

UNIT-I

Signal Analysis

SIGNALS & SYSTEMS

Signal : A signal describes a time varying physical phenomenon which is intended to convey information. (or) Signal is a function of time or any other variable of interest. (or) Signal is a function of one or more independent variables, which contain some information.

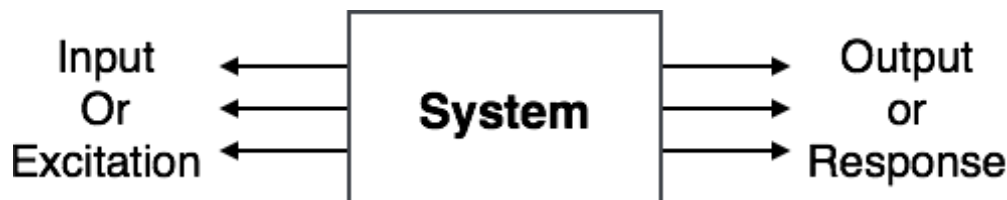
Example: voice signal, video signal, signals on telephone wires, EEG, ECG etc.

Signals may be of continuous time or discrete time signals.

System : System is a device or combination of devices, which can operate on signals and produces corresponding response. Input to a system is called as excitation and output from it is called as response. (or) System is a combination of sub units which will interact with each other to achieve a common interest.

For one or more inputs, the system can have one or more outputs.

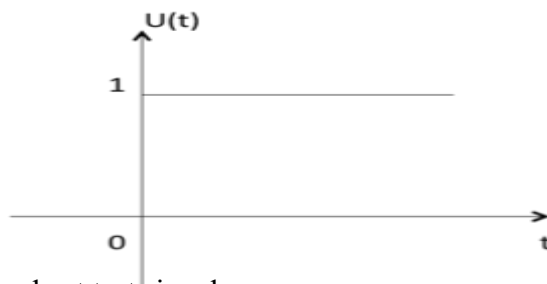
Example: Communication System



Elementary Signals or Basic Signals:

Unit Step Function

Unit step function is denoted by $u(t)$. It is defined as $u(t) = 1$ when $t \geq 0$ and 0 when $t < 0$

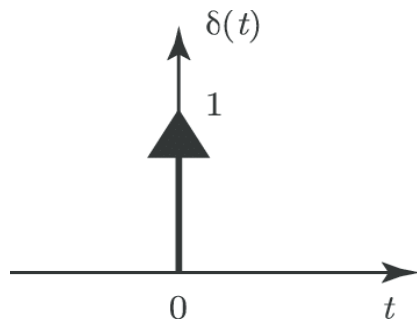


- It is used as best test signal.
- Area under unit step function is unity.

Unit Impulse Function

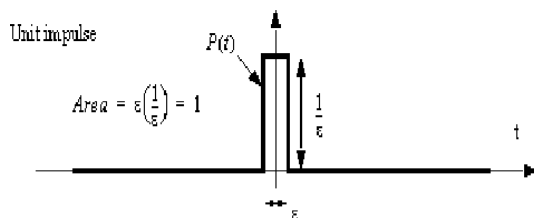
Impulse function is denoted by $\delta(t)$, and it is defined as

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(t) dt = u(t)$$

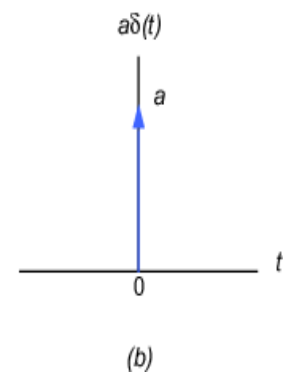
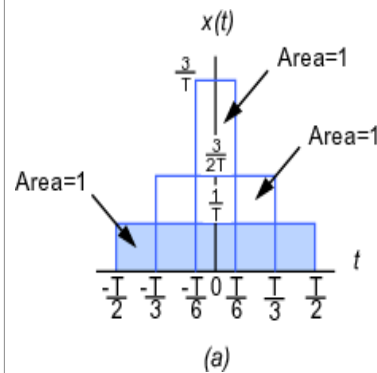
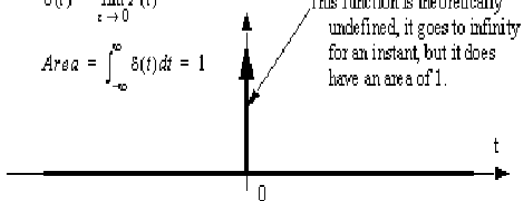
$$\delta(t) = \frac{du(t)}{dt}$$



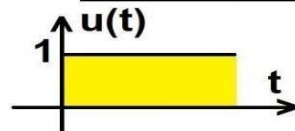
Dirac delta function

$$\delta(t) = \lim_{\epsilon \rightarrow 0} P(t)$$

$$\text{Area} = \int_{-\infty}^{\infty} \delta(t) dt = 1$$



The Derivative of an Unit Step Function



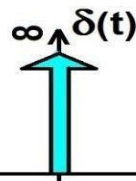
$$u(t) = 0 \quad t < 0$$

$$u(t) = 1 \quad t > 0$$

$$\frac{d}{dt} [u(t)] = ? \delta(t)$$

unit impulse function = delta function

$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases}$$

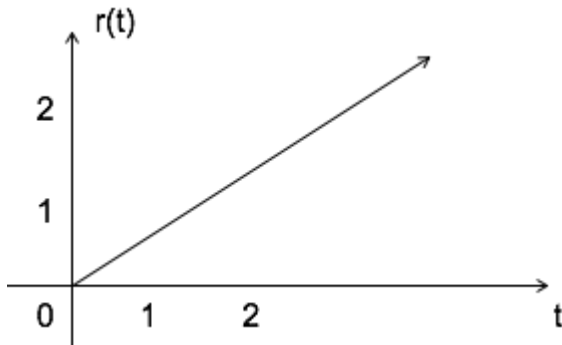


$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{-0}^{+0} \delta(t) dt$$

Ramp Signal

Ramp signal is denoted by $r(t)$, and it is defined as $r(t) =$

$$\begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



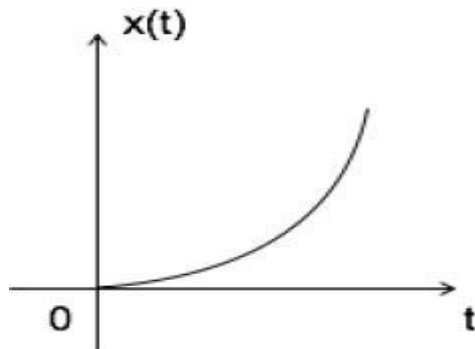
$$\int u(t) = \int 1 = t = r(t)$$

$$u(t) = \frac{dr(t)}{dt}$$

Area under unit ramp is unity.

Parabolic Signal

Parabolic signal can be defined as $x(t) = \begin{cases} t^2/2 & t \geq 0 \\ 0 & t < 0 \end{cases}$



$$\iint u(t)dt = \int r(t)dt = \int tdt = \frac{t^2}{2} = \text{parabolic signal}$$

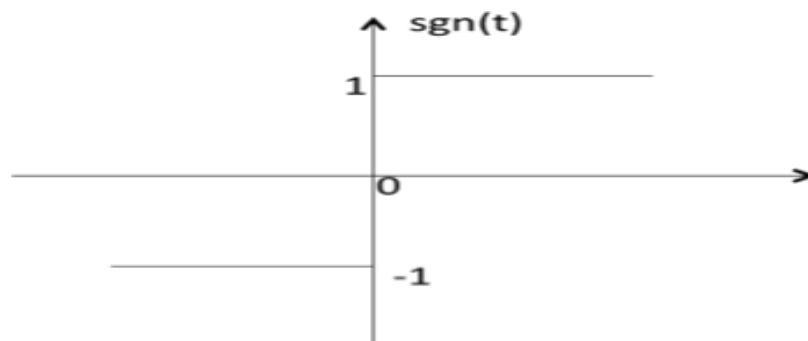
$$\Rightarrow u(t) = \frac{d^2 x(t)}{dt^2}$$

$$\Rightarrow r(t) = \frac{dx(t)}{dt}$$

Signum Function

$$\begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

Signum function is denoted as $\text{sgn}(t)$. It is defined as $\text{sgn}(t) =$



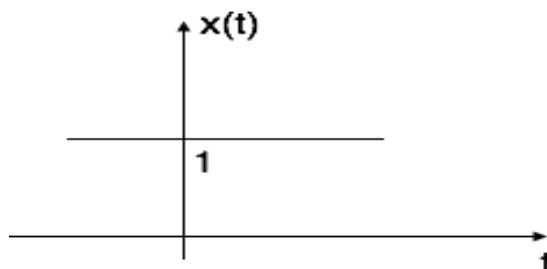
$$\text{sgn}(t) = 2u(t) - 1$$

Exponential Signal

Exponential signal is in the form of $x(t) = e^{\alpha t}$

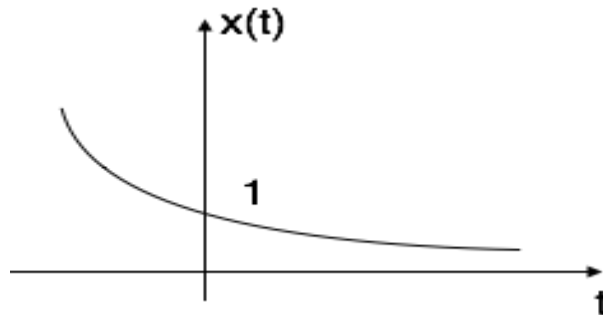
The shape of exponential can be defined by α

Case: if $\alpha = 0 \rightarrow x(t) = e^0 = 1$



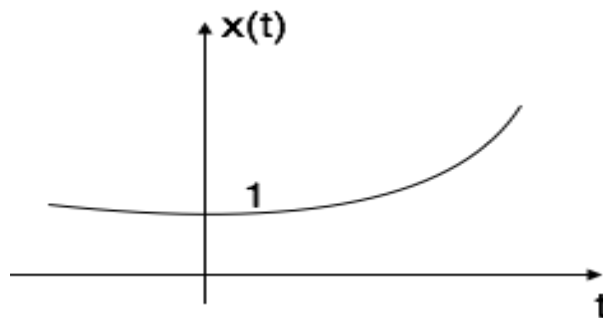
Caseii: if $\alpha < 0$ i.e. -ve then $x(t) = e^{-\alpha t}$

. The shape is called decaying exponential.



Caseiii: if $\alpha > 0$ i.e. +ve then $x(t) = e^{\alpha t}$

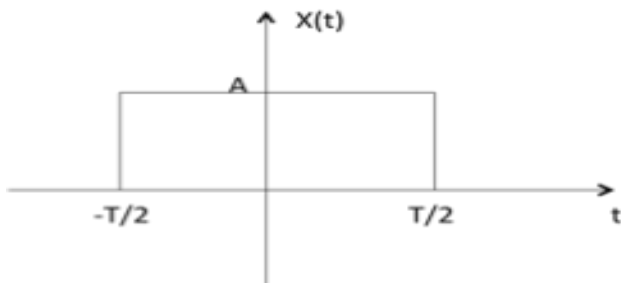
. The shape is called raising exponential.



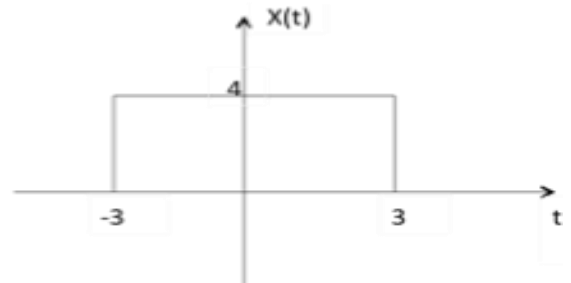
Rectangular Signal

Let it be denoted as $x(t)$ and it is defined as

$$x(t) = A \operatorname{rect} \left[\frac{t}{T} \right]$$



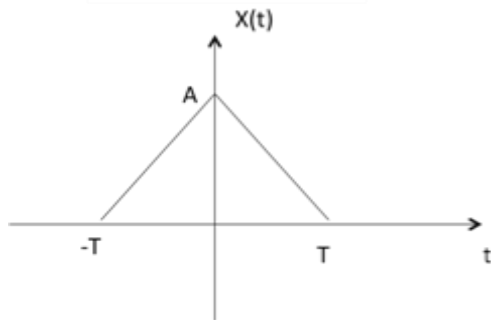
$$\text{ex: } 4 \operatorname{rect} \left[\frac{t}{6} \right]$$



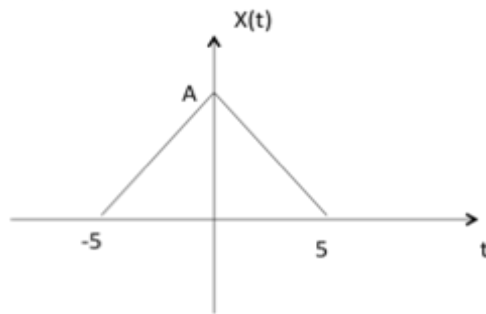
Triangular Signal

Let it be denoted as $x(t)$

$$x(t) = A \left[1 - \frac{|t|}{T} \right]$$

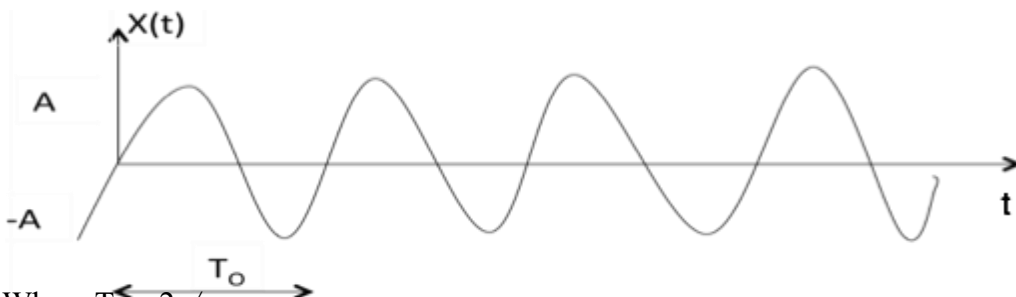


$$\text{ex: } x(t) = A \left[1 - \frac{|t|}{5} \right]$$



Sinusoidal Signal

Sinusoidal signal is in the form of $x(t) = A \cos(\omega_0 t \pm \phi)$ or $A \sin(\omega_0 t \pm \phi)$



Where $T_0 = \frac{2\pi}{\omega_0}$

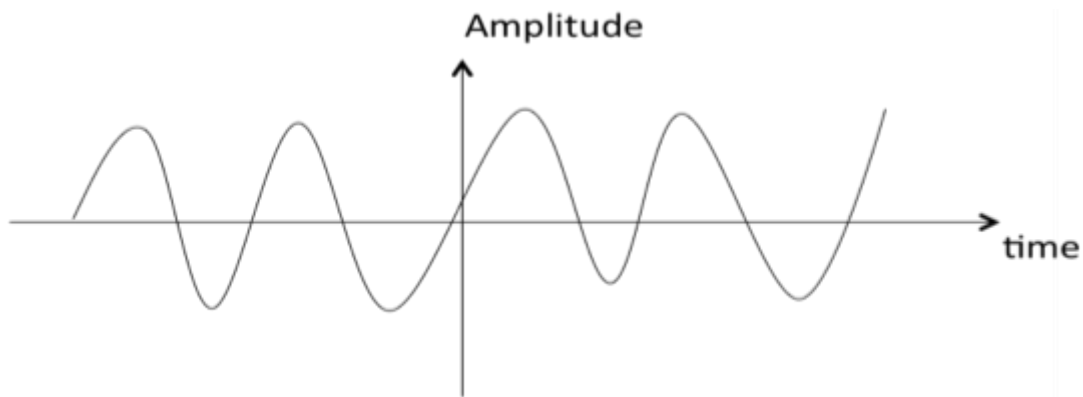
Classification of Signals:

Signals are classified into the following categories:

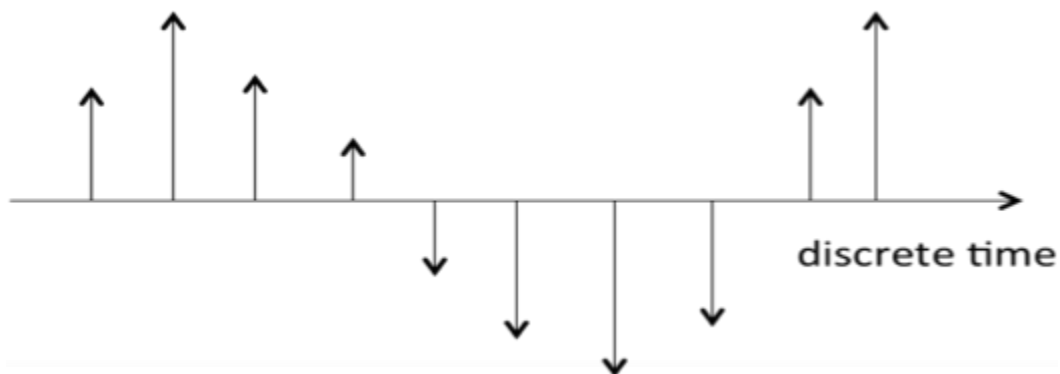
- Continuous Time and Discrete Time Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

Continuous Time and Discrete Time Signals

A signal is said to be continuous when it is defined for all instants of time.

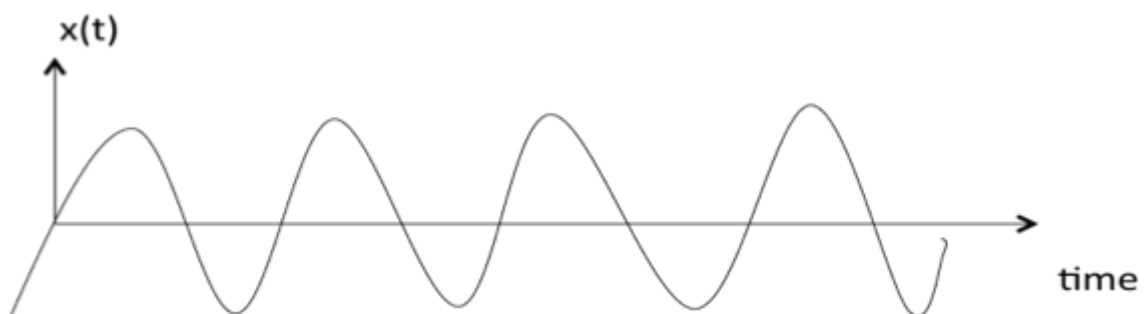


A signal is said to be discrete when it is defined at only discrete instants of time/

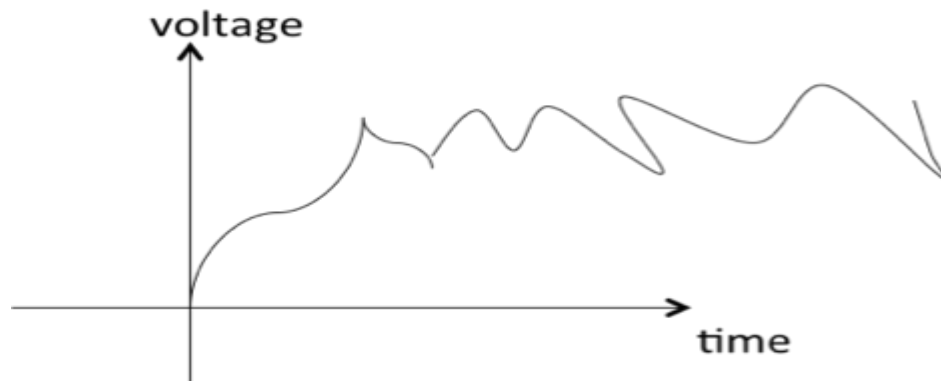


Deterministic and Non-deterministic Signals

A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.



A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.



Even and Odd Signals

A signal is said to be even when it satisfies the condition $x(t) = x(-t)$

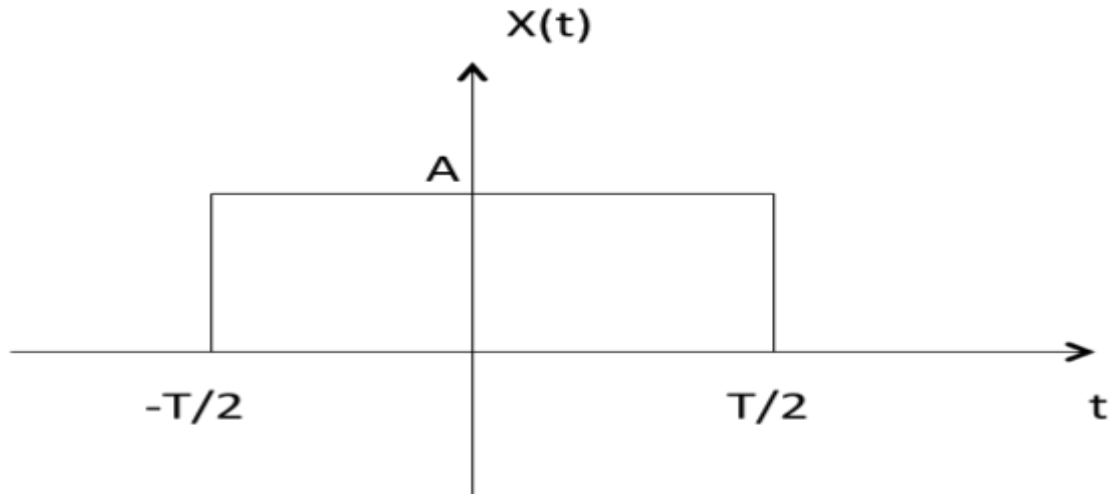
Example 1: $t^2, t^4 \dots$ cost etc.

$$\text{Let } x(t) = t^2$$

$$x(-t) = (-t)^2 = t^2 = x(t)$$

$\therefore t^2$ is even function

Example 2: As shown in the following diagram, rectangle function $x(t) = x(-t)$ so it is also even function.



A signal is said to be odd when it satisfies the condition $x(t) = -x(-t)$

Example: $t, t^3 \dots$ And $\sin t$

$$\text{Let } x(t) = \sin t$$

$$x(-t) = \sin(-t) = -\sin t = -x(t)$$

$\therefore \sin t$ is odd function.

Any function $f(t)$ can be expressed as the sum of its even function $f_e(t)$ and odd function $f_o(t)$.

$$f(t) = f_e(t) + f_o(t)$$

where

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

Periodic and Aperiodic Signals

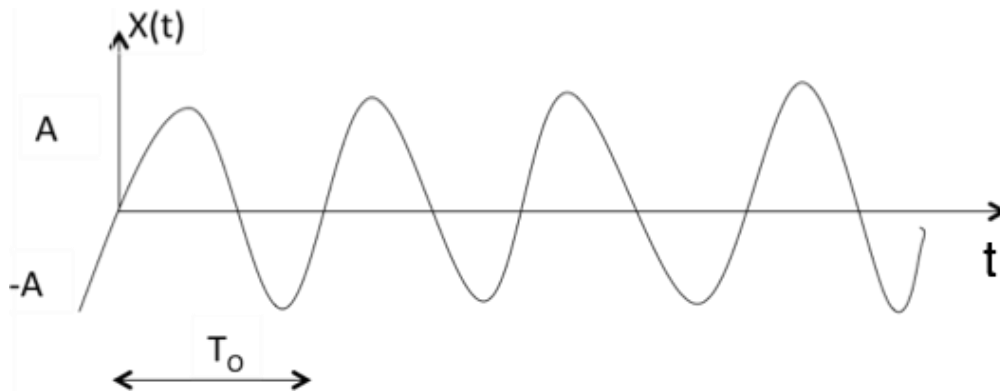
A signal is said to be periodic if it satisfies the condition $x(t) = x(t + T)$ or $x(n) = x(n + N)$.

Where

T = fundamental time period,

$1/T = f$ = fundamental

frequency.



The above signal will repeat for every time interval T_0 hence it is periodic with period T_0 .

Energy and Power Signals

A signal is said to be energy signal when it has finite energy.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2(t) dt$$

A signal is said to be power signal when it has finite power.

$$\text{Power } P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt$$

NOTE: A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0

Energy of power signal = ∞

Real and Imaginary Signals

A signal is said to be real when it satisfies the condition $x(t) = x^*(t)$

A signal is said to be odd when it satisfies the condition $x(t) = -x^*(t)$

Example:

If $x(t) = 3$ then $x^*(t) = 3^* = 3$ here $x(t)$ is a real signal.

If $x(t) = 3j$ then $x^*(t) = 3j^* = -3j = -x(t)$ hence $x(t)$ is an odd signal.

Note: For a real signal, imaginary part should be zero. Similarly for an imaginary signal, real part should be zero.

Basic operations on Signals:

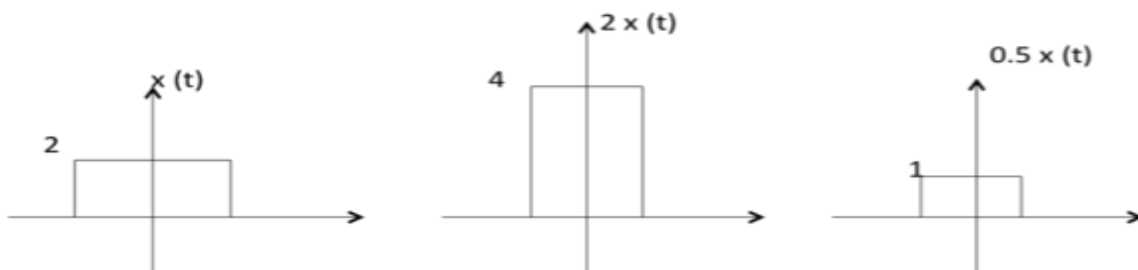
There are two variable parameters in general:

1. Amplitude
2. Time

(1) The following operation can be performed with amplitude:

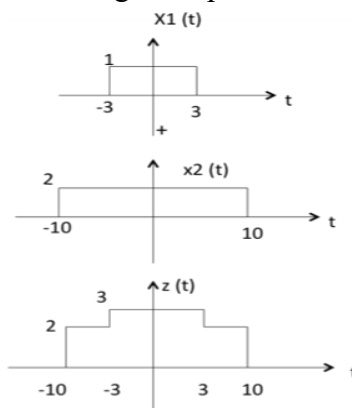
Amplitude Scaling

$C x(t)$ is a amplitude scaled version of $x(t)$ whose amplitude is scaled by a factor C .



Addition

Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:



As seen from the previous diagram,

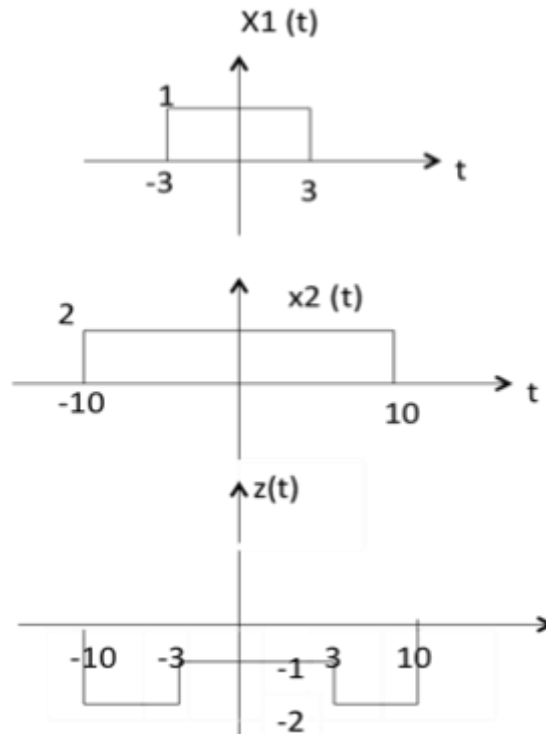
$$-10 < t < -3 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 0 + 2 = 2$$

$$-3 < t < 3 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 1 + 2 = 3$$

$$3 < t < 10 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 0 + 2 = 2$$

Subtraction

subtraction of two signals is nothing but subtraction of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

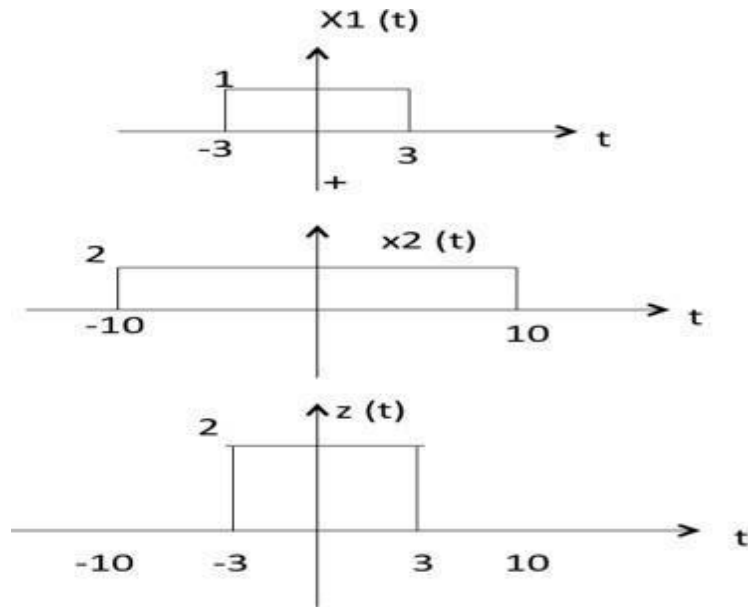
$$-10 < t < -3 \text{ amplitude of } z(t) = x_1(t) - x_2(t) = 0 - 2 = -2$$

$$-3 < t < 3 \text{ amplitude of } z(t) = x_1(t) - x_2(t) = 1 - 2 = -1$$

$$3 < t < 10 \text{ amplitude of } z(t) = x_1(t) - x_2(t) = 0 - 2 = -2$$

Multiplication

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

$-10 < t < -3$ amplitude of $z(t) = x_1(t) \times x_2(t) = 0 \times 2 = 0$

$-3 < t < 3$ amplitude of $z(t) = x_1(t) \times x_2(t) = 1 \times 2 = 2$

$3 < t < 10$ amplitude of $z(t) = x_1(t) \times x_2(t) = 0 \times 2 = 0$

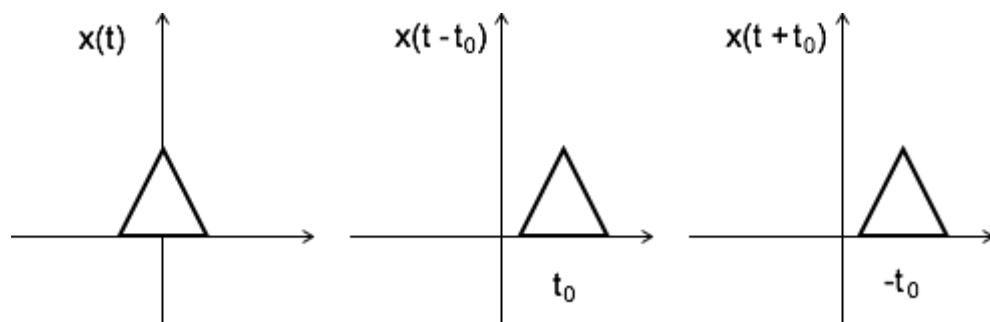
(2) The following operations can be performed with time:

Time Shifting

$x(t \pm t_0)$ is time shifted version of the signal $x(t)$.

$x(t + t_0) \rightarrow$ negative shift

$x(t - t_0) \rightarrow$ positive shift

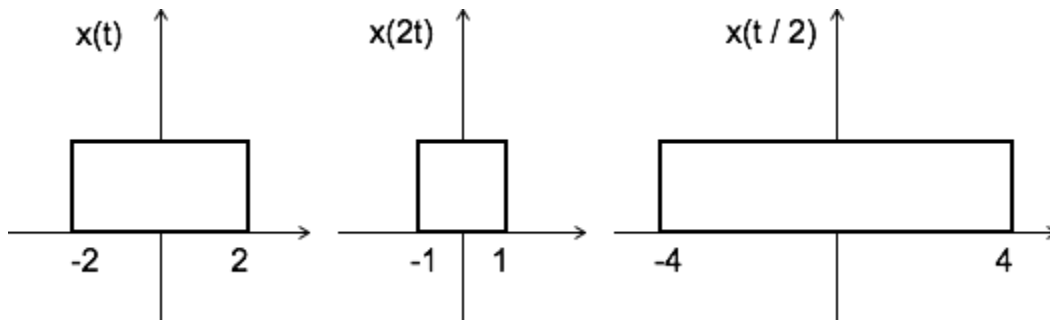


Time Scaling

$x(At)$ is time scaled version of the signal $x(t)$, where A is always positive.

$|A| > 1 \rightarrow$ Compression of the signal

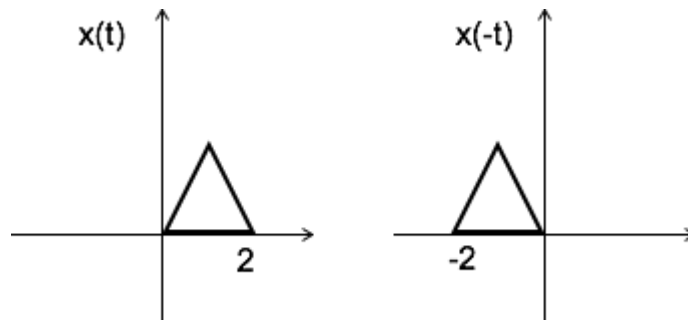
$|A| < 1 \rightarrow$ Expansion of the signal



Note: $u(at) = u(t)$ time scaling is not applicable for unit step function.

Time Reversal

$x(-t)$ is the time reversal of the signal $x(t)$.



Classification of Systems:

Systems are classified into the following categories:

- Liner and Non-liner Systems
- Time Variant and Time Invariant Systems
- Liner Time variant and Liner Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

Linear and Non-linear Systems

A system is said to be linear when it satisfies superposition and homogeneity principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogeneity principles,

$$T [a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

$$\therefore T [a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

From the above expression, it is clear that response of overall system is equal to response of individual system.

Example:

$$y(t) = x^2(t)$$

Solution:

$$y_1(t) = T[x_1(t)] = x_1^2(t)$$

$$y_2(t) = T[x_2(t)] = x_2^2(t)$$

$$T [a_1 x_1(t) + a_2 x_2(t)] = [a_1 x_1(t) + a_2 x_2(t)]^2$$

Which is not equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be non linear.

Time Variant and Time Invariant Systems

A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant.

The condition for time invariant system is:

$$y(n, t) = y(n-t)$$

The condition for time variant system is:

$$y(n, t) \neq y(n-t)$$

Where $y(n, t) = T[x(n-t)]$ = input change

$y(n-t)$ = output change

Example:

$$y(n) = x(-n)$$

$$y(n, t) = T[x(n-t)] = x(-n-t)$$

$$y(n-t) = x(-(n-t)) = x(-n + t)$$

$\therefore y(n, t) \neq y(n-t)$. Hence, the system is time variant.

Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system.

Example 1: $y(t) = 2 x(t)$

For present value $t=0$, the system output is $y(0) = 2x(0)$. Here, the output is only dependent upon present input. Hence the system is memory less or static.

Example 2: $y(t) = 2 x(t) + 3 x(t-3)$

For present value $t=0$, the system output is $y(0) = 2x(0) + 3x(-3)$.

Here $x(-3)$ is past value for the present input for which the system requires memory to get this output. Hence, the system is a dynamic system.

Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.

For non causal system, the output depends upon future inputs also.

Example 1: $y(n) = 2 x(n) + 3 x(n-3)$

For present value $t=1$, the system output is $y(1) = 2x(1) + 3x(-2)$.

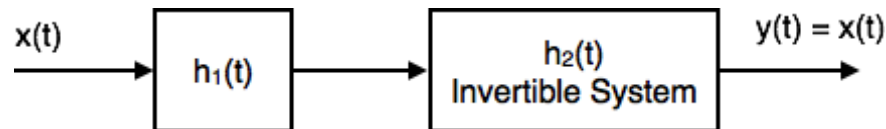
Here, the system output only depends upon present and past inputs. Hence, the system is causal.

Example 2: $y(n) = 2x(n) + 3x(n-3) + 6x(n+3)$

For present value $t=1$, the system output is $y(1) = 2x(1) + 3x(-2) + 6x(4)$ Here, the system output depends upon future input. Hence the system is non-causal system.

Invertible and Non-Invertible systems

A system is said to be invertible if the input of the system appears at the output.



$$Y(S) = X(S) H_1(S) H_2(S)$$

$$= X(S) H_1(S) \cdot 1/H_1(S)$$

$$\text{Since } H_2(S) = 1/(H_1(S))$$

$$\therefore Y(S) = X(S)$$

$$\rightarrow y(t) = x(t)$$

Hence, the system is invertible.

If $y(t) \neq x(t)$, then the system is said to be non-invertible.

Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.

Note: For a bounded signal, amplitude is finite.

Example 1: $y(t) = x^2(t)$

Let the input is $u(t)$ (unit step bounded input) then the output $y(t) = u^2(t) = u(t)$ = bounded output.

Hence, the system is stable.

Example 2: $y(t) = \int x(t)dt$

Let the input is $u(t)$ (unit step bounded input) then the output $y(t) = \int u(t)dt =$ ramp signal (unbounded because amplitude of ramp is not finite it goes to infinite when $t \rightarrow$ infinite).

Hence, the system is unstable.

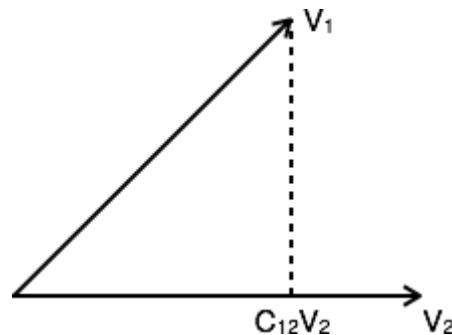
Analogy Between Vectors and Signals:

There is a perfect analogy between vectors and signals.

Vector

A vector contains magnitude and direction. The name of the vector is denoted by bold face type and their magnitude is denoted by light face type.

Example: V is a vector with magnitude V . Consider two vectors V_1 and V_2 as shown in the following diagram. Let the component of V_1 along with V_2 is given by $C_{12}V_2$. The component of a vector V_1 along with the vector V_2 can be obtained by taking a perpendicular from the end of V_1 to the vector V_2 as shown in diagram:



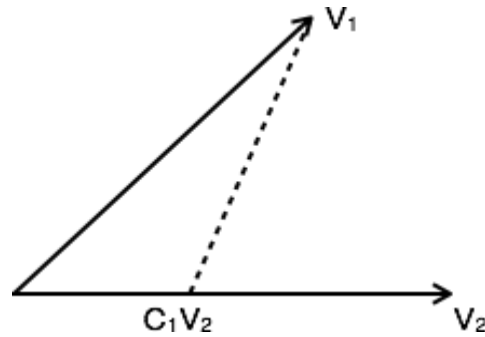
The vector V_1 can be expressed in terms of vector V_2

$$V_1 = C_{12}V_2 + V_e$$

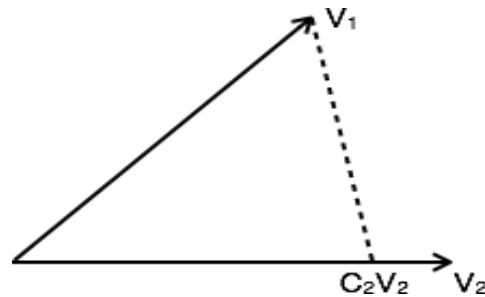
Where V_e is the error vector.

But this is not the only way of expressing vector V_1 in terms of V_2 . The alternate possibilities are:

$$V_1 = C_1V_2 + V_{e1}$$



$$V_2 = C_2 V_2 + V_{e2}$$



The error signal is minimum for large component value. If $C_{12}=0$, then two signals are said to be orthogonal.

Dot Product of Two Vectors

$$V_1 \cdot V_2 = V_1 V_2 \cos \theta$$

θ = Angle between V_1 and V_2

$$V_1 \cdot V_2 = V_2 \cdot V_1$$

From the diagram, components of V_1 along $V_2 = C_{12} V_2$

$$\frac{V_1 \cdot V_2}{V_2} = C_{12} V_2$$

$$\Rightarrow C_{12} = \frac{V_1 \cdot V_2}{V_2^2}$$

Signal

The concept of orthogonality can be applied to signals. Let us consider two signals $f_1(t)$ and $f_2(t)$. Similar to vectors, you can approximate $f_1(t)$ in terms of $f_2(t)$ as

$$f_1(t) = C_{12} f_2(t) + f_e(t) \text{ for } (t_1 < t < t_2)$$

$$\Rightarrow f_e(t) = f_1(t) - C_{12} f_2(t)$$

One possible way of minimizing the error is integrating over the interval t_1 to t_2 .

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)] dt$$
$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)] dt$$

However, this step also does not reduce the error to appreciable extent. This can be corrected by taking the square of error function.

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$
$$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t) - C_{12} f_2]^2 dt$$

Where ε is the mean square value of error signal. The value of C_{12} which minimizes the error, you need to calculate $d\varepsilon/dC_{12}=0$

$$\Rightarrow \frac{d}{dC_{12}} \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2 dt \right] = 0$$
$$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[\frac{d}{dC_{12}} f_1^2(t) - \frac{d}{dC_{12}} 2f_1(t)C_{12}f_2(t) + \frac{d}{dC_{12}} f_2^2(t)C_{12}^2 \right] dt = 0$$

Derivative of the terms which do not have C_{12} term are zero.

$$\Rightarrow \int_{t_1}^{t_2} -2f_1(t)f_2(t)dt + 2C_{12} \int_{t_1}^{t_2} [f_2^2(t)]dt = 0$$

If $C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$ component is zero, then two signals are said to be orthogonal.

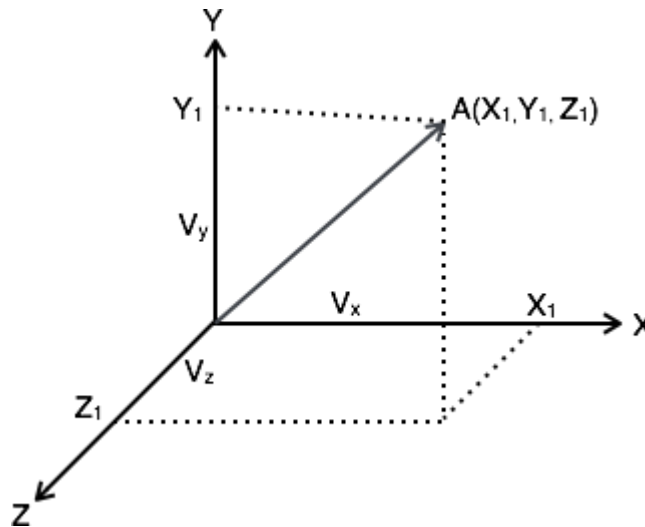
Put $C_{12} = 0$ to get condition for orthogonality.

$$0 = \frac{\int_{t_1}^{t_2} f_1(t)f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$$

$$\int_{t_1}^{t_2} f_1(t)f_2(t)dt = 0$$

Orthogonal Vector Space

A complete set of orthogonal vectors is referred to as orthogonal vector space. Consider a three dimensional vector space as shown below:



Consider a vector A at a point (X₁, Y₁, Z₁). Consider three unit vectors (V_x, V_y, V_z) in the direction of X, Y, Z axis respectively. Since these unit vectors are mutually orthogonal, it satisfies that

$$V_X \cdot V_X = V_Y \cdot V_Y = V_Z \cdot V_Z = 1$$

$$V_X \cdot V_Y = V_Y \cdot V_Z = V_Z \cdot V_X = 0$$

We can write above conditions as

$$V_a \cdot V_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

The vector A can be represented in terms of its components and unit vectors as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z \dots \dots \dots (1)$$

Any vectors in this three dimensional space can be represented in terms of these three unit vectors only.

If you consider n dimensional space, then any vector A in that space can be represented as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z + \dots + N_1 V_N \dots \dots (2)$$

As the magnitude of unit vectors is unity for any vector A

The component of A along x axis = $A \cdot V_X$

The component of A along Y axis = $A \cdot V_Y$

The component of A along Z axis = $A \cdot V_Z$

Similarly, for n dimensional space, the component of A along some G axis

$$= A \cdot V_G \dots \dots \dots (3)$$

Substitute equation 2 in equation 3.

$$\Rightarrow CG = (X_1 V_X + Y_1 V_Y + Z_1 V_Z + \dots + G_1 V_G \dots + N_1 V_N) V_G$$

$$= X_1 V_X V_G + Y_1 V_Y V_G + Z_1 V_Z V_G + \dots + G_1 V_G V_G \dots + N_1 V_N V_G$$

$$= G_1 \quad \text{since } V_G V_G = 1$$

$$\text{If } V_G V_G \neq 1 \text{ i.e. } V_G V_G = k$$

$$AV_G = G_1 V_G V_G = G_1 K$$

$$G_1 = \frac{(AV_G)}{K}$$

Orthogonal Signal Space

Let us consider a set of n mutually orthogonal functions $x_1(t)$, $x_2(t)$... $x_n(t)$ over the interval t_1 to t_2 . As these functions are orthogonal to each other, any two signals $x_j(t)$, $x_k(t)$ have to satisfy the orthogonality condition. i.e.

$$\int_{t_1}^{t_2} x_j(t)x_k(t)dt = 0 \text{ where } j \neq k$$

$$\text{Let } \int_{t_1}^{t_2} x_k^2(t)dt = k_k$$

Let a function $f(t)$, it can be approximated with this orthogonal signal space by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t) + f_e(t)$$

$$= \sum_{r=1}^n C_r x_r(t)$$

$$f(t) = f(t) - \sum_{r=1}^n C_r x_r(t)$$

$$\text{Mean square error } \varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - \sum_{r=1}^n C_r x_r(t)]^2 dt$$

The component which minimizes the mean square error can be found by

$$\frac{d\varepsilon}{dC_1} = \frac{d\varepsilon}{dC_2} = \dots = \frac{d\varepsilon}{dC_k} = 0$$

$$\text{Let us consider } \frac{d\varepsilon}{dC_k} = 0$$

$$\frac{d}{dC_k} \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - \sum_{r=1}^n C_r x_r(t)]^2 dt \right] = 0$$

All terms that do not contain C_k is zero. i.e. in summation, $r=k$ term remains and all other terms are zero.

$$\int_{t_1}^{t_2} -2f(t)x_k(t)dt + 2C_k \int_{t_1}^{t_2} [x_k^2(t)]dt = 0$$

$$\Rightarrow C_k = \frac{\int_{t_1}^{t_2} f(t)x_k(t)dt}{\int_{t_1}^{t_2} x_k^2(t)dt}$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)x_k(t)dt = C_k K_k$$

Mean Square Error:

The average of square of error function $f_e(t)$ is called as mean square error. It is denoted by ε (epsilon).

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t) - \sum_{r=1}^n C_r x_r(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} [f_e^2(t)]dt + \sum_{r=1}^n C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt - 2\sum_{r=1}^n C_r \int_{t_1}^{t_2} x_r(t)f(t)dt \right]$$

You know that $C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt = C_r \int_{t_1}^{t_2} x_r(t)f(t)dt = C_r^2 K_r$

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} [f^2(t)]dt + \sum_{r=1}^n C_r^2 K_r - 2\sum_{r=1}^n C_r^2 K_r \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} [f^2(t)]dt - \sum_{r=1}^n C_r^2 K_r \right]$$

$$\therefore \varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} [f^2(t)]dt + (C_1^2 K_1 + C_2^2 K_2 + \dots + C_n^2 K_n) \right]$$

The above equation is used to evaluate the mean square error.

Closed and Complete Set of Orthogonal Functions:

Let us consider a set of n mutually orthogonal functions $x_1(t), x_2(t) \dots x_n(t)$ over the interval t_1 to t_2 . This is called as closed and complete set when there exist no function $f(t)$ satisfying the condition

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

If this function is satisfying the equation

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

For $k=1,2,\dots$ then $f(t)$ is said to be orthogonal to each and every function of orthogonal set. This set is incomplete without $f(t)$. It becomes closed and complete set when $f(t)$ is included.

$f(t)$ can be approximated with this orthogonal set by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t) + f_e(t)$$

If the infinite series $C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t)$ converges to $f(t)$ then mean square error is zero.

Orthogonality in Complex Functions:

If $f_1(t)$ and $f_2(t)$ are two complex functions, then $f_1(t)$ can be expressed in terms of $f_2(t)$ as

$f_1(t) = C_{12}f_2(t)$.. with negligible error

$$\text{Where } C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2^*(t)dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt}$$

Where $f_2^*(t)$ is the complex conjugate of $f_2(t)$

If $f_1(t)$ and $f_2(t)$ are orthogonal then $C_{12} = 0$

$$\frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t) dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} f_1(t) f_2^*(t) dt = 0$$

The above equation represents orthogonality condition in complex functions.