** of the two conductor system denoted as C.

$$\therefore \qquad C = \frac{Q}{V_{12}} \qquad \text{... (1)}$$

In general,
$$C = \frac{Q}{V} \qquad \text{... (2)}$$

where $\quad Q$ = Charge in coulombs

V = Potential difference in volts

The capacitance is measured in **farads (F)** and

$$1 \text{ Farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}}$$

As charge Q resides only on the surface of the conductor, it can be obtained from the Gauss's law as,

$$Q = \oint_S \bar{D} \cdot d\bar{S} = \oint_S \varepsilon_0 \varepsilon_r \bar{E} \cdot d\bar{S} = \oint_S \varepsilon \bar{E} \cdot d\bar{S}$$

While V is the work done in moving unit positive charge from negative to the positive surface and can be obtained as,

$$V = -\int_{l} \overline{E} \cdot \overline{dL} = -\int_{-}^{+} \overline{E} \cdot \overline{dL}$$

Hence capacitance can be expressed as,

$$C = \frac{Q}{V} = \frac{\oint_{S} \varepsilon \overline{E} \cdot d\overline{S}}{-\int_{-}^{+} \overline{E} \cdot \overline{dL}} \text{ F} \qquad \dots (3)$$

If the charge Q is increased, then $\overline{E}$ and $\overline{D}$ get increased by same factor. The voltage V also increases by same factor. Thus the ratio Q to V remains constant as C. Hence capacitance is not the function of charge, field intensity, flux density and potential difference.

5.12 Capacitors in Parallel

Key Point: *When capacitors are in parallel, the same voltage exists across them, but charges are different.*

$$\boxed{Q_1 = C_1 V, \quad Q_2 = C_2 V, \quad Q_3 = C_3 V}$$

The total charge stored by the parallel bank of capacitors Q is given by,

$$Q = Q_1 + Q_2 + Q_3$$

$$= C_1 V + C_2 V + C_3 V = (C_1 + C_2 + C_3) V \qquad \dots (1)$$

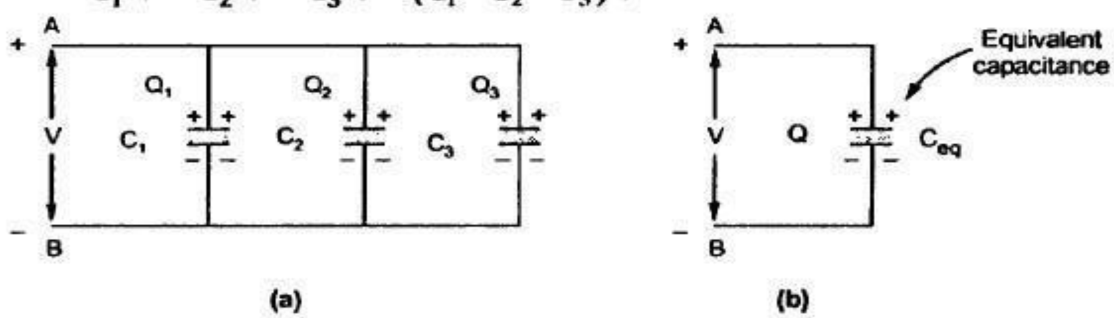


Fig. 5.15 Capacitors in parallel

An equivalent capacitor which stores the same charge Q at the same voltage V, will have

$$Q = C_{eq} V \qquad \dots (2)$$

Comparing equation (1) and equation (2),

As $\qquad C_{eq} = C_1 + C_2 + C_3$

$\therefore \qquad Q = C_1 V + C_2 V + C_3 V$

It is easy to find Q_1, Q_2 and Q_3 if V is known.

$\boxed{\text{For 'n' capacitors in parallel, } C_{eq} = C_1 + C_2 + \dots + C_n}$

5.13 Parallel Plate Capacitor

A parallel plate capacitor is shown in the Fig. 5.16. It consists of two parallel metallic plates separated by distance 'd'. The space between the plates is filled with a dielectric of permittivity ε. The lower plate, plate 1 carries the positive charge and is distributed over it with a charge density $+\rho_S$. The upper plate, plate 2 carries the negative charge and is distributed over its surface with a charge density $-\rho_S$. The plate 1 is placed in z = 0 i.e. xy plane hence normal to it is z-direction. While upper plate 2 is in z = d plane, parallel to xy plane.

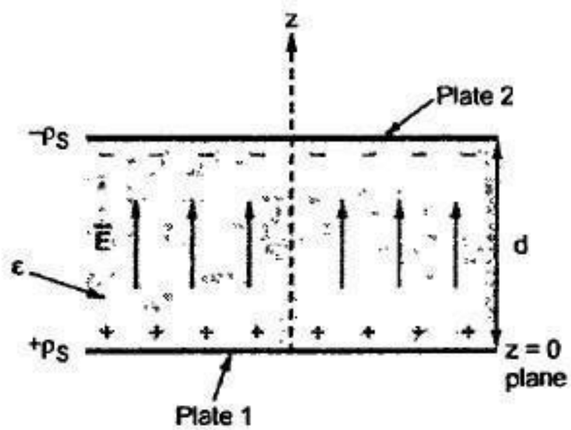


Fig. 5.16

Let A = Area of cross section of the plates in m^2.

$$\therefore \qquad Q = \rho_S \, A \quad C \qquad \dots (1)$$

This is magnitude of charge on any one plate as charge carried by both is equal in magnitude. To find potential difference, let us obtain $\bar{E}$ between the plates.

Assuming plate 1 to be infinite sheet of charge,

$$\bar{E}_1 = \frac{\rho_S}{2\varepsilon}\bar{a}_N = \frac{\rho_S}{2\varepsilon}\bar{a}_z \text{ V/m} \quad \text{... (2)}$$

The $\bar{E}_1$ is **normal** at the boundary between conductor and dielectric without any tangential component.

While for plate 2, we can write

$$\bar{E}_2 = \frac{-\rho_S}{2\varepsilon}\bar{a}_N = \frac{-\rho_S}{2\varepsilon}(-\bar{a}_z) \text{ V/m} \quad \text{... (3)}$$

The direction of $\bar{E}_2$ is downwards i.e. in $-\bar{a}_z$ direction.

In between the plates,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_S}{2\varepsilon}\bar{a}_z + \frac{\rho_S}{2\varepsilon}\bar{a}_z = \frac{\rho_S}{\varepsilon}\bar{a}_z \quad \text{... (4)}$$

The potential difference is given by,

$$V = -\int_{-}^{+} \bar{E}\cdot d\bar{L} = -\int_{\text{upper}}^{\text{lower}} \frac{\rho_S}{\varepsilon}\bar{a}_z \cdot d\bar{L}$$

Now $\bar{dL} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$ in Cartesian system.

$$\therefore \quad V = -\int_{z=d}^{z=0} \frac{\rho_S}{\varepsilon}\, \bar{a}_z \cdot [dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z]$$

$$= -\int_{z=d}^{z=0} \frac{\rho_S}{\varepsilon}\, dz = -\frac{\rho_S}{\varepsilon}\,[z]_d^0 = -\frac{\rho_S[-d]}{\varepsilon}$$

$$\therefore \quad V = \frac{\rho_S\, d}{\varepsilon}\ \text{V}$$

The capacitance is the ratio of charge Q to voltage V,

$$C = \frac{Q}{V} = \frac{\rho_S A}{\frac{\rho_S d}{\varepsilon}} = \frac{\varepsilon A}{d}\ \text{F} \qquad \text{... (5)}$$

Thus if, $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r A}{d}\ \text{F}} \qquad \text{... (6)}$$

It can be seen that the value of capacitance depends on,

1. The permittivity of the dielectric used.
2. The area of cross section of the plates.
3. The distance of separation of the plates.

It is not dependant on the charge or the potential difference between the plates.

5.14 Capacitance of a Co-axial Cable

Consider a co-axial cable or co-axial capacitor as shown in the Fig. 5.17.

Let a = Inner radius

b = Outer radius

The two concentric conductors are separated by dielectric of permittivity ε.

The length of the cable is L m.

The inner conductor carries a charge density $+\rho_L$ C/m on its surface then equal and opposite charge density $-\rho_L$ C/m exists on the outer conductor.

$$\therefore \quad Q = \rho_L \times L \qquad \text{... (1)}$$

Assuming cylindrical co-ordinate system, $\bar{E}$ will be radial from inner to outer conductor, and for infinite line charge it is given by,

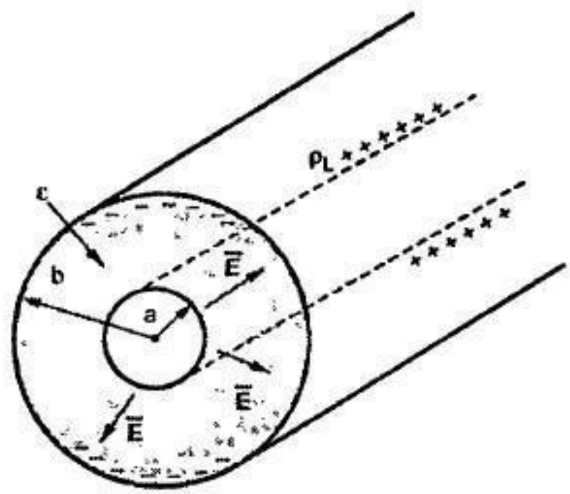


Fig. 5.17 Co-axial cable

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r}\,\bar{a}_r \qquad \text{... (2)}$$

$\bar{E}$ is directed from inner conductor to the outer conductor. The potential difference is work done in moving unit charge against $\bar{E}$ i.e. from r = b to r = a.

To find potential difference, consider $d\bar{L}$ in radial direction whch is $dr\bar{a}_r$

$$\therefore \quad V = -\int_{-}^{+} \bar{E}\cdot d\bar{L} = -\int_{r=b}^{r=a} \frac{\rho_L}{2\pi\varepsilon r}\bar{a}_r \cdot dr\bar{a}_r$$

$$= -\frac{\rho_L}{2\pi\varepsilon}[ln\, r]_b^a = -\frac{\rho_L}{2\pi\varepsilon}\, ln\left[\frac{a}{b}\right]$$

$$\therefore \quad V = \frac{\rho_L}{2\pi\varepsilon}\, ln\left[\frac{b}{a}\right] V \qquad \text{... (3)}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{\rho_L \times L}{\frac{\rho_L}{2\pi\varepsilon}\, ln\left[\frac{b}{a}\right]}$$

$$\therefore \quad \boxed{C = \frac{2\pi\varepsilon L}{ln\left[\frac{b}{a}\right]}\ F} \qquad \text{... (4)}$$

5.15 Spherical Capacitor

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b. The capacitor is shown in the Fig. 5.18.

The radius of outer sphere is 'b' while that of inner sphere is 'a'. Thus b > a. The region between the two spheres is filled with a dielectric of permittivity ε. The inner sphere is given a positive charge + Q while for the outer sphere it is – Q.

Considering Gaussian surface as a sphere of radius r, it can be obtained that $\bar{E}$ is in radial direction and given by,

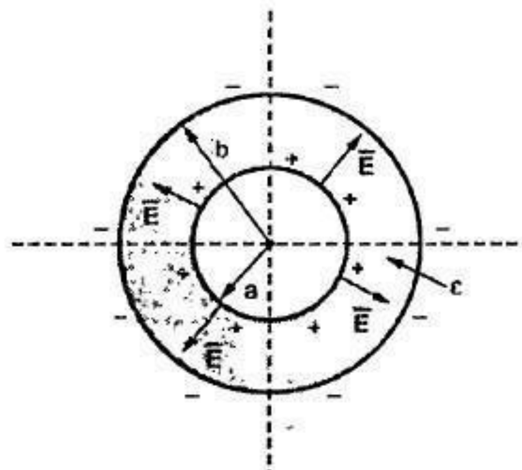


Fig. 5.18 Spherical capacitor

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2}\bar{a}_r \text{ V/m} \qquad \text{... (1)}$$

The potential difference is work done in moving unit positive charge against the direction of $\bar{E}$ i.e. from r = b to r = a.

$$\therefore \quad V = -\int_{-}^{+} \bar{E}\cdot d\bar{L} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\,\bar{a}_r \cdot d\bar{L} \qquad \text{... (2)}$$

Now $\quad d\bar{L} = dr\,\bar{a}_r$ ... In radial direction

$$\therefore \quad V = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\bar{a}_r \cdot dr\,\bar{a}_r = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\,dr$$

$$= -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{r=b}^{r=a} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_{r=b}^{r=a}$$

$$\therefore \quad V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right] V \qquad \text{... (3)}$$

Now $\quad C = \dfrac{Q}{V} = \dfrac{Q}{\dfrac{Q}{4\pi\varepsilon}\left[\dfrac{1}{a}-\dfrac{1}{b}\right]}$

$$\therefore \quad \boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}\ F} \qquad \text{... (4)}$$

5.17 Energy Stored in a Capacitor

It is seen that capacitor can store the energy. Let us find the expression for the energy stored in a capacitor.

Consider a parallel plate capacitor as shown in the Fig. 5.25. It is supplied with the voltage V.

Let $\bar{a}_N$ is the direction normal to the plates.

$$\therefore \quad \bar{E} = \frac{V}{d}\bar{a}_N \qquad \text{... (1)}$$

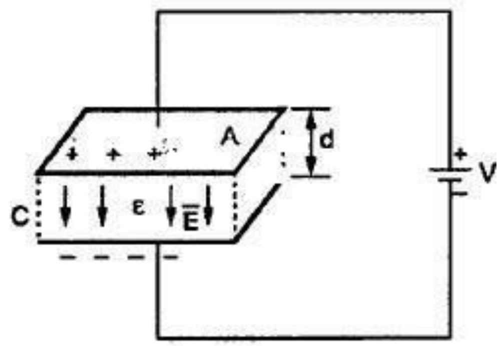


Fig. 5.25 Parallel plate capacitor

The energy stored is given by,

$$W_E = \frac{1}{2}\int_{vol} \bar{D}\cdot\bar{E}\,dv$$

$$= \frac{1}{2}\int_{vol} \varepsilon\bar{E}\cdot\bar{E}\,dv \qquad \text{but } \bar{E}\cdot\bar{E} = |\bar{E}|^2$$

$$= \frac{1}{2}\int_{vol} \varepsilon|\bar{E}|^2\,dv \qquad \text{but } |\bar{E}| = \frac{V}{d}$$

$$= \frac{1}{2}\varepsilon\frac{V^2}{d^2}\int_{vol} dv \qquad \text{but } \int_{vol} dv = \text{Volume} = A\times d$$

$$= \frac{1}{2}\varepsilon\frac{V^2 Ad}{d^2} = \frac{1}{2}\frac{\varepsilon A}{d}V^2$$

$$\boxed{W_E = \frac{1}{2} C V^2 \ \text{J}}$$

If the dielectric is free space then there is increase in the stored energy if free space is replaced by other dielectric having $\varepsilon_r > 1$.

5.17.1 Energy-Density

As seen in earlier chapter, energy density is the energy stored per unit volume as volume tends to zero.

$$\therefore \qquad W_E = \frac{1}{2} \varepsilon \int_{Vol} |\bar{E}|^2 \, dv$$

$$\therefore \qquad W_E = \frac{1}{2} \varepsilon |\bar{E}|^2 \ \text{J/m}^3 = \text{Energy density}$$

Using $\qquad |\bar{D}| = \varepsilon |\bar{E}|$ we can write,

$$\boxed{W_E = \frac{1}{2} \frac{|\bar{D}|^2}{\varepsilon} = \frac{1}{2} |\bar{D}| |\bar{E}| \ \text{J/m}^3}$$