UNIT V

Sampling theorem
&
Correlation

Sampling Theorem and its Importance:

Statement of Sampling Theorem:

A band limited signal can be reconstructed exactly if it is sampled at a rate atleast twice the maximum frequency component in it."

$$fs \leq 2fm.$$

The following figure shows a signal g(t) that is bandlimited.

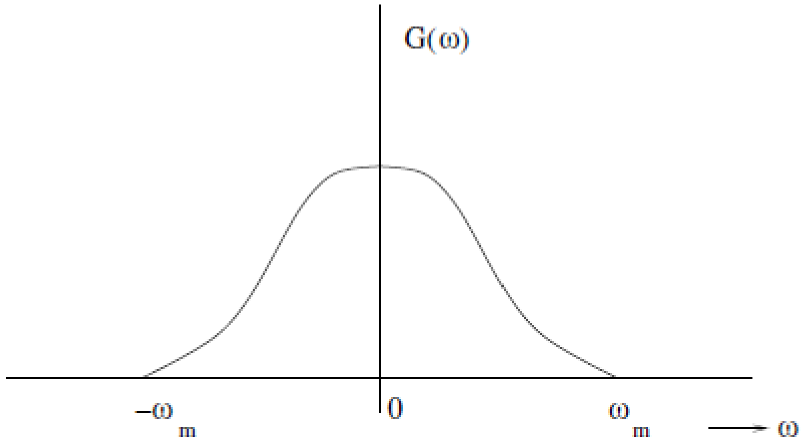


Figure1: Spectrum of band limited signal g(t)

The maximum frequency component of g(t) is fm. To recover the signal g(t) exactly from its samples it has to be sampled at a rate fs ≥ 2fm.

The minimum required sampling rate fs = 2fm is called "Nyquist rate".

<u>Proof:</u>

Let g(t) be a bandlimited signal whose bandwidth is f_m ($\omega_m = 2\pi f_m$).

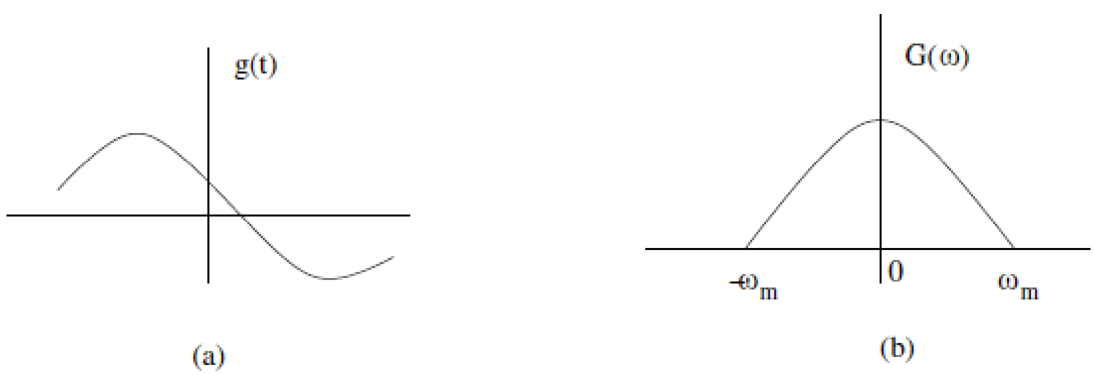


Figure 2: (a) Original signal g(t) (b) Spectrum G(ω)

$\delta_T(t)$ is the sampling signal with $f_s = 1/T > 2f_m$.

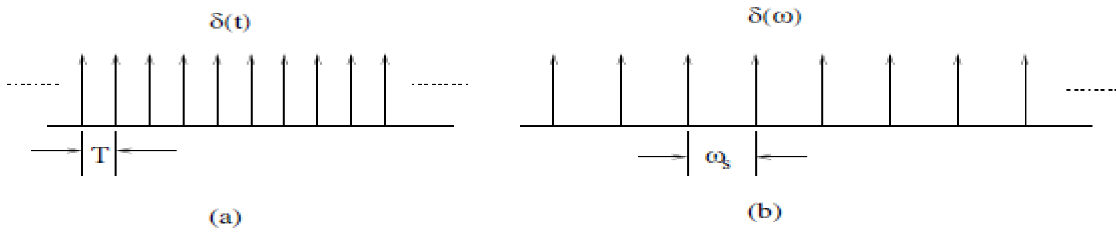


Figure 3: (a) sampling signal $\delta_T(t)$ (b) Spectrum $\delta_T(\omega)$

Let $g_s(t)$ be the sampled signal. Its Fourier Transform $G_s(\omega)$ is given by

$$
\begin{aligned}
\mathcal{F}(g_s(t)) &= \mathcal{F}\left[g(t)\delta_T(t)\right] \\
&= \mathcal{F}\left[g(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right] \\
&= \frac{1}{2\pi}\left[G(\omega)*\omega_0\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_0)\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty}G(\omega)*\delta(\omega-n\omega_0) \\
G_s(\omega) &= \mathcal{F}\left[g(t)+2g(t)\cos(\omega_0 t)+2g(t)\cos(2\omega_0 t)+\cdots\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty}G(\omega-n\omega_0)
\end{aligned}
$$

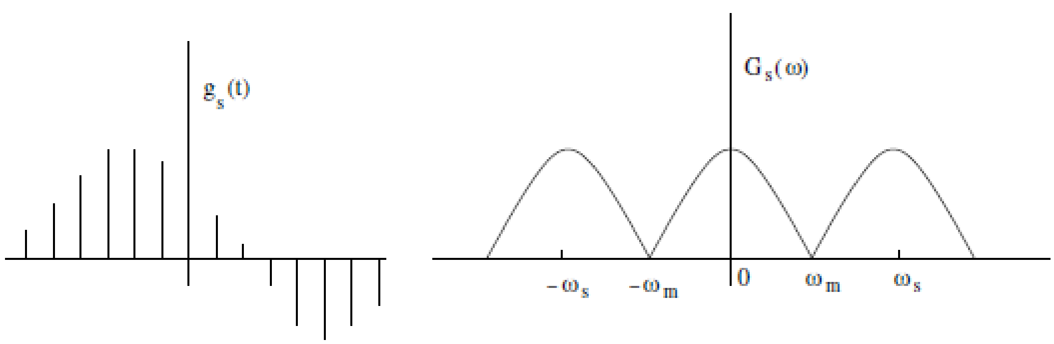


Figure 4: (a) sampled signal $g_s(t)$ (b) Spectrum $G_s(\omega)$

If $\omega_s = 2\omega_m$, i.e., $T = 1/2f_m$. Therefore, $G_s(\omega)$ is given by

$$G_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega - n\omega_m)$$

To recover the original signal $G(\omega)$:

1. Filter with a Gate function, $H_{2\omega_m}(\omega)$ of width $2\omega_m$.

2. Scale it by T.

$$G(\omega) = TG_s(\omega)H_{2\omega_m}(\omega).$$

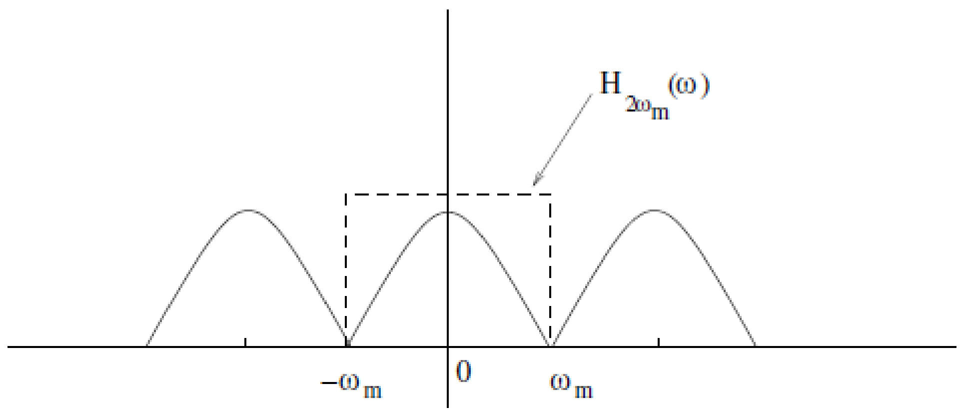


Figure 5: Recovery of signal by filtering with a filter of width $2\omega_m$

Aliasing:

Aliasing is a phenomenon where the high frequency components of the sampled signal interfere with each other because of inadequate sampling $\omega_s<\omega_m$

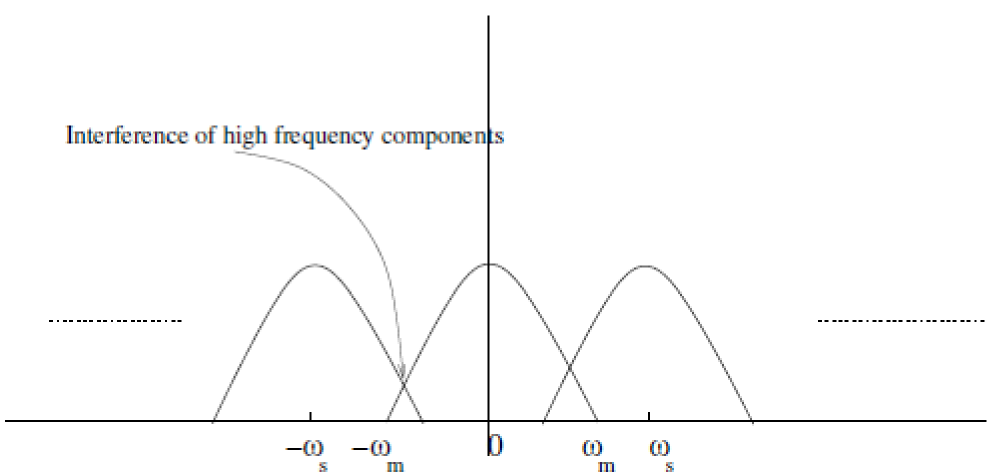


Figure 6: Aliasing due to inadequate sampling

Aliasing leads to distortion in recovered signal. This is the reason why sampling frequency should be atleast twice the bandwidth of the signal.

Oversampling:

In practice signal are oversampled, where fs is significantly higher than Nyquist rate to avoid aliasing.

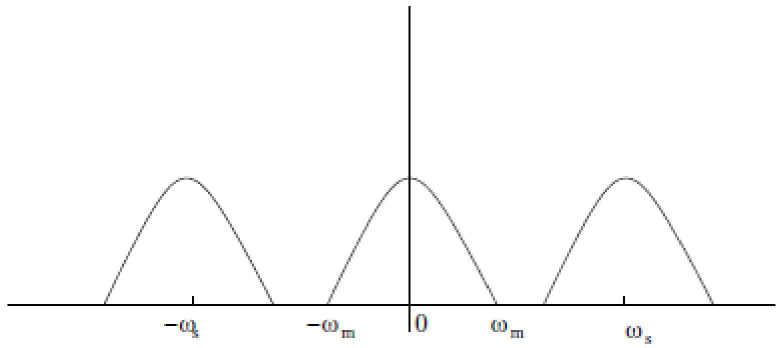
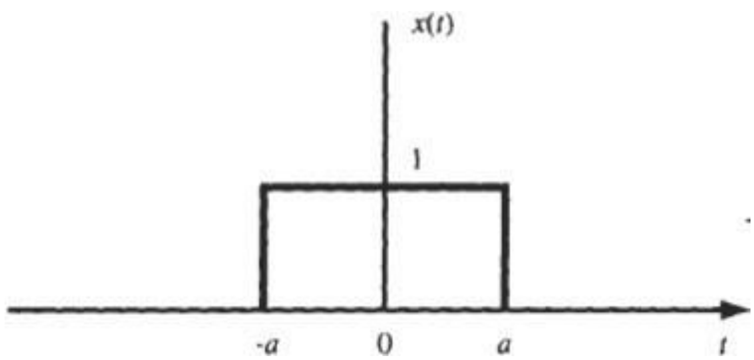
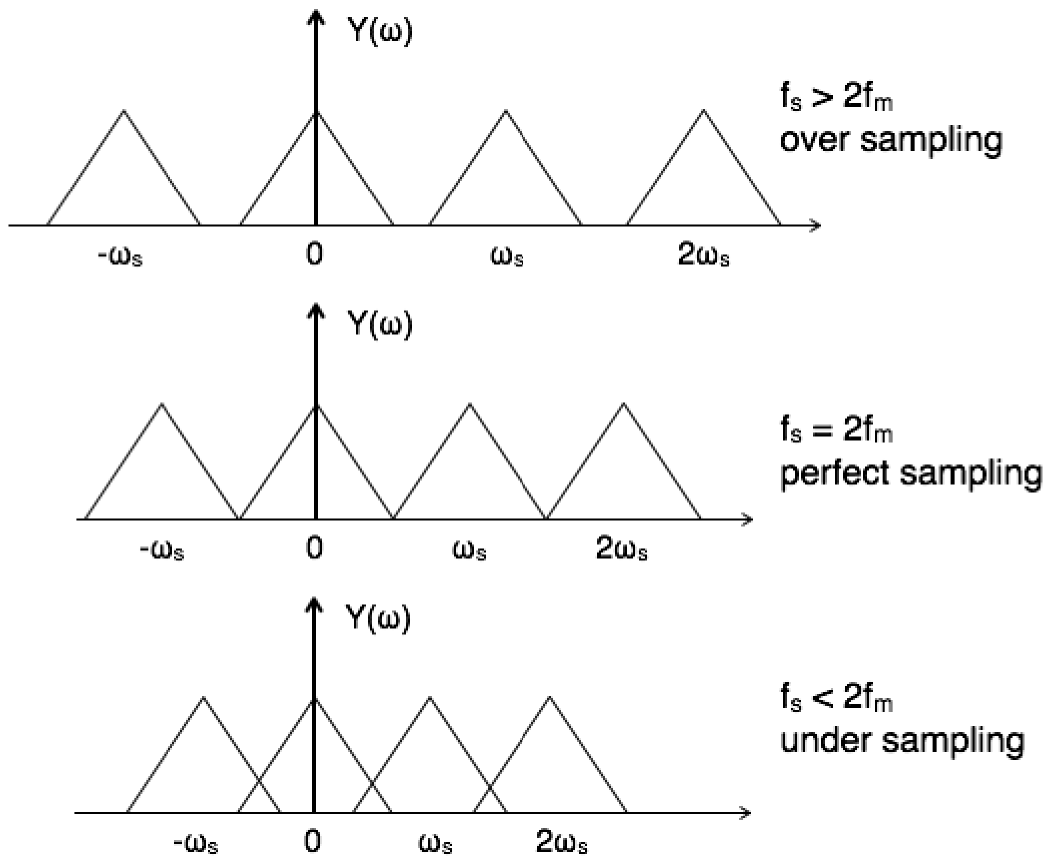


Figure 7: Oversampled signal-avoids aliasing

$$x(t) = p_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$





SIGNALS SAMPLING TECHNIQUES

There are three types of sampling techniques:

1. Impulse sampling.
2. Natural sampling.
3. Flat Top sampling.

Impulse Sampling

Impulse sampling can be performed by multiplying input signal x*t* with impulse train $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ of period 'T'. Here, the amplitude of impulse changes with respect to amplitude of input signal x*t*. The output of sampler is given by

$$y(t) = x(t) \times \text{ impulse train}$$

$$= x(t) \times \Sigma^{\infty} \quad \delta(t - nT)$$

$$y(t) = y_\delta(t) = \sum_{n=-\infty}^{\infty} x(nt)\delta(t - nT) \ldots\ldots\ldots.1$$

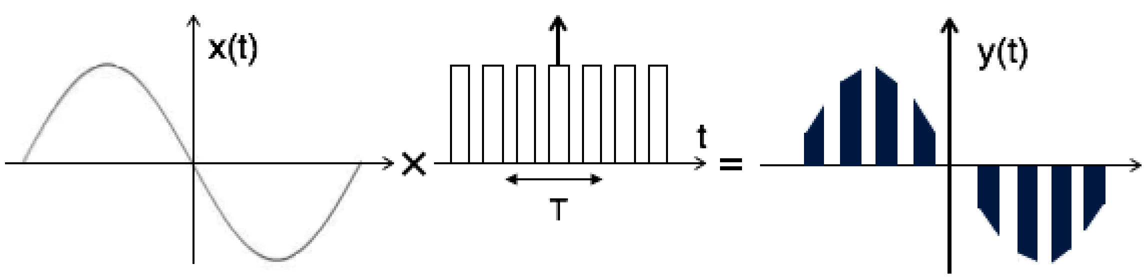
To get the spectrum of sampled signal, consider Fourier transform of equation 1 on both sides

$$Y(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

This is called ideal sampling or impulse sampling. You cannot use this practically because pulse width cannot be zero and the generation of impulse train is not possible practically.

Natural Sampling

Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. you multiply input signal x*t* to pulse train $\Sigma^{\infty} \quad P(t - nT)$ as shown below



The output of sampler is

$$y(t) = x(t) \times \text{pulse train}$$

- $= x(t) \times p(t)$
- $= x(t) \times \sum_{n=-\infty}^{\infty} P(t - nT) \ldots\ldots\ldots.(1)$

The exponential Fourier series representation of pt can be given as

$$p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \text{ .... } (2)$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t}$$

Where $F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} p(t) e^{-jn\omega_s t} dt$

$$= \frac{1}{T} P(n\omega_s)$$

Substitute F_n value in equation 2

$$\therefore p(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} P(n\omega_s) e^{jn\omega_s t}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) e^{jn\omega_s t}$$

Substitute pt in equation 1

$$y(t) = x(t) \times p(t)$$

$$= x(t) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, e^{jn\omega_s t}$$

$$y(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}$$

To get the spectrum of sampled signal, consider the Fourier transform on both sides.

$$F.T\,[y(t)] = F.T\left[\frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}\right]$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, F.T\,[x(t)\, e^{jn\omega_s t}]$$

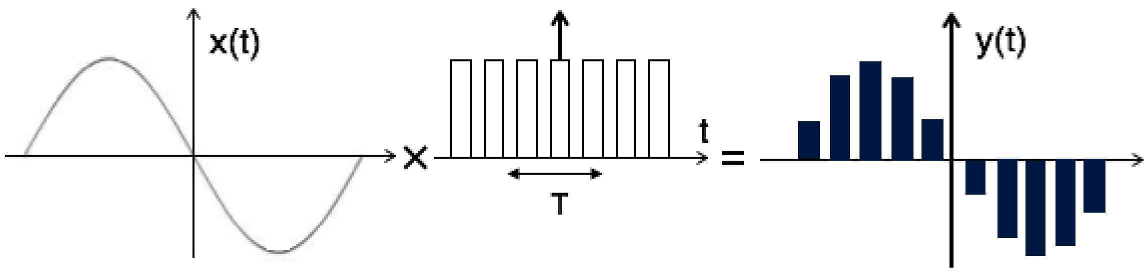
According to frequency shifting property

$$F.T\,[x(t)\, e^{jn\omega_s t}] = X[\omega - n\omega_s]$$

$$\therefore Y[\omega] = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, X[\omega - n\omega_s]$$

Flat Top Sampling

During transmission, noise is introduced at top of the transmission pulse which can be easily removed if the pulse is in the form of flat top. Here, the top of the samples are flat i.e. they have constant amplitude. Hence, it is called as flat top sampling or practical sampling. Flat top sampling makes use of sample and hold circuit.



Theoretically, the sampled signal can be obtained by convolution of rectangular pulse p*t* with ideally sampled signal say yδ*t* as shown in the diagram

i.e. $y(t) = p(t) \times y\delta(t)$ .... (1)

To get the sampled spectrum, consider Fourier transform on both sides for equation 1

Y [ω] = F. T [P (t) × yδ(t)]

By the knowledge of convolution property,

Y [ω] = P (ω) Yδ (ω)

Here $P(\omega) = TSa(\omega T) = 2 \sin \omega T / \omega$

Nyquist Rate

It is the minimum sampling rate at which signal can be converted into samples and can be recovered back without distortion.

Nyquist rate $f_N = 2f_m$ hz

Nyquist interval = $1 / f_N = 1 / 2f_m$ seconds

Samplings of Band Pass Signals

In case of band pass signals, the spectrum of band pass signal X[ω] = 0 for the frequencies outside the range $f_1 \le f \le f_2$. The frequency f_1 is always greater than zero. Plus, there is no aliasing effect when $f_s > 2f_2$. But it has two disadvantages:

The sampling rate is large in proportion with f2. This has practical limitations. The sampled signal spectrum has spectral gaps.

To overcome this, the band pass theorem states that the input signal x*t* can be converted into its samples and can be recovered back without distortion when sampling frequency $f_s < 2f_2$.

Also,

$$f_s = \frac{1}{T} = \frac{2f_2}{m}$$

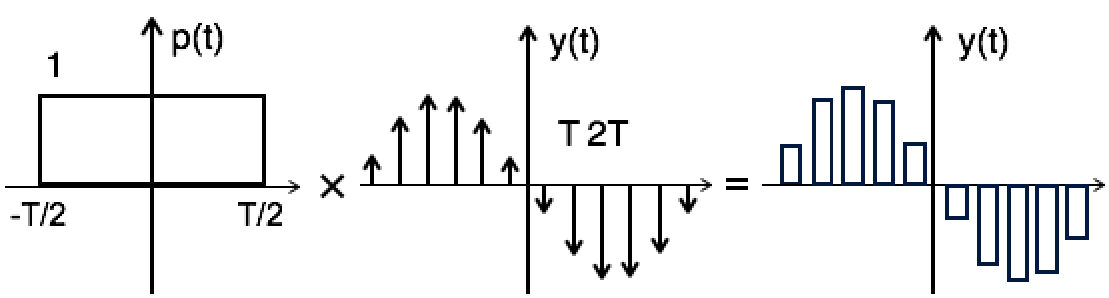
Where m is the largest integer

and B is the bandwidth of the signal

If f2=KB, then

$$f_s = \frac{1}{T} =$$

For band pass signals of bandwidth 2fm and the minimum sampling rate fs= 2 B = 4fm, the spectrum of sampled signal is given by $Y[\omega] = 1\ \Sigma\infty\ X[\omega - 2nB]$



Convolution
Convolution is a mathematical operation used to express the relation between input and output of an LTI system. It relates input, output and impulse response of an LTI system as
$y(t) = x(t) * h(t)$
Where y t = output of LTI
x t = input of LTI
h t = impulse response of LTI
There are two types of convolutions:
1. Continuous convolution 2.Discrete convolution
Continuous Convolution

$$y(t) = x(t) * h(t)$$
$$= \int^{\infty} x(\tau)h(t-\tau)d\tau$$
$$or$$
$$= \int^{\infty} x(t-\tau)h(\tau)d\tau$$

Discrete Convolution

$$y(n) = x(n) * h(n)$$
$$= \Sigma_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$or$$
$$= \Sigma_{k=-\infty}^{\infty} x(n-k)h(k)$$

By using convolution we can find zero state response of the system.
Properties of Convolution
1. **Commutative Property**
   $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
2. **Distributive Property**
   $x1(t) * [x2(t) + x3(t)] = [x1(t) * x2(t)] + [x1(t) * x3(t)]$
3. **Associative Property**
   $x1(t) * [x2(t) * x3(t)] = [x1(t) * x2(t)] * x3(t)$
4. **Shifting Property**
   $x1(t) * x2(t) = y(t)$
   $x1(t) * x2(t - t0) = y(t - t0)$
   $x1(t - t0) * x2(t) = y(t - t0)$
   $x1(t - t0) * x2(t - t1) = y(t - t0 - t1)$

Convolution with Impulse
$x1(t) * \delta(t) = x(t)$
$x1(t) * \delta(t - t0) = x(t - t0)$
Convolution of Unit Steps
$u(t) * u(t) = r(t)$
$u(t - T1) * u(t - T2) = r(t - T1 - T2)$
$u(n) * u(n) = [n + 1]u(n)$

Scaling Property

If $x(t) * h(t) = y(t)$

then $x(at) * h(at) = \frac{1}{|a|} y(at)$

Differentiation of Output

if $y(t) = x(t) * h(t)$

then $\frac{dy(t)}{dt} = \frac{dx(t)}{dt} * h(t)$

or

$\frac{dy(t)}{dt} = x(t) * \frac{dh(t)}{dt}$

Note:

1. Convolution of two causal sequences is causal.
2. Convolution of two anti causal sequences is anti causal.
3. Convolution of two unequal length rectangles results a trapezium.
4. Convolution of two equal length rectangles results a triangle.
5. A function convoluted itself is equal to integration of that function.

   Example: You know that $u(t) * u(t) = r(t)$

   According to above note, $u(t) * u(t) = \int u(t)dt = \int 1dt = t = r(t)$

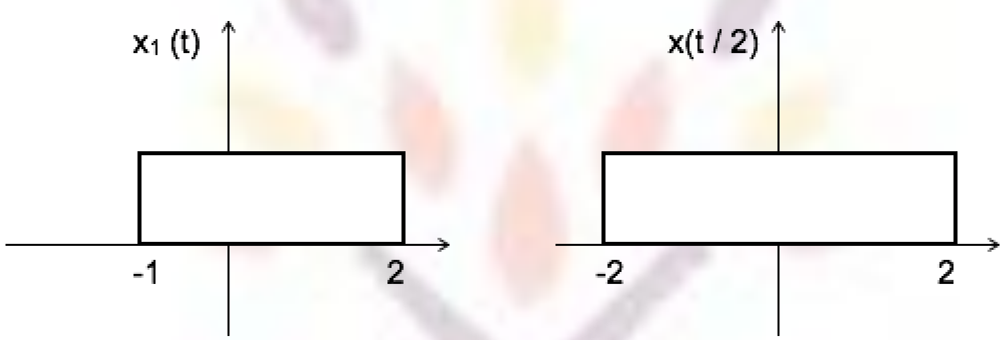
   Here, you get the result just by integrating $u(t)$.

Limits of Convoluted Signal

If two signals are convoluted then the resulting convoluted signal has following range:

Sum of lower limits < t < sum of upper limits

Ex: find the range of convolution of signals given below



Here, we have two rectangles of unequal length to convolute, which results a trapezium. The range of convoluted signal is:

Sum of lower limits < t < sum of upper limits

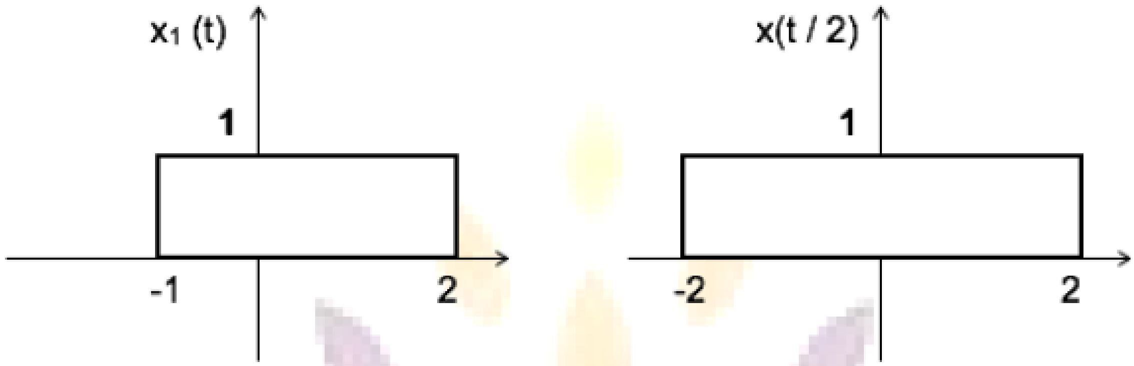
$-1 + -2 < t < 2 + 2$
$-3 < t < 4$
Hence the result is trapezium with period 7.
Area of Convoluted Signal
The area under convoluted signal is given by $Ay = Ax\ Ah$
Where Ax = area under input signal
Ah = area under impulse response
Ay = area under output signal
Proof: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Take integration on both sides

$$\int y(t)dt = \int \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt$$

$$= \int x(\tau)d\tau \int_{-\infty}^{\infty} h(t-\tau)dt$$

We know that area of any signal is the integration of that signal itself.
∴ Ay = Ax Ah
DC Component
DC component of any signal is given by
DC component = area of the signal
period of the signal
Ex: what is the dc component of the resultant convoluted signal given below?



Here area of $x_1 t$ = length $\times$ breadth = $1 \times 3 = 3$

area of $x_2 t$ = length $\times$ breadth = $1 \times 4 = 4$

area of convoluted signal = area of $x_1 t$ $\times$ area of $x_2 t$

$= 3 \times 4 = 12$

Duration of the convoluted signal = sum of lower limits < t < sum of upper limits

$= -1 + -2 < t < 2+2$

$= -3 < t < 4$

Period=7

$\therefore$ Dc component of the convoluted signal $= \dfrac{\text{area of the signal}}{\text{period of the signal}}$

Dc component $= \dfrac{12}{7}$

Correlation

Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

There are two types of correlation:

1.Auto correlation

2.Cros correlation

Auto Correlation Function

It is defined as correlation of a signal with itself. Auto correlation function is a measure of similarity between a signal & its time delayed version. It is represented with R$\tau$$.

Consider a signals xt. The auto correlation function of xt with its time delayed version is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad \text{[-ve shift]}$$

Where τ = searching or scanning or delay parameter.

If the signal is complex then auto correlation function is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x*(t-\tau)dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x(t+\tau)x*(t)dt \qquad \text{[-ve shift]}$$

Properties of Auto-correlation Function of Energy Signal

- Auto correlation exhibits conjugate symmetry i.e. R $\tau$$ = R*−$\tau$$
- Auto correlation function of energy signal at origin i.e. at τ=0 is equal to total energy of that signal, which is given as:

  R (= E = $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$
- Auto correlation function $\frac{1}{\tau}$,
- Auto correlation function is maximum at τ=0 i.e |R $\tau$$ | ≤ R 0 ∀ τ
- Auto correlation function and energy spectral densities are Fourier transform pairs. i.e.

  $F.T\,[R(\tau)] = \Psi(\omega)$

  $\Psi(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$
- $R(\tau) = x(\tau) * x(-\tau)$

Auto Correlation Function of Power Signals

The auto correlation function of periodic power signal with period T is given by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)x*(t-\tau)dt$$

Properties

- Auto correlation of power signal exhibits conjugate symmetry i.e. $R(\tau) = R*(-\tau)$
- Auto correlation function of power signal at $\tau = 0$ *atorigin*is equal to total power of that signal. i.e.

  $R(0) = \rho$
- Auto correlation function of power signal $\frac{1}{\tau}$,
- Auto correlation function of power signal is maximum at $\tau = 0$ i.e.,

  $|R(\tau)| \leq R(0)\ \forall\ \tau$
- Auto correlation function and power spectral densities are Fourier transform pairs. i.e.,

  $F.T[R(\tau)] = s(\omega)$

  $s(\omega) = \int^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$
- $R(\tau) = x(\tau) * x(-\tau)$

Density Spectrum

Let us see density spectrums:

Energy Density Spectrum

Energy density spectrum can be calculated using the formula:

$$E = \int_{-\infty}^{\infty} |\,x(f)\,|^2 df$$

Power Density Spectrum

Power density spectrum can be calculated by using the formula:

$$P = \Sigma_{n=-\infty}^{\infty} |\,C_n|^2$$

Cross Correlation Function

Cross correlation is the measure of similarity between two different signals.

Consider two signals x_1t and x_2t. The cross correlation of these two signals $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)\,dt \qquad \text{[+ve shift]}$$

∞

If signals are complex then

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2(t)\, dt \qquad \text{[-ve shift]}$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)\, dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_1^*(t)\, dt \qquad \text{[-ve shift]}$$

$$R_{21}(\tau) = \int_{-\infty} x_2(t)x_1^*(t-\tau)\, dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_2(t+\tau)x_1^*(t)\, dt \qquad \text{[-ve shift]}$$

Properties of Cross Correlation Function of Energy and Power Signals

- Auto correlation exhibits conjugate symmetry i.e. $R_{12}(\tau) = R^*(-\tau)$.
- Cross correlation is not commutative like convolution i.e.

$$R_{12}(\tau) \neq R_{21}(-\tau)$$

- If $R_{12}0 = 0$ means, if $\int^{\infty} x_1(t)x^*(t)dt = 0$, then the two signals are said to be orthogonal.

  For power signal if $\lim_{T\to\infty} \frac{1}{} \int_{\frac{}{2}}^{\frac{T}{2}} x(t)x^*(t)\, dt$ then two signals are said to be orthogonal.

- Cross correlation function corresponds to the multiplication of spectrums of one signal to the complex conjugate of spectrum of another signal. i.e.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega)X^*(\omega)$$

  This also called as correlation theorem.

Parsvel's Theorem

Parsvel's theorem for energy signals states that the total energy in a signal can be obtained by the spectrum of the signal as

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Note: If a signal has energy E then time scaled version of that signal $x at$ has energy E/a.