UNIT – II

Fourier series
&
Fourier Transforms

Fourier series:

To represent any periodic signal x(t), Fourier developed an expression called Fourier series. This is in terms of an infinite sum of sines and cosines or exponentials. Fourier series uses orthoganality condition.

Jean Baptiste Joseph Fourier, a French mathematician and a physicist; was born in Auxerre, France. He initialized Fourier series, Fourier transforms and their applications to problems of heat transfer and vibrations. The Fourier series, Fourier transforms and Fourier's Law are named in his honour.

Fourier Series Representation of Continuous Time Periodic Signals

A signal is said to be periodic if it satisfies the condition x (t) = x (t + T) or x (n) = x (n + N).

Where T = fundamental time period,

ω_0= fundamental frequency = 2π/T

There are two basic periodic signals:
$x(t)=\cos\omega_0 t$(sinusoidal)
&$x(t)=e^{j\omega_0 t}$(complex exponential)
These two signals are periodic with period $T=2\pi/\omega_0$

. A set of harmonically related complex exponentials can be represented as $\{\phi_k(t)\}$

$$\phi_k(t) = \{e^{jk\omega_0 t}\} = \{e^{jk(\frac{2\pi}{T})t}\} \text{where}\, k = 0 \pm 1, \pm 2..n \;\; .....(1)$$

All these signals are periodic with period T

According to orthogonal signal space approximation of a function x (t) with n, mutually orthogonal functions is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots(2)$$

$$= \sum_{k=-\infty}^{\infty} a_k k e^{jk\omega_0 t}$$

Where a_k= Fourier coefficient = coefficient of approximation.

This signal x(t) is also periodic with period T.

Equation 2 represents Fourier series representation of periodic signal x(t).

The term k = 0 is constant.

The term $k=\pm1$ having fundamental frequency ω_0 , is called as 1st harmonics.

The term $k=\pm2$ having fundamental frequency $2\omega_0$, is called as 2nd harmonics, and so on... The term $k=\pm n$ having fundamental frequency $n\omega_0$, is called as nth harmonics.

Deriving Fourier Coefficient

We know that

$$x(t) = \Sigma_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots(1)$$

Multiply $e^{-jn\omega_0 t}$ onbothsides.Then

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}$$

Consider integral on both sides.

$$\int_0^T x(t)e^{jk\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}\,dt$$

$$= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} .\,dt$$

$$\int_0^T x(t)e^{jk\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt. \;\ldots\ldots(2)$$

by Euler's formula,

$$\int_0^T e^{j(k-n)\omega_0 t} dt. = \int_0^T \cos(k-n)\omega_0 dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt. = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

Hence in equation 2, the integral is zero for all values of k except at k = n. Put k = n in equation 2.

$$\Rightarrow \int_0^T x(t)e^{-jn\omega_0 t}\,dt = a_n T$$

$$\Rightarrow a_n = \frac{1}{T}\int_0^T e^{-jn\omega_0 t}\,dt$$

Replace n by k

$$\Rightarrow a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\,dt$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$$\text{where} a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\,dt$$

<u>Properties of Fourier series:</u>

Linearity Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$ & $y(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{yn}$

then linearity property states that

$\mathrm{a}\, x(t) + \mathrm{b}\, y(t) \xleftrightarrow{fourier\ series\ \ coefficient} \mathrm{a}\, f_{xn} + \mathrm{b}\, f_{yn}$

Time Shifting Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then time shifting property states that

$x(t - t_0) \xleftrightarrow{fourier\ series\ \ coefficient} e^{-jn\omega_0 t_0} f_{xn}$

Frequency Shifting Property

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then frequency shifting property states that

$$e^{jn\omega_0 t_0} \,.\, x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{x(n-n_0)}$$

Time Reversal Property

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then time reversal property states that

If $x(-t) \xleftrightarrow{fourier\ series\quad coefficient} f_{-xn}$

Time Scaling Property

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then time scaling property states that

If $x(at) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

Time scaling property changes frequency components from ω_0 to $a\omega_0$.

Differentiation and Integration Properties

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then differentiation property states that

If $\frac{dx(t)}{dt} \xleftrightarrow{fourier\ series\ \ coefficient} jn\omega_0 . f_{xn}$

& integration property states that

If $\int x(t)dt \xleftrightarrow{fourier\ series\ \ coefficient} \frac{f_{xn}}{jn\omega_0}$

Multiplication and Convolution Properties

f $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$ & $y(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{yn}$

:hen multiplication property states that

$x(t). y(t) \xleftrightarrow{fourier\ series\ \ coefficient} Tf_{xn} * f_{yn}$

& convolution property states that

$x(t) * y(t) \xleftrightarrow{fourier\ series\ \ coefficient} Tf_{xn}. f_{yn}$

Conjugate and Conjugate Symmetry Properties

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

Then conjugate property states that

$x * (t) \xleftrightarrow{fourier\ series\ \ coefficient} f *_{xn}$

Conjugate symmetry property for real valued time signal states that

$$f *_{xn} = f_{-xn}$$

& Conjugate symmetry property for imaginary valued time signal states that

$$f *_{xn} = -f_{-xn}$$

Trigonometric Fourier Series (TFS)

sin$n\omega_0 t$ and sin$m\omega_0 t$ are orthogonal over the interval (t_0,t_0+$2\pi\omega_0$). So sin$\omega_0 t$,sin$2\omega_0 t$forms an orthogonal set. This set is not complete without {cos$n\omega_0 t$ } because this cosine set is also orthogonal to sine set. So to complete this set we must include both cosine and sine terms. Now the complete orthogonal set contains all cosine and sine terms i.e. {sin$n\omega_0 t$,cos$n\omega_0 t$} where n=0, 1,2...

$\therefore$ Any function xt in the interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ can be represented as

$$\begin{aligned} x(t) &= a_0 \cos 0\omega_0 t + a_1 \cos 1\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots \\ &\quad + b_0 \sin 0\omega_0 t + b_1 \sin 1\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots \\ &= a_0 + a_1 \cos 1\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots \\ &\quad + b_1 \sin 1\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots \end{aligned}$$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad (t_0 < t < t_0 + T)$$

The above equation represents trigonometric Fourier series representation of x(t).

$$\text{Where } a_0 = \frac{\int_{t_0}^{t_0+T} x(t) \cdot 1 dt}{\int_{t_0}^{t_0+T} 1^2 dt} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} x(t) dt$$

$$a_n = \frac{\int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t \, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t \, dt}$$

$$b_n = \frac{\int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t \, dt}{\int_{t_0}^{t_0+T} \sin^2 n\omega_0 t \, dt}$$

$$\text{Here } \int_{t_0}^{t_0+T} \cos^2 n\omega_0 t \, dt = \int_{t_0}^{t_0+T} \sin^2 n\omega_0 t \, dt = \frac{T}{2}$$

$$\therefore a_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t \, dt$$

Exponential Fourier Series (EFS):

Consider a set of complex exponential functions $\left\{e^{jn\omega_0 t}\right\} (n = 0, \pm1, \pm2...)$ which is orthogonal over the interval (t_0, t_0+T). Where $T=2\pi/\omega_0$. This is a complete set so it is possible to represent any function f(t) as shown below

$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad (t_0 < t < t_0 + T) \ldots\ldots (1)$$

Equation 1 represents exponential Fourier series representation of a signal f(t) over the interval (t_0, t_0+T). The Fourier coefficient is given as

$$F_n = \frac{\int_{t_0}^{t_0+T} f(t)(e^{jn\omega_0 t})^* dt}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t}(e^{jn\omega_0 t})^* dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} e^{-jn\omega_0 t} e^{jn\omega_0 t} dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} 1\, dt} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

$$\therefore F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

Relation Between Trigonometric and Exponential Fourier Series:

Consider a periodic signal x(t), the TFS & EFS representations are given below respectively

$$x(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\ldots\ldots(1)$$

$$x(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

$$= F_0 + F_1(\cos\omega_0 t + j\sin\omega_0 t) + F_2(cos2\omega_0 t + j\sin 2\omega_0 t) + \ldots + F_n(\cos n\omega_0 t + j\sin n\omega_0 t) + .$$

$$. + F_{-1}(\cos\omega_0 t - j\sin\omega_0 t) + F_{-2}(\cos 2\omega_0 t - j\sin 2\omega_0 t) + \ldots + F_{-n}(\cos n\omega_0 t - j\sin n\omega_0 t) + \ldots$$

$$= F_0 + (F_1 + F_{-1})\cos\omega_0 t + (F_2 + F_{-2})\cos 2\omega_0 t + \ldots + j(F_1 - F_{-1})\sin\omega_0 t + j(F_2 - F_{-2})\sin 2\omega_0 t + \ldots$$

$$\therefore x(t) = F_0 + \Sigma_{n=1}^{\infty}((F_n + F_{-n})\cos n\omega_0 t + j(F_n - F_{-n})\sin n\omega_0 t)\ldots\ldots(2)$$

Compare equation 1 and 2.

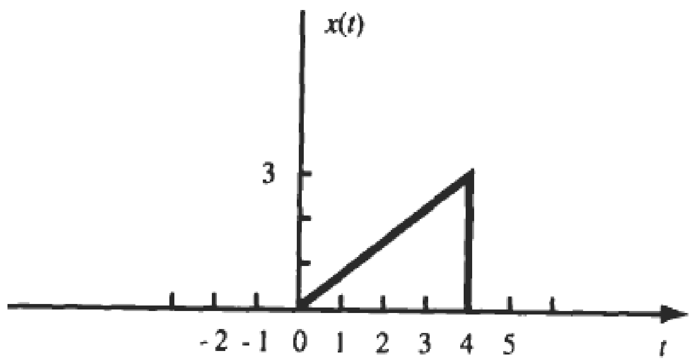
$$a_0 = F_0$$
$$a_n = F_n + F_{-n}$$
$$b_n = j(F_n - F_{-n})$$

Similarly,

$$F_n = \frac{1}{2}(a_n - jb_n)$$
$$F_{-n} = \frac{1}{2}(a_n + jb_n)$$

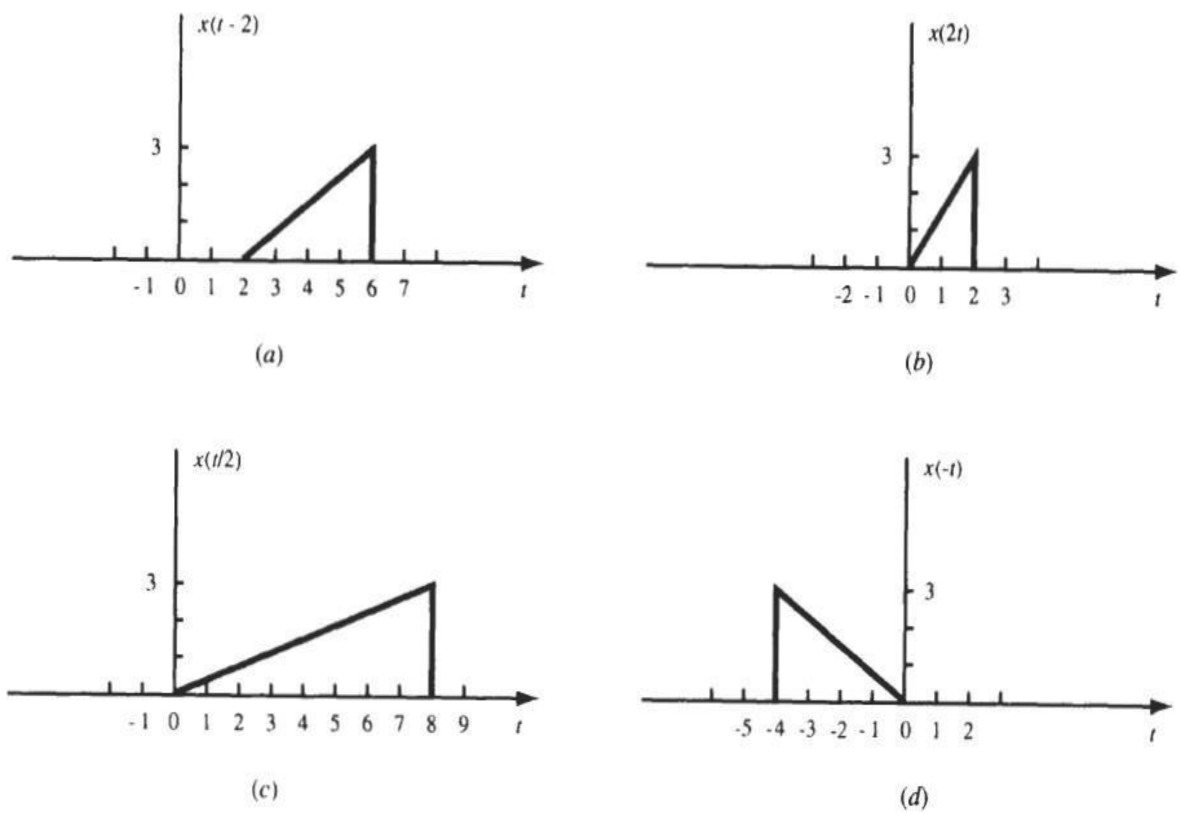
Problems

1. **A continuous-time signal *x(t)* is shown in the following figure. Sketch and label each of the following signals.**



(a) x(t-2); (b)x(2t); (c)x(t/2); (d) x (- t)

Sol:



2. **Determine whether the following signals are energy signals, power signals, or neither.**

$(a)\quad x(t)=e^{-at}u(t),\quad a>0 \qquad (b)\quad x(t)=A\cos(\omega_0 t+\theta)$

$(c)\quad x(t)=tu(t) \qquad (d)\quad x[n]=(-0.5)^n u[n]$

$(e)\quad x[n]=u[n] \qquad (f)\quad x[n]=2e^{j3n}$

(a) $E=\int_{-\infty}^{\infty}|x(t)|^2 dt=\int_0^{\infty}e^{-2at}\,dt=\frac{1}{2a}<\infty$

Thus, $x(t)$ is an energy signal.

(b) The sinusoidal signal $x(t)$ is periodic with $T_0=2\pi/\omega_0$. Then by the result from Prob. 1.18, the average power of $x(t)$ is

$$P=\frac{1}{T_0}\int_0^{T_0}[x(t)]^2\,dt=\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}A^2\cos^2(\omega_0 t+\theta)\,dt$$

$$=\frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{1}{2}[1+\cos(2\omega_0 t+2\theta)]\,dt=\frac{A^2}{2}<\infty$$

Thus, $x(t)$ is a power signal. Note that periodic signals are, in general, power signals.

(c) $E=\lim_{T\to\infty}\int_{-T/2}^{T/2}|x(t)|^2 dt=\lim_{T\to\infty}\int_0^{T/2}t^2\,dt=\lim_{T\to\infty}\frac{(T/2)^3}{3}=\infty$

$P=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt=\lim_{T\to\infty}\frac{1}{T}\int_0^{T/2}t^2\,dt=\lim_{T\to\infty}\frac{1}{T}\frac{(T/2)^3}{3}=\infty$

Thus, $x(t)$ is neither an energy signal nor a power signal.

(d) we know that energy of a signalis

$$E=\sum_{n=-\infty}^{\infty}|x[n]|^2$$

And by using

$$\sum_{n=0}^{\infty}\alpha^n=\frac{1}{1-\alpha}\qquad |\alpha|<1$$

we obtain

$$E=\sum_{n=-\infty}^{\infty}|x[n]|^2=\sum_{n=0}^{\infty}0.25^n=\frac{1}{1-0.25}=\frac{4}{3}<\infty$$

Thus, ***x [n]*** is a power signal.

(e) By the definition of power of signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1^2 = \lim_{N\to\infty} \frac{1}{2N+1}(N+1) = \frac{1}{2} < \infty$$

Thus, $x[n]$ is a power signal.

(f) Since $|x[n]| = |2e^{j3n}| = 2|e^{j3n}| = 2$,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 2^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} 4(2N+1) = 4 < \infty$$

Thus, $x[n]$ is a power signal.

3. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$ (b) $x(t) = \sin \frac{2\pi}{3} t$

(c) $x(t) = \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t$ (d) $x(t) = \cos t + \sin \sqrt{2}\, t$

(e) $x(t) = \sin^2 t$ (f) $x(t) = e^{j[(\pi/2)t - 1]}$

(g) $x[n] = e^{j(\pi/4)n}$ (h) $x[n] = \cos \frac{1}{4} n$

(i) $x[n] = \cos \frac{\pi}{3} n + \sin \frac{\pi}{4} n$ (j) $x[n] = \cos^2 \frac{\pi}{8} n$

Sol:

(*a*) $x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \longrightarrow \omega_0 = 1$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

(*b*) $x(t) = \sin\frac{2\pi}{3}t \longrightarrow \omega_0 = \frac{2\pi}{3}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

(*c*) $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t = x_1(t) + x_2(t)$

where $x_1(t) = \cos(\pi/3)t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

(*d*) $x(t) = \cos t + \sin\sqrt{2}\,t = x_1(t) + x_2(t)$

where $x_1(t) = \cos t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}\,t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\,\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(*e*) Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we can write

$$x(t) = \sin^2 t = \tfrac{1}{2} - \tfrac{1}{2}\cos 2t = x_1(t) + x_2(t)$$

where $x_1(t) = \frac{1}{2}$ is a dc signal with an arbitrary period and $x_2(t) = -\frac{1}{2}\cos 2t = -\frac{1}{2}\cos\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \pi$. Thus, $x(t)$ is periodic with fundamental period $T_0 = \pi$.

(*f*) $x(t) = e^{j[(\pi/2)t - 1]} = e^{-j}e^{j(\pi/2)t} = e^{-j}e^{j\omega_0 t} \longrightarrow \omega_0 = \dfrac{\pi}{2}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 4$.

(*g*) $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \longrightarrow \Omega_0 = \dfrac{\pi}{4}$

Since $\Omega_0/2\pi = \frac{1}{8}$ is a rational number, $x[n]$ is periodic, and by Eq. (*1.55*) the fundamental period is $N_0 = 8$.

(*h*) $x[n] = \cos \frac{1}{4}n = \cos \Omega_0 n \longrightarrow \Omega_0 = \frac{1}{4}$

Since $\Omega_0/2\pi = 1/8\pi$ is not a rational number, $x[n]$ is nonperiodic.

(*i*) $x[n] = \cos \dfrac{\pi}{3}n + \sin \dfrac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos \frac{\pi}{3}n = \cos \Omega_1 n \longrightarrow \Omega_1 = \frac{\pi}{3}$$

$$x_2[n] = \sin \frac{\pi}{4}n = \cos \Omega_2 n \longrightarrow \Omega_2 = \frac{\pi}{4}$$

Since $\Omega_1/2\pi = \frac{1}{6}$ (= rational number), $x_1[n]$ is periodic with fundamental period $N_1 = 6$, and since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, from the result of Prob. 1.15, $x[n]$ is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is, $N_0 = 24$.

(*j*) Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we can write

$$x[n] = \cos^2 \frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos \frac{\pi}{4}n = x_1[n] + x_2[n]$$

where $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$ and $x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos \Omega_2 n \longrightarrow \Omega_2 = \pi/4$. Since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, $x[n]$ is periodic with fundamental period $N_0 = 8$ (the least common multiple of N_1 and N_2).

1. Determine the even and odd components of the following signals:

$(a)\quad x(t) = u(t)$

$(b)\quad x(t) = \sin\left(\omega_0 t + \dfrac{\pi}{4}\right)$

$(c)\quad x[n] = e^{j(\Omega_0 n + \pi/2)}$

$(d)\quad x[n] = \delta[n]$

Ans. $(a)\quad x_e(t) = \frac{1}{2},\ x_o(t) = \frac{1}{2}\operatorname{sgn} t$

$(b)\quad x_e(t) = \dfrac{1}{\sqrt{2}}\cos\omega_0 t,\ x_o(t) = \dfrac{1}{\sqrt{2}}\sin\omega_0 t$

$(c)\quad x_e[n] = j\cos\Omega_0 n,\ x_o[n] = -\sin\Omega_0 n$

$(d)\quad x_e[n] = \delta[n],\ x_o[n] = 0$

FOURIER TRANSFORM

INTRODUCTION:

The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as non periodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called 'Fourier transform'.

Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

Deriving Fourier transform from Fourier series:

Consider a periodic signal f(t) with period T. The complex Fourier series representation of f(t) is given as

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \ldots\ldots(1)$$

Let $\frac{1}{T_0} = \Delta f$, then equation 1 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\Delta f t} \ldots\ldots(2)$$

but you know that

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t} dt$$

Substitute in equation 2.

$$2 \Rightarrow f(t) = \Sigma_{k=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t}\, dt\, e^{j2\pi k\Delta f t}$$

Let $t_0 = \frac{T}{2}$

$$= \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-j2\pi k\Delta f t}\, dt]\, e^{j2\pi k\Delta f t} . \Delta f$$

In the limit as T→∞,Δf approaches differential df, kΔf becomes a continuous variable f, and summation becomes integration

$$f(t) = lim_{T\to\infty} \left\{ \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-j2\pi k\Delta f t}\, dt]\, e^{j2\pi k\Delta f t} . \Delta f \right\}$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t) e^{-j2\pi f t}\, dt] e^{j2\pi f t}\, df$$

$$f(t) = \int_{-\infty}^{\infty} F[\omega] e^{j\omega t}\, d\omega$$

$$\text{Where } F[\omega] = [\int_{-\infty}^{\infty} f(t) e^{-j2\pi f t}\, dt]$$

Fourier transform of a signal

$$f(t) = F[\omega] = [\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt]$$

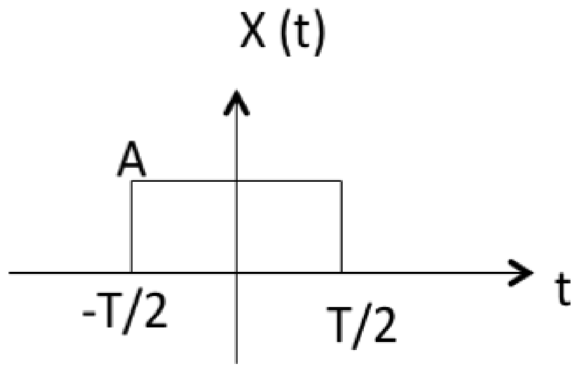
Inverse Fourier Transform is

$$f(t) = \int_{-\infty}^{\infty} F[\omega] e^{j\omega t}\, d\omega$$

Fourier Transform of Basic Functions:

Let us go through Fourier Transform of basic functions:

FT of GATE Function



$$\boxed{F[\omega] = ATSa(\frac{\omega T}{2})}$$

FT of Impulse Function:

$$FT[\omega(t)] = [\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt]$$

$$= e^{-j\omega t} \mid t = 0$$

$$= e^{0} = 1$$

$$\boxed{\therefore \delta(\omega) = 1}$$

FT of Unit Step Function:

$$U(\omega) = \pi\delta(\omega) + 1/j\omega$$

FT of Exponentials:

$$e^{-at}u(t) \stackrel{\text{F.T}}{\longleftrightarrow} 1/(a + j\omega)$$

$$e^{-at}u(t) \stackrel{\text{F.T}}{\longleftrightarrow} 1/(a + j\omega)$$

$$e^{-a|t|} \stackrel{\text{F.T}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$e^{j\omega_0 t} \stackrel{\text{F.T}}{\longleftrightarrow} \delta(\omega - \omega_0)$$

FT of Signum Function :

$$sgn(t) \stackrel{\text{F.T}}{\longleftrightarrow} \frac{2}{j\omega}$$

Conditions for Existence of Fourier Transform:

Any function f(t) can be represented by using Fourier transform only when the function satisfies Dirichlet's conditions. i.e.

- The function f(t) has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f(t),in the given interval of time.
- It must be absolutely integrable in the given interval of time i.e.

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Properties of Fourier Transform:

Here are the properties of Fourier Transform:

Linearity Property:

$$\text{If } x(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(\omega)$$

$$\& \ y(t) \stackrel{\text{F.T}}{\longleftrightarrow} Y(\omega)$$

Then linearity property states that

$$ax(t) + by(t) \stackrel{\text{F.T}}{\longleftrightarrow} aX(\omega) + bY(\omega)$$

Time Shifting Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time shifting property states that

$$x(t - t_0) \overset{\text{F.T}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$$

Frequency Shifting Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then frequency shifting property states that

$$e^{j\omega_0 t} . x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega - \omega_0)$$

Time Reversal Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time reversal property states that

$$x(-t) \overset{\text{F.T}}{\longleftrightarrow} X(-\omega)$$

Time Scaling Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time scaling property states that

$$x(at) \frac{1}{|\,a\,|} X \frac{\omega}{a}$$

Differentiation and Integration Properties:

$$\text{If } x(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Differentiation property states that

$$\frac{dx(t)}{dt} \stackrel{\text{F.T}}{\longleftrightarrow} j\omega. X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \stackrel{\text{F.T}}{\longleftrightarrow} (j\omega)^n. X(\omega)$$

and integration property states that

$$\int x(t)\, dt \stackrel{\text{F.T}}{\longleftrightarrow} \frac{1}{j\omega} X(\omega)$$

$$\iiint \ldots \int x(t)\, dt \stackrel{\text{F.T}}{\longleftrightarrow} \frac{1}{(j\omega)^n} X(\omega)$$

<u>Multiplication and Convolution Properties:</u>

$$\text{If } x(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(\omega)$$

$$\&\ y(t) \stackrel{\text{F.T}}{\longleftrightarrow} Y(\omega)$$

Then multiplication property states that

$$x(t). y(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(\omega) * Y(\omega)$$

and convolution property states that

$$x(t) * y(t) \stackrel{\text{F.T}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega). Y(\omega)$$

Problems

1. Find the Fourier transform of the rectangular pulse signal *x(t)* definedby

Sol: By definition of Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} p_a(t)\, e^{-j\omega t}\, dt = \int_{-a}^{a} e^{-j\omega t}\, dt$$

$$= \frac{1}{j\omega}\left(e^{j\omega a} - e^{-j\omega a}\right) = 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$

Hence we obtain

$$p_a(t) \longleftrightarrow 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$

The following figure shows the Fourier transform of the given signal x(t)

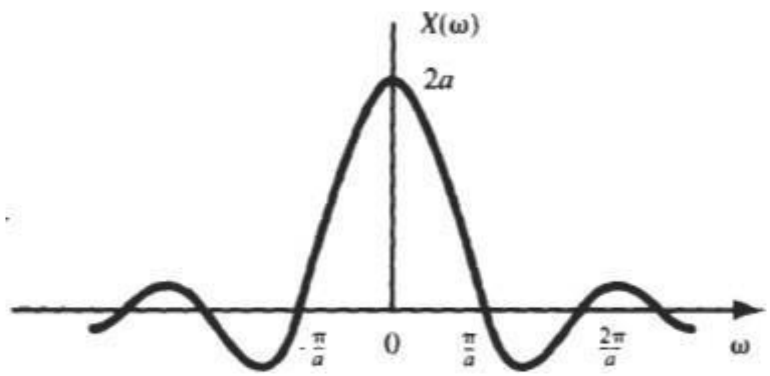


Figure: Fourier transform of the given signal

2. Find the Fourier transform of the following signal x(t)

$$x(t) = e^{-a|t|} \qquad a > 0$$

Sol: Signal x(t) can be rewritten as

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$$

Then $$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

Hence, we get

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2 + \omega^2}$$

The Fourier transform *X(w)* of *x(t)* is shown in the following figures

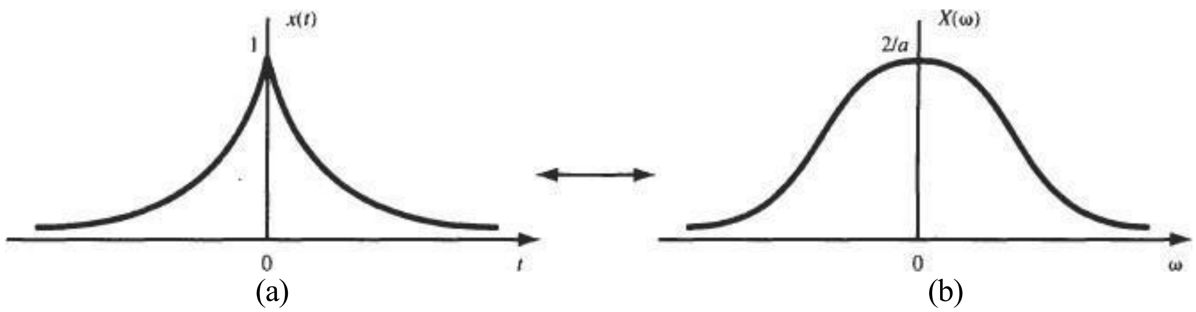


Fig: (a) Signal x(t) (b) Fourier transform *X(w)* of *x(t)*

4. Find the Fourier transform of the periodic impulse train

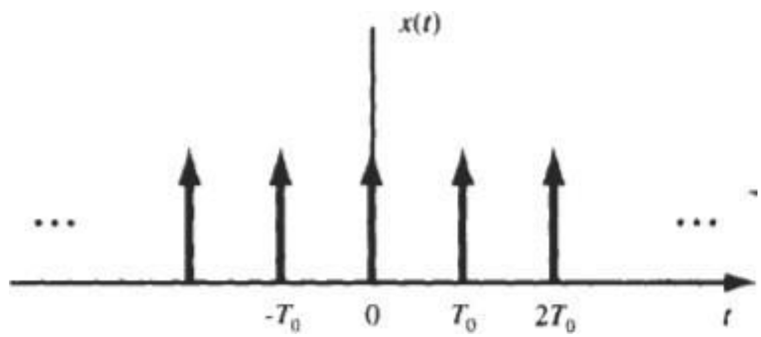


Fig: Train of impulses

Sol: Given signal can be written as

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

the complex exponential Fourier series of $\delta_{T_0}(t)$ is given by

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$\mathscr{F}\left[\delta_{T_0}(t)\right] = \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$= \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \omega_0 \delta_{\omega_0}(\omega)$$

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_0) \longleftrightarrow \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Thus, the Fourier transform of a unit impulse train is also a similar impulse train. The following figure shows the Fourier transform of a unit impulse train

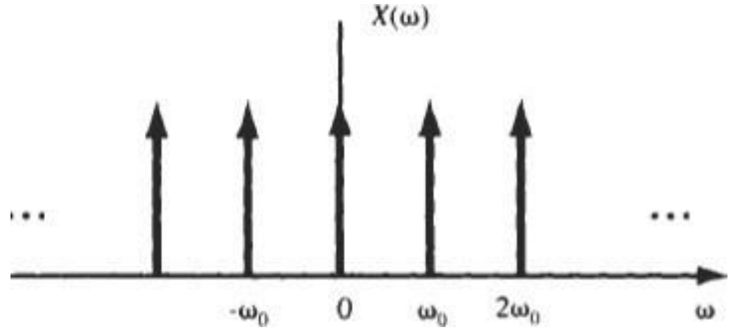


Figure: Fourier transform of the given signal

5.Find the Fourier transform of the *signum*function

Sol: Signum function is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

The signum function, sgn(t), can be expressed as

Sgn(t)= 2u(t)-1

We know that

$$\frac{d}{dt}\operatorname{sgn}(t) = 2\delta(t)$$

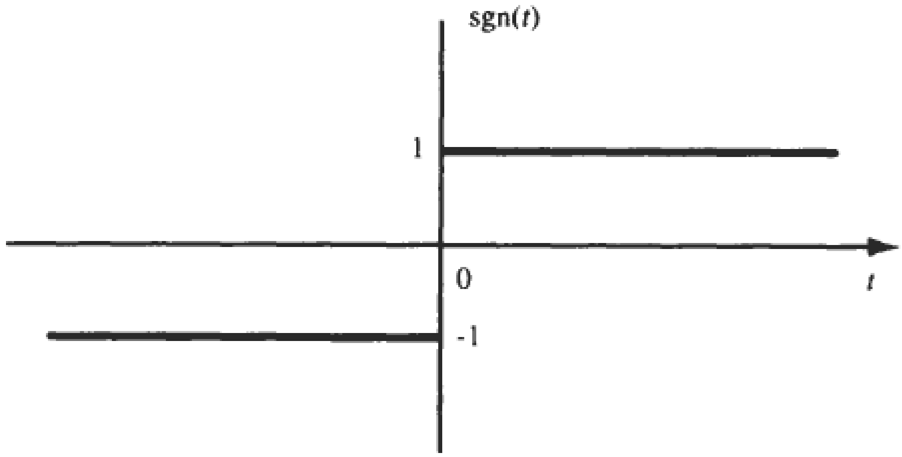


Fig: Signum function

Let

$$\operatorname{sgn}(t) \longleftrightarrow X(\omega)$$

Then applying the differentiation property , we have

$$j\omega X(\omega) = \mathscr{F}[2\delta(t)] = 2 \longrightarrow X(\omega) = \frac{2}{j\omega}$$

$$\operatorname{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

Note that sgn(t) is an odd function, and therefore its Fourier transform is a pure imaginary function of w

Properties of Fourier Transform:

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$
	$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$j\mathcal{I}m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transform of Basic Functions:

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

DISCRETE TIME FOURIER TRANSFORM

Discrete Time Fourier Transforms (DTFT)

Here we take the exponential signals to be $\{e^{j\omega n}\}$ where ω is a real number.The representation is motivated by the Harmonic analysis, but instead of following the historical development of the representation we give directly the defining equation.Let $\{x[n]\}$ be discrete time signal such that $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ that is sequence $\{x[n]\}$ is absolutely summable.
The sequence $\{x[n]\}$ can be represented by a Fourier integral of the form.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \quad \text{.......................(1)}$$

Where

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{........................(2)}$$

Equation (1) and (2) give the Fourier representation of the signal. Equation (1) is referred as synthesis equation or the inverse discrete time Fourier transform (IDTFT) and equation (2)is Fourier transform in the analysis equation. Fourier transform of a signal in general is a complex valued function, we can write

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

where $|X(e^{j\omega})|$ is magnitude and $\angle X(e^{j\omega})$ is the phase of. We also use the term Fourier spectrum or simply, the spectrum to refer to. Thus $|X(e^{j\omega})|$ is called the magnitude spectrum and $\angle X(e^{j\omega})$ is called the phase spectrum. From equation (2) we can see that $X(e^{j\omega})$ is a periodic function with period 2π i.e.. We can interpret (1) as Fourier coefficients in the representation of a periodic function. In the Fourier series analysis our attention is on the periodic function, here we are concerned with the representation of the signal. So the roles of the two equation are interchanged compared to the Fourier series analysis of periodic signals.

Now we show that if we put equation (2) in equation (1) we indeed get the signal.
Let

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{+j\omega n}\,d\omega$$

where we have substituted $X(e^{j\omega})$ from (2) into equation (1) and called the result as.
Since we have used n as index on the left hand side we have used m as the index variable forthe

sum defining the Fourier transform. Under our assumption that $\{x[n]\}$ sequence is absolutely summable we can interchange the order of integration and summation.Thus

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{+j\omega(n-m)} d\omega \right)$$

Example: Let

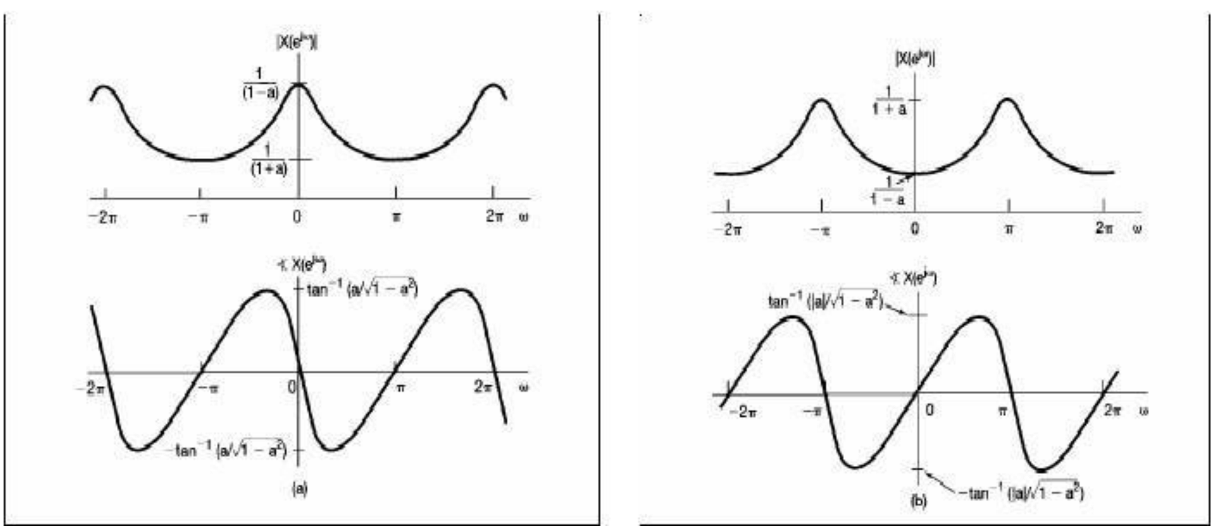
$$\{x[n]\} = \{a^n u[n]\}$$

Fourier transform of this sequence will exist if it is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$

The magnitude and phase for this example are show in the figure below, where $a > 0$ and $a < 0$ are shown in (a) and (b).

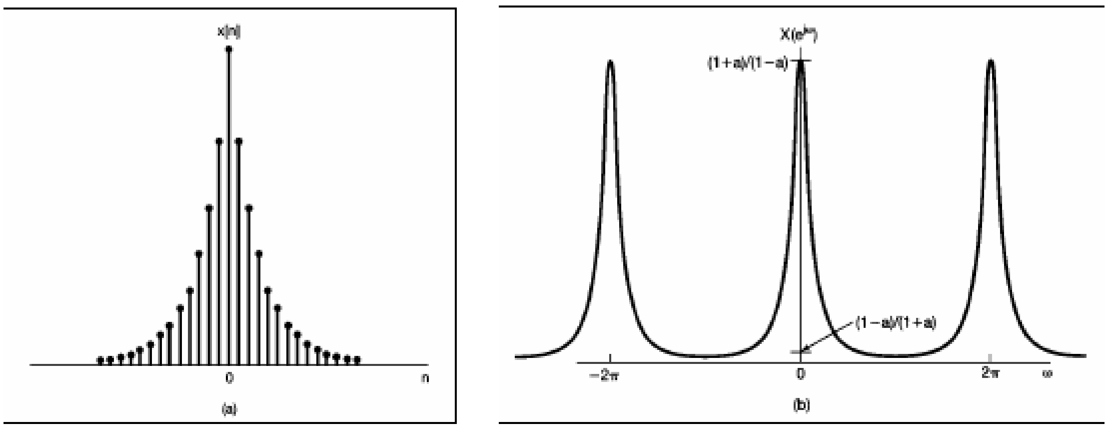


$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

Let $m = -n$ in the first summation, we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{m=1}^{\infty} a^{m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

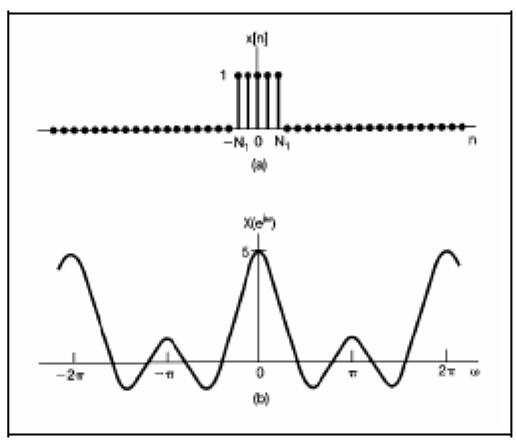
$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$



Example: Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & |n| > 2 \end{cases}, \qquad (5.14)$$

$$X(j\omega) = \sum_{n=-2}^{2} e^{-j\omega n} = \frac{\sin\omega(N_1 + 1/2)}{\sin(\omega/2)}. \qquad (5.15)$$



This function is the discrete counterpart of the sic function, which appears in the Fourier transform of the continuous-time pulse.

The difference between these two functions is that the discrete one is periodic (see figure) with period of 2π, whereas the sinc function is aperiodic.

Fourier transform of Periodic Signals

For a periodic discrete-time signal,

$$x[n] = e^{j\omega_0 n},$$

its Fourier transform of this signal is periodic in *w* with period 2∏ , and is given

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l).$$

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}.$$

The Fourier transform is

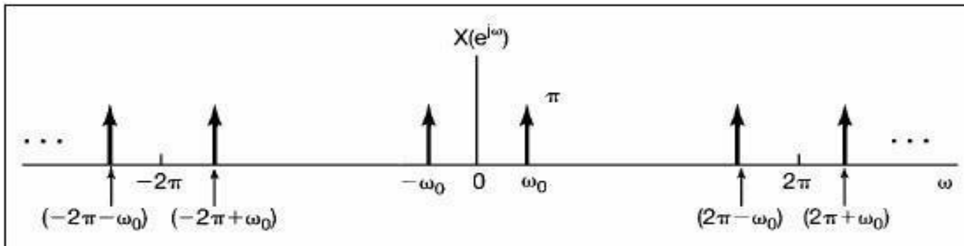
$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N}).$$

Example: The Fourier transform of the periodic signal

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \text{ with } \omega_0 = \frac{2\pi}{3},$$

is given as

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{3}\right) + \pi\delta\left(\omega + \frac{2\pi}{3}\right), \qquad -\pi \le \omega < \pi.$$



Discrete-time Fourier transform of $x[n] = \cos\omega_0 n$.

Example: The periodic impulse train

$$x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN].$$

The Fourier series coefficients for this signal can be calculated

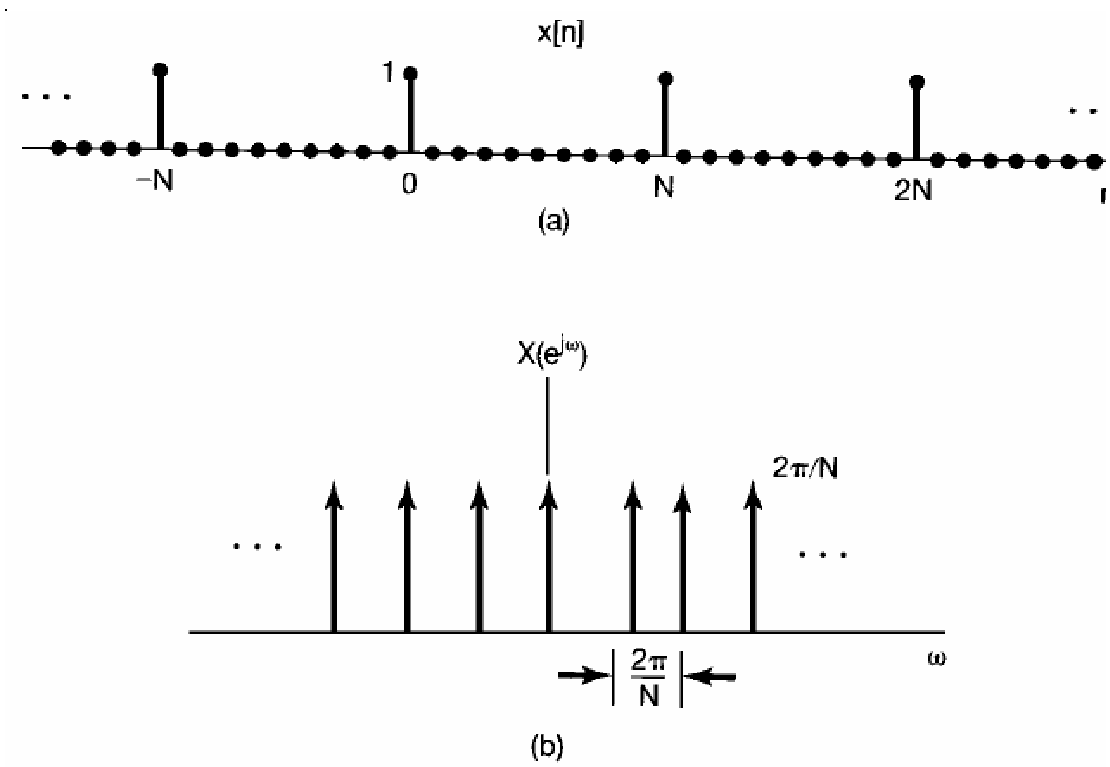
$$a_k=\sum_{n=\langle N\rangle}x[n]e^{-jk(2\pi/N)n}.$$

Choosing the interval of summation as $0\le n\le N-1$, we have

$$a_k=\frac{1}{N}.$$

The Fourier transform is

$$X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right).$$



(a) Discrete-time periodic impulse train; (b) its Fourier transform.

Properties of the Discrete Time Fourier Transform:

Let $\{x[n]\}$ and $\{y[n]\}$be two signal, then their DTFT is denoted by $X(e^{j\omega})$and. The notation

$$\{x[n]\} \leftrightarrow X(e^{j\omega})$$

is used to say that left hand side is the signal x[n] whose DTFT is $X(e^{j\omega})$is given at right hand side.

1. Periodicity of the DTFT:

The discrete-time Fourier transform is always periodic in ω with period 2π, i.e.,

$X\left(e^{j(\omega+2\pi)}\right)=X\left(e^{j\omega}\right).$

2. Linearity of the DTFT:

If $x_1[n] \xleftrightarrow{F} X_1(e^{j\omega})$, and $x_2[n] \xleftrightarrow{F} X_2(e^{j\omega})$,

then

$$\boxed{ax_1[n]+bx_2[n] \xleftrightarrow{F} aX_1(e^{j\omega})+bX_2(e^{j\omega})}$$

3. Time Shifting and Frequency Shifting:

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})}$$

and

$$\boxed{e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})}$$

4. Conjugation and Conjugate Symmetry:

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})}$$

If $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega})}$$

From this, it follows that Re□ $X(e^{jw})$□ is an even function of w and Im□ $X(e^{jw})$□ is an odd function of w . Similarly, the **magnitude** of $X(e^{jw})$ is an even function and the phase angle is

an odd function. Furthermore,

$$Ev\{x[n]\} \xleftrightarrow{F} \mathrm{Re}\{X(e^{j\omega})\},$$

and

$$Od\{x[n]\} \xleftrightarrow{F} j\,\mathrm{Im}\{X(e^{j\omega})\}.$$

5. Differencing and Accumulation

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[n] - x[n-1] \xleftrightarrow{F} \left(1 - e^{-j\omega}\right) X(e^{j\omega})}.$$

For signal

$$y[n] = \sum_{m=-\infty}^{n} x[m],$$

its Fourier transform is given as

$$\boxed{\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - 2\pi k)}.$$

The impulse train on the right-hand side reflects the dc or average value that can result from summation.

For example, the Fourier transform of the unit step $x[n]$ □$u[n]$ can be obtained by using the accumulation property.

We know $g[n] = \delta[n] \xleftrightarrow{F} G(e^{j\omega}) = 1$, so

$$x[n] = \sum_{m=-\infty}^{n} g[m] \xleftrightarrow{F} \frac{1}{(1-e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) = \frac{1}{(1-e^{-j\omega})} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k).$$

6. Time Reversal

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[-n] \xleftrightarrow{F} X(-e^{j\omega})}.$$

7. Time Expansion

For continuous-time signal, we have

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

For discrete-time signals, however, a should be an integer. Let us define a signal with k a positive integer,

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}.$$

$x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal.

The Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk} = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$\boxed{x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})}.$$

For k □ 1, the signal is spread out and slowed down in time, while its Fourier transform is compressed.

Example: Consider the sequence $x[n]$ displayed in the figure (a) below. This sequence can be related to the simpler sequence $y[n]$ as shown in (b).

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

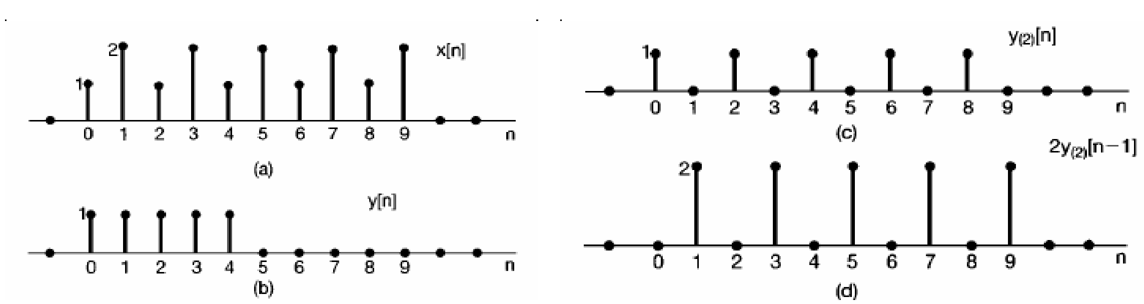
where

$$y_2[n] = \begin{cases} y[n/2], & if\ \ n\ is\ even \\ 0, & if\ \ n\ is\ odd \end{cases}$$

The signals $y_{(2)}[n]$ and $2y_{(2)}[n-1]$ are depicted in (c) and (d).

As can be seen from the figure below, $y[n]$ is a rectangular pulse with 2 $_1 N$ □ , its Fourier transform is given by

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$



Using the time-expansion property, we then obtain

$$y_{(2)}[n] \xleftrightarrow{F} e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \xleftrightarrow{F} 2e^{-j5\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

Combining the two, we have

$$X(e^{j\omega}) = e^{-j4\omega}(1+2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$

8. Differentiation in Frequency

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

Differentiate both sides of the analysis equation $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of the above equation is the Fourier transform of $jnx[n]$. Therefore, multiplying both sides by j, we see that

$$\boxed{nx[n] \xleftrightarrow{F} j\frac{dX(e^{j\omega})}{d\omega}}.$$

9. Parseval's Relation

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$, then we have

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega}$$

Properties of the Discrete Time Fourier Transform:

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Basic Discrete Time Fourier Transform Pa

Signal	Fourier Transform
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π
$\delta[n]$	1
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$