UNIT– IV

LAPLACE TRANSFORM

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THE LAPLACE TRANSFORM:

we know that for a continuous-time LTI system with impulse response h(t), the output **y(t)** of the system to the complex exponential input of the form e^{st} is

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

A. Definition:

The function H(s) is referred to as the Laplace transform of h(t). For a general continuous-time signal x(t), the Laplace transform X(s) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

The variable s is generally complex-valued and is expressed as

$$s = \sigma + j\omega$$

Relation between Laplace and Fourier transforms:

Laplace transform of x(t)

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Substitute s= σ + jω in above equation.

$$\rightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$

$$\therefore X(S) = F.T[x(t)e^{-\sigma t}]$$

$$X(S) = X(\omega) \qquad for\ s = j\omega$$

Inverse Laplace Transform:

We know that

$$X(S) = F.T[x(t)e^{-\sigma t}]$$

$$\rightarrow x(t)e^{-\sigma t} = F.T^{-1}[X(S)] = F.T^{-1}[X(\sigma + j\omega)]$$

$$= \frac{1}{2}\pi \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$x(t) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\,d\omega$$

Here, $\sigma + j\omega = s$

$$jd\omega = ds \rightarrow d\omega = ds/j$$

$$\therefore x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}\,ds \ldots\ldots$$

Conditions for Existence of Laplace Transform:

Dirichlet's conditions are used to define the existence of Laplace transform. i.e.

- The function f has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f ,in the given interval of time.
- It must be absolutly integrable in the given interval of time. i.e.

$$\int_{-\infty}^{\infty} |\,f(t)|\,dt < \infty$$

Initial and Final Value Theorems

If the Laplace transform of an unknown function x(t) is known, then it is possible to determine the initial and the final values of that unknown signal i.e. x(t) at $t=0^+$ and $t=\infty$.

Initial Value Theorem

Statement: If x(t) and its 1st derivative is Laplace transformable, then the initial value of x(t) is given by

$$x(0^+) = \lim_{s \to \infty} SX(S)$$

Final Value Theorem

Statement: If x(t) and its 1st derivative is Laplace transformable, then the final value of x(t) is given by

$$x(\infty) = \lim_{s \to \infty} SX(S)$$

<u>Properties of Laplace transform:</u>

The properties of Laplace transform are:

Linearity Property

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

& $y(t) \xleftrightarrow{\text{L.T}} Y(s)$

Then linearity property states that

$$ax(t) + by(t) \xleftrightarrow{\text{L.T}} aX(s) + bY(s)$$

Time Shifting Property

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then time shifting property states that

$$x(t - t_0) \xleftrightarrow{\text{L.T}} e^{-st_0} X(s)$$

Frequency Shifting Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then frequency shifting property states that

$$e^{s_0 t} . x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s - s_0)$$

Time Reversal Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then time reversal property states that

$$x(-t) \overset{\text{L.T}}{\longleftrightarrow} X(-s)$$

Time Scaling Property

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then time scaling property states that

$$x(at) \overset{\text{L.T}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{s}{a})$$

Differentiation and Integration Properties

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

Then differentiation property states that

$$\frac{dx(t)}{dt} \overset{\text{L.T}}{\longleftrightarrow} s . X(s)$$

$$\frac{d^n x(t)}{dt^n} \overset{\text{L.T}}{\longleftrightarrow} (s)^n . X(s)$$

The integration property states that

$$\int x(t)dt \overset{\text{L.T}}{\longleftrightarrow} \frac{1}{s}X(s)$$

$$\iiint \ldots \int x(t)dt \overset{\text{L.T}}{\longleftrightarrow} \frac{1}{s^n}X(s)$$

Multiplication and Convolution Properties

If $x(t) \overset{\text{L.T}}{\longleftrightarrow} X(s)$

and $y(t) \overset{\text{L.T}}{\longleftrightarrow} Y(s)$

Then multiplication property states that

$$x(t).\, y(t) \overset{\text{L.T}}{\longleftrightarrow} \frac{1}{2\pi j}X(s) * Y(s)$$

The convolution property states that

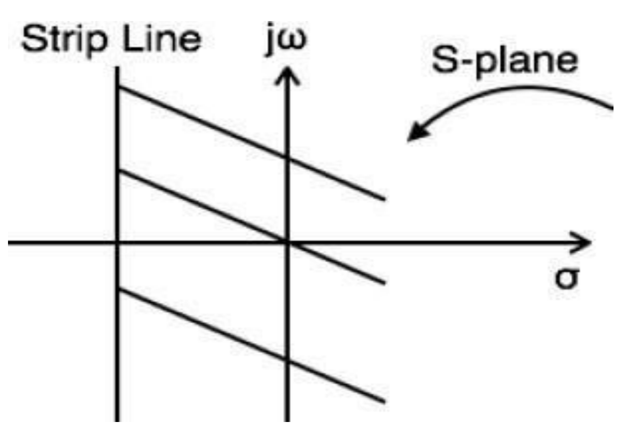
$$x(t) * y(t) \overset{\text{L.T}}{\longleftrightarrow} X(s).\, Y(s)$$

Region of convergence.

The range variation of σ for which the Laplace transform converges is called region of convergence.

Properties of ROC of Laplace Transform

- ROC contains strip lines parallel to jω axis in s-plane.



 If x(t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.

If x(t) is a right sided sequence then ROC : Re{s} >σ_o.

If x(t) is a left sided sequence then ROC : Re{s} <σ_o.

 If x(t) is a two sided sequence then ROC is the combination of two regions.

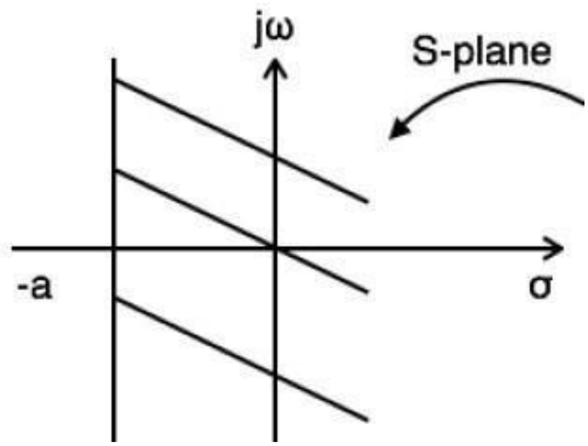
ROC can be explained by making use of examples given below:

Example 1: Find the Laplace transform and ROC of x(t)=e− at u(t)x(t)=e^{-at}u(t)

$$L.T[x(t)] = L.T[e^{-at}\ u(t)] = \frac{1}{S+a}$$

$$Re > -a$$

$$ROC : Res >> -a$$

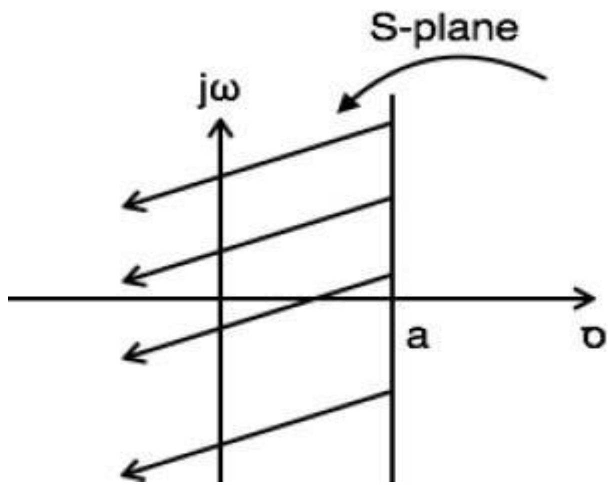


Example 2: Find the Laplace transform and ROC of x(t)=e at u(−t)x(t)=e^{at}u(−t)

$$L.T[x(t)] = L.T[e^{at}u(t)] = \frac{1}{S-a}$$

$$Res < a$$

$$ROC : Res < a$$

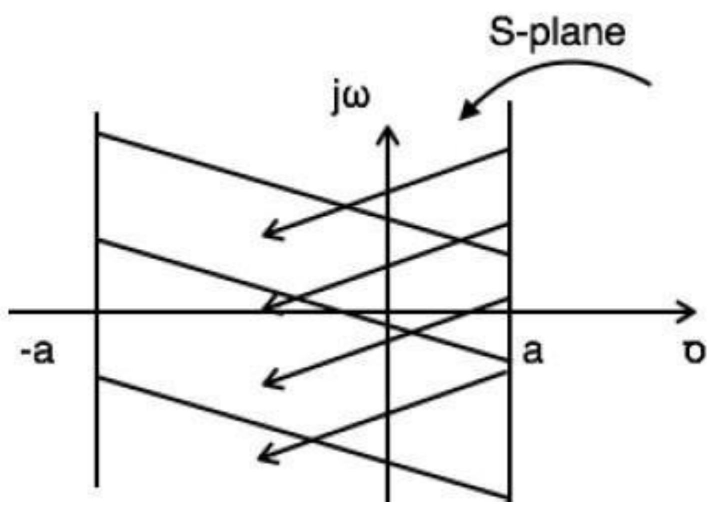


Example 3: Find the Laplace transform and ROC of x(t)=e −at u(t)+e at u(−t) $x(t)=e^{-at}u(t)+e^{at}u(-t)$

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{S+a} + \frac{1}{S-a}$$

For $\frac{1}{S+a} Re\{s\} > -a$

For $\frac{1}{S-a} Re\{s\} < a$

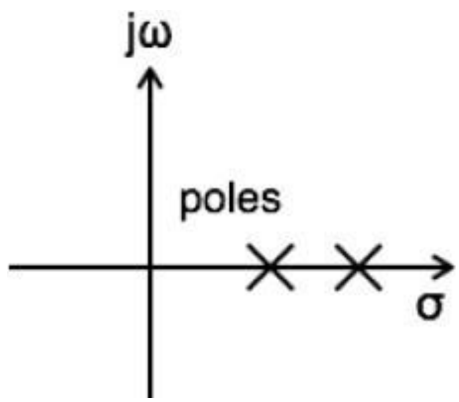


Referring to the above diagram, combination region lies from –a to a. Hence,

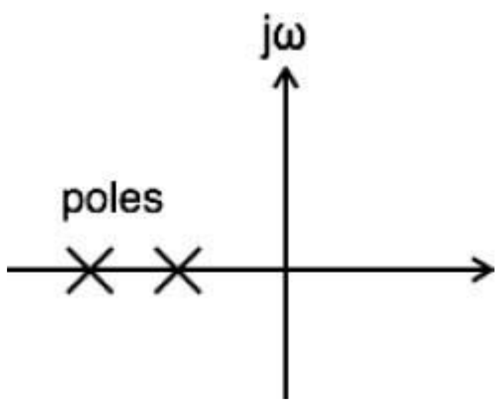
ROC: $-a < Res < a$

Causality and Stability

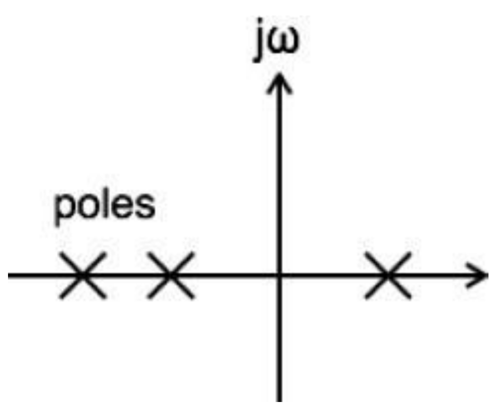
- For a system to be causal, all poles of its transfer function must be right half of s-plane.



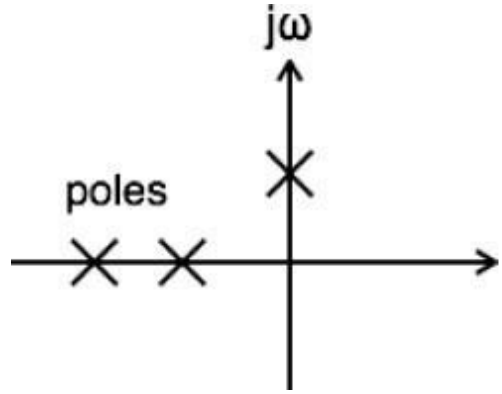
- A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



- A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



- A system is said to be marginally stable when at least one pole of its transferfunction lies on the jωaxis ofs-plane



LAPLACE TRANSFORMS OF SOME COMMON SIGNALS

A. Unit Impulse Function δ(*t*):

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1 \qquad \text{all } s$$

B. Unit Step Function ***u(t)*****):**

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \text{Re}(s) > 0$$

where $0^+ = \lim_{\varepsilon \to 0}(0 + \varepsilon)$.

Some Laplace Transforms Pairs:

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$\cos \omega_0 t u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos \omega_0 t u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at}\sin \omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

Z-Transform

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Z-transform may exist for some signals for which Discrete Time Fourier Transform (DTFT) does not exist.

Concept of Z-Transform and Inverse Z-Transform

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

$$X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n} \ldots \ldots (1)$$

If $Z = re^{j\omega}$ then equation 1 becomes

$$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n) [re^{j\omega}]^{-n}$$

$$= \Sigma_{n=-\infty}^{\infty} x(n) [r^{-n}] e^{-j\omega n}$$

$$X(re^{j\omega}) = X(Z) = F.T[x(n) r^{-n}] \ldots \ldots (2)$$

The above equation represents the relation between Fourier transform and Z-transform

$$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$$

Inverse Z-transform:

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$

$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$

$$x(n) = r^n F.T^{-1}[X(re^{j\omega})]$$

$$= r^n \frac{1}{2\pi} \int X(re^j\omega)e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int X(re^j\omega)[re^{j\omega}]^n d\omega \ldots\ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega} d\omega = jzd\omega$$

$$d\omega = \frac{1}{j} z^{-1} dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \frac{1}{2\pi} \int X(z)z^n \frac{1}{j} z^{-1} dz = \frac{1}{2\pi j} \int X(z)z^{n-1} dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z)z^{n-1} dz$$

Z-Transform Properties:

Z-Transform has following properties:

Linearity Property:

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \overset{\text{Z.T}}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

Time Shifting Property:

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then Time shifting property states that

$$x(n-m) \overset{\text{Z.T}}{\longleftrightarrow} z^{-m} X(Z)$$

Multiplication by Exponential Sequence Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n \,.\, x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z/a)$$

Time Reversal Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{\text{Z.T}}{\longleftrightarrow} X(1/Z)$$

Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

Convolution Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$$

Correlation Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z^{-1})$$

Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1}[z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z-transform

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain anypoles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC isentire z-plane except at z =0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROCis entire z-plane except at z =∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radiusa. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle withradius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except atz = 0 & z = ∞.

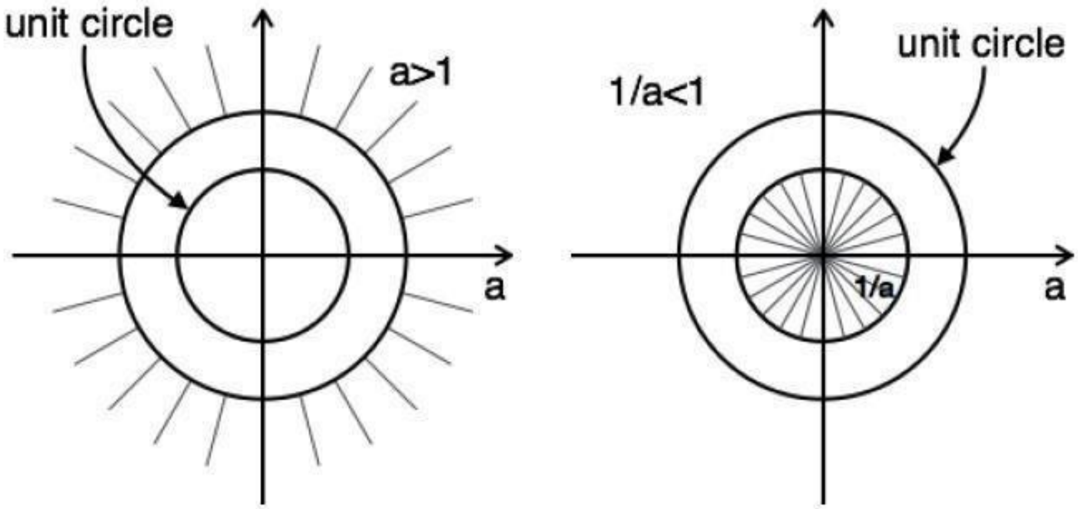
The concept of ROC can be explained by the following example:

Example 1: Find z-transform and ROC of $a^{n}u[n]+a^{-n}u[-n-1]$ $a^{n}u[n]+a^{-n}u[-n-1]$

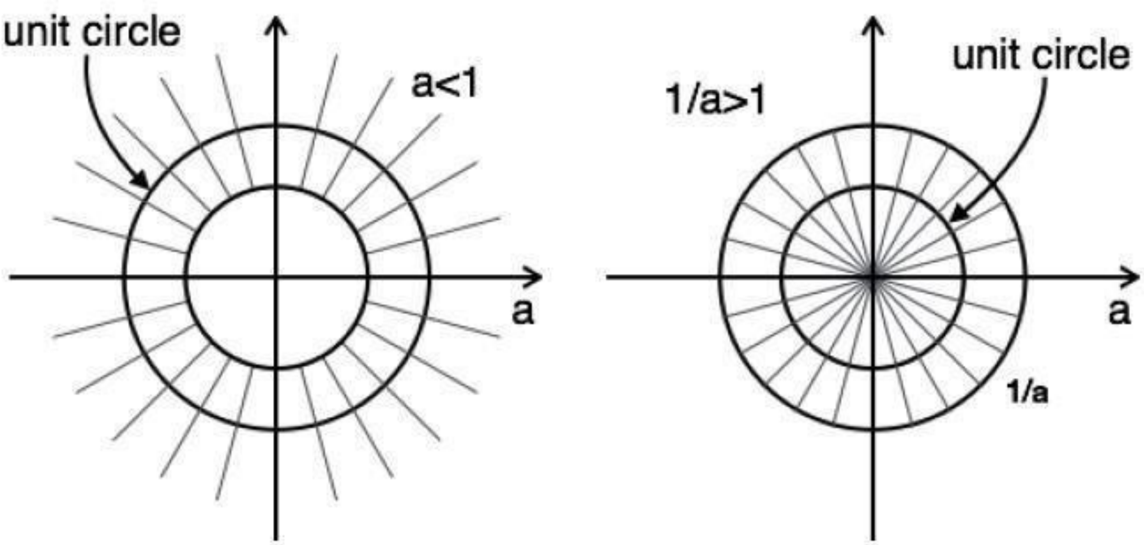
$$Z.T[a^n u[n]] + Z.T[a^{-n}u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

$$ROC: |z| > a \qquad ROC: |z| < \frac{1}{a}$$

The plot of ROC has two conditions as a > 1 and a < 1, as we do not know a.



In this case, there is no combination ROC.



Here, the combination of ROC is from a<|z|<1/a

Hence for this problem, z-transform is possible when a < 1.

Causality and Stability

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

- ROC is outside the outer most pole.
- In The transfer function H[Z], the order of numerator cannot be greater than the order of denominator.

Stability Condition for Discrete Time LTI Systems

A discrete time LTI system is stable when

- its system function H[Z] include unit circle |z|=1.
- all poles of the transfer function lay inside the unit circle |z|=1.

Z-Transform of Basic Signals

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$	$\lvert z\rvert > \lvert a\rvert$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\lvert z\rvert > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\lvert z\rvert > 1$
$(r^n \cos \Omega_0 n)u[n]$	$\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\lvert z\rvert > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\lvert z\rvert > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Some Properties of the Z- Transform:

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	$R' \supset R \cap \{0<\lvert z\rvert<\infty\}$
Multiplication by z_0^n	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \lvert z_0\rvert R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}}X(z)$	$R' \supset R \cap \{\lvert z\rvert > 1\}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$

Inverse Z transform:

Three different methods are:

1. Partial fraction method
2. Power series method
3. Long division method

Partial fraction method:

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Usually $M < N$

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $B(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \xleftrightarrow{z} \delta[n]$$

$$z^{-n_o} \xleftrightarrow{z} \delta[n-n_o]$$

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}
- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

where d_k are the poles of $X(z)$

For distinct poles

- For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair
- We get

$$A_k (d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$

$$\text{with ROC} \quad z > d_k \qquad \text{OR}$$

$$-A_k (d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$

$$\text{with ROC} \quad z < d_k$$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

For Repeated poles

- If pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_iz^{-1}}, \frac{A_{i_2}}{(1-d_iz^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_iz^{-1})^r}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \xleftrightarrow{z} \frac{A}{(1-d_iz^{-1})^m}, \qquad \text{with ROC}|z| > d_i$$

- If the ROC is of the form $|z| < d_i$, the left-sided inverse z-transform is chosen, ie.

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \xleftrightarrow{z} \frac{A}{(1-d_iz^{-1})^m}, \quad \text{with ROC}|z| < d_i$$

Deciding ROC

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.
- In order to chose the correct inverse z-transform, we must infer the ROC of each term from the ROC of $X(z)$.
- By comparing the location of each pole with the ROC of $X(z)$.
- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term
- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued

- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will be complex conjugate of each other

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen
- If the signal is stable, then t is absolutely summable and has DTFT
- Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the z-plane, ie. $|z| = 1$
- The inverse z-transform is determined by comparing the poles and the unit circle
- If the pole is inside the unit circle then the right-sided inverse z-transform is chosen
- If the pole is outside the unit circle then the left-sided inverse z-transform is chosen

Power series expansion method

- Express $X(z)$ as a power series in z^{-1} or z as given in z-transform equation
- The values of the signal $x[n]$ are then given by coefficient associated with z^{-n}
- Main disadvantage: limited to one sided signals

- Signals with ROCs of the form $|z| > a$ or $|z| < a$
- If the ROC is $|z| > a$, then express $X(z)$ as a power series in z^{-1} and we get right sided signal
- If the ROC is $|z| < a$, then express $X(z)$ as a power series in z and we get left sided signal

Long division method:

- Find the z-transform of

$$X(z) = \frac{2+z^{-1}}{1-\frac{1}{2}z^{-1}}, \text{with ROC} \quad |z| > \frac{1}{2}$$

- Solution is: use long division method to write $X(z)$ as a power series in z^{-1}, since ROC indicates that $x[n]$ is right sided sequence
- We get

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \dots$$

- Compare with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We get

$$x[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \dots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand $X(z)$ as a power series in z using long division method
- We get

$$X(z) = -2 - 8z - 16z^2 - 32z^3 + \dots$$

- We can write $x[n]$ as

$$x[n] = -2\delta[n] - 8\delta[n+1] - 16\delta[n+2]$$
$$- 32\delta[n+3] + \dots$$

- Find the z-transform of

$$X(z) = e^{z^2}, \text{with ROC all } z \text{ except } |z| = \infty$$

- Solution is: use power series expansion for e^a and is given by

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!}$$
$$X(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

- We can write $x[n]$ as

$$x[n] = \begin{cases} 0 & n > 0 \text{ or } n \text{ is odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

Example: A finite sequence *x [n]* is defined as

$$x[n] = \{5, 3, -\underset{\uparrow}{2}, 0, 4, -3\}$$

Find X(z) and its ROC.

Sol: We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-2}^{3} x[n] z^{-n}$$

$$= x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$
$$= 5z^2 + 3z - 2 + 4z^{-2} - 3z^{-3}$$

For z not equal to zero or infinity, each term in $X(z)$ will be finite and consequently $X(z)$ will converge. Note that $X(z)$ includes both positive powers of z and negative powers of z. Thus, from the result we conclude that the ROC of $X(z)$ is **0** <lzl< m.

Example: Consider the sequence

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1,\ a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $X(z)$ **and plot the poles and zeros of X(z). Sol:**

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

From the above equation we see that there is a pole of $(N-1)^{th}$order at $z = 0$ and a pole at $z = a$. Since x[n] is a finite sequence and is zero for $n < 0$, the ROC is $IzI>0$. The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 0, 1, \ldots, N-1$$

The root at k = **0** cancels the pole at $z = a$. The remaining zeros of $X(z)$ are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 1, \ldots, N-1$$

The pole-zero plot is shown in the following figure with N=8