UNIT-2
MOMENTS OF RANDOM VARIABLE

Moments about origin

Moments About the Origin

The expected value of X^n, $n = 0,1,2, \dots\dots\dots$ is given by

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$$

gives the moments about the origin of the random variable X. These are also called standard moments and are denoted as m_n

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Moments about mean

Moments About the Mean

The expected value of $(X - \bar{X})^n$, $n = 0,1,2, \dots \dots \dots$ is given by

$$E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x) dx$$

gives the moments about the mean of the random variable X. These are also called central moments and are denoted as μ_n

Characteristic function

Characteristic Function

The *characteristic function* of a random variable $X$is defined by

$$\Phi_x(\omega) = E\left[e^{j\omega x}\right] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)dx$$

where $= \sqrt{-1}$. It is a function of the real number $-\infty < \omega < \infty$.

$\Phi_x(\omega)$ is seen as the *Fourier transform* (with the sign of ω reversed) of $f_x(x)$

Moment generating function

Moment Generating Function

The *moment generating function* of a random variable X is defined by

$$M_X(v) = E[e^{vX}] = \int_{-\infty}^{\infty} f_X(x) e^{vx} dx$$

Where v is a real number $-\infty < v < \infty$.

Moment generating function

- Moments are related to $M_x(v)$ by the expression

$$m_x = (-j)^n \frac{d^n \mathrm{M}_x(v)}{dv^n}\bigg|_{v=0}$$

- The main disadvantage of the moment generating function is that it may not exist for all random variables.
- In fact, $M_x(v)$ exists only if all the moments exist

Monotonically increasing RV

Transformations of A Random Variable

- Quite often one may wish to transform one random variable X into a new random variable Y by means of a transformation

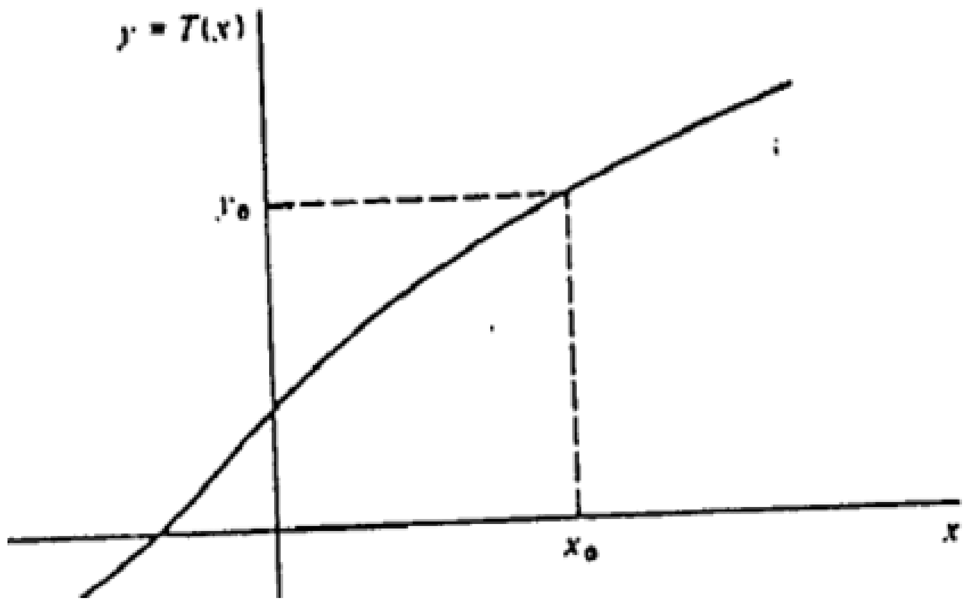
$$\underset{f_X(x)}{\xrightarrow{X}} \boxed{Y = T(X)} \underset{f_Y(y)}{\xrightarrow{Y}}$$

- Typically, the density function $f_X(x)$ or distribution function $F_X(x)$ of X is known , and the problem is to determine either the density function $f_Y(y)$ or distribution function $F_Y(y)$ of Y.
- The transformation T can be linear, nonlinear, segmented ,staircase, etc

Monotonically increasing RV (contd..)

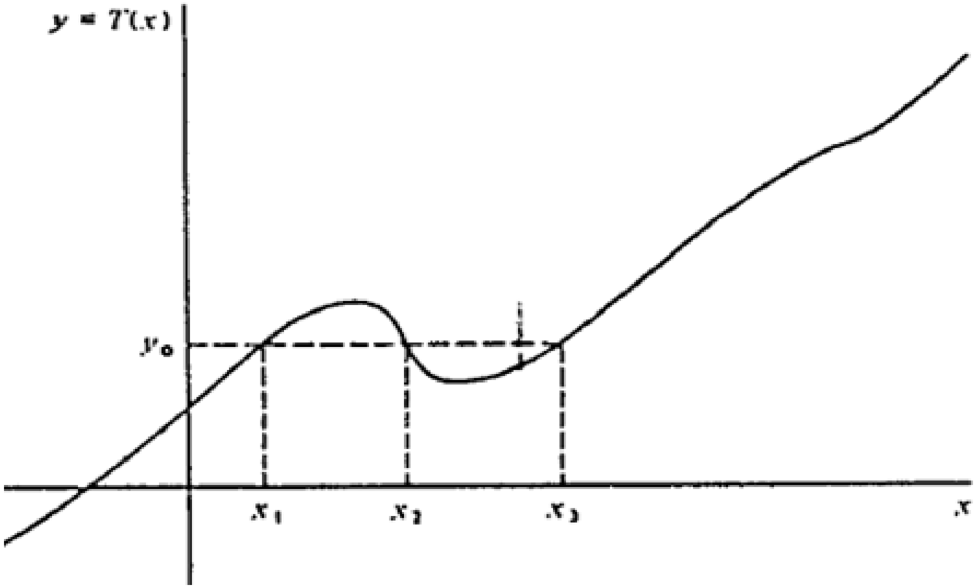
Monotonic Transformation of a Continuous Random variable

- A transformation T is called *monotonically increasing* if $T(x_1) < T(x_2)$ for any $x_1 < x_2$.



Nonmonotonic Transformation of a RV

Nonmonotonic transformations of a continuous random variable



- In this case, there may be more than one interval of values of X that correspond to the event $\{Y \leq y_0\}$ corresponds to the event $\{X \leq x_1 \; and \; x_2 \leq X \leq x_3\}$

Nonmonotonic Transformation of a RV (contd..)

- Thus, the probability of the event $\{Y \leq y_0\}$ now equals the probability of the event $\{x \text{ values yielding } Y \leq y_0\}$, which we shall write as $\{x|Y \leq y_0\}$ i.e.,

$$F_y(y_0) = p\{Y \leq y_0\} = p\{x|Y \leq y_0\} = \int_{\{x|Y \leq y_0\}} f_x(x)dx$$

- Differentiating we get the density function of Y as

$$f_y(y_0) = \frac{d}{dy_0} \int_{\{x|Y \leq y_0\}} f_x(x)\, dx$$

Nonmonotonic Transformation of a RV (contd..)

- The density function is also given by

$$f_y(y) = \sum_n \frac{f_x(x_n)}{\left| \frac{dT(x)}{dx} \Big|_{x=x_0} \right|}$$

where the sum is taken so as to include all the roots $x_n, n = 1, 2, \ldots \ldots$,which are the real solutions of the equation

$$y = T(x)$$

Transformation of a DiscreteRV

Transformation of a Discrete Random Variable

- If X is a discrete random variable

$$f_x(x) = \sum_n p(x_n)\delta(x - x_n)$$

$$F_x(x) = \sum_n p(x_n)u(x - x_n)$$

where the sum is taken to include all the possible values x_n , $n = 1, 2, \dots\dots,$ of X.

- If the transformation $Y = T(X)$ is continuous and monotonic, there is a one-to-one correspondence between X and Y so that a set $\{x_n\}$, through the equation $y_n = T\{x_n\}$ so that $P\{y_n\} = P\{x_n\}$.

Transformation of a DiscreteRV (contd..)

- If the transformation $Y = T(X)$ is continuous and monotonic, there is a one-to-one correspondence between X and Y so that a set $\{x_n\}$, through the equation $y_n = T\{x_n\}$ so that $P\{y_n\} = P\{x_n\}$.
- Thus, we have

$$f_y(y) = \sum_n p(y_n)\delta(y - y_n)$$

$$F_y(y) = \sum_n p(y_n)u(y - y_n) \text{ where } y_n = T(x_n)$$

- If T is not monotonic , the above procedure remains same, but $P(y_n)$ will equal the sum of the probabilities of the various x_n for which $y_n = T(x_n)$

Expected value of a RV

Expected Value of a Random variable

In general, the expected value of any random variable X is defined by

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x) dx$$

Expected value of a RV (contd..)

If X is discrete with N possible values x_i having probabilities $P(x_i)$ of occurrence, then

$$f_x(x) = \sum_{i=1}^{N} x_i P(x_i)\delta(x - x_i)$$

Then we have

$$E[x] = \sum_{i=1}^{N} x_i P(x_i)$$

If the density is symmetrical about a line $x = a$ i.e.

$$f_x(x + a) = f_x(-x + a)$$

then

$$E[x] = a$$

Conditional Expected value of a RV

Conditional Expected Value

If $f_X(x|B)$ is the conditional density where B is any event defined on the sample space of X, then the *conditional expected value* of X, is given by

$$E[X|B] = \int_{-\infty}^{\infty} x f_X(x|B)\,dx$$

Conditional Expected value of a RV (contd..)

If the event $B = \{X \leq b\}$, $\infty < b < \infty$

$$f_X(x|X \leq b) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)dx} & x < b \\ 0 & x \geq b \end{cases}$$

Then, the conditional expected value is given by

$$E[x|X \leq b] = \frac{\int_{-\infty}^{b} x f_X(x)dx}{\int_{-\infty}^{b} f_X(x)dx}$$

which is the mean value of X when X is constrained to the set $\{X \leq b\}$.

Moments about origin

Moments About the Origin

The expected value of X^n, $n = 0,1,2, \dots \dots \dots$ is given by

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

gives the moments about the origin of the random variable X. These are also called standard moments and are denoted as m_n

Moments about origin (contd..)

For $n = 0$,

$$m_0 = E[X^0] = \int_{-\infty}^{\infty} x^0 f_X(x)dx = \int_{-\infty}^{\infty} f_X(x)dx$$

is the area of under the function $f_X(x)$.

For $n = 1$,

$$m_1 = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \bar{X}$$

is the expected value of X.

Moments about mean

Moments About the Mean

The expected value of $(X-\bar{X})^n,\ n = 0,1,2,\ldots\ldots\ldots$ is given by

$$E[(X-\bar{X})^n] = \int_{-\infty}^{\infty} (x-\bar{X})^n f_X(x)\,dx$$

gives the moments about the mean of the random variable X. These are also called central moments and are denoted as μ_n

Moments about mean (contd..)

For $n = 0$,

$$\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f_x(x)dx$$

$$\mu_0 = E[(X - \bar{X})^0] = \int_{-\infty}^{\infty} f_x(x)dx$$

is the area of under the function $f_x(x)$.

For $n = 1$,

$$\mu_1 = E[(X - \bar{X})] = E[X] - \bar{X}=0$$

Variance

Variance

The second central moment μ_2 is given by

$$\mu_2 = E[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (X - \bar{X})^2 f_x(x) dx$$

1. It is popularly known as the *variance* σ_x^2 of the random variable X.
2. The positive square root σ_x of variance is called the standard deviation of X.
3. It is a measure of the spread in the function $f_x(x)$ about the mean.

Variance (contd..)

The second central moment is given by

$$\mu_2 = E\left[(X - \bar{X})^2\right]$$

By expanding we get

$$\begin{aligned}\mu_2 &= E\left[X^2 - 2\bar{X}X + \bar{X}^2\right] \\ &= E\left[X^2\right] - 2\bar{X}E\left[X\right] + \bar{X}^2 \\ &= E\left[X^2\right] - \bar{X}^2 = m_2 - {\mu_1}^2\end{aligned}$$

Skew

The third central moment is given by

$$\mu_3 = E[(X - \bar{X})^3]$$

$$\mu_3 = E[X^3 - 3X^2\bar{X} + 3X\bar{X}^2 - \bar{X}^3]$$

$$= E[X^3] - 3E[X^2]\bar{X} + 3\bar{X}^2E[X] - \bar{X}^3$$

$$= m_3 - 3m_2\mu_1 + 3\mu_1{}^3 - \mu_1{}^3$$

$$= m_3 - 3m_2\mu_1 + 2\mu_1{}^3$$

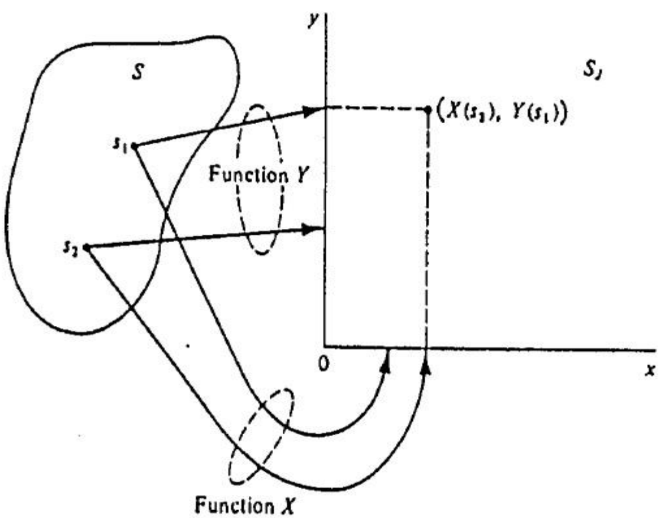
Skew (contd..)

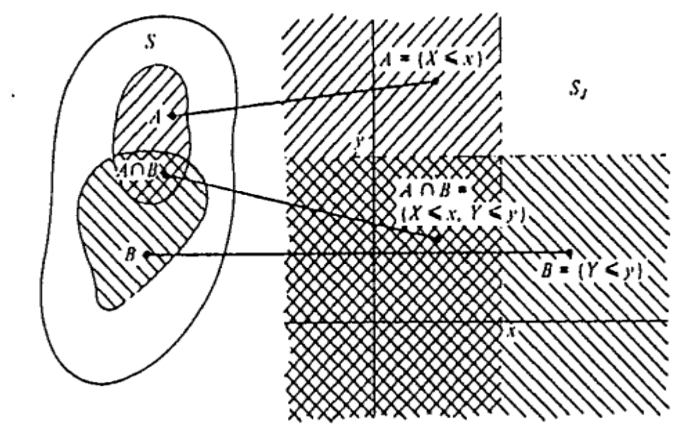
- μ_3 is a measure of asymmetry of $f_x(x)$ about the mean.
- It will be called the *skew* of the density function.
- If a density is symmetric about $x = \bar{X}$, it has zero skew. For this case, $\mu_n = 0$ for all odd values of n.
- The normalized third central moment $\mu_3/{\sigma_x}^3$ is known as the coefficient of *skewness*.

UNIT-II
Multiple Random Variables and Operations

Vector random variables

- There are many cases where the outcome is a vector of numbers. We have already seen one such experiment, in, where a dart is thrown at random on a dartboard of radius r. The outcome is a pair (X, Y) of random variables that are such that $X^2 + Y^2 \leq r^2$.
- we measure voltage and current in an electric circuit with known resistance. Owing to random fluctuations and measurement error, we can view this as an outcome (V, I)of a pair of random variables.
- Mapping the sample space to joint sample space





Comparision of sample space s with sj

Joint distribution function

- Let X and Y be random variables. The pair (X, Y) is then called a (two-dimensional) random vector.
- The joint distribution function (joint cdf) of (X, Y) is defined as $F(x, y) = P(X \leq x, Y \leq y)$ for $x, y \in R$.
- Assume the joint sample space S_J has only three possible elements (1,1),(2,1),(3,3).The probabilities of the elements are to be P(1,1)=0.2,P(2,1)=0.3 ,P(3,3)=0.5.We find $F_{X,Y}(X,Y)$
- In constructing joint distribution function we observe that has no elements for $x<1,y<1$.only at the point (1,1)does the function assume a step value.
- So long as $x \geq 1, y \geq 1$ this probability is maintained.For larger x and y the point(2,1) produces a second stair step of 0.3 which holds the region $x \geq 2, y \geq 1$.The second step is added to the first.Finally third step of 0.5 is added to the two for $x \geq 3, y \geq 3$

Properties of Joint Distribution

- Properties:

1) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

   Note that $\{X \leq -\infty, Y \leq y\} \subseteq \{X \leq -\infty\}$

2) $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$

   If $x_1 < x_2$ and $y_1 < y_2$,

   $\{X \leq x_1, Y \leq y_1\} \subseteq \{X \leq x_2, Y \leq y_2\}$

   $\therefore P\{X \leq x_1, Y \leq y_1\} \leq P\{X \leq x_2, Y \leq y_2\}$

   $\therefore F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$

3) $F_{X,Y}(\infty, \infty) = 1$

4) $F_{X,Y}(x, y)$ is right continuous in both the variables

$F_X(x) = F_{XY}(x, +\infty)$

Properties of joint distribution

5) If $x_1 < x_2$ and $y_1 < y_2$

$$P(\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$$

$$F_{X,Y}(x,y),\ -\infty < x < \infty, -\infty < y < \infty$$

6)

$$F_X(x) = F_{XY}(x,+\infty)$$

$$\{X \le x\} = \{X \le x\} \cap \{Y \le +\infty\}$$

$$\therefore F_X(x) = P(\{X \le x\}) = P(\{X \le x, Y \le \infty\}) = F_{X,Y}(x,+\infty)$$

$$F_Y(y) = F_{XY}(\infty, y)$$

$$F_{X,Y}(x,y),\ -\infty < x < \infty, -\infty < y < \infty$$

$F_X(x)$ and $F_Y(y)$ Called marginal cumulative distribution function

Marginal distribution functions

- The distribution of one random variable can be obtained by setting the other value to infinity in $F_{X,Y}(x,y)$. The functions obtained in this manner $F_X(x), F_Y(y)$ are called marginal distribution functions.
- Example:

  $F_{X,Y}(x,y)=P(1,1)u(x-1)u(y-1)+P(2,1)u(x-2)u(y-1)+P(3,3)u(x-3)u(y-3)$

  $P(1,1)=0.2$, $P(2,1)=0.3$, $P(3,3)=0.5$ if we set $y=\infty$ then

  $F_X(x)=0.2u(x-1)+0.3u(x-2)+0.5u(x-3)$

  similarly

  $F_Y(y)=0.2u(y-1)+0.3u(y-1)+0.5u(y-3)$

  $=0.5u(y-1)+0.5u(y-3)$

Marginal distribution functions

- Consider two jointly distributed random variables and with the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-2x})(1-e^{-y}) & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

1) Find the marginal CDFs

2) Find the probability P(1<x≤2, 1<y≤2)

Marginal distribution functions

a)

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y) = \begin{cases} 1 - e^{-2x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y) = \begin{cases} 1 - e^{-y} & \text{y} \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P\{1 < X \leq 2,\ \ 1 < Y \leq 2\} = F_{X,Y}(2,2) + F_{X,Y}(1,1) - F_{X,Y}(1,2) - F_{X,Y}(2,1)$$

$$= (1-e^{-4})(1-e^{-2}) + (1-e^{-2})(1-e^{-1}) - (1-e^{-2})(1-e^{-2}) - (1-e^{-4})(1-e^{-1})$$

$$= 0.0272$$

Joint Probability Density Function

- If and are two continuous random variables and their joint distribution function is continuous in both and then we can define joint probability density function by

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y),$$

provided it exists.

Clearly

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$$

Marginal density function

- The marginal CDF and pdf are same as the CDF and pdf of the concerned single random variable. The marginal term simply refers that it is derived from the corresponding joint distribution or density function of two or more jointly random variables.
- With the help of the two-dimensional Dirac Delta function, we can define the joint pdf of two discrete jointly random variables. Thus for discrete jointly random variables and

$$f_{X,Y}(x,y) = \sum_{(x_i, y_j) \in R_X \times R_Y} \sum p_{X,Y}(x,y)\delta(x - x_i, y - y_j)$$

Marginal density function

- The joint density function

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-2x})(1-e^{-y}) & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

$$= \frac{\partial^2}{\partial x \partial y}[(1-e^{-2x})(1-e^{-y})] \quad x \geq 0, y \geq 0$$

$$= 2e^{-2x}e^{-y} \quad x \geq 0, y \geq 0$$

Conditional distribution

- We discussed the conditional CDF and conditional PDF of a random variable conditioned on some events defined in terms of the same random variable. We observed that

$$F_X(x/B) = \frac{P(\{X \le x\} \cap B)}{P(B)} \qquad P(B) \ne 0$$

$$f_X(x/B) = \frac{d}{dx} F_X(x/B)$$

Conditional density function

- Suppose and are two discrete jointly random variable with the joint PMF fxy(x,y) . The conditional PMF of y given x=x is denoted by and defined as

$$f_{y/x}(y/x)$$

$$p_{Y/X}(y/x) = P(\{Y=y\}/\{X=x\})$$

$$= \frac{P(\{X=x\} \cap \{Y=y\})}{P\{X=x\}}$$

$$= \frac{p_{X,Y}(x,y)}{p_X(x)} \quad \text{provided} \quad p_X(x) \neq 0$$

Thus,

$$p_{Y/X}(y/x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \quad \text{provided} \quad p_X(x) \neq 0$$

Conditional Probability Distribution Function

- Consider two continuous jointly random variables and with the joint probability distribution function We are interested to find the conditional distribution function of one of the random variables on the condition of a particular value of the other random variable.
- We cannot define the conditional distribution function of the random variable on the condition of the event by the relation

$$F_{Y/X}(y/x) = P(Y \le y / X = x)$$

$$= \frac{P(Y \le y, X = x)}{P(X = x)}$$

Point conditioning

- First consider the case when X and Y are both discrete. Then the marginal pdf's
- $f_Y(y)=P(Y=y)$ $f_X(x)=P(X=x)$
- The joint pdf is, similarly

  $f_{X,Y}(x,y)=P(X\leq x,Y\leq y)$

- Conditional density function is given by

  $f_X(x/B)= \frac{dFX(x|B)}{dx}$

Point conditioning (contd..)

- The conditional pdf of the conditional distribution Y|X is

$$F_{Y/X}(y/x) = P(Y \le y / X = x)$$

$$= \frac{P(Y \le y, X = x)}{P(X = x)}$$

- Distribution function of one random variable X conditioned by that second variable Y has some specific values of y. This is called point conditioning
- B={y-Δy<Y≤y+Δy}

  Where Δy is a small quantity that we eventually let approach 0.

Point conditioning (contd..)

$$F_X(x / y-\Delta y<Y\leq y+\Delta y) = \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1\,d\xi_2}{\int_{y-\Delta y}^{y+\Delta y} f_Y(\xi)\,d\xi}$$

$$F_{X,Y}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i, y_j)\delta(x-x_i)\delta(y-y_j)$$

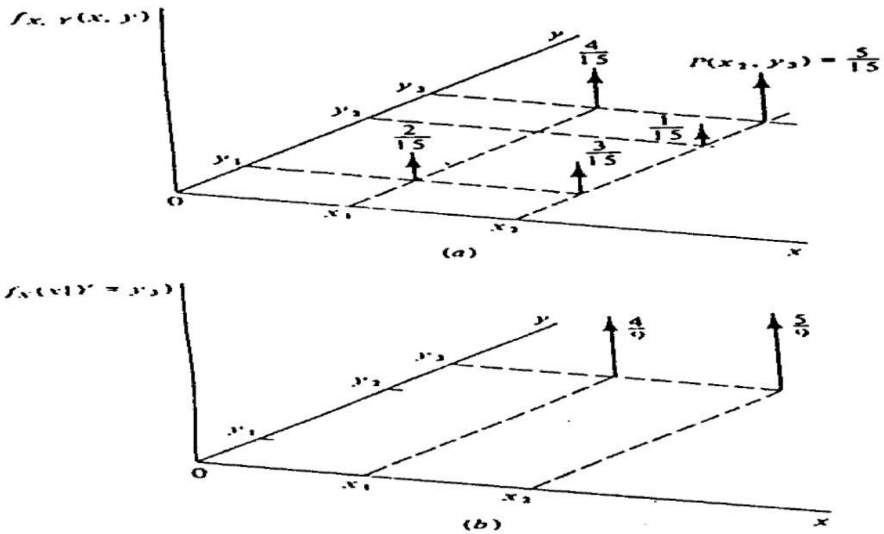
Now the specific value of y of interest is y_k

$$Fx(x/Y = yk) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} u(x-x_i)$$

$$fx(x/Y = yk) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} \delta(x-x_i)$$

Interval Conditioning

- Distribution function of one random variable X conditioned by that second variable Y has some specific values of y. This is called point conditioning $B=\{y_a<Y\leq y_b\}$
- $P(x_1,y_1)=2/15$, $P(x_2,y_1)=3/15$.etc.since $P(y_3)=4/15+5/15=9/15$ find $f_x(x/y=y_3)$



Statistical independence

- Let and be two random variables characterized by the joint distribution function

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$

and the corresponding joint density function

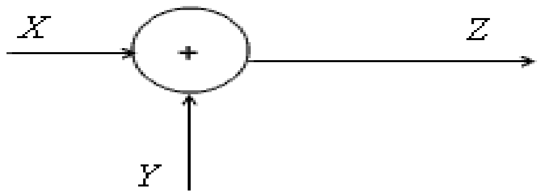
$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{X,Y}(x,y)$$

Sum of two random variables

- We are often interested in finding out the probability density function of a function of two or more RVs

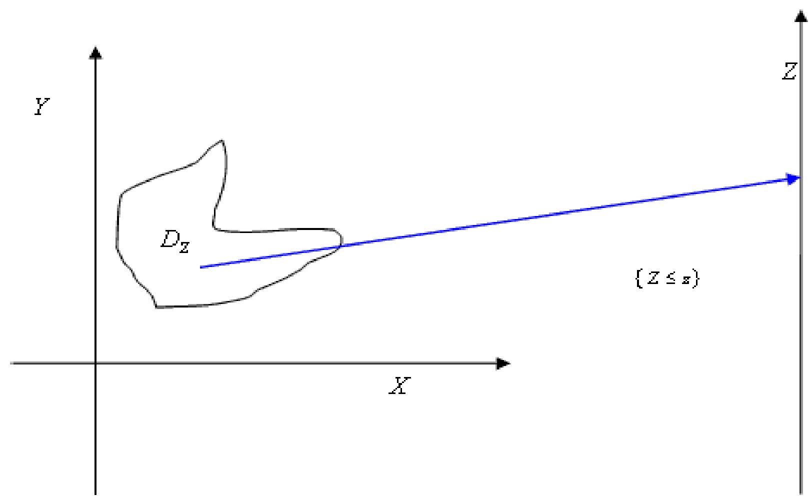
- The received signal by a communication receiver is given by

$$Z = X + Y$$



- where is received signal which is the superposition of the message signal and the noise.

corresponding to each z. $\{Z \le z\}$ We can find a variable subset $D_Z = \{(x,y) \mid g(x,y) \le z\}$



$$\begin{aligned} \therefore F_Z(z) &= P(\{Z \le z\}) \\ &= P\{(x,y) \mid (x,y) \in D_z\} \\ &= \iint_{(x,y)\in D_z} f_{X,Y}(x,y)\,dy\,dx \end{aligned}$$

Central Limit Theorem

- Consider n **independent** random variables $x_1,x_2,x_3......xn$,The mean and variance of each of the random variables are assumed to be known. Suppose $E[x]=\mu_x$ $var(x)=\varsigma_x^{\;2}$ and . Form a random variable

$$Y_N=X_1+X_2+.......X_N$$

The mean and variance of YN are given by

$$E[yn]= \mu_{x\,1} + \mu_{x\,2} + \mu_{x\,3}......... + \mu_{x\,n}$$

$$\text{var}(Y_N) = \sigma^2_{Y_N} = E\{\sum_{i=1}^{N}(X_i - \mu_{X_i})\}^2$$

$$= \sum_{i=1}^{N} E(X_i - \mu_{X_i})^2 + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} E(X_i - \mu_i)(X_j - \mu_j)$$

$$= \sigma^2_{X_1} + \sigma^2_{X_2} + \ldots + \sigma^2_{X_n}$$

$$\because X_i \text{ and } X_j \text{ are independent for } i \neq j.$$

Central Limit Theorem (contd..)

The CLT states that under very general conditions $\left\{Y_n = \sum_{i=1}^{n} X_i\right\}$ converges in distribution to $Y \sim N(\mu_Y, \sigma_Y^2)$ as $n \to \infty$

1. The random variables are independent and identically distributed.
2. The random variables are independent with same mean and variance, but not identically distributed.
3. The random variables are independent with different means and same variance and not identically distributed.
4. The random variables are independent with different means and each variance being neither too small nor too large.

Expected Values of Random Variables

- If g(x,y) is a function of a continuous random variables X and Y then then the expected value of is given by

$$\overline{g} = E\left[g(X,Y)\right] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy & \text{Continuous} \\ \sum_{i}\sum_{k} g(x_i,y_k) P_{X,Y}(x_i,y_k) & \text{Discrete} \end{cases}$$

Example

•Consider the discrete random variables x and y. The joint probability mass function of the random variables are tabulated in Table . Find the joint expectation of g(x,y)=xy.

X \ Y	0	1	2	$p_Y(y)$
0	0.25	0.1	0.15	0.5
1	0.14	0.35	0.01	0.5
$p_X(x)$	0.39	0.45	0.16	

$$E[XY] = \sum_x \sum_y g(x,y)\, p_{XY}(x,y)$$

$$= 1 \times 1 \times 0.35 + 1 \times 2 \times 0.01$$

$$= 0.37$$

Properties

- Expectation is a linear operator. We can generally write
  $E[a_1 g_1(x,y)+a_2 g_2(x,y)=a_1 E(g_1(x,y)+a_2 E(g_2(x,y))$
  $E[xy+5\log_e xy]=E[xy]+5E[\log_e xy]$
- If x and y are independent random variables and $g(x,y)=g1(x,y)\times g2(x,y)$ then $E[g(x,y)]=E[g1(x,y)]\times E[g2(x,y]$

$$
\begin{aligned}
Eg(X,Y) &= Eg_1(X)g_2(Y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)g_2(Y)f_{X,Y}(x,y)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)g_2(Y)f_X(x)f_Y(y)dxdy \\
&= \int_{-\infty}^{\infty} g_1(X)f_X(x)dx\int_{-\infty}^{\infty} g_2(Y)f_Y(y)dy \\
&= Eg_1(X)Eg_2(Y)
\end{aligned}
$$

Joint moments about the origin

For two continuous random variables X and Y, the joint moment of order m+n is defined as

$$E(X^m Y^n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n f_{XY}(x,y)dxdy$$

And the joint central moment of order m+n is defined as

$$E(X-\mu_x)^m E(Y-\mu_y)^n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)^m (y-\mu_y)^n f_{XY}(x,y)dxdy$$

$$\mu_x = E[x]$$

$$\mu_y = E[y]$$

Covariance of two random variables

The covariance of two random variables X and Y is defined as

$$Cov(X,Y)=E(X-\mu_x)E(Y-\mu_y)$$

Cov(X, Y) is also denoted as ς_{XY}.

$$Cov(X,Y) = E(X-\mu_x)^m E(Y-\mu_y)^n$$

$$= E(XY - \mu_y X - \mu_x Y + \mu_x \mu_y)$$

$$= E(XY) - \mu_Y E(X) - \mu_X E(y) + \mu_X \mu_Y$$

$$= E(XY) - \mu_x \mu_y$$

Uncorrelated random variables

Two random variables are called *uncorrelated* if

$$Cov(X,Y)=0$$

Which also means $E(XY)=\mu_x\mu_y$

If are independent random variables, then

$$f_{XY}(x,y)=f_X(x)f_Y(y)$$

Thus two independent random variables are always uncorrelated.

joint characteristic function

The joint characteristic function of two random variables X and Y is defined by

$$\phi_{XY}(\omega_1,\omega_2) = E[e^{j\omega_1 x + j\varpi_2 y}]$$

If and are jointly continuous random variables, then

$$\phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)e^{j\omega_1 x + j\varpi_2 y}dxdy$$

Joint moments about the origin

For two discrete random variables X and Y, the joint moment of order m+n is defined as

$$E(X^m Y^n) = \sum_x \sum_y x^m y^n f_{XY}(x, y)dxdy$$

And the joint central moment of order m+n is defined as

$$E(X - \mu^x)^m E(Y - \mu^y)^n = \sum_x \sum_y (x - \mu_x)^m (y - \mu_y)^n f_{XY}(x, y)$$

$$\mu_x = E[x]$$

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Covariance of two random variables

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$$Cov(X,Y) = E(X-\mu_x)^m E(Y-\mu_y)^n$$

$$= E(XY-\mu_y X-\mu_x Y+\mu_x\mu_y)$$

$$= E(XY)-\mu_Y E(X)-\mu_X E(y)+\mu_X\mu_Y$$

$$= E(XY)-\mu_x\mu_y$$

Two Random variables

Two random variables X and Y are called jointly Gaussian if their joint probability density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{XY}^2}} e^{-\frac{1}{2(1-\rho_{XY}^2)}\left[\frac{(x-\mu_x)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_y)^2}{\sigma_Y^2}\right]}$$

$-\infty < x < \infty, -\infty < y < \infty$

means μ_x and μ_y
variances ς_x^2 ς_y^2
correlation coefficient ρ_{XY}
We denote the jointly Gaussian random variables and with these parameters as $(X,Y) \sim N(\mu_x, \mu_y, \varsigma_x^2, \varsigma_y^2, \rho_{XY})$

Transformations of multiple random variables

The joint density function of new random variable $Y_i=T(X_1,X_2,......X_N)$ $i=1,2,3....n$

The random variable Xj can be obtained from inverse transformation

$X_j=T_j^{-1}(Y_1,Y_2,.....Y_N)$

$$\left.\begin{aligned} x_1 &= g_1^{-1}(y_1, y_2, \ldots, y_k) \\ x_2 &= g_2^{-1}(y_1, y_2, \ldots, y_k) \\ &\vdots \\ x_n &= g_n^{-1}(y_1, y_2, \ldots, y_{k=n}) \end{aligned}\right\} **$$

Transformations of multiple random variables

- Assuming that the partial derivatives $\partial g_i^{-1} / \partial y_i$ exist at every point $(y_1, y_2, \ldots, y_{k=n})$. Under these assumptions, we have the following determinant J

$$J = det \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial y_1} & \cdots & \frac{\partial g_1^{-1}}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n^{-1}}{\partial y_1} & \cdots & \frac{\partial g_n^{-1}}{\partial y_n} \end{bmatrix}$$

called as the Jacobian of the transformation specified by (**). Then, the joint pdf of $Y_1, Y_2, \ldots, Y_k$ can be obtained by using the change of variable technique of multiple variables.

Transformations of multiple random variables

- As a result, the new p.d.f. is defined as follows:

$$g(y_1, y_2, \ldots, y_n) = \begin{cases} f_{X_1,\ldots,X_n}\left(g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}\right) |J|, \text{ for } (y_1, y_2, \ldots, y_n) \in \Psi \\ 0, \textit{otherwise} \end{cases}$$

Linearly transformation of Gaussian RV

- Linearly transforming set of Gaussian random variables $X_1,X_2,.....X_N$ for which the joint density function exists. The new variables $Y_1,Y_2,.....Y_N$ are
- $Y_1=a_{11}X_1+a_{12}X_2+......+a_{1N}X_N$
- $Y_2=a_{21}X_1+a_{22}X_2+......+a_{2N}X_{N.}$
- $Y_N=a_{N1}X_1+a_{N2}X_2+......+a_{NN}X_N$

$$[T]=\begin{bmatrix} a11 & a12\ldots. & a1N \\ a21 & a22\ldots & a2N \\ aN1 & aN2\ldots & aNN \end{bmatrix}$$

$$[Y] = \begin{bmatrix} Y1 \\ \vdots \\ YN \end{bmatrix}$$

$X_i=Ti^{-1}(Y_1.....Y_N)=a^{i1}Y_1+a^{i2}y_2+....+a^{iN}Y_N$

[Y]=[T][X]