# Example

$$X(t) = \text{Poisson r.p.}$$

$$0 < t_1 < t_2 < t_3$$

$$0 \le k_1 \le k_2 \le k_3 \quad \Rightarrow$$

$$P[X(t_1) = k_1, X(t_2) = k_2, X(t_3) = k_3]$$

$$= P[X(t_1) = k_1, X(t_2) - X(t_1) = k_2 - k_1, X(t_3) - X(t_2) = k_3 - k_2]$$

$$= P[X(t_1) = k_1]P[X(t_2) - X(t_1) = k_2 - k_1]P[X(t_3) - X(t_2) = k_3 - k_2]$$

$$= \frac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda t_1} \frac{[\lambda(t_2 - t_1)]^{(k_2 - k_1)}}{(k_2 - k_1)!} e^{-\lambda(t_2 - t_1)} \frac{[\lambda(t_3 - t_2)]^{(k_3 - k_2)}}{(k_3 - k_2)!} e^{-\lambda(t_3 - t_2)}$$

$$= \frac{(\lambda t_1)^{k_1}[\lambda(t_2 - t_1)]^{(k_2 - k_1)}[\lambda(t_3 - t_2)]^{(k_3 - k_2)}}{k_1!(k_2 - k_1)!(k_3 - k_2)!} e^{-\lambda t_3}$$

# UNIT-IV
# Stochastic Processes: Spectral Characteristics

# Introduction to Power density spectrum

- Fourier integral

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau ] e^{j\omega t} d\omega$$

- Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

# Introduction (Contd..)

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & o/w \end{cases}$$

Assume $\int_{-T}^{T} \left| x_T(t) \right| dt < \infty$, for all finite $T$.

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt = \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

- Energy contained in x(t) in the interval (-T,T)

$$E(T) = \int_{-\infty}^{\infty} x_T(t)^2 dt = \int_{-T}^{T} x(t)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| X_T(\omega) \right|^2 d\omega$$

# Introduction (Contd..)

- Average power in x(t) in the interval (-T,T)

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left|X_T(\omega)\right|^2}{2T}\,d\omega$$

$x(t) \rightarrow X(t)$, take expectation, let $T \rightarrow \infty$.

- Average power in random process x(t)

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X(t)^2]\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[\left|X_T(\omega)\right|^2]}{2T}\,d\omega$$

$$P_{XX} = A\{E[X(t)^2]\} \qquad P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$S_{XX} = \lim_{T\to\infty} \frac{E[\left|X_T(\omega)\right|^2]}{2T}$$ power density spectrum

# Example-1

$$P_{XX} = A\{E[X(t)^2]\}$$

$$\text{w.s.s.} \Rightarrow \quad P_{XX} = R_{XX}(0)$$

- Example-1 $X(t) = A_0 \cos(\omega_0 t + \Theta)$ $\quad \Theta$ -- uniformly distributed on $(0, \frac{\pi}{2})$

$$E[X(t)^2] = E[A_0^2 \cos^2(\omega_0 t + \Theta)] = E[\frac{A_0^2}{2} + \frac{A_0^2}{2}\cos(2\omega_0 t + 2\Theta)]$$

$$= \frac{A_0^2}{2} + \frac{A_0^2}{2}\int_0^{\frac{\pi}{2}} \frac{2}{\pi}\cos(2\omega_0 t + 2\theta)\,d\theta = \frac{A_0^2}{2} + \frac{A_0^2}{2\pi}\sin(2\omega_0 t + 2\theta)\Big|_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{A_0^2}{2} - \frac{A_0^2}{\pi}\sin(2\omega_0 t)$$

$$P_{XX} = A\{E[X(t)^2]\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\frac{A_0^2}{2} - \frac{A_0^2}{\pi}\sin(2\omega_0 t)]dt = \frac{A_0^2}{2}$$

# Example-2

- Example-2 $$X(t) = A_0 \cos(\omega_0 t + \Theta)$$

$$X_T(\omega) = \int_{-T}^{T} A_0 \cos(\omega_0 t + \Theta) e^{-j\omega t} dt = \int_{-T}^{T} A_0 \frac{1}{2}[e^{j\Theta} e^{j\omega_0 t} + e^{-j\Theta} e^{-j\omega_0 t}] e^{-j\omega t} dt$$

$$= \frac{A_0}{2} e^{j\Theta} \int_{-T}^{T} e^{j(\omega_0 - \omega)t} dt + \frac{A_0}{2} e^{-j\Theta} \int_{-T}^{T} e^{-j(\omega_0 + \omega)t} dt$$

$$= A_0 T e^{j\Theta} \frac{\sin[(\omega - \omega_0)T]}{(\omega - \omega_0)T} + A_0 T e^{-j\Theta} \frac{\sin[(\omega + \omega_0)T]}{(\omega + \omega_0)T}$$

$$\int_{-T}^{T} e^{j\beta t} dt = \frac{1}{j\beta} e^{j\beta t} \Big|_{t=-T}^{T} = \frac{e^{j\beta T} - e^{-j\beta T}}{j\beta} = 2T \frac{\sin(\beta T)}{\beta T}$$

$$S_{XX} = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$ power density spectrum

# Example-2 (Contd..)

$$X_T(\omega) = A_0 T e^{j\Theta} \frac{\sin[(\omega - \omega_0)T]}{(\omega - \omega_0)T} + A_0 T e^{-j\Theta} \frac{\sin[(\omega + \omega_0)T]}{(\omega + \omega_0)T}$$

$$X_T(\omega)^* = A_0 T e^{-j\Theta} \frac{\sin[(\omega - \omega_0)T]}{(\omega - \omega_0)T} + A_0 T e^{j\Theta} \frac{\sin[(\omega + \omega_0)T]}{(\omega + \omega_0)T}$$

$$\left|X_T(\omega)\right|^2 = X_T(\omega) X_T(\omega)^* = A_0^2 [T^2 \frac{\sin^2[(\omega - \omega_0)T]}{(\omega - \omega_0)^2 T^2} + T^2 \frac{\sin^2[(\omega + \omega_0)T]}{(\omega + \omega_0)^2 T^2}]$$

$$+ A_0^2 T^2 (e^{j2\Theta} + e^{-j2\Theta}) \frac{\sin[(\omega - \omega_0)T]}{(\omega - \omega_0)T} \frac{\sin[(\omega + \omega_0)T]}{(\omega + \omega_0)T}$$

$$E[e^{j2\Theta} + e^{-j2\Theta}] = E[2\cos 2\Theta] = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} 2\cos 2\theta \, d\theta = \frac{2}{\pi} \sin 2\theta \Big|_0^{\pi/2} = 0$$

$$\frac{E[\left|X_T(\omega)\right|^2]}{2T} = \frac{A_0^2 \pi}{2} [\frac{T}{\pi} \frac{\sin^2[(\omega - \omega_0)T]}{(\omega - \omega_0)^2 T^2} + \frac{T}{\pi} \frac{\sin^2[(\omega + \omega_0)T]}{(\omega + \omega_0)^2 T^2}]$$

# Example-2 (Contd..)

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{T}{\pi} \frac{\sin^2(\alpha T)}{(\alpha T)^2} d\alpha = \int_{-\infty}^{\infty} \frac{T}{\pi} \frac{\sin^2 x}{x^2} \frac{1}{T} dx = 1 \qquad \text{(a)}$$

$$\lim_{T \to \infty} \frac{T}{\pi} \frac{\sin^2(\alpha T)}{(\alpha T)^2} = \begin{cases} \infty, & \text{if } \alpha = 0 \\ 0, & \text{if } \alpha \neq 0 \end{cases} \qquad \text{(b)}$$

$$\text{(a) \& (b)} \Rightarrow \quad \lim_{T \to \infty} \frac{T}{\pi} \frac{\sin^2(\alpha T)}{(\alpha T)^2} = \delta(\alpha)$$

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T} = \frac{A_0^2 \pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0^2 \pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] d\omega = \frac{A_0^2}{2}$$

# Properties Power density spectrum

Properties of the power density spectrum:

(1) $S_{XX}(\omega) \geq 0$

(2) $X(t)$ real $\Rightarrow$ $S_{XX}(-\omega) = S_{XX}(\omega)$

(3) $S_{XX}(\omega)$ is real

(4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = A\{E[X(t)^2]\}$

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T}$$

PF of (2): $X_T(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t}dt$

$$X_T(\omega)^* = \int_{-T}^{T} X(t)^* e^{j\omega t}dt = \int_{-T}^{T} X(t)e^{j\omega t}dt = X_T(-\omega)$$

$$S_{XX}(-\omega) = \lim_{T\to\infty}\frac{E[X_T(-\omega)X_T(-\omega)^*]}{2T} = \lim_{T\to\infty}\frac{E[X_T(\omega)^* X_T(\omega)]}{2T} = S_{XX}(\omega)$$

# Properties Power density spectrum

Properties of the power density spectrum

$$(5) \qquad S_{XX}(\omega) = \omega^2 S_{XX}(\omega) \qquad\qquad \frac{d}{dt}X(t) = \lim_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

PF of (5):

$$\dot{X}_T(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{X(t+\varepsilon) - X(t)}{\varepsilon}, & -T < t < T \\ 0, & \text{o/w} \end{cases}$$

$$f(t-a) \xleftrightarrow{FT} F(\omega)e^{-j\omega a}$$

$$\dot{X}_T(t) \xleftrightarrow{F} \lim_{\varepsilon \to 0} \frac{X_T(\omega)e^{j\omega\varepsilon} - X_T(\omega)}{\varepsilon} = j\omega X_T(\omega)$$

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T \to \infty} \frac{E[|\dot{X}_T(\omega)|^2]}{2T} = \lim_{T \to \infty} \frac{E[|j\omega X_T(\omega)|^2]}{2T} = \omega^2 \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T} = \omega^2 S_{XX}(\omega)$$

# Properties Power density spectrum

Bandwidth of the power density spectrum

$X(t)$ real $\Rightarrow$ $S_{XX}(\omega)$ even

$S_{XX}(\omega)$ lowpass form $\Rightarrow$

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega}$$

root mean square Bandwidth

$S_{XX}(\omega)$ bandpass form $\Rightarrow$

$$\overline{\omega}_0 = \frac{\int_0^{\infty} \omega S_{XX}(\omega) d\omega}{\int_0^{\infty} S_{XX}(\omega) d\omega}$$

mean frequency

$$W_{rms}^2 = \frac{4\int_0^{\infty} (\omega - \overline{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_0^{\infty} S_{XX}(\omega) d\omega}$$

rms BW

# Example

$$S_{XX}(\omega) = \frac{10}{[1+(\omega/10)^2]^2} \qquad S_{XX}(\omega)\ \text{lowpass form}$$

$$\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = \int_{-\infty}^{\infty} \frac{10}{[1+(\omega/10)^2]^2} d\omega = \int_{-\pi/2}^{\pi/2} \frac{10}{[1+\tan^2\theta]^2} 10\sec^2\theta\ d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{100}{\sec^2\theta} d\theta = \int_{-\pi/2}^{\pi/2} 100\cos^2\theta\ d\theta = \int_{-\pi/2}^{\pi/2} 100\frac{1+\cos 2\theta}{2} d\theta = 50\pi$$

$$\omega = 10\tan\theta \quad \Rightarrow \quad d\omega = 10\sec^2\theta\ d\theta$$

$$\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)d\omega = \int_{-\infty}^{\infty} \frac{10\omega^2}{[1+(\omega/10)^2]^2} d\omega = \int_{-\pi/2}^{\pi/2} \frac{10^3\tan^2\theta}{[1+\tan^2\theta]^2} 10\sec^2\theta\ d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{10^4\tan^2\theta}{\sec^2\theta} d\theta = \int_{-\pi/2}^{\pi/2} 10^4\sin^2\theta\ d\theta = \int_{-\pi/2}^{\pi/2} 10^4\frac{1-\cos 2\theta}{2} d\theta = 5000\pi$$

# Example

$$W_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega} = 100$$

rms BW $\quad W_{\text{rms}} = 10 \text{ rad/sec}$

$$S_{XX}(\omega) = \frac{10}{[1 + (\omega/10)^2]^2}$$

# Relationship between PSD and autocorrelation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = A[R_{XX}(t,t+\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]e^{-j\omega\tau}d\tau$$

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{E[X_T(\omega)^* X_T(\omega)]}{2T} = \lim_{T\to\infty}\frac{1}{2T}E[\int_{-T}^{T} X(t_1)e^{j\omega t_1}dt_1 \int_{-T}^{T} X(t_2)e^{-j\omega t_2}dt_2]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1\, e^{j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau+t_1-t_2)}d\omega\, dt_2 dt_1$$

# Relationship between PSD and autocorrelation

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) \delta(\tau + t_1 - t_2) dt_2 dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t_1, t_1 + \tau) dt_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t, t + \tau) dt$$

$$= A[R_{XX}(t, t + \tau)]$$

$$A[R_{XX}(t, t + \tau)] \xleftrightarrow{FT} S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

- $X(t)$ w.s.s. $\Rightarrow$ $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$

$$R_{XX}(\tau) \xleftrightarrow{FT} S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

# Cross-power density spectrum

$$W(t) = X(t) + Y(t)$$

$$R_{WW}(t, t+\tau) = E[W(t)W(t+\tau)] = E\{[X(t)+Y(t)][X(t+\tau)+Y(t+\tau)]\}$$

$$= R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau)$$

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + F\{A[R_{XY}(t, t+\tau)]\} + F\{A[R_{YX}(t, t+\tau)]\}$$

# Cross-power density spectrum

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & o/w \end{cases} \qquad y_T(t) = \begin{cases} y(t), & -T < t < T \\ 0, & o/w \end{cases}$$

$$\text{Assume} \quad \int_{-T}^{T} |x_T(t)|\,dt < \infty \;\;\&\;\; \int_{-T}^{T} |y_T(t)|\,dt < \infty, \text{ for all finite } T.$$

$$x_T(t) \xleftrightarrow{\text{FT}} X_T(\omega) \qquad y_T(t) \xleftrightarrow{\text{FT}} Y_T(\omega)$$

Cross Power contained in $x(t), y(t)$ in the interval $(-T, T)$

$$P_{XY}(T) = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(t)\,y_T(t)\,dt = \frac{1}{2T}\int_{-T}^{T} x(t)\,y(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T(\omega)^{*}\,Y_T(\omega)}{2T}\,d\omega$$

Parseval's theorem

# Cross-power density spectrum

average Cross Power contained in $X(t), Y(t)$ in the interval $(-T, T)$

$$\bar{P}_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[X_T(\omega)^* Y_T(\omega)]}{2T}\,d\omega$$

total average Cross Power contained in $X(t), Y(t)$

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[X_T(\omega)^* Y_T(\omega)]}{2T}\,d\omega$$

cross-power density spectrum $\quad S_{XY}(\omega) = \lim_{T\to\infty} \dfrac{E[X_T(\omega)^* Y_T(\omega)]}{2T}$

# Cross-power density spectrum

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\,d\omega$$

$$S_{YX}(\omega) = \lim_{T\to\infty}\frac{E[Y_T(\omega)^* X_T(\omega)]}{2T}$$

$$P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\,d\omega = P_{XY}^*$$

$$\text{Total cross power} = P_{XY} + P_{YX}$$

$$X(t), Y(t) \ \text{ orthogonal} \Rightarrow P_{XY} = P_{YX} = 0$$

# Properties of cross-power density spectrum

$X(t), Y(t)$ real

Properties of the cross-power density spectrum:

(1) $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}(\omega)^*$

PF of (1): $X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt$

$$X_T(\omega)^* = \int_{-T}^{T} X(t)^* e^{j\omega t} dt = \int_{-T}^{T} X(t) e^{j\omega t} dt = X_T(-\omega)$$

$$S_{YX}(-\omega) = \lim_{T\to\infty} \frac{E[Y_T(-\omega)^* X_T(-\omega)]}{2T} = \lim_{T\to\infty} \frac{E[Y_T(\omega) X_T(\omega)^*]}{2T} = S_{XY}(\omega)$$

$$S_{YX}(-\omega) = \lim_{T\to\infty} \frac{E[Y_T(-\omega)^* X_T(-\omega)]}{2T} = \lim_{T\to\infty} \frac{E[Y_T(\omega) X_T(\omega)^*]}{2T} = S_{YX}(\omega)^*$$

# Properties of cross-power density spectrum

(2) $\mathrm{Re}[S_{XY}(\omega)]$ & $\mathrm{Re}[S_{YX}(\omega)]$ -- even

(3) $\mathrm{Im}[S_{XY}(\omega)]$ & $\mathrm{Im}[S_{YX}(\omega)]$ -- odd

$$A\left[R_{XY}(t, t+\tau)\right] \xleftrightarrow{FT} S_{XY}(\omega)$$

$$A\left[R_{YX}(t, t+\tau)\right] \xleftrightarrow{FT} S_{YX}(\omega)$$

(4) $X(t)$ & $Y(t)$ orthogonal $\Rightarrow$ $S_{XY}(\omega) = S_{YX}(\omega) = 0$

$X(t)$ & $Y(t)$ orthogonal $\Rightarrow$ $R_{XY}(t, t+\tau) = 0$ $\Rightarrow$ $\mathrm{A}[R_{XY}(t, t+\tau)] = 0$

(5) $X(t)$ & $Y(t)$ uncorrelated & have constant mean $\overline{X}, \overline{Y}$

$$\Rightarrow \quad S_{XY}(\omega) = S_{YX}(\omega) = 2\pi\, \overline{X}\,\overline{Y}\,\delta(\omega)$$

# Properties of cross-power density spectrum

PF of (5): $R_{XY}(t, t+\tau) = \overline{X}\,\overline{Y} \quad \Rightarrow \quad A[R_{XY}(t, t+\tau)] = \overline{X}\,\overline{Y}$

$$\Rightarrow \quad S_{XY}(\omega) = 2\pi\, \overline{X}\,\overline{Y}\,\delta(\omega) = S_{YX}(\omega)^*$$

$X(t), Y(t)$ -- jointly w.s.s. $\Rightarrow \quad R_{XY}(\tau) \xleftrightarrow{\text{FT}} S_{XY}(\omega)$

$$R_{YX}(\tau) \xleftrightarrow{\text{FT}} S_{YX}(\omega)$$

# Relationship between C-PSD and cross-correlation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega = A[R_{XY}(t,t+\tau)]$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t,t+\tau)]e^{-j\omega\tau}d\tau$$

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{E[X_T(\omega)^* Y_T(\omega)]}{2T} = \lim_{T\to\infty}\frac{1}{2T}E[\int_{-T}^{T} X(t_1)e^{j\omega t_1}dt_1 \int_{-T}^{T} Y(t_2)e^{-j\omega t_2}dt_2]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)Y(t_2)]e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1\ e^{j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau+t_1-t_2)}d\omega\, dt_2 dt_1$$

# Relationship between C-PSD and cross-correlation

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XY}(t_1, t_2) \delta(\tau + t_1 - t_2) dt_2 dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t_1, t_1 + \tau) dt_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t+\tau) dt$$

$$= A[R_{XY}(t, t+\tau)]$$

$$A[R_{XY}(t, t+\tau)] \xleftrightarrow{FT} S_{XY}(\omega)$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

# Example

Example:

$$R_{XY}(t, t+\tau) = \frac{AB}{2}\{\sin(\omega_0\tau) + \cos[\omega_0(2t+\tau)]\}$$

$$A[R_{XY}(t, t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)dt$$

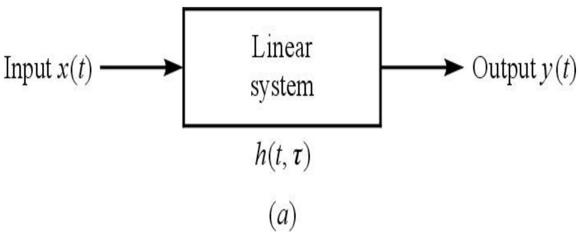
$$= \frac{AB}{2}\sin(\omega_0\tau) + \frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos[\omega_0(2t+\tau)]dt$$

$$= \frac{AB}{2}\sin(\omega_0\tau) = \frac{AB}{4j}[e^{j\omega_0\tau} - e^{-j\omega_0\tau}]$$

$$S_{XY}(\omega) = \frac{AB}{4j}[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)] = \frac{-j\pi AB}{2}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

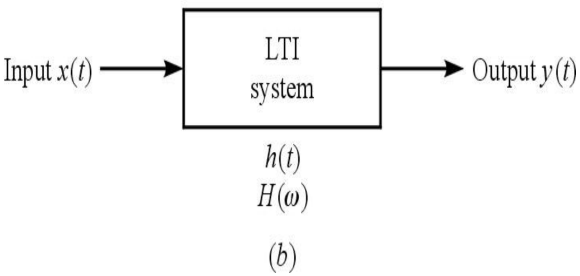
# Linear system fundamentals

Linear System $$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t,\xi)d\xi$$




$\delta(t-\xi) \rightarrow h(t,\xi)$ impulse response

Linear Time-Invariant System (LTI system)




$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t-\xi)d\xi = \int_{-\infty}^{\infty} h(\xi)x(t-\xi)d\xi$$

$y(t) = x(t) * h(t) = h(t) * x(t)$ convolution integral

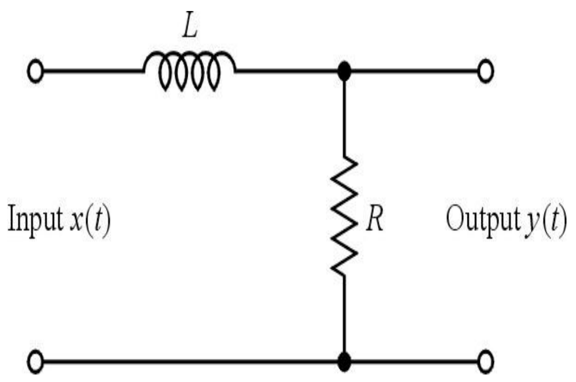
$$Y(\omega) = X(\omega)H(\omega)$$

$$x(t) = e^{j\omega t} \quad \Rightarrow \quad \frac{y(t)}{x(t)} = \frac{\int_{-\infty}^{\infty} h(\xi)e^{j\omega(t-\xi)}d\xi}{e^{j\omega t}} = \int_{-\infty}^{\infty} h(\xi)e^{-j\omega\xi}d\xi = H(\omega)$$

# Linear system fundamentals

Example-1: $$H(s) = \frac{R}{sL + R}$$

$$H(\omega) = \frac{R}{j\omega L + R}$$




LTI causal $\Leftrightarrow$ $h(t) = 0 \quad \text{for } t < 0$

LTI stable $\Leftrightarrow$ $\int_{-\infty}^{\infty} \left| h(t) \right| dt < \infty$

# Linear system fundamentals

Ideal lowpass filter

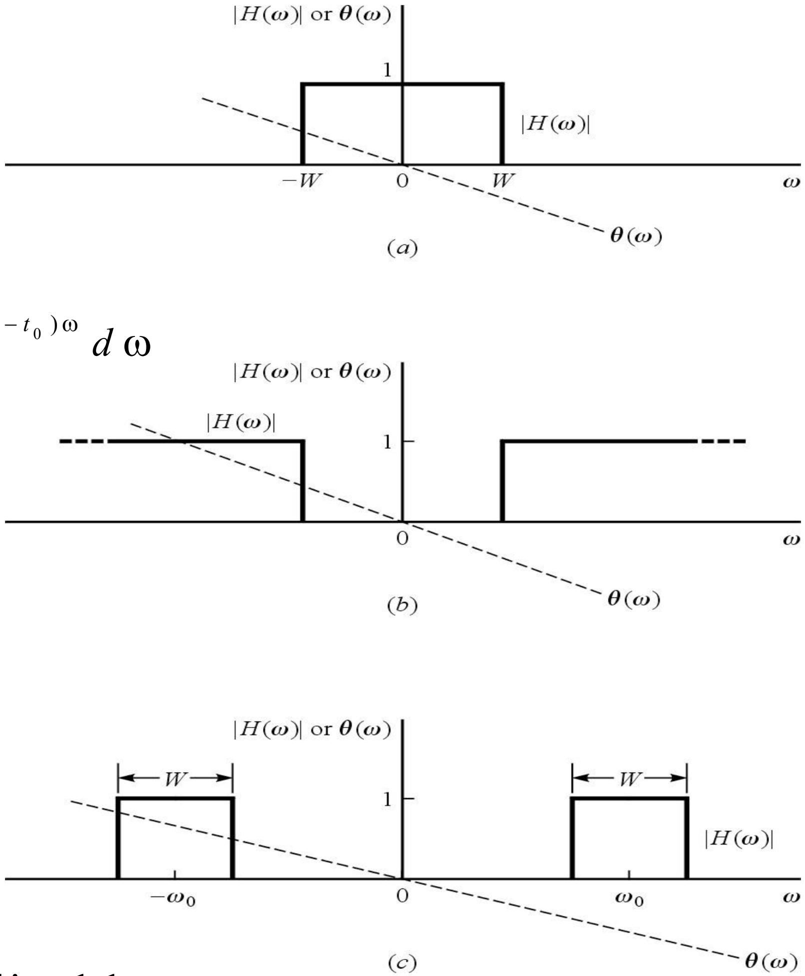
$$H(\omega) = \begin{cases} e^{-jt_0\omega}, & |\omega| < W \\ 0, & \text{o/w} \end{cases}$$

$$h(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{-jt_0\omega} e^{jt\omega}\, d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j(t-t_0)\omega}\, d\omega$$

$$= \frac{1}{2\pi}\frac{1}{j(t-t_0)} e^{j(t-t_0)\omega}\Big|_{-W}^{W}$$

$$= \frac{1}{2\pi}\frac{e^{j(t-t_0)W} - e^{-j(t-t_0)W}}{j(t-t_0)}$$

$$= \frac{W}{\pi}\frac{\sin[(t-t_0)W]}{(t-t_0)W}$$




Not causal $\Rightarrow$ Not physically realizable

# Random signal response of linear systems

$X(t)$ -- w.s.s. random input $\qquad Y(t) = \int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi$

$$E[Y(t)] = E[\int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi] = \int_{-\infty}^{\infty} h(\xi)E[X(t-\xi)]d\xi$$

$$= \overline{X}\int_{-\infty}^{\infty} h(\xi)d\xi = \overline{Y}$$

$$R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= E[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)X(t+\tau-\xi_2)d\xi_2]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\xi_1)X(t+\tau-\xi_2)]h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$X(t)$ w.s.s. $\Rightarrow$ $Y(t)$ w.s.s.

# Random signal response of linear systems

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} R_{XX}(\tau + \xi_1 - \xi_2)h(\xi_1)d\xi_1]h(\xi_2)d\xi_2$$

$$= \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} R_{XX}(\tau - \xi_2 - \xi_1)h(-\xi_1)d\xi_1]h(\xi_2)d\xi_2$$

$$= \int_{-\infty}^{\infty} R_{XX}(\xi_1) * h(-\xi_1)\Big|_{\xi_1=\tau-\xi_2} h(\xi_2)d\xi_2$$

$$= R_{XX}(\tau) * h(-\tau) * h(\tau)$$

$$E[Y(t)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1 - \xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

Example-1: white noise $X(t)$ $\qquad R_{XX}(\tau) = (N_0/2)\delta(\tau)$

$$E[Y(t)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (N_0/2)\delta(\xi_1 - \xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$= (N_0/2)\int_{-\infty}^{\infty} h(\xi_2)^2 d\xi_2$$

# Random signal response of linear systems

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = E[X(t)\int_{-\infty}^{\infty} h(\xi)X(t+\tau-\xi)d\xi]$$

$$= \int_{-\infty}^{\infty} E[X(t)X(t+\tau-\xi)]h(\xi)d\xi$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)h(\xi)d\xi$$

$$= R_{XX}(\tau) * h(\tau) = R_{XY}(\tau)$$

$$R_{YX}(\tau) = R_{XY}(-\tau) = R_{XX}(-\tau) * h(-\tau) = R_{XX}(\tau) * h(-\tau)$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)h(-\xi)d\xi$$

$$X(t) \ \text{w.s.s.} \quad \Rightarrow \quad X(t) \ \& \ Y(t) \ \text{jointly w.s.s.}$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{YX}(\tau) * h(\tau)$$

# Random signal response of linear systems

Example-2: white noise $X(t)$ $R_{XX}(\tau) = (N_0/2)\delta(\tau)$

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau - \xi) h(\xi) d\xi$$

$$= \int_{-\infty}^{\infty} (N_0/2)\delta(\tau - \xi) h(\xi) d\xi = (N_0/2) h(\tau)$$

$$R_{YX}(\tau) = R_{XY}(-\tau) = (N_0/2) h(-\tau)$$

# Spectral characteristics of system response

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) \qquad S_{XY}(\omega) = S_{XX}(\omega)H(\omega)$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau) \qquad S_{YX}(\omega) = S_{XX}(\omega)H(-\omega) = S_{XX}(\omega)H(\omega)^*$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

$$S_{YY}(\omega) = S_{XY}(\omega)H(\omega)^* = S_{XX}(\omega)H(\omega)H(\omega)^* = S_{XX}(\omega)\left|H(\omega)\right|^2$$

$$h(\tau) \xleftrightarrow{FT} H(\omega)$$

$$h(\tau) \ \text{real} \Rightarrow \quad h(-\tau) \xleftrightarrow{FT} H(-\omega) = H(\omega)^*$$

# Spectral characteristics of system response

average power $$P_{YY}=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{YY}(\omega)\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)\left|H(\omega)\right|^{2}d\omega$$

Example-1: $$S_{XX}(\omega)=\frac{N_0}{2}\qquad H(\omega)=\frac{1}{1+(j\omega L/R)}$$

$$S_{YY}(\omega)=S_{XX}(\omega)\left|H(\omega)\right|^{2}=\frac{N_0/2}{1+(\omega L/R)^{2}}$$

$$P_{YY}=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{YY}(\omega)\,d\omega=\frac{N_0}{4\pi}\int_{-\infty}^{\infty}\frac{1}{1+(\omega L/R)^{2}}\,d\omega$$

$$=\frac{N_0}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{1+\tan^{2}\theta}\,\frac{R}{L}\sec^{2}\theta\,d\theta=\frac{N_0 R}{4\pi L}\int_{-\pi/2}^{\pi/2}d\theta=\frac{N_0 R}{4L}$$

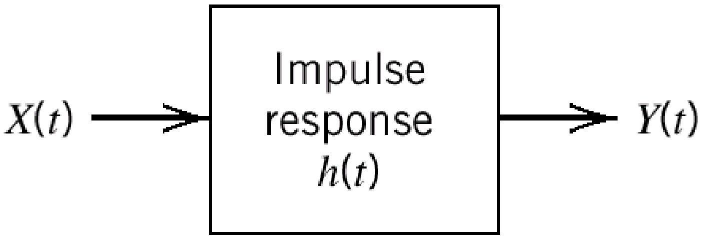
# Spectral characteristics of system response

$$h(t) = (R/L)u(t)e^{-Rt/L} \xleftrightarrow{FT} H(\omega) = \frac{1}{1 + (j\omega L/R)}$$

By Example-1,

$$P_{YY} = \frac{N_0}{2}\int_{-\infty}^{\infty} h(t)^2\,dt = \frac{N_0}{2}\int_0^{\infty} (R/L)^2 e^{-2Rt/L}\,dt = \frac{N_0 R}{4L} e^{-2Rt/L}\Big|_0^{\infty} = \frac{N_0 R}{4L}$$

# Random process through a LTI System




$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1)\, d\tau_1$$

where h(t) is the impulse response of the system

$$\mu_Y(t) = E[Y(t)]$$

If $E[X(t)]$ is finite and system is stable

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1) E[x(t - \tau_1)]\, d\tau_1$$

If $X(t)$ is stationary,
$H(0)$ :System DC response.

$$= \int_{-\infty}^{\infty} h(\tau_1) \mu_X(t - \tau_1)\, d\tau_1$$

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 = \mu_X H(0),$$

# Random process through a LTI System

Consider autocorrelation function of $Y(t)$:

$$R_Y(t,\mu) = E\left[Y(t)Y(\mu)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(\mu-\tau_2)\,d\tau_2\right]$$

If $\mathrm{E}[X^2(t)]$ is finite and the system is stable,

$$R_Y(t,\mu) = \int_{-\infty}^{\infty} d\tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2) R_X(t-\tau_1, \mu-\tau_2)$$

If $R_X(t-\tau_1, \mu-\tau_2) = R_X(t-\mu-\tau_1+\tau_2)$ (stationary)

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2$$

Stationary input, Stationary output

$$R_Y(0) = E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

# Power Spectral Density (PSD)

Consider the Fourier transform of *g*(t),

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)\,dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

Let *H*(*f* ) denote the frequency response,

$$\tau = \tau_2 - \tau_1$$

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f)\exp(j2\pi f\tau_1)\,df$$

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(f)\exp(j2\pi f\tau_1)\,df\right] h(\tau_2) R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

$$= \int_{-\infty}^{\infty} df\; H(f)\int_{-\infty}^{\infty} d\tau_2 h(\tau_2)\int_{-\infty}^{\infty} R_X(\tau_2-\tau_1)\exp(j2\pi f\tau_1)\,d\tau_1$$

$$= \int_{-\infty}^{\infty} df\; H(f)\underbrace{\int_{-\infty}^{\infty} d\tau_2 h(\tau_2)\exp(j2\pi f\tau_2)}_{H^*(f)}\int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)\,d\tau$$

$H^*(f)$ (complex conjugate response of the filter)

# Power Spectral Density (PSD)

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} df \left|H(f)\right|^2 \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)\, d\tau$$

$\left|H(f)\right|$ : the magnitude response

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-2\pi f\tau)\, d\tau$$

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} \left|H(f)\right|^2 S_X(f)\, df$$

Define: Power Spectral Density ( Fourier Transform of $R(\tau)$)

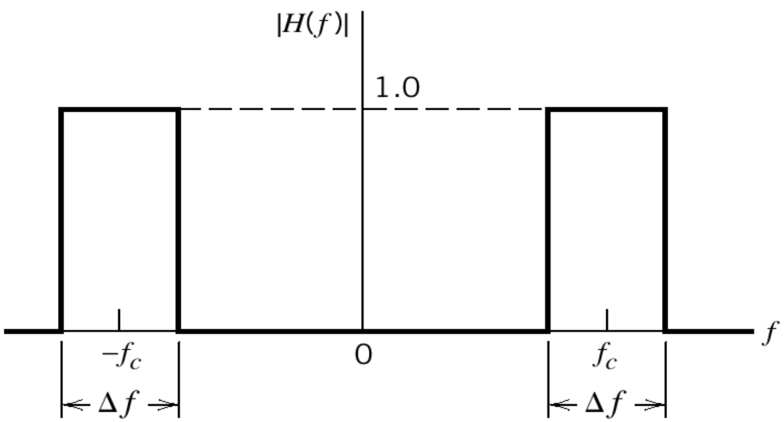
$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

Recall

$$|H(f)| = \begin{cases} 1, & \left|f \pm f_c\right| < \frac{1}{2}\Delta f \\ 0, & \left|f \pm f_c\right| > \frac{1}{2}\Delta f \end{cases}$$

Let $\left|H(f)\right|$ be the magnitude response of an ideal narrowband filter

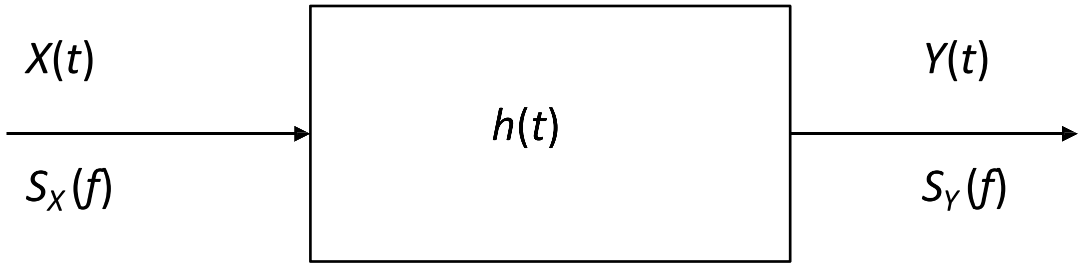
D$f$ : Filter Bandwidth




If $\Delta f \ll f_c$ and $S_X(f)$ is continuous ,

$$E\left[Y^2(t)\right] \approx 2\Delta f\, S_X(f_c) \quad \text{in W/Hz}$$

# The PSD of the Input and Output Random Processes




$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)\, d\tau_1\, d\tau_2$$

$$S_Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)\exp(-j2\pi f\tau)\, d\tau_1\, d\tau_2\, d\tau$$

$$Let\ \tau - \tau_1 + \tau_2 = \tau_0 \quad , \ \text{or}\ \tau = \tau_0 + \tau_1 - \tau_2$$

$$\begin{aligned} S_Y(f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_1)\exp(j2\pi f\tau_0)\exp(-j2\pi f\tau_2)\exp(-j2\pi f\tau_0)\, d\tau_1\, d\tau_2\, d\tau_0 \\ &= S_X(f)H(f)H^*(f) \\ &= \left|H(f)\right|^2 S_X(f) \end{aligned}$$

# Relation Among The PSD and The Magnitude Spectrum of a Sample Function

Let $x(t)$ be a sample function of a stationary and ergodic Process $X(t)$.

In general, the condition for Fourier transformable is

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

This condition can never be satisfied by any stationary $x(t)$ with infinite duration.

We may write $X(f,T) = \int_{-T}^{T} x(t)\exp(-j2\pi ft)\, dt$

$$\text{Ergodic} \Rightarrow \text{Take time average}$$

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)\, dt$$

If $x(t)$ is a power signal (finite average power)

$$\frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt \Leftrightarrow \frac{1}{2\mathrm{T}}|X(f,T)|^2$$

Time-averaged autocorrelation periodogram function

# Relation Among The PSD and The Magnitude Spectrum of a Sample Function

Take inverse Fourier Transform

$$\frac{1}{2\pi}\int_{-T}^{T} x(t+\tau)x(t)\,dt = \int_{-\infty}^{\infty}\frac{1}{2T}\left|X(f,T)\right|^2 \exp(j2\pi\; fT)\,df$$

we have

$$R_X(\tau) = \lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{1}{2T}\left|X(f,T)\right|^2 \exp(j2\pi\; f\tau)\,df$$

Note that for any given x(t) periodogram does not converge as $T \to \infty$

Since x(t) is ergodic

$$E\left[R_X(\tau)\right] = R_X(\tau) = \lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{1}{2T}E\left[\left|X(f-T)\right|^2\right]\exp(\; j2\pi f\tau)\,df$$

$$R_X(\tau) = \int_{-\infty}^{\infty}\left\{\lim_{T\to\infty}\frac{1}{2T}E\left[\left|X(f-T)\right|^2\right]\right\}\exp(\; j2\pi f\tau)\,df$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(\; j2\pi f\tau)\,df$$

$$S_X(f) = \lim_{T\to\infty}\frac{1}{2T}E\left[\left|X(f,T)\right|^2\right]$$

is used to estimate the PSD of *x*(*t*)

$$= \lim_{T\to\infty}\frac{1}{2T}E\left[\left|\int_{-T}^{T} x(t)\exp(-j2\pi f t)\,dt\right|^2\right]$$

# Cross-Spectral Densities

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau)\exp(-j2\pi f\tau)d\tau$$

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau)\exp(-j2\pi f\tau)d\tau$$

$S_{XY}(f)$ and $S_{YX}(f)$ may not be real.

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(f)\exp(j2\pi f\tau)df$$

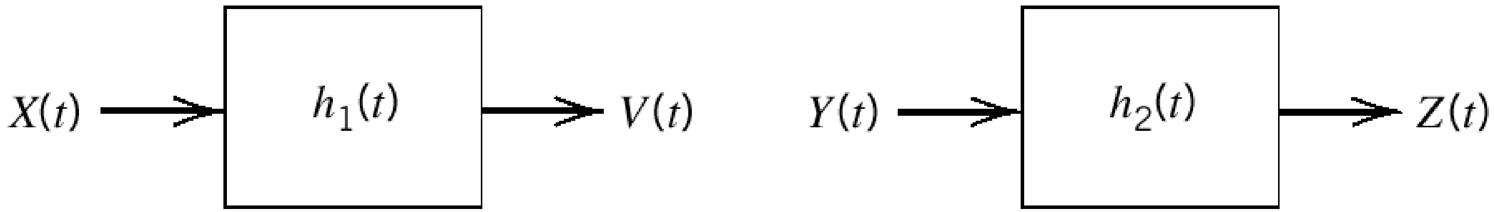
$$R_{YX}(\tau) = \int_{-\infty}^{\infty} S_{YX}(f)\exp(j2\pi f\tau)df$$

$$\square\, R_{XY}(\tau) = R_{YX}(-\tau)$$

$$S_{XY}(f) = S_{YX}(-f) = S_{YX}^{*}(f)$$

# Cross-Spectral Densities Example

Example: X($t$) and Y($t$) are jointly stationary.




$$R_{VZ}(t,u) = E\left[V(t)Z(u)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h_1(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2)Y(u-\tau_2)d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)R_{XY}(t-\tau_1, u-\tau_2)d\tau_1 d\tau_2$$

$$\text{Let} \quad \tau = t - u$$

$$R_{VZ}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)R_{XY}(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2$$

$$\xrightarrow{F} S_{VY}(f) = H_1(f)H_2^*(f)S_{XY}(f)$$

# Cross-Spectral Densities

**Output Statistics:** the mean of the output process is given by

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{+\infty} E\{X(\tau)h(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} \mu_X(\tau)h(t-\tau)d\tau = \mu_X(t) * h(t).$$

Similarly the cross-correlation function between the input and output processes is given by

$$R_{XY}(t_1, t_2) = E\{X(t_1)Y^*(t_2)\}$$

$$= E\{X(t_1)\int_{-\infty}^{+\infty} X^*(t_2-\alpha)h^*(\alpha)d\alpha\}$$

$$= \int_{-\infty}^{+\infty} E\{X(t_1)X^*(t_2-\alpha)\}h^*(\alpha)d\alpha$$

$$= \int_{-\infty}^{+\infty} R_{XX}(t_1, t_2-\alpha)h^*(\alpha)d\alpha$$

$$= R_{XX}(t_1, t_2) * h^*(t_2).$$

Finally the output autocorrelation function is given by

# Cross-Spectral Densities

$$R_{YY}(t_1,t_2) = E\{Y(t_1)Y^*(t_2)\}$$

$$= E\{\int_{-\infty}^{+\infty} X(t_1-\beta)h(\beta)d\beta \; Y^*(t_2)\}$$

$$= \int_{-\infty}^{+\infty} E\{X(t_1-\beta)Y^*(t_2)\}\, h(\beta)d\beta$$

$$= \int_{-\infty}^{+\infty} R_{XY}(t_1-\beta,t_2)h(\beta)d\beta$$

or

$$= R_{XY}(t_1,t_2) * h(t_1),$$

$$R_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) * h^*(t_2) * h(t_1).$$

$$\mu_X(t) \longrightarrow \boxed{h(t)} \longrightarrow \mu_Y(t)$$

(a)

$$R_{XX}(t_1,t_2) \longrightarrow \boxed{h^*(t_2)} \xrightarrow{R_{XY}(t_1,t_2)} \boxed{h(t_1)} \longrightarrow R_{YY}(t_1,t_2)$$

(b)

# Cross-Spectral Densities

In particular if $X(t)$ is wide-sense stationary, then we have $\mu_X(t) = \mu_X$ so that $\mu_Y(t) = \mu_X \int_{-\infty}^{+\infty} h(\tau)d\tau = \mu_X c, \quad a \ constant.$

Also $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$ so that reduces to

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XX}(t_1 - t_2 + \alpha)h^*(\alpha)d\alpha$$

$$= R_{XX}(\tau) * h^*(-\tau) \triangleq R_{XY}(\tau), \quad \tau = t_1 - t_2.$$

Thus $X(t)$ and $Y(t)$ are jointly w.s.s. Further, the output autocorrelation simplifies to

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XY}(t_1 - \beta - t_2)h(\beta)d\beta, \quad \tau = t_1 - t_2$$

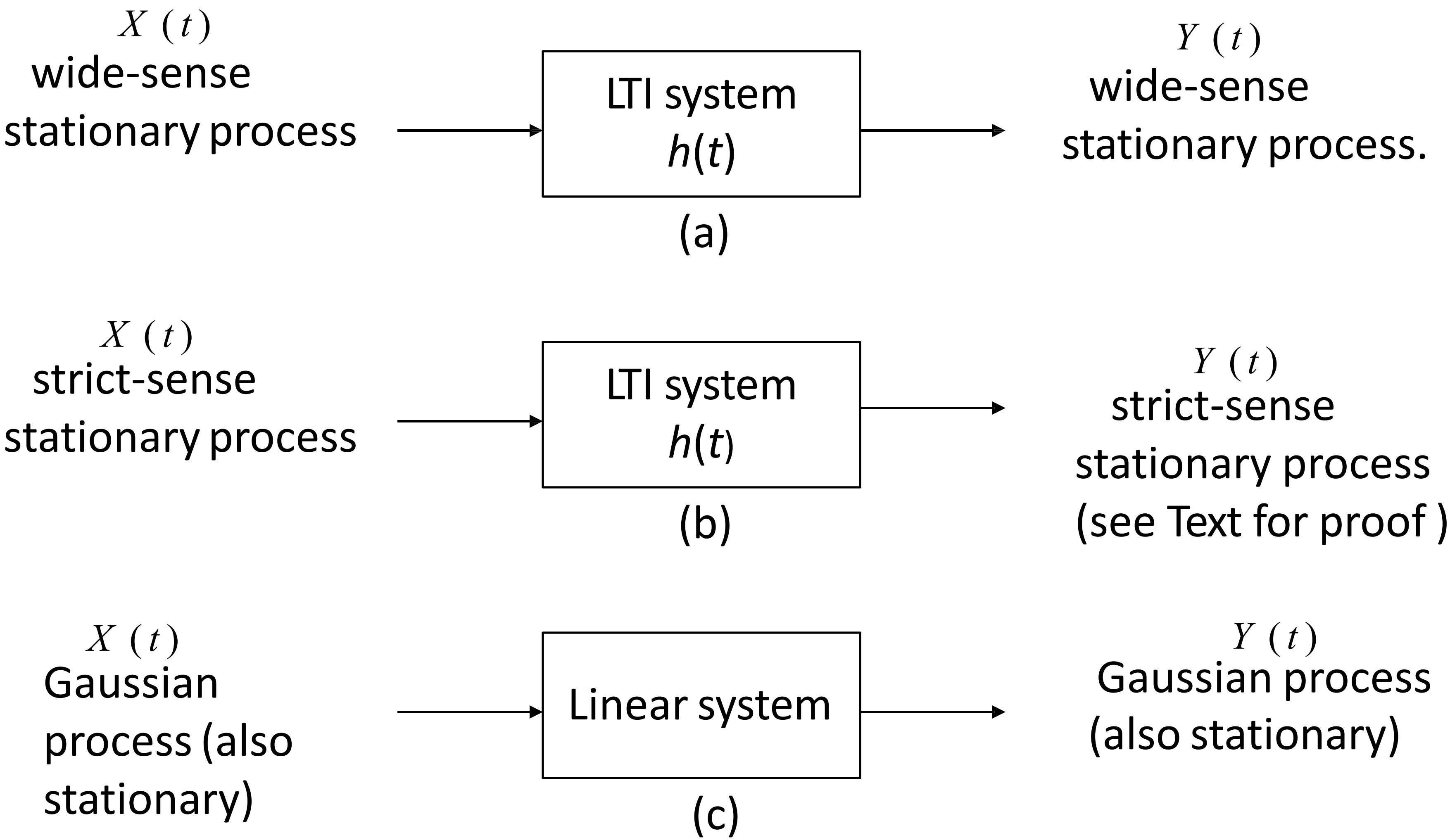
$$= R_{XY}(\tau) * h(\tau) = R_{YY}(\tau).$$

we obtain

$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau).$$

# Cross-Spectral Densities

the output process is also wide-sense stationary. This gives rise to the following representation



# White Noise Process

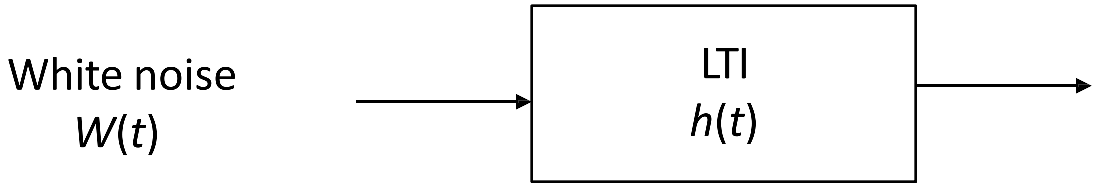
$W(t)$ is said to be a white noise process if

$$R_{WW}(t_1, t_2) = q(t_1)\delta(t_1 - t_2),$$

i.e., $E[W(t_1)\, W^*(t_2)] = 0$ unless $t_1 = t_2$.
$W(t)$ is said to be wide-sense stationary (w.s.s) white noise if $E[W(t)]$ = constant, and

$$R_{WW}(t_1, t_2) = q\delta(t_1 - t_2) = q\delta(\tau).$$

If $W(t)$ is also a Gaussian process (white Gaussian process), then all of its samples are independent random variables




Colored noise

$$N(t) = h(t) * W(t)$$

For w.s.s. white noise input $W(t)$, we have

# White Noise Process

$$E[N(t)] = \mu_W \int_{-\infty}^{+\infty} h(\tau)d\tau, \quad a\ constant$$

and

$$R_{nn}(\tau) = q\delta(\tau) * h^*(-\tau) * h(\tau)$$

$$= qh^*(-\tau) * h(\tau) = q\rho(\tau)$$

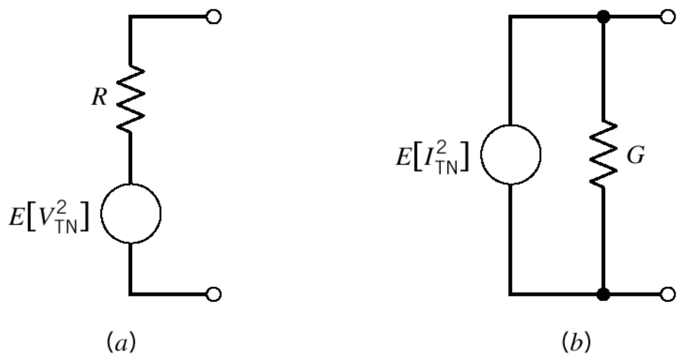
where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{+\infty} h(\alpha)h^*(\alpha+\tau)d\alpha.$$

Thus the output of a white noise process through an LTI system represents a (colored) noise process.
Note: White noise need not be Gaussian.
"White" and "Gaussian" are two different concepts!

# Noise

- Shot noise
- Thermal noise

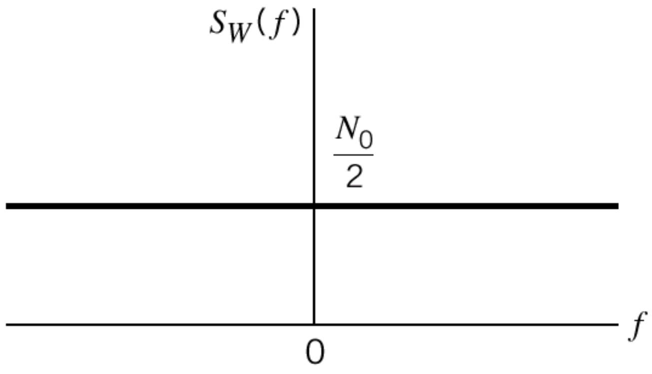



$$E\left[V_{TN}^2\right] = 4kTR\,\Delta f \qquad \text{volts}^2$$

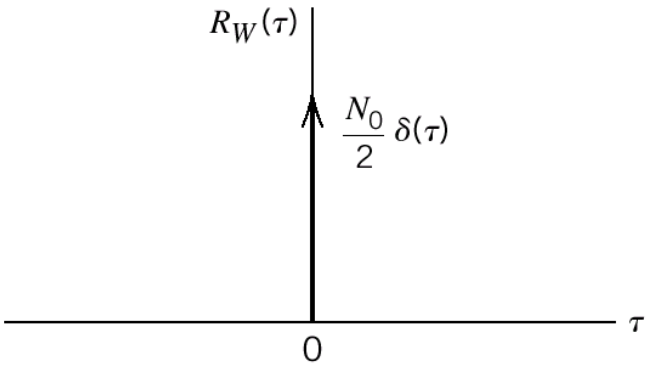
$$E\left[I_{TN}^2\right] = \frac{1}{R^2} E\left[V_{TN}^2\right] = 4kT\frac{1}{R}\Delta f = 4kTG\,\Delta f \qquad \text{amps}^2$$

$k$: Boltzmann's constant = 1.38 x $10^{-23}$ joules/K, $T$ is the absolute temperature in degree Kelvin.

# White noise




(a)




(b)

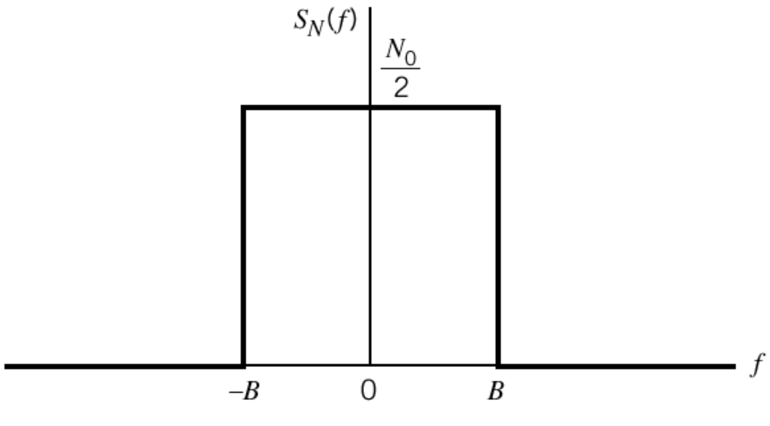
$$S_W(f) = \frac{N_0}{2}$$

$$N_0 = kT_e$$

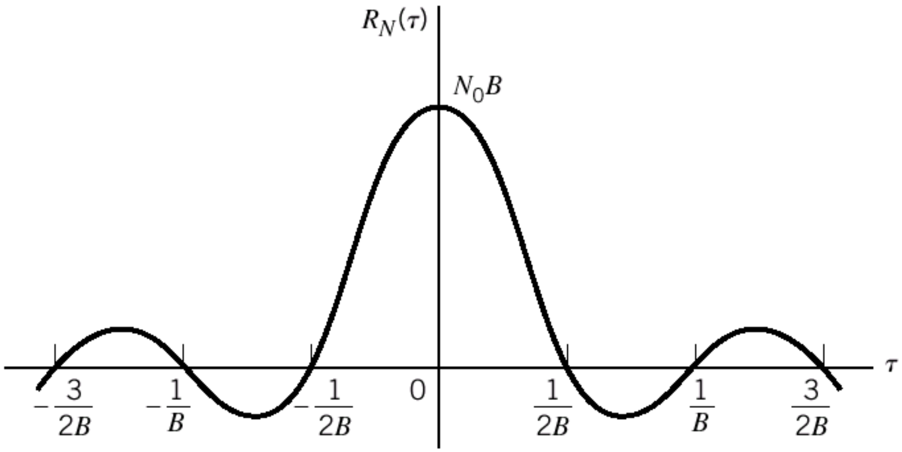
$T_e$ : equivalent noise temperature of the receiver

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau)$$

# Ideal Low-Pass Filtered White Noise




(a)




(b)

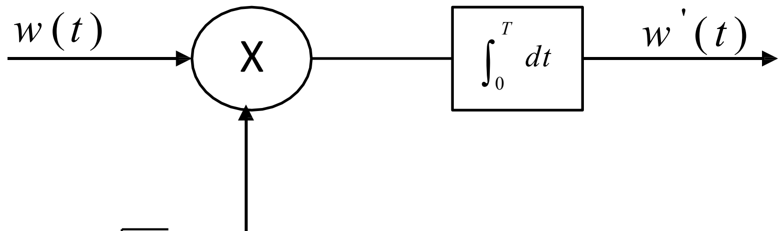
$$S_N(f) = \begin{cases} \frac{N_0}{2} & -B < f < B \\ 0 & |f| > B \end{cases}$$

$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f\tau)\, df$$

$$= N_0 B \operatorname{sinc}(2B\tau)$$

# Correlation of White Noise with a Sinusoidal Wave

White noise




$$\sqrt{\frac{2}{T}}\cos(2\pi f_c t) \quad , \quad f_c = \frac{k}{T} \quad , \quad k \text{ is integer}$$

$$w'(t) = \sqrt{\frac{2}{T}}\int_0^T w(t)\cos(2\pi f_c t)\,dt$$

The varance of $w'(t)$ is

$$\sigma^2 = E\left[\frac{2}{T}\int_0^T\int_0^T w(t_1)\cos(2\pi f_c t_1)w(t_2)\cos(2\pi f_c t_2)\,dt_1\,dt_2\right]$$

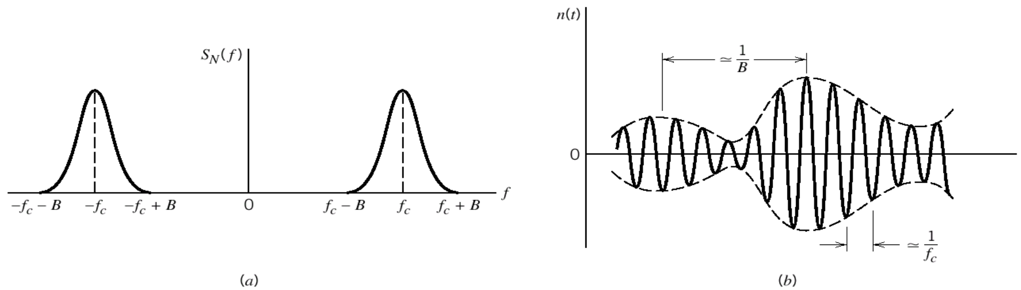
$$= \frac{2}{T}\int_0^T\int_0^T E[w(t_1)w(t_2)]\cos(2\pi f_c t_1)\cos(2\pi f_c t_2)\,dt_1\,dt_2$$

$$= \frac{2}{T}\int_0^T\int_0^T R_W(t_1,t_2)\cos(2\pi f_c t_1)\cos(2\pi f_c t_2)\,dt_1\,dt_2$$

$$\sigma^2 = \frac{2}{T}\int_0^T\int_0^T \frac{N_0}{2}\delta(t_1-t_2)\cos(2\pi f_c t_1)\cos(2\pi f_c t_2)\,dt_1\,dt_2$$

$$= \frac{N_0}{T}\int_0^T \cos^2(2\pi f_c t)\,dt = \frac{N_0}{2}$$

# Narrowband Noise (NBN)




Two representations

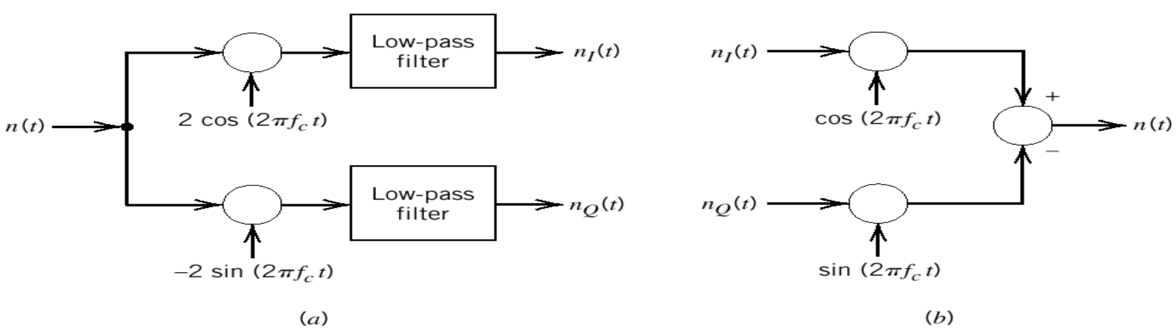
a. in-phase and quadrature components ($\cos(2\pi f_c t)$, $\sin(2\pi f_c t)$)

b. envelope and phase

**In-phase and quadrature representation**

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$n_I(t) \text{ and } n_Q(t) \text{ are low-pass signals}$$



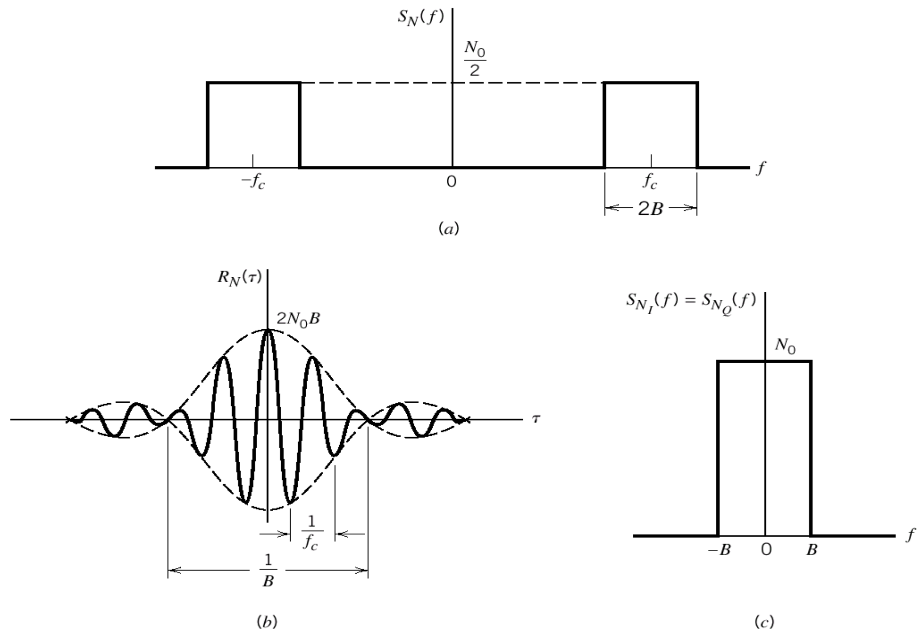
# Important Properties

1. $n_I(t)$ and $n_Q(t)$ have zero mean.
2. If $n(t)$ is Gaussian then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian.
3. If $n(t)$ is stationary then $n_I(t)$ and $n_Q(t)$ are jointly stationary.
4. $$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c), & -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$
5. $n_I(t)$ and $n_Q(t)$ have the same variance $\frac{N_0}{2}$.
6. Cross-spectral density is purely imaginary.

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f)$$
$$= \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

7. If $n(t)$ is Gaussian, its PSD is symmetric about $f_c$, then $n_I(t)$ and $n_Q(t)$ are statistically independent.

# Ideal Band-Pass Filtered White Noise




$$R_N(\tau) = \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau)\,df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau)\,df$$

$$= N_0 B\,\mathrm{sinc}(2B\tau)\left[\exp(-j2\pi f_c\tau)\exp(j2\pi f_c\tau)\right]$$

$$= 2N_0 B\,\mathrm{sinc}(2B\tau)\cos(2\pi f_c\tau)$$

Compare (a factor of $\tau$),

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = 2N_0 B\,\mathrm{sinc}(2B\tau).$$