# UNIT-III
# Stochastic Processes: Temporal Characteristics

# Random Process

❑ The concept of random variable was defined previously as mapping from the **Sample Space S** to the real line as shown below

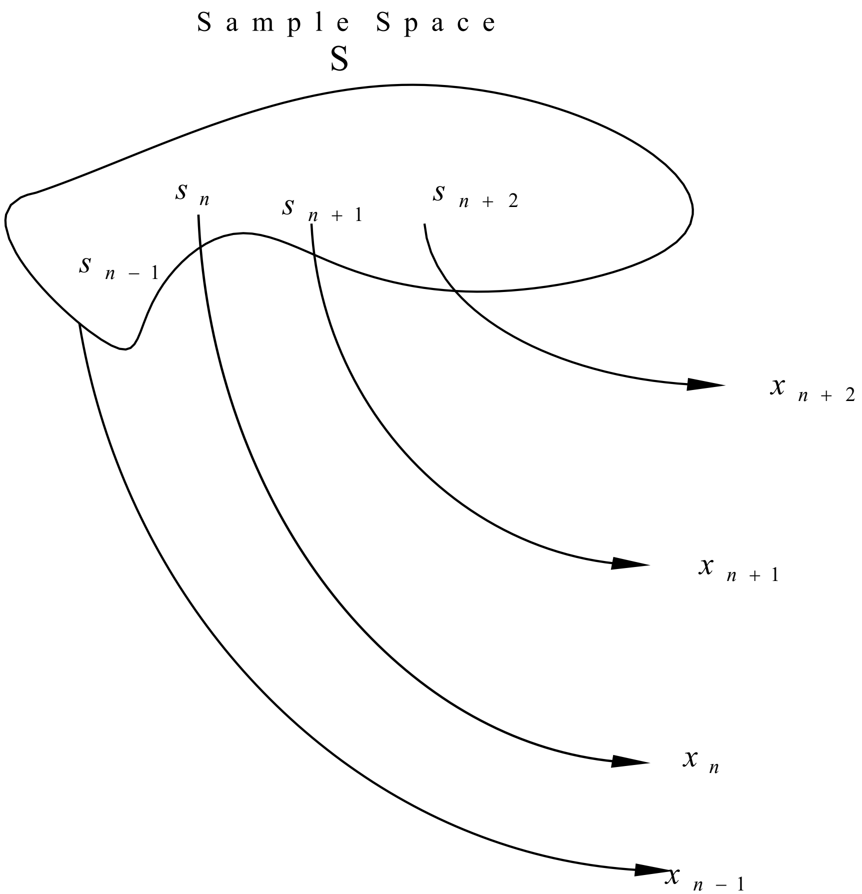
❑A random process is a process (i.e., variation in time or one dimensional space) whose behavior is not completely predictable and can be characterized by statistical laws.

❑ Examples of random processes

Daily stream flow

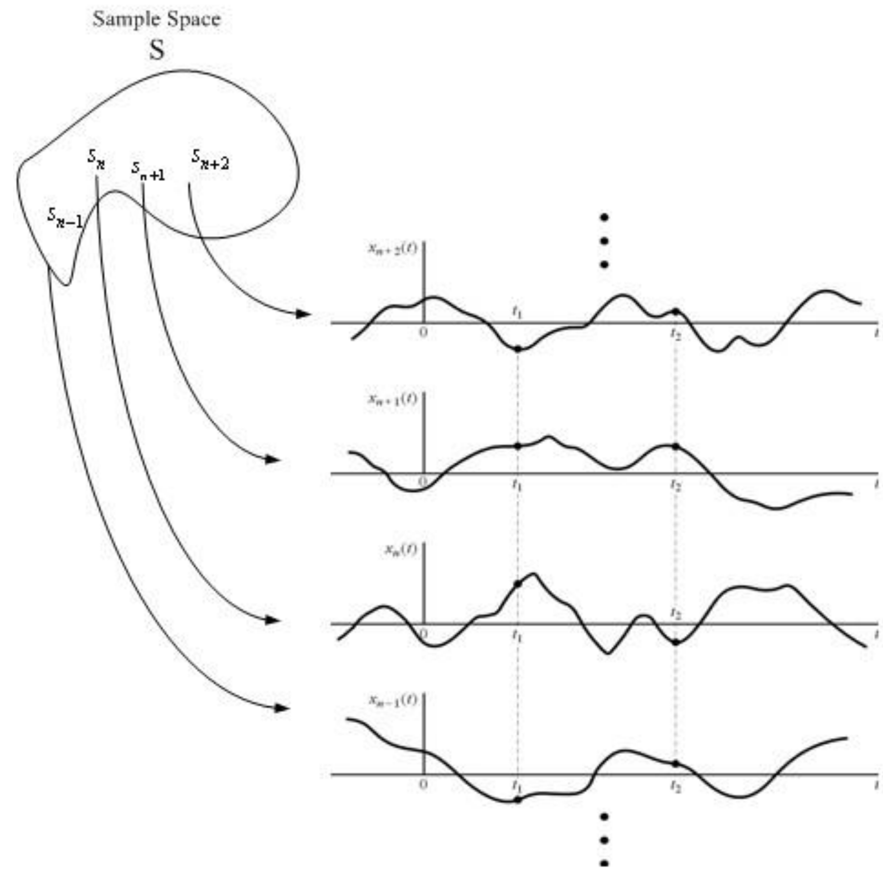
Hourly rainfall of storm events

Stock index



# Random Process (Contd..)

- The concept of random process can be extended to include time and the outcome will be random functions of time as shown beside $x(t, s)$
- Where s is the outcome of an experiment




- The functions

$$\ldots \; x_{n+2}(t), x_{n+1}(t), x_n(t), x_{n-1}(t), \ldots$$

are one realizations of many of the random process X(t)

- A random process also represents a random variable when time is fixed

$X(t_1)$ is a random variable

# Classification of Random Process

- Classification of random process
  - Continuous random process
  - Discrete random process
  - Continuous random sequence
  - Discrete random sequence

Continuous time t => $x(t)$ = Random process
Discrete time n => $x[n]$ = Random sequence

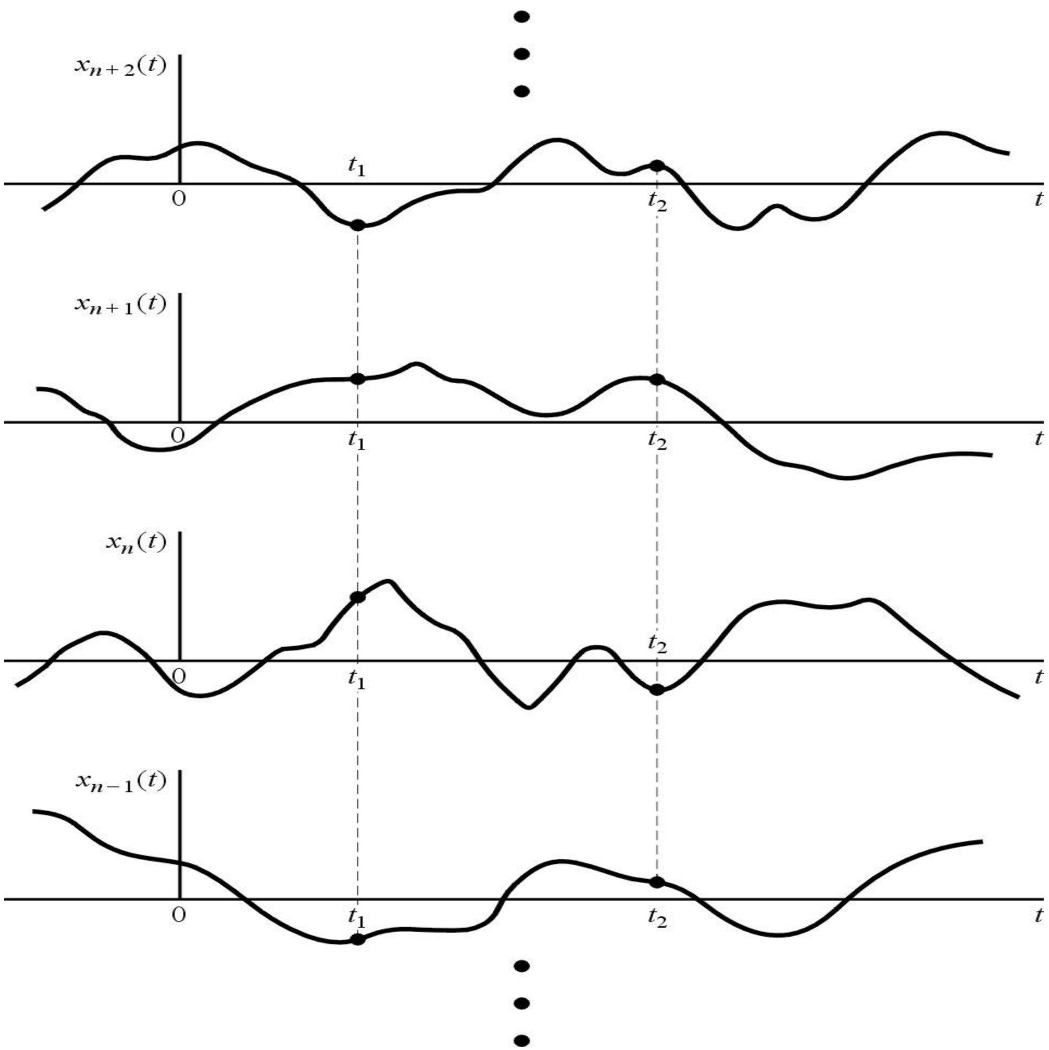
# Continuous Random Process

- Continuous random process

  Continuous time t

  x(t) = Continuous

  Random process

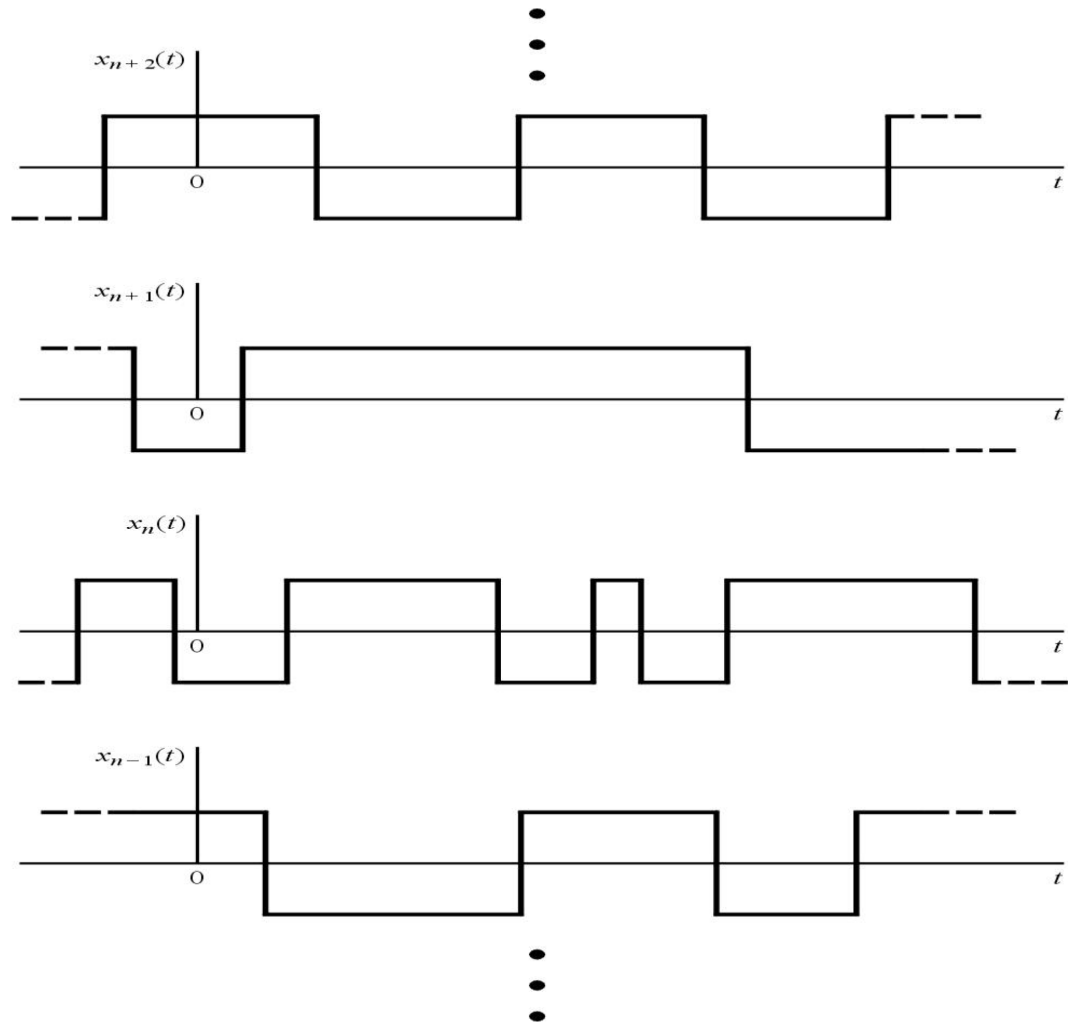


# Discrete Random Process

- Discrete random process

  Continuous time t

  x(t) = Discrete Random process



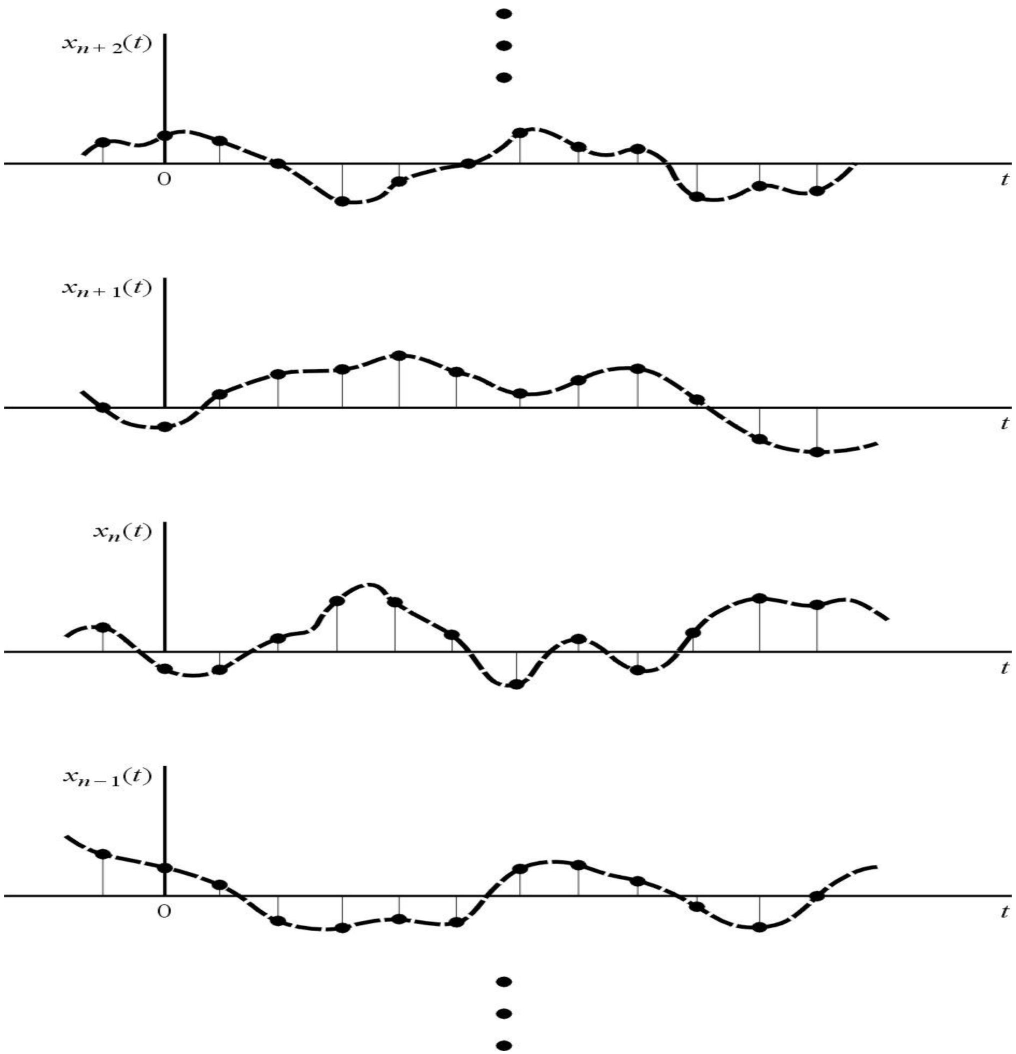
# Continuous Random Sequence

- Continuous random sequence

  discrete time n

  x(n) = Continuous

  Random sequence

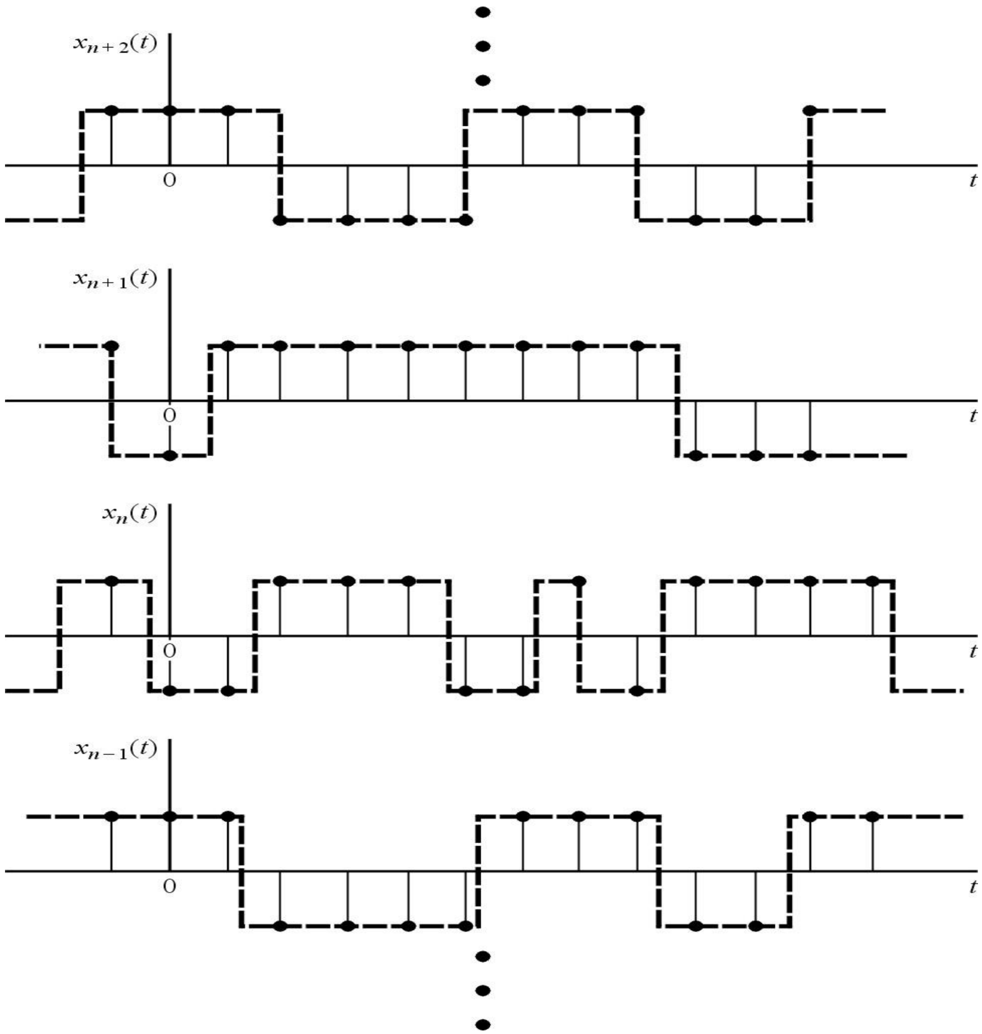


# Discrete Random Sequence

- Discrete random sequence

  discrete time n

  x(n) = discrete Random sequence



# Random Process Concept

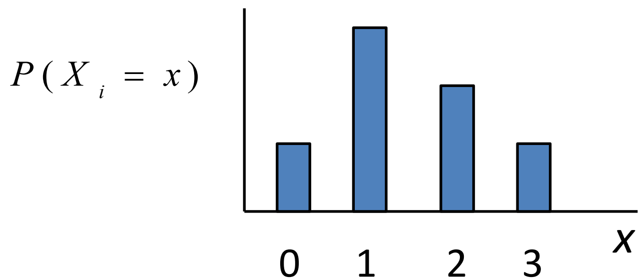
- Deterministic random process
  - Future values of any sample function can be predicted exactly from the past values

$$X(t) = A\cos(\omega_0 t + \theta), \qquad A, \omega_0, \theta : \quad \text{r.v.'s}$$

- Non deterministic random process
  - Future values of any sample function can not be predicted exactly from the past values

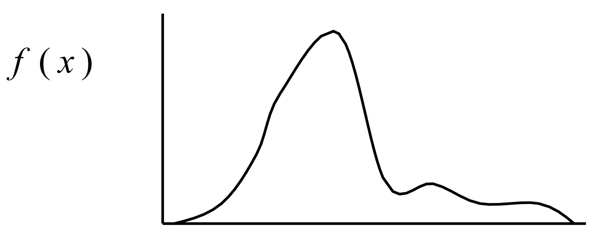
# What is a distribution and density?

- A distribution characterises the probability (mass) associated with each possible outcome of a stochastic process
- Distributions of discrete data characterised by **probability mass functions**




$$\sum_i P(X = x_i) = 1$$

- Distributions of continuous data are characterised by **probability density functions** (pdf)



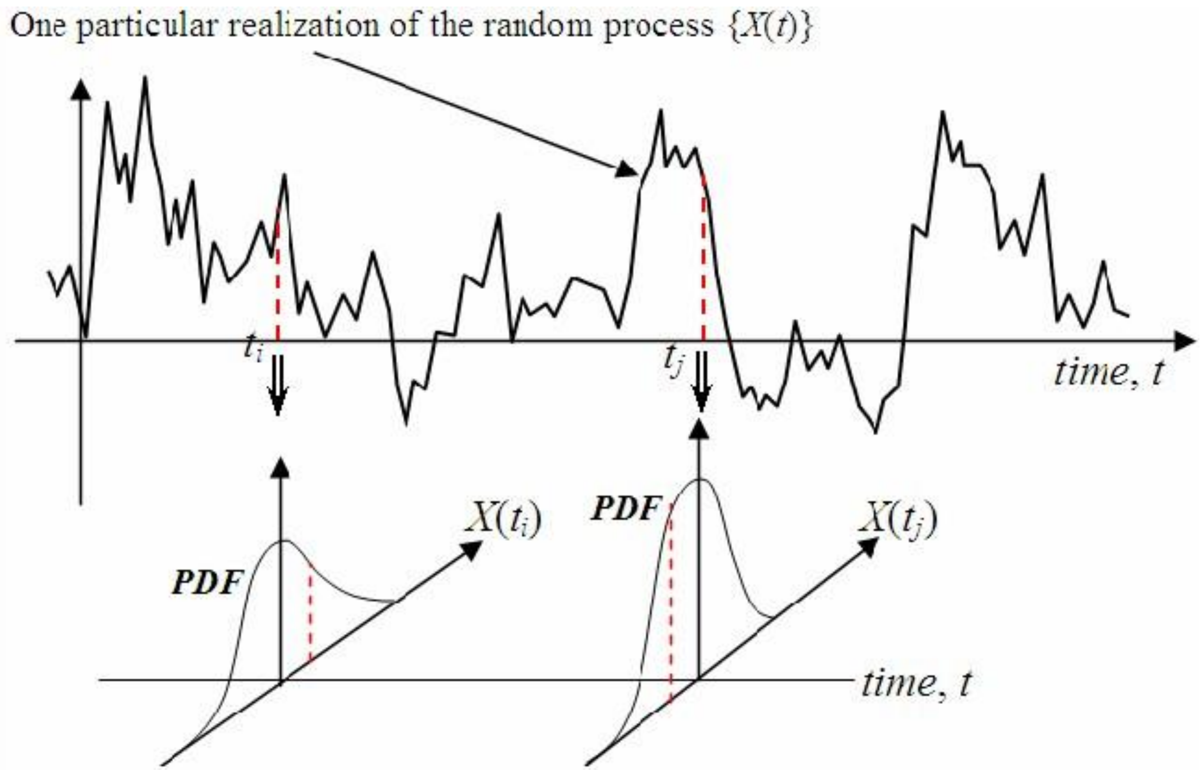

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

- For RVs that map to the integers or the real numbers, the **cumulative density function** (cdf) is a useful alternative representation

# Stationary and Independence

- ❑ Stationary Random Process
  - ❑ all its statistical properties do not change with time

- ❑ Non Stationary Random Process
  - ❑ not stationary



# Stationary and Independence (Contd..)

- First-order densities of a random process
  - A stochastic process is defined to be completely or totally characterized if the joint densities for the random variables $X(t_1), X(t_2), \ldots X(t_n)$ are known for all times $t_1, t_2, \ldots, t_n$ and all n.

- For a specific t, X(t) is a random variable with distribution

$$F(x,t) = p[X(t) \leq x]$$

- The function F(x,t) is defined as the first-order distribution of the random variable X(t). Its derivative with respect to x

$$f(x,t) = \frac{\partial F(x,t)}{\partial x}$$

is the first-order density of X(t).

# Stationary and Independence (Contd..)

- If the first-order densities defined for all time t, i.e. f(x,t), are all the same, then f(x,t) does not depend on t and we call the resulting density the first-order density of the random process {x(t)} ; otherwise, we have a family of first-order densities.

- The first-order densities (or distributions) are only a partial characterization of the random process as they do not contain information that specifies the joint densities of the random variables defined at two or more different times.

# Stationary and Independence (Contd..)

- For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively. Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

and

$$f_X(x_1, x_2, t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \, \partial x_2}$$

represents the second-order density function of the process $X(t)$.

- Similarly $f_X(x_1, x_2, \cdots x_n, \; t_1, t_2 \cdots, t_n)$ represents the $n^{th}$ order density function of the process $X(t)$.

# Mean and variance of a random process

- The first-order density of a random process, $f(x,t)$, gives the probability density of the random variables $X(t)$ defined for all time $t$. The mean of a random process, $m_X(t)$, is thus a function of time specified by

$$m_X(t) = E[X(t)] = E[X_t] = \int_{-\infty}^{+\infty} x_t f(x_t, t)\, dx_t$$

- For the case where the mean of $X(t)$ does not depend on $t$, we have

$$m_X(t) = E[X(t)] = m_X \quad \text{(a constant)}$$

- The variance of a random process, also a function of time, is defined by

$$\sigma_X^2(t) = E\left\{[X(t) - m_X(t)]^2\right\} = E[X_t^2] - [m_X(t)]^2$$

# Stationary and Independence

- The random process X(t) can be classified as follows:
- **First-order stationary**
- A random process is classified as **first-order stationary** if its first-order probability density function remains equal regardless of any shift in time to its time origin.
- If we $X_{t1}$let represent a given value at time t1then we define a first-order stationary as one that satisfies the following equation:

$$f_X(x_{t1}) = f_X(x_{t1} + \tau)$$

- The physical significance of this equation is that our density function,

$f_X(x_{t1})$ is completely independent of t1 and thus any time shift t

- For first-order stationary the mean is a constant, independent of any time shift

# Stationary and Independence (Contd..)

## ❑Second-order stationary

❑A random process is classified as **second-order stationary** if its second-order probability density function does not vary over any time shift applied to both values.

❑ In other words, for values $X_{t1}$ and $X_{t2}$ then we will have the following be equal for an arbitrary time shift t

$$f_X(x_{t1}, x_{t2}) = f_X(x_{t1+\tau}, x_{t2+\tau})$$

❑From this equation we see that the absolute time does not affect our functions, rather it only really depends on the time difference between the two variables.

# Stationary and Independence (Contd..)

- For a second-order stationary process, we need to look at the **autocorrelation function** ( will be presented later) to see its most important property.

- Since we have already stated that a second-order stationary process depends only on the time difference, then all of these types of processes have the following property:

$$R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)]$$

$$= R_{XX}(\tau)$$

# Wide-Sense Stationary (WSS)

- A process that satisfies the following:
- The mean is a constant and the autocorrelation function depends only on the difference between the time indices

$$E\left[X(t)\right] = \overline{X} = \text{constant}$$

$$E\left[X(t)X(t+\tau)\right] = R_{XX}(\tau)$$

is a Wide-Sense Stationary (WSS)

Second-order stationary $\Rightarrow$ Wide-Sense Stationary

The converse is not true in general

# Wide-Sense Stationary (Example)

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0, 2\pi).$$

This gives

$$\mu_X(t) = E\{X(t)\} = aE\{\cos(\omega_0 t + \varphi)\}$$

$$= a\cos\omega_0 t\, E\{\cos\varphi\} - a\sin\omega_0 t\, E\{\sin\varphi\} = 0,$$ Constant

since $E\{\cos\varphi\} = \frac{1}{2\pi}\int_0^{2\pi}\cos\varphi\, d\varphi = 0 = E\{\sin\varphi\}.$

Similarly

$$R_{XX}(t_1, t_2) = a^2 E\{\cos(\omega_0 t_1 + \varphi)\cos(\omega_0 t_2 + \varphi)\}$$

$$= \frac{a^2}{2} E\{\cos\omega_0(t_1 - t_2) + \cos(\omega_0(t_1 + t_2) + 2\varphi)\}$$

$$= \frac{a^2}{2}\cos\omega_0(t_1 - t_2).$$

So given X(t) is WSS

# Nth order and Strict-Sense Stationary

- In strict terms, the statistical properties are governed by the joint probability density function. Hence a process is n$^{th}$-order **Strict-Sense Stationary (S.S.S)** if

$$f_X(x_1, x_2, \cdots x_n, t_1, t_2 \cdots, t_n) \equiv f_X(x_1, x_2, \cdots x_n, t_1 + c, t_2 + c \cdots, t_n + c) \rightarrow (1)$$

- For *any c*, where the left side represents the joint density function of the random variables $X_1 = X(t_1),\ X_2 = X(t_2),\ \cdots,\ X_n = X(t_n)$ and the right side corresponds to the joint density function of the random variables $X_1' = X(t_1 + c),\ X_2' = X(t_2 + c),\ \cdots,\ X_n' = X(t_n + c)$.

- A process *X*(*t*) is said to be **strict-sense stationary** if equation (1) true for all $t_i,\ i = 1, 2, \cdots, n,\ \ n = 1, 2, \cdots$ and *any* $c$.

# Ergodic Process

A stationary random process for which time averages equal ensemble averages is called an ergodic process:

$$\langle x[n] \rangle = m_x$$

$$\langle x[n+m]x[n]^* \rangle = \phi_{xx}[m]$$

# Ergodic Process (Contd..)

It is common to assume that a given sequence is a sample sequence of an ergodic random process, so that averages can be computed from a single sequence.

In practice, we cannot compute with the limits, but instead the quantities.

Similar quantities are often computed as estimates of the mean, variance, and autocorrelation.

$$\hat{m}_x = \frac{1}{L}\sum_{n=0}^{L-1} x[n]$$

$$\sigma_x^2 = \frac{1}{L}\sum_{n=0}^{L-1} \left(x[n] - \hat{m}_x\right)^2$$

$$\left\langle x[n+m]x^*[n]\right\rangle_L = \frac{1}{L}\sum_{n=0}^{L-1} x[n+m]x^*[n]$$

# Time Average and Ergodicity

- The time average of a quantity is defined as

$$A[\bullet] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\bullet] dt$$

Here A is used to denote time average in a manner analogous to E for the statistical average.

- The time average is taken over all time because, as applied to random processes, sample functions of processes are presumed to exist for all time.

# Time Average and Ergodicity (Contd..)

- Let x(t) be a sample of the random process X(t) were the lower case letter imply a sample function.
- We define the mean value $\overline{x} = A\left[x(t)\right]$

( a lowercase letter is used to imply a sample function)
and the time autocorrelation function $\Re_{xx}(\tau)$ as follows:

$$\overline{x} = A\left[x(t)\right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

$$\Re_{xx}(\tau) = A\left[x(t)x(t+\tau)\right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\, dt$$

- For any one sample function ( i.e., x(t) ) of the random process X(t), the last two integrals simply produce two numbers.
- A number for the average $\overline{x}$ and a number for $\Re_{xx}(\tau)$ for a specific value of $\tau$

# Time Average and Ergodicity (Contd..)

- Since the sample function x(t) is one out of other samples functions of the random process X(t),
- The average $\overline{x}$ and the autocorrelation $\Re_{xx}(\tau)$ are random variables
- By taking the expected value for $\overline{x}$ and $\Re_{xx}(\tau)$ ,we obtain

$$E[\overline{x}] = E[A[x(t)]] = E\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[x(t)]\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\overline{X}\,dt = \lim_{T\to\infty}\overline{X}(1) = \overline{X}$$

$$E[\Re_{xx}(\tau)] = E\,[A[x(t)x(t+\tau)]\,] = E\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt\right]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[x(t)x(t+\tau)]\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{xx}(\tau)\,dt = R_{xx}(\tau)$$

# Time Average and Ergodicity (Contd..)

- Time cross correlation

$$\Re_{xy}(\tau) = A[x(t)y(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)\,dt$$

- Ergodic =>

$$\overline{x} = \overline{X}$$

$$\Re_{xx}(\tau) = R_{XX}(\tau)$$

- Jointly Ergodic => Ergodic X(t) and Y(t)

$$\Re_{xy}(\tau) = R_{XY}(\tau)$$

# Introduction to Autocorrelation

- Autocorrelation occurs in time-series studies when the errors associated with a given time period carry over into future time periods.

- For example, if we are predicting the growth of stock dividends, an overestimate in one year is likely to lead to overestimates in succeeding years.

- Times series data follow a natural ordering over time.

- It is likely that such data exhibit intercorrelation, especially if the time interval between successive observations is short, such as weeks or days.

# Introduction (contd..)

- We expect stock market prices to move or move down for several days in succession.
- We experience autocorrelation when

$$E(u_i u_j) \neq 0$$

- Tintner defines autocorrelation as ‘lag correlation of a given series within itself, lagged by a number of times units’ whereas serial correlation is the ‘lag correlation between two different series’.

# Autocorrelation and its Properties

- The autocorrelation function of a random process X(t) is the correlation $E[X_1 X_2]$ of two random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ by the process at times t1 and t2

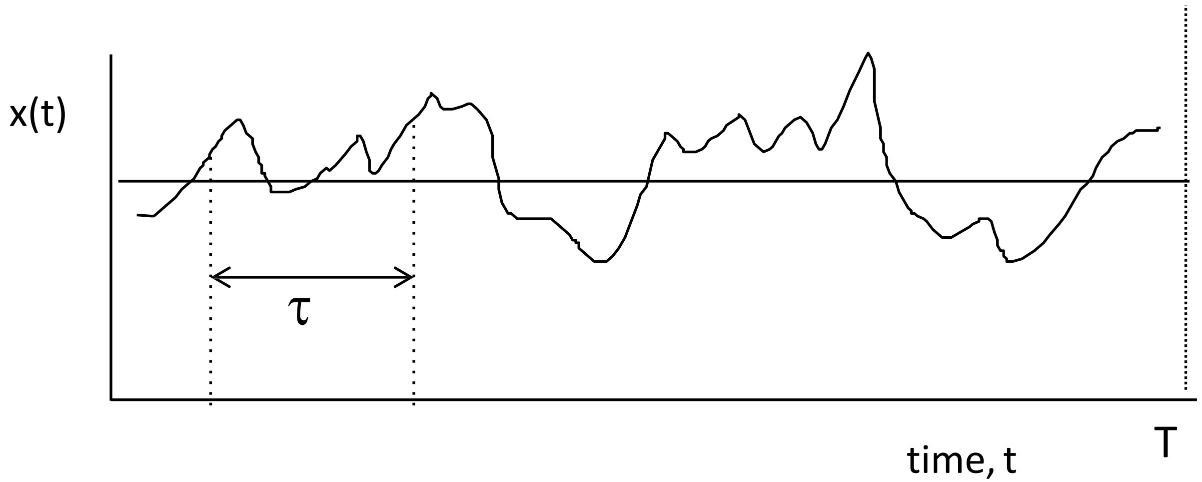
$$R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]$$

- Assuming a second-order stationary process

$$R_{XX}(t, t + \tau) = E[X(t) X(t + \tau)] \qquad R_{XX}(\tau) = E[X(t) X(t + \tau)]$$

# Autocorrelation and its Properties (Contd..)

- Autocorrelation :

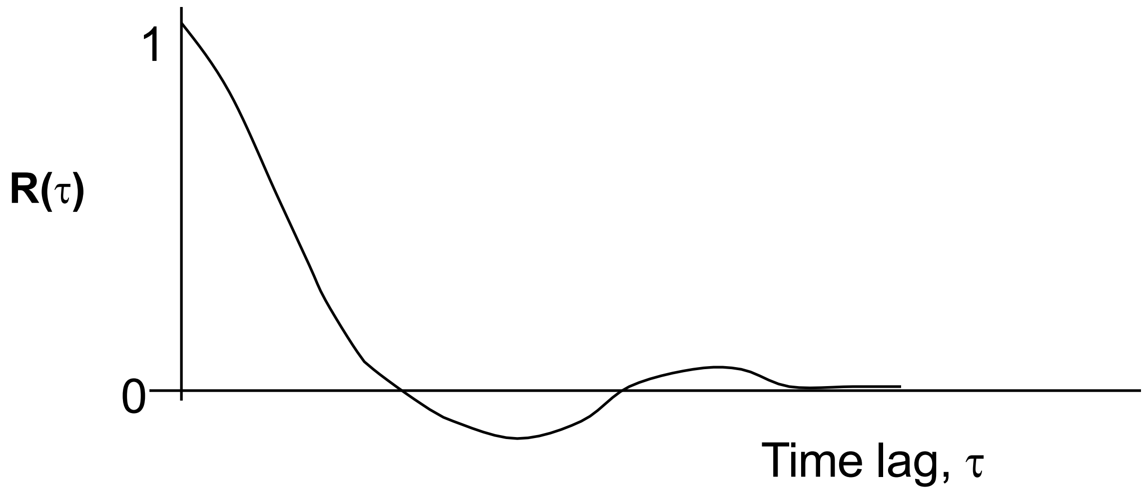



- The autocorrelation, or auto covariance, describes the general dependency of x(t) with its value at a short time later, $x(t+\tau)$

$$\rho_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[x(t) - \bar{x}\right] . \left[x(t+\tau) - \bar{x}\right] dt$$

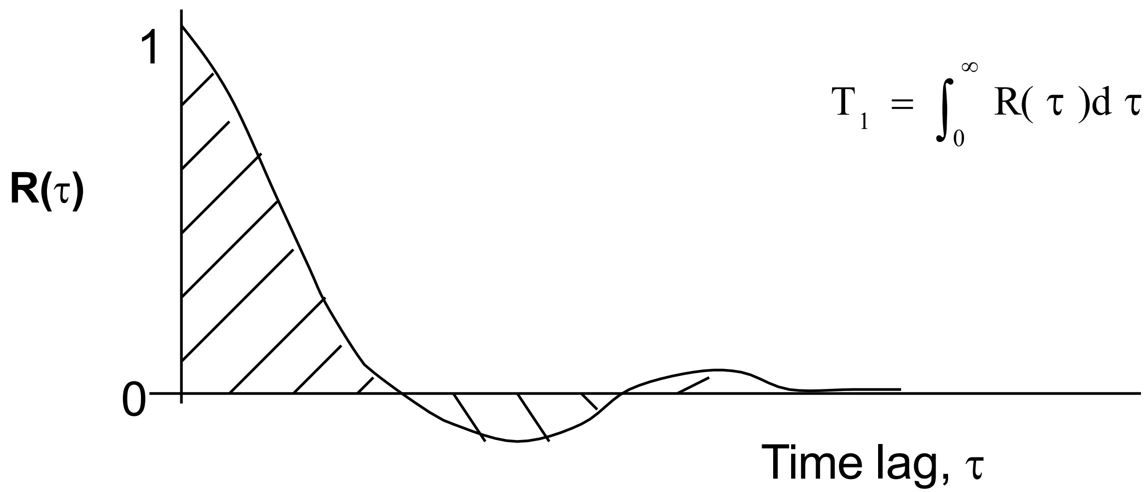
- The value of $\rho_x(\tau)$ at $\tau$ equal to 0 is the variance, $\sigma_x^2$
- Normalized auto-correlation : $R(\tau) = \rho_x(\tau)/\sigma_x^2$ $R(0)= 1$

# Autocorrelation and its Properties (Contd..)




- The autocorrelation for a random process eventually decays to zero at large $\tau$
- The autocorrelation for a sinusoidal process (deterministic) is a cosine function which does not decay to zero

# Autocorrelation and its Properties (Contd..)




- The area under the normalized autocorrelation function for the fluctuating wind velocity measured at a point is a measure of the average time scale of the eddies being carried passed the measurement point, say $T_1$
- If we assume that the eddies are being swept passed at the mean velocity, $\overline{U}.T_1$ is a measure of the average length scale of the eddies. This is known as the 'integral length scale', denoted by $l_u$

# Autocorrelation and its Properties (Contd..)

- **Properties of Autocorrelation function**

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

(1) $\left| R_{XX}(\tau) \right| \le R_{XX}(0)$

(2) $R_{XX}(-\tau) = R_{XX}(\tau)$

(3) $R_{XX}(0) = E[X(t)^2]$

(4) stationary & ergodic $X(t)$ with no periodic components

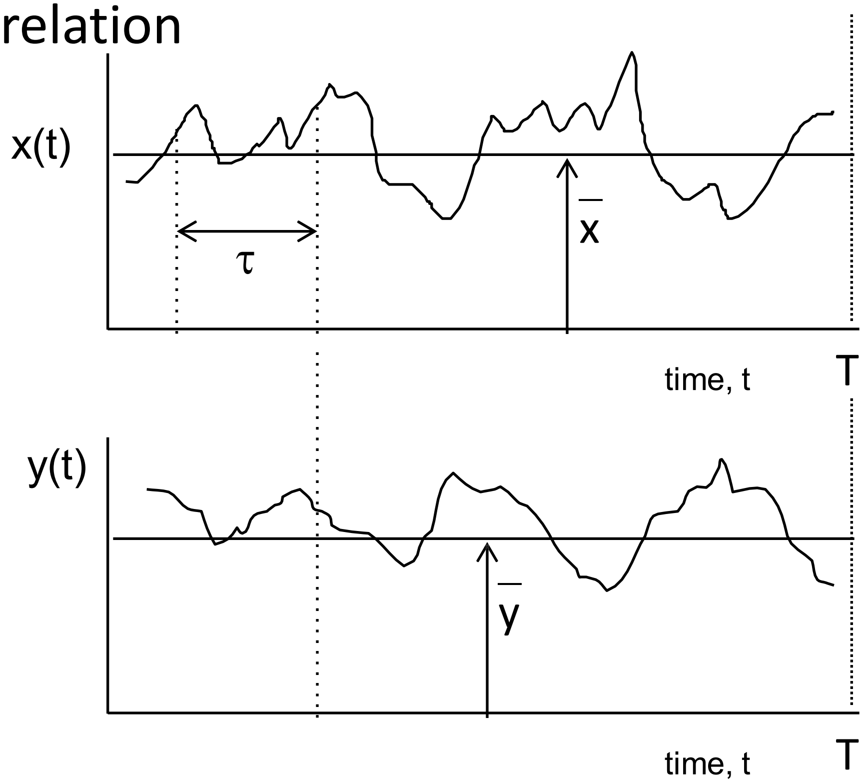
$$\Rightarrow \lim_{|\tau| \to \infty} R_{XX}(\tau) = \overline{X}^2$$

(5) stationary $X(t)$ has a periodic component

$\Rightarrow$ $R_{XX}(\tau)$ has a periodic component with the same period.

# Cross-correlation

- Cross-correlation




- The cross-correlation function describes the general dependency of x(t) with another random process y(t+τ), delayed by a time delay, τ

$$c_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[x(t) - \bar{x}\right]\left[y(t+\tau) - \bar{y}\right] dt$$

# Correlation coefficient

- Correlation coefficient
  - The correlation coefficient, ρ, is the covariance normalized by the standard deviations of x and y

$$\rho = \frac{\overline{x'(t).y'(t)}}{\sigma_x.\sigma_y}$$

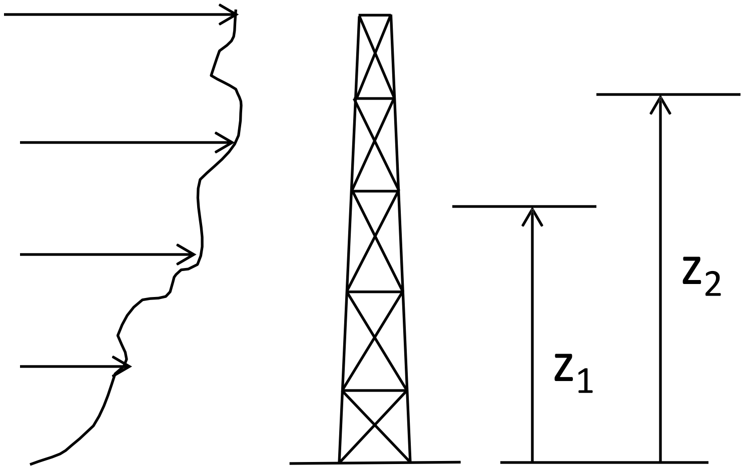
When x and y are identical to each other, the value of ρ is +1 (full correlation)

When y(t)=–x(t), the value of ρ is – 1

In general, $-1 < \rho < +1$

# Application of correlation

- Correlation - application :
  - The fluctuating wind loading of a tower depends on the correlation coefficient between wind velocities and hence wind loads, at various heights




For heights, $z_1$, and $z_2$ :

$$\rho(z_1, z_2) = \frac{\overline{u'(z_1).u'(z_2)}}{\sigma_u(z_1).\sigma_u(z_2)}$$

# Properties of Cross Correlation

Properties of cross-correlation function of jointly w.s.s. r.p.'s:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

(1) $R_{XY}(-\tau) = R_{YX}(\tau)$

(2) $\left|R_{XY}(\tau)\right| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$

(3) $\left|R_{XY}(\tau)\right| \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$

$$E[\{Y(t+\tau) + \alpha X(t)\}^2] \geq 0, \quad \forall \alpha$$

$$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

# Example of Cross Correlation

$$A, B : \text{r.v.'s} \qquad \omega_0 = \text{const}$$

$$E[A] = E[B] = 0, \quad E[AB] = 0, \quad E[A^2] = E[B^2] = \sigma^2$$

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t), \quad Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

$$E[X(t)] = E[A\cos(\omega_0 t) + B\sin(\omega_0 t)] = E[A]\cos(\omega_0 t) + E[B]\sin(\omega_0 t) = 0$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E[A^2\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) + AB\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)$$

$$+ AB\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) + B^2\sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)]$$

$$= \sigma^2\{\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) + \sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)\} = \sigma^2\cos(\omega_0\tau)$$

$$\Rightarrow \quad X(t) : \text{w.s.s.}$$

# Example of Cross Correlation

$$Y(t): \text{w.s.s.}$$

$$\begin{aligned}
R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\
&= E\{[A\cos(\omega_0 t) + B\sin(\omega_0 t)][B\cos(\omega_0(t+\tau)) - A\sin(\omega_0(t+\tau))]\} \\
&= E[AB\cos(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) + B^2 \sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) \\
&\qquad - A^2\cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau) - AB\sin(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)] \\
&= \sigma^2[\sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) - \cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)] \\
&= -\sigma^2 \sin(\omega_0 \tau)
\end{aligned}$$

$$\Rightarrow \quad X(t)\ \&\ Y(t): \text{jointly w.s.s.}$$

# Covariance

- Covariance
  - The covariance is the cross correlation function with the time delay, $\tau$, set to zero

$$c_{xy}(0) = \overline{x'(t).\, y'(t)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[x(t) - \overline{x}\right].\left[y(t) - \overline{y}\right] dt$$

  - Note that here x'(t) and y'(t) are used to denote the fluctuating parts of x(t) and y(t) (mean parts subtracted)

# Auto Covariance

- The auto covariance Cx(t1,t2) of a random process X(t) is defined as the covariance of X(t1) and X(t2)

  Cx(t1,t2)=E[{X(t1)-mx(t1)}{X(t2)-mx(t2)}]

  Cx(t1,t2) = Rx(t1,t2)-mx(t1)mx(t2)

- The variance of X(t) can be obtained from Cx(t1,t2)

  VAR[X(t)] = E[(X(t)-mx(t))2] = Cx(t,t)

- The correlation coefficient of X(t) is given by

$$\rho_x(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)}\sqrt{C_X(t_2, t_2)}}$$

$$\left|\rho_x(t_1, t_2)\right| \le 1$$

# Auto Covariance Example#1

**Example:**

Let X(t) = Acos2πt, where A is some random variable

The mean of X(t) is given by

$$m_X(t) = E[A\cos 2\pi t] = E[A]\cos 2\pi t$$

The autocorrelation is

$$R_X(t_1, t_2) = E[A\cos(2\pi t_1) A\cos(2\pi t_2)]$$

$$R_X(t_1, t_2) = E[A^2]\cos(2\pi t_1)\cos(2\pi t_2)$$

And the autocovariance

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$

$$C_X(t_1, t_2) = \{E[A_2] - E[A]_2\}\cos(2\pi t_1)\cos(2\pi t_2)$$

$$C_X(t_1, t_2) = VAR[A]\cos(2\pi t_1)\cos(2\pi t_2)$$

# Auto Covariance Example#2

**Example:**

Let X(t) = cos(ωt+θ), where θ is uniformly distributed in the interval (-π, π). The mean of X(t) is given by

$$m_X(t) = E[\cos(\omega t + \theta)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta) = 0$$

The autocorrelation and autocovariance are then

$$C_X(t_1, t_2) = R_X(t_1, t_2) = E[\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)]$$

$$C_X(t_1, t_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\{\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\theta)\} d\theta$$

$$C_X(t_1, t_2) = \frac{1}{2}\cos(\omega(t_1 - t_2))$$

# Cross Covariance

- The cross covariance $C_{x,y}(t_1,t_2)$ of a random process $X(t)$ and $Y(t)$ is defined as

$$C_{x,y}(t_1,t_2)=E[\{X(t_1)-m_x(t_1)\}\{Y(t_2)-m_y(t_2)\}]$$

$$C_x(t_1,t_2) = R_{x,y}(t_1,t_2)-m_x(t_1)m_y(t_2)$$

- The process $X(t)$ and $Y(t)$ are said to be uncorrelated if

$C_{x,y}(t_1,t_2) = 0$ for all $t_1$, $t_2$

# Random sequence

Random Sequence (=Discrete-time R.P)

$$X(nT_s) = X[n]$$

Mean $= E(X[n])$

$$R_{XX}(n, n+k) = E(X[n]X[n+k])$$

$$C_{XX}(n, n+k) = E\{(X[n] - \overline{X}[n])(X[n+k] - \overline{X}[n+k])\}$$
$$= R_{XX}(n, n+k) - \overline{X}[n]\overline{X}[n+k]$$

$$R_{XY}(n, n+k) = E(X[n]Y[n+k])$$

$$C_{XY}(n, n+k) = E\{(X[n] - \overline{X}[n])(Y[n+k] - \overline{Y}[n+k])\}$$
$$= R_{XY}(n, n+k) - \overline{X}[n]\overline{Y}[n+k]$$

# Gaussian Random Process

- Let X(t) be a random process and let X(t1), X(t2), ….X(tn) be the random variables obtained from X(t) at t=t1,t2……..tn sec respectively

- Let all these random variables be expressed in the form of a matrix

$$X = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{bmatrix}$$

- Then, X(t) is referred to as normal or Gaussian process if all the elements of X are jointly Gaussian

# Gaussian Random Process

- continuous r.p. $X(t), \quad -\infty < t < \infty$

$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = \frac{1}{\sqrt{(2\pi)^N \left|C_X\right|}} \exp\{-\frac{1}{2}[x - \bar{X}]^t C_X^{-1} [x - \bar{X}]\}$$

$$\bar{X}_i = E[X(t_i)] \qquad C_{ik} = C_{XX}(t_i, t_k)$$

stationary $\Rightarrow \quad E[X(t)] = \bar{X}$ (const) & $R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

w.s.s. Gaussian $\Rightarrow$ strictly stationary

# Gaussian Random Process

w.s.s. gaussian r.p. $X(t)$

$$\overline{X} = 4 \qquad R_{XX}(\tau) = 25e^{-3|\tau|} \qquad t_i = t_0 + \frac{i-1}{2}, \quad i = 1,2,3.$$

$$C_{ik} = C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - \overline{X}^2 = 25e^{-3\frac{|k-i|}{2}} - 16$$

$$C_X = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 25-16 & 25e^{-\frac{3}{2}}-16 & 25e^{-3}-16 \\ 25e^{-\frac{3}{2}}-16 & 25-16 & 25e^{-\frac{3}{2}}-16 \\ 25e^{-3}-16 & 25e^{-\frac{3}{2}}-16 & 25-16 \end{bmatrix}$$

# Properties of Gaussian Process

- If a gaussian process X(t) is applied to a stable linear filter, then the random process Y(t) developed at the output of the filter is also gaussian.
- Considering the set of random variables or samples X(t1), X(t2),.....X(tn) obtained by observation of a random process X(t) at instants t1,t2,.......tn, if the process X(t) is gaussian, then this set of random variables are jointly gaussian for any n, with their n-fold joint p.d.f. being completely determined by the set of means.

$$mx(ti) = E[X(ti)] \text{ for } i=1,2,....n$$

and the set of auto covariance function

$$Cxx(t1,t2) = E[\{X(t1)-E[X(t1)]\}\{X(t2)-E[X(t2)]\}]$$

- If a gaussian process is wide sense stationary, then the process is also stationary in the strict sense
- If the set of random variables X(t1),X(t2)...X(tn) are uncorrelated then they are statistically independent

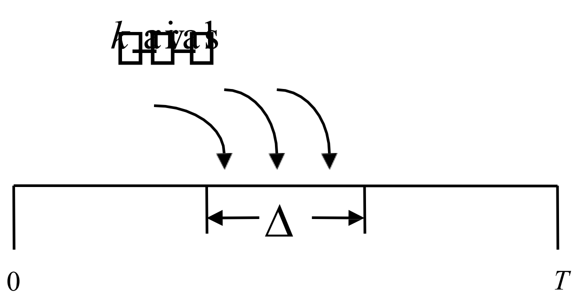
# Poisson Random Process

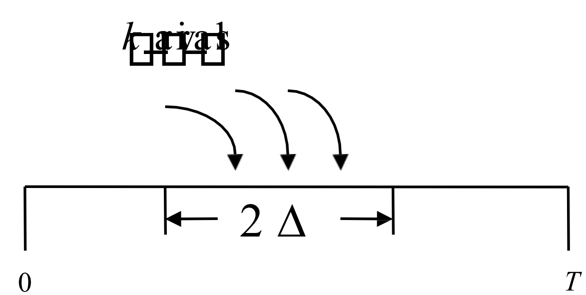
❑we introduced Poisson arrivals as the limiting behavior of Binomial random variables

where

$$P\left\{\begin{array}{l} \text{" } k \text{ arrivals occur in an} \\ \text{ interval of duration } \Delta \text{ "} \end{array}\right\} = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

$$\lambda = np = \mu T \cdot \frac{\Delta}{T} = \mu \Delta$$



# Poisson Random Process (contd..)

❑ It follows that

$$P\left\{ \begin{array}{l} \text{"}k \text{ arrivals occur in an} \\ \text{interval of duration } 2\Delta\text{"} \end{array} \right\} = e^{-2\lambda} \frac{(2\lambda)^k}{k!}, \qquad k = 0, 1, 2, \cdots,$$

since in that case

$$np_1 = \mu T \cdot \frac{2\Delta}{T} = 2\mu\Delta = 2\lambda.$$

❑ From the above equations, Poisson arrivals over an interval form a Poisson random variable whose parameter depends on the duration of that interval.

❑The Bernoulli nature of the underlying basic random arrivals, events over non overlapping intervals are independent. We shall use these two key observations to define a Poisson process formally.

# Poisson Random Process (contd..)

- **Definition:** $X(t) = n(0, t)$ represents a Poisson process if

(i) the number of arrivals $n(t_1, t_2)$ in an interval $(t_1, t_2)$ of length $t = t_2 - t_1$ is a Poisson random variable with parameter $\lambda t$.

Thus

$$P\{n(t_1, t_2) = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, 2, \ldots, \qquad t = t_2 - t_1$$

and

(ii) If the intervals $(t_1, t_2)$ and $(t_3, t_4)$ are non overlapping, then the random variables $n(t_1, t_2)$ and $n(t_3, t_4)$ are independent.

Since $n(0, t) \sim P(\lambda t)$, we have

$$E[X(t)] = E[n(0, t)] = \lambda t$$

and

$$E[X^2(t)] = E[n^2(0, t)] = \lambda t + \lambda^2 t^2.$$

# Poisson Random Process (contd..)

- To determine the autocorrelation function $R_{XX}(t_1, t_2)$, let $t_2 > t_1$, then from (ii) above $n(0, t_1)$ and $n(t_1, t_2)$ are independent Poisson random variables with parameters $\lambda t_1$ and $\lambda(t_2 - t_1)$ respectively. Thus

$$E[n(0,t_1)n(t_1,t_2)] = E[n(0,t_1)]E[n(t_1,t_2)] = \lambda^2 t_1(t_2 - t_1).$$

But

$$n(t_1,t_2) = n(0,t_2) - n(0,t_1) = X(t_2) - X(t_1)$$

and hence the left side of above equation can be rewritten as

$$E[X(t_1)\{X(t_2) - X(t_1)\}] = R_{XX}(t_1,t_2) - E[X^2(t_1)].$$

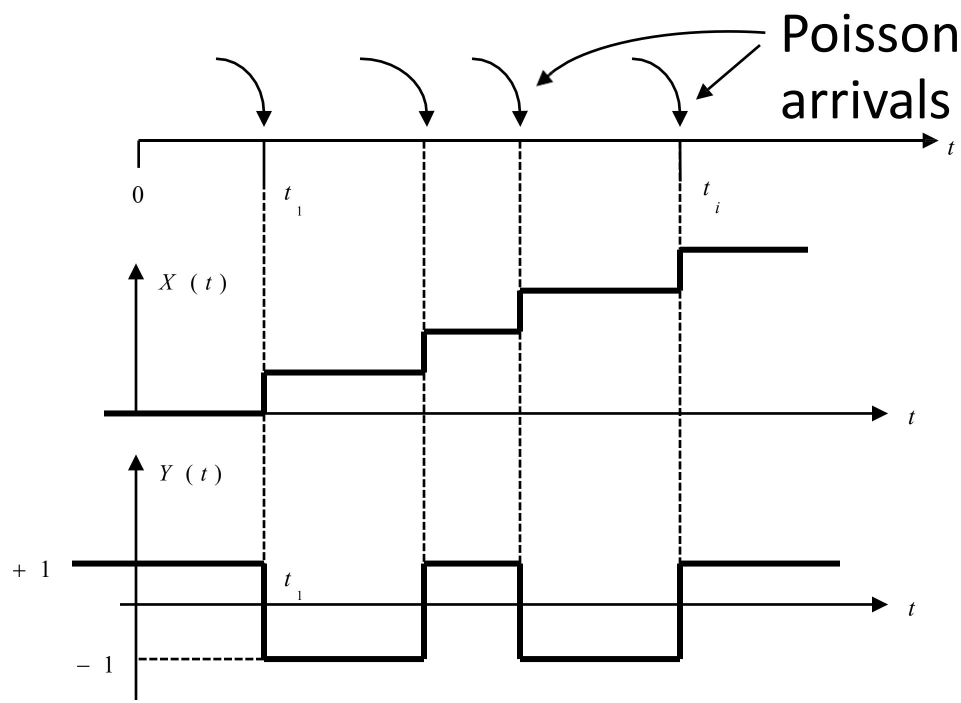
$$R_{XX}(t_1,t_2) = \lambda^2 t_1(t_2 - t_1) + E[X^2(t_1)]$$

$$= \lambda t_1 + \lambda^2 t_1 t_2, \qquad t_2 \geq t_1.$$

Similarly $$R_{XX}(t_1,t_2) = \lambda t_2 + \lambda^2 t_1 t_2, \qquad t_2 < t_1.$$

Thus $$R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2).$$

# Poisson Random Process (contd..)




- Notice that the Poisson process X(t) does not represent a wide sense stationary process.

- Define a binary level process

$$Y(t) = (-1)^{X(t)}$$

that represents a telegraph signal Notice that the transition instants $\{t_i\}$ are random Although $X(t)$ does not represent a wide sense stationary process,

# Poisson Random Process (contd..)

its derivative $X'(t)$ *does* represent a wide sense stationary process.

$$X(t) \longrightarrow \boxed{\frac{d(\cdot)}{dt}} \longrightarrow X'(t)$$

(Derivative as a LTI system)

From there

$$\mu_{X'}(t) = \frac{d\mu_X(t)}{dt} = \frac{d\lambda t}{dt} = \lambda, \quad a\ constant$$

and

$$R_{XX'}(t_1, t_2) = \frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} = \begin{cases} \lambda^2 t_1 & t_1 \le t_2 \\ \lambda^2 t_1 + \lambda & t_1 > t_2 \end{cases}$$

$$= \lambda^2 t_1 + \lambda\, U(t_1 - t_2)$$

and

$$R_{X'X'}(t_1, t_2) = \frac{\partial R_{XX'}(t_1, t_2)}{\partial t_1} = \lambda^2 + \lambda\, \delta(t_1 - t_2).$$

# Poisson Random Process (contd..)

Define the processes

$$Y(t) = \sum_{i=1}^{X(t)} N_i \quad ; \quad Z(t) = \sum_{i=1}^{X(t)} (1 - N_i) = X(t) - Y(t)$$

we claim that both *Y*(*t*) and *Z*(*t*) are independent Poisson processes with parameters $\lambda pt$ and $\lambda qt$ respectively.

**Proof:**

$$Y(t) = \sum_{n=k}^{\infty} P\{Y(t) = k \mid X(t) = n\} P\{X(t) = n)\}.$$

But given *X*(*t*) = *n*, we have $Y(t) = \sum_{i=1}^{n} N_i \sim B(n, p)$ so that

$$P\{Y(t) = k \mid X(t) = n\} = \binom{n}{k} p^k q^{n-k}, \qquad 0 \le k \le n,$$

and

$$P\{X(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

# Poisson Random Process (contd..)

$$P\{Y(t)=k\} = e^{-\lambda t}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!\,k!}p^k q^{n-k}\,\frac{(\lambda t)^n}{n!} = \frac{p^k e^{-\lambda t}}{k!}(\lambda t)^k \underbrace{\sum_{n=k}^{\infty}\frac{(q\lambda t)^{n-k}}{(n-k)!}}_{e^{q\lambda t}}$$

$$= (\lambda pt)^k\,\frac{e^{-(1-q)\lambda t}}{k!} = e^{-\lambda pt}\,\frac{(\lambda pt)^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

$$\sim \quad P(\lambda pt).$$

More generally,

$$P\{Y(t)=k, Z(t)=m\} = P\{Y(t)=k, X(t)-Y(t)=m\}$$

$$= P\{Y(t)=k, X(t)=k+m\}$$

$$= P\{Y(t)=k \mid X(t)=k+m\}\,P\{X(t)=k+m\}$$

$$= \binom{k+m}{k} p^k q^m \cdot e^{-\lambda t}\,\frac{(\lambda t)^{k+m}}{(k+m)!} = \underbrace{e^{-\lambda pt}\,\frac{(\lambda pt)^n}{k!}}_{P(Y(t)=k)}\;\underbrace{e^{-\lambda qt}\,\frac{(\lambda qt)^n}{m!}}_{P(Z(t)=m)}$$

$$= P\{Y(t)=k\}\,P\{Z(t)=m\},$$ which completes the proof.

# Poisson Random Process (contd..)

-- integer-valued discrete r.p. $X(t), \quad -\infty < t < \infty$

$$X(0) = 0 \qquad t_b < t_a \quad \Rightarrow \quad X(t_b) \le X(t_a)$$

$$P[X(t_a) - X(t_b) = k] = \frac{[\lambda(t_a - t_b)]^k}{k!} e^{-\lambda(t_a - t_b)}, \quad k = 0,1,2,\square$$

$$t_d < t_c \le t_b < t_a \quad \Rightarrow \quad X(t_a) - X(t_b) \;\&\; X(t_c) - X(t_d) \text{ are indep.}$$

$$\overline{X}(t) = E[X(t)] = \lambda t \qquad R_{XX}(t,t) = E[X(t)^2] = \lambda t + (\lambda t)^2$$

$$C_{XX}(t,t) = \lambda t$$

# Poisson Random Process (contd..)

$$0 < t_1 < t_2 \quad \Rightarrow$$

$$P[X(t_1) = k_1, X(t_2) = k_2] = P[X(t_1) = k_1, X(t_2) - X(t_1) = k_2 - k_1]$$

$$= \begin{cases} \dfrac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda t_1} \dfrac{[\lambda(t_2 - t_1)]^{(k_2-k_1)}}{(k_2 - k_1)!} e^{-\lambda(t_2 - t_1)}, & k_2 \geq k_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{(\lambda t_1)^{k_1}[\lambda(t_2 - t_1)]^{(k_2-k_1)}}{k_1!(k_2 - k_1)!} e^{-\lambda t_2}, & k_2 \geq k_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

# Poisson Random Process (contd..)

$$0 < t_1 < t_2 \quad \Rightarrow$$

$$P[X(t_2) = k_2 \mid X(t_1) = k_1] = P[X(t_2) - X(t_1) = k_2 - k_1 \mid X(t_1) = k_1]$$

$$= P[X(t_2) - X(t_1) = k_2 - k_1]$$

$$= \begin{cases} \dfrac{[\lambda(t_2 - t_1)]^{(k_2 - k_1)}}{(k_2 - k_1)!} e^{-\lambda(t_2 - t_1)}, & k_2 \geq k_1 \\ 0, & \text{otherwise} \end{cases}$$