UNIT-4

STOCHASTIC PROCESSES—SPECTRAL CHARACTERISTICS

Definition of Power Spectral Density of a WSS Process

Let us define the truncated random process as follows

$$X_T(t) = X(t) \quad -T < t < T$$
$$= 0 \quad \text{otherwise}$$
$$= X(t)rect(\frac{t}{2T})$$

$\{X_T(t)\}$

where is the unity-amplitude rectangular pulse of width centering the origin. As will represent the random process define the mean-square integral

Applying the Pareseval's theorem we find the energy of the signal

$rect(\frac{t}{2T})$ $2T$

$t \to \infty,\ \{X_T(t)\}$ $\{X(t)\}$

Therefore, the power associated with is

$$FTX_T(\omega) = \int_{-T}^{T} X_T(t)e^{-j\omega t}dt$$

And

The average power is given by

$$\int_{-T}^{T} X_T^2(t)dt = \int_{-\infty}^{\infty} |FTX_T(\omega)|^2 d\omega$$

$\{X_T(t)\}$

Where the contribution to the average is power at frequency w and represents the power spectral density of As the left-hand side in the above expression represents the

$$\frac{1}{2T}\int_{-T}^{T} X_T^2(t)dt = \frac{1}{2T}\int_{-\infty}^{\infty} |FTX_T(\omega)|^2 d\omega$$

$$\frac{1}{2T}E\int_{-T}^{T} X_T^2(t)dt = \frac{1}{2T}E\int_{-\infty}^{\infty} |FTX_T(\omega)|^2 d\omega = E\int_{-\infty}^{\infty} \frac{|FTX_T(\omega)|^2}{2T} d\omega$$

$$\frac{E|FTX_T(\omega)|^2}{2T}$$

$\{X_T(t)\}$ $T \to \infty$

$\{X(t)\}$

average power of $S_X(\omega)$ $\{X(t)\}$
Therefore, the PSD of the process is defined in the limiting sense by

$$S_X(\omega) = \lim_{T\to\infty} \frac{E\left|FTX_T(\omega)\right|^2}{2T}$$

Relation between the autocorrelation function and PSD: Wiener-Khinchin-Einstein theorem

We have

$$E\frac{|FTX_T(\omega)|^2}{2T} = E\frac{FTX_T(\omega)FTX_T{}^*(\omega)}{2T}$$
$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} EX_T(t_1)X_T(t_2)e^{-j\omega t_1}e^{+j\omega t_2}dt_1dt_2$$
$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

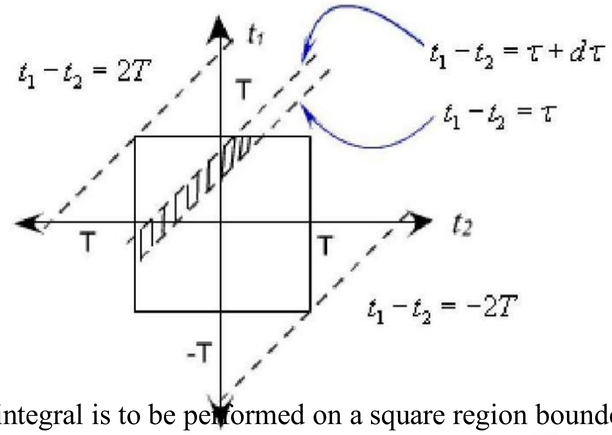


Figure 1

Note that the above integral is to be performed on a square region bounded by and as illustrated in Figure 1.Substitute so that is a family of straight lines parallel to The differential area in terms of is given by the shaded area and equal to The double integral is now replaced by a single integral in

$t_2 = \pm T$

$t_1 = \pm T$

$t_1 - t_2 = \tau$ $t_1 = t_2 + \tau$

$t_1 - t_2 = 0.$ τ

$(2T - |\tau|)d\tau.$ τ

Therefore, $E\dfrac{FTX_T(\omega)X_T^*(\omega)}{2T} = \dfrac{1}{2T}\int_{-2T}^{2T} R_x(\tau)e^{-j\omega\tau}(2T-|\tau|)d\tau$

$$= \int_{-2T}^{2T} R_X(\tau)e^{-j\omega\tau}(1-\frac{|\tau|}{2T})d\tau$$

If $R_X(\tau)$ is integral then the right hand integral converges to $\int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$ as $T\to\infty$

$$\therefore \lim_{T\to\infty}\frac{E|FTX_T(\omega)|^2}{2T} = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$

As we have noted earlier, the power spectral density $S_X(\omega) = \lim_{T\to\infty}\dfrac{E|FTX_T(\omega)|^2}{2T}$ is the contribution to the average power at frequency ω and is called the power spectral density of $\{X(t)\}$. Thus ,

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$

and using the inverse Fourier transform

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}d\omega$$

Example 1 The autocorrelation function of a WSS process $\{X(t)\}$ is given by $R_X(\tau) = a^2e^{-b|\tau|} \quad b>0$ Find the power spectral density of the process.

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} a^2e^{-b|\tau|}e^{-j\omega\tau}d\tau$$

$$= \int_{-\infty}^{0} a^2e^{b\tau}e^{-j\omega\tau}d\tau + \int_{0}^{\infty} a^2e^{-b\tau}e^{-j\omega\tau}d\tau$$

$$= \frac{a^2}{b-j\omega} + \frac{a^2}{b+j\omega}$$

$$= \frac{2a^2b}{b^2+\omega^2}$$

The autocorrelation function and the PSD are shown in Figure 2

Figure 2

Example 3 Find the PSD of the amplitude-modulated random-phase sinusoid

$$X(t) = M(t)\cos(\omega_c t + \Phi), \qquad \Phi \sim U[0, 2\pi]$$

Where M(t) is a WSS process independent of Φ.

$$\begin{aligned} R_X(\tau) &= E\, M(t+\tau)\ \cos(\omega_c(t+\tau)+\Phi)\ \ M(t)\cos(\omega_c\tau+\Phi) \\ &= E\, M(t+\tau)\, M(t)\ E\cos(\omega_c(t+\tau)+\Phi)\ \cos(\omega_c\tau+\Phi) \\ &\qquad\qquad (\text{Using the independence of } M(t) \text{ and the sinusoid}) \\ &= R_M(\tau)\,\frac{A^2}{2}\cos\omega_c\tau \end{aligned}$$

$$\therefore\ S_X(\omega) = \frac{A^2}{4}\left(S_M(\omega+\omega_c) + S_M(\omega-\omega_c)\right)$$

where $S_M(\omega)$ is the PSD of $M(t)$.

Figure 4 illustrates the above result.

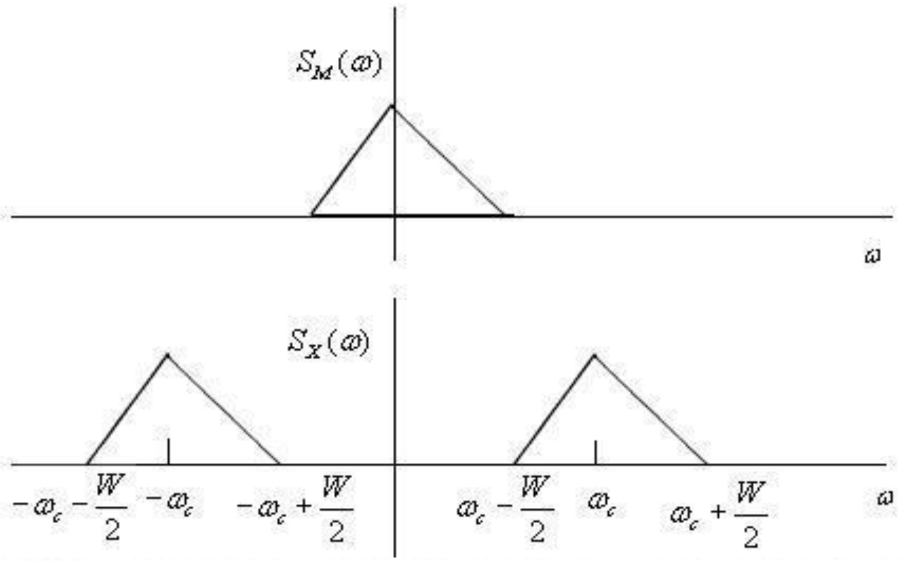


Figure 4

Properties of the PSD

$S_X(\omega)$ $R_X(\tau)$

being the Fourier transform of it shares the properties of the Fourier transform. Here we discuss important properties of $S_X(\omega)$

the average power of a random process is $X(t)$

$$EX^2(t) = R_X(0)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)dw$$

If is real, is a real and even function of .Therefore,

$X(t)$ $R_X(\tau)$ τ

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$
$$= \int_{-\infty}^{\infty} R_X(\tau)(\cos\omega\tau + j\sin\omega\tau)d\tau$$
$$= \int_{-\infty}^{\infty} R_X(\tau)\cos\omega\tau d\tau$$

Thus for a real WSS process, the PSD is always real.

$$= 2\int_{0}^{\infty} R_X(\tau)\cos\omega\tau d\tau$$

Thus is a real and even function of .

From the definition is always non-negative. Thus

If has a periodic component, is periodic and so will have impulses.

$S_X(\omega)$ ω

Cross Power Spectral Density

$$S_X(\omega) = \lim_{T\to\infty}\frac{E\left|X_T(\omega)\right|^2}{2T}$$

Consider a random process which is sum of two real jointly WSS random processes As we have seen earlier, $S_X(\omega) \geq 0$.

$X(t)$ $R_X(\tau)$ $S_X(\omega)$

$\{Z(t)\}$

$\{X(t)\}$ and $\{Y(t)\}$

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

If we take the Fourier transform of both sides,

$$S_Z(\omega) = S_X(\omega) + S_Y(\omega) + FT(R_{XY}(\tau)) + FT(R_{YX}(\tau))$$

Where $FT(.)$ stands for the Fourier transform.

Thus we see that $S_Z(\omega)$ includes contribution from the Fourier transform of the cross- correlation functions
These Fourier transforms represent cross power spectral densities.

Definition of Cross Power Spectral Density

$R_{XY}(\tau)$ and $R_{YX}(\tau)$.

Given two real jointly WSS random processes the cross power spectral density (CPSD) is defined as

$\{X(t)\}$ and $\{Y(t)\}$

Where $S_{XY}(\omega)$ are the Fourier transform of the truncated processes

respectively and denotes the complex

$$\boxed{S_{XY}(\omega) = \lim_{T\to\infty} E\frac{FTX_T^*(\omega)FTY_T(\omega)}{2T}}$$

$FTX_T(\omega)$ and $FTY_T(\omega)$

$$X_T(t) = X(t)rect(\frac{t}{2T}) \text{ and } Y_T(t) = Y(t)rect(\frac{t}{2T})$$

Proceeding in the same way as the derivation of the Wiener-Khinchin-Einstein theorem for the WSS process, it can be shown that

$S_{YX}(\omega)$

and $$\boxed{S_{YX}(\omega) = \lim_{T\to\infty} E\frac{FTY_T^*(\omega)FTX_T(\omega)}{2T}}$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j\omega\tau}d\tau$$

The cross-correlation function and the cross-power spectral density form a Fourier transform pair and we can write

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

and

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} S_{YX}(\omega) e^{j\omega\tau} d\omega$$

Properties of the CPSD

The CPSD is a complex function of the frequency 'w'. Some properties of the CPSD of two jointly WSS processes
are listed below:
$\{X(t)\}$ and $\{Y(t)\}$
(1)

Note that $S_{XY}(\omega) = S_{YX}^{*}(\omega)$

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

$$\begin{aligned} \therefore S_{XY}(\omega) &= \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} R_{YX}(-\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} R_{YX}(\tau) e^{j\omega\tau} d\tau \\ &= S_{YX}^{*}(\omega) \end{aligned}$$

We have $\mathrm{Re}(S_{XY}(\omega))$ is an even function of ω and $\mathrm{Im}(S_{XY}(\omega))$ is an odd function of ω.

$$\begin{aligned} S_{XY}(\omega) &= \int_{-\infty}^{\infty} R_{XY}(\tau)(\cos\omega\tau + j\sin\omega\tau) d\tau \\ &= \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau d\tau + j\int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau) d\tau \\ &= \mathrm{Re}(S_{XY}(\omega)) + j\,\mathrm{Im}(S_{XY}(\omega)) \end{aligned}$$

where

$$\mathrm{Re}(S_{XY}(\omega)) = \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau d\tau \text{ is an even function of } \omega \text{ and}$$

$$\mathrm{Im}(S_{XY}(\omega)) = \int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau d\tau \text{ is an odd function of } \omega \text{ and}$$

$\{X(t)\}$ and $\{Y(t)\}$

$S_{XY}(\omega) = S_{YX}(\omega) = \mu_X \mu_Y \delta(\omega)$

Ifare uncorrelated and have constant means, then

$\partial(\omega)$

Where is the Dirac delta function?

Observe that

$$\begin{aligned} R_{XY}(\tau) &= EX(t+\tau)Y(t) \\ &= EX(t+\tau)EY(t) \\ &= \mu_X \mu_Y \\ &= \mu_Y \mu_X \\ &= R_{YX}(\tau) \end{aligned}$$

$$\therefore S_{XY}(\omega) = S_{YX}(\omega) = \mu_X \mu_Y \delta(\omega)$$

If are orthogonal, then

$\{X(t)\}$ and $\{Y(t)\}$

$S_{XY}(\omega) = S_{YX}(\omega) = 0$

If are orthogonal, we have

$\{X(t)\}$ and $\{Y(t)\}$

$$\begin{aligned} R_{XY}(\tau) &= EX(t+\tau)Y(t) \\ &= 0 \\ &= R_{XY}(\tau) \end{aligned}$$

the cross power between is defined by

$$\therefore S_{XY}(\omega) = S_{YX}(\omega) = 0$$

P_{XY} $\{X(t)\}$ and $\{Y(t)\}$

Applying Parseval's theorem, we get

$$P_{XY} = \lim_{T \to \infty} \frac{1}{2T} E \int_{-T}^{T} X(t)Y(t)\,dt$$

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} E \int_{-T}^{T} X(t)Y(t)dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} E \int_{-\infty}^{\infty} X_T(t)Y_T(t)dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} E \frac{1}{2\pi} \int_{-\infty}^{\infty} FTX_T^*(\omega)\, FTY_T(\omega)d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{EFTX_T^*(\omega)\, FTY_T(\omega)}{2T} d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

$$\therefore\ P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

Similarly,

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}^*(\omega)d\omega$$
$$= P_{XY}^*$$

Example 1 Consider the random process given by $Z(t) = X(t) + Y(t)$ discussed in the beginning of the lecture. Here $\{Z(t)\}$ is the sum of two jointly WSS orthogonal random processes $\{X(t)\}$ and $\{Y(t)\}$

We have,

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

Taking the Fourier transform of both sides,

$$S_Z(\omega) = S_X(\omega) + S_Y(\omega) + S_{XY}(\omega) + S_{YX}(\omega)$$
$$\therefore \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Z(\omega)d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega)d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)d\omega$$

Therefore,

$$\boxed{P_Z(\omega) = P_X(\omega) + P_Y(\omega) + P_{XY}(\omega) + P_{YX}(\omega)}$$

Wiener-Khinchin-Einstein theorem

The Wiener-Khinchin-Einstein theorem is also valid for discrete-time random processes. The power spectral density $S_X(\omega)$ of the WSS process $\{X[n]\}$ is the discrete-time Fourier transform of autocorrelation sequence.

$$S_X(\omega) = \sum_{m=-\infty}^{\infty} R_X[m] e^{-j\omega m} \qquad -\pi \le w \le \pi$$

$R_x[m]$ Is related to $S_X(\omega)$ by the inverse discrete-time Fourier transform and given by

$$R_X[m]) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega) e^{j\omega m} d\omega$$

Thus $R_x[m]$ and $S_X(\omega)$ forms a discrete-time Fourier transform pair. A generalized PSD can be defined in terms of $z - \text{transform}$ as follows

clearly, $S_X(z) = \sum_{m=-\infty}^{\infty} R_x[m] z^{-m}$

$$S_X(\omega) = S_X(z)\big|_{z=e^{j\omega}}$$

Linear time-invariant systems

In many applications, physical systems are modeled as linear time-invariant (LTI) systems. The dynamic behavior of an LTI system to deterministic inputs is described by linear differential equations. We are familiar with time and transform domain (such as Laplace transform and Fourier transform) techniques to solve these differential equations. In this lecture, we develop the technique to analyze the response of an LTI system to WSS random process.

The purpose of this study is two-folds:

Analysis of the response of a system
Finding an LTI system that can optimally estimate an unobserved random process from an observed process. The observed random process is statistically related to the unobserved random process. For example, we may have to find LTI system (also called a filter) to estimate the signal from the noisy observations.

Basics of Linear Time Invariant Systems

$x(t)$

A system is modeled by a transformation T that maps an input signal to an output signal y(t) as shown in Figure 1. We can thus write, $y(t) = T\left[x(t)\right]$

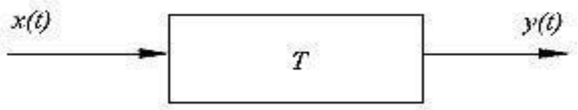


Figure 1

Linear system

The system is called linear if the principle of superposition applies: the weighted sum of inputs results in the weighted sum of the corresponding outputs. Thus for a linear system

Example 1 Consider the output of a differentiator, given by

$$T\left[a_1x_1(t)+a_2x_2(t)\right] = a_1T\left[x_1(t)\right]+a_2T\left[x_2(t)\right]$$

Then,

$$y(t) = \frac{d\,x(t)}{dx}$$

$$\frac{d}{dt}\left(a_1x_1(t) + a_2x_2(t) \right)$$

$$= a_1\frac{d}{dt}x_1(t) + a_2\frac{d}{dt}x_2(t)$$

Hence the linear differentiator is a linear system.

Linear time-invariant system

Consider a linear system with y (t) = T x (t). The system is called time-invariant if

It is easy to check that that the differentiator in the above example is a linear time - invariant system.

$$T\,x(t-t_0) = y(t-t_0) \qquad \forall \quad t_0$$

Response of a linear time-invariant system to deterministic input

As shown in Figure 2, a linear system can be characterised by its impulse response where is the Dirac delta function.

$h(t) = T\delta(t)$ $\delta(t)$

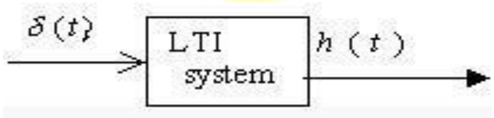


Figure 2

Recall that any function x(t) can be represented in terms of the Dirac delta function as follows

If x(t) is input to the linear system y (t) = T x (t), then

$$x(t) = \int_{-\infty}^{\infty} x(s)\, \delta(t-s)\, ds$$

$$y(t) = T \int_{-\infty}^{\infty} x(s)\, \delta(t-s)\, ds$$

$$= \int_{-\infty}^{\infty} x(s)\, T\delta(t-s)\, ds \quad [\text{Using the linearity property}]$$

Where is the response at time t due to the shifted impulse?

$$= \int_{-\infty}^{\infty} x(s)\, h(t,s)\, ds$$

If the system is time invariant,

$h(t,s) = T\delta(t-s)$ $\delta(t-s)$

Therefore for a linear-time invariant system,

$$h(t,s) = h(t-s)$$

where * denotes the convolution operation. We also note that

$$y(t) = \int_{-\infty}^{\infty} x(s)\, h(t-s)\, ds = x(t) * h(t)$$

$$x(t) * h(t) = h(t) * x(t).$$

Thus for a LTI System, $y(t) = x(t) * h(t) = h(t) * x(t)$

Taking the Fourier transform, we get

$$Y(\omega) = H(\omega)\, X(\omega)$$

where $H(\omega) = FT\, h(t) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$ is the frequency response of the system

Figure 3 shows the input-output relationship of an LTI system in terms of the impulse response and the frequency response.

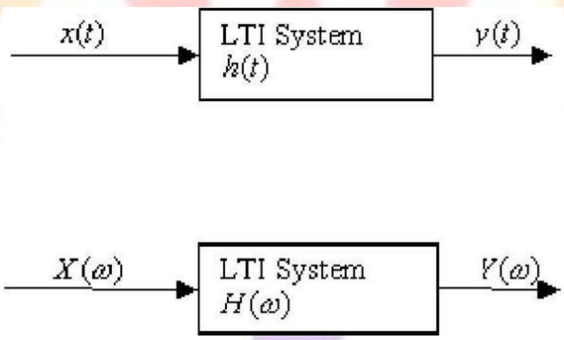


Figure 3

Response of an LTI System to WSS input

Consider an LTI system with impulse response h (t). Suppose $\{X(t)\}$ is a WSS process input to the system. The output $\{Y(t)\}$ of the system is given by

$$Y(t) = \int_{-\infty}^{\infty} h(s)\, X(t-s)\, ds = \int_{-\infty}^{\infty} h(t-s)\, X(s)\, ds$$

Where we have assumed that the integrals exist in the mean square sense.

Mean and autocorrelation of the output process $\{Y(t)\}$

$$
\begin{aligned}
EY(t) &= E\int_{-\infty}^{\infty} h(s)X(t-s)\,ds \\
&= \int_{-\infty}^{\infty} h(s)EX(t-s)\,ds \\
&= \int_{-\infty}^{\infty} h(s)\mu_X\,ds \\
&= \mu_X \int_{-\infty}^{\infty} h(s)\,ds \\
&= \mu_X H(0)
\end{aligned}
$$

Where $H(0)$ is the frequency response at 0 frequency ($\omega = 0$) and given by

$$H(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt\Big|_{\omega=0} = \int_{-\infty}^{\infty} h(t)\,dt$$

T The Cross correlation of the input {X(t)} and the out put {Y (t)} is given by

$$
\begin{aligned}
E\big(X(t+\tau)Y(t)\big) &= E\ X(t+\tau)\int_{-\infty}^{\infty} h(s)\ X(t-s)\ ds \\
&= \int_{-\infty}^{\infty} h(s)\ \ E\ X(t+\tau)\ X(t-s)\ ds \\
&= \int_{-\infty}^{\infty} h(s)\ \ R_X(\tau+s)\,ds \\
&= \int_{-\infty}^{\infty} h(-u)\ \ R_X(\tau-u)\,du \qquad [\ \text{Put } s = -u\ ] \\
&= h(-\tau) * R_X(\tau)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\quad & R_{XY}(\tau) = h(-\tau) * R_X(\tau) \\
also\quad & R_{YX}(\tau) = R_{XY}(-\tau) = h(\tau) * R_X(-\tau) \\
& \qquad = h(\tau) * R_X(\tau)
\end{aligned}
$$

Therefore, the mean of the output process $\{Y(t)\}$ is a constant

The Cross correlation of the input {X (t)} and the out put {Y (t)} is given by

$$E\big(X(t+\tau)Y(t)\big) = E\, X(t+\tau) \int_{-\infty}^{\infty} h(s)\, X(t-s)\, ds$$
$$= \int_{-\infty}^{\infty} h(s)\;\; E\, X(t+\tau)\, X(t-s)\, ds$$
$$= \int_{-\infty}^{\infty} h(s)\;\; R_X(\tau+s)\, ds$$
$$= \int_{-\infty}^{\infty} h(-u)\;\; R_X(\tau-u)\, du \qquad [\text{ Put } s = -u\ ]$$
$$= h(-\tau) * R_X(\tau)$$

$$\therefore \quad R_{XY}(\tau) = h(-\tau) * R_X(\tau)$$
$$also \quad R_{YX}(\tau) = R_{XY}(-\tau) = h(\tau) * R_X(-\tau)$$
$$= h(\tau) * R_X(\tau)$$

Thus we see that $R_{XY}(\tau)$ is a function of lag τ only. Therefore, $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary.

The autocorrelation function of the output process {Y(t)} is given by,

$$\therefore \quad E\big(Y(t+\tau)Y(t)\big) = E \int_{-\infty}^{\infty} h(s)\; X(t+\tau-s)\, ds\, Y(t)$$
$$= \int_{-\infty}^{\infty} h(s)\; E\, X(t+\tau-s)\; Y(t)\, ds$$
$$= \int_{-\infty}^{\infty} h(s)\; R_{XY}(\tau-s)\, ds$$
$$= h(\tau) * R_{XY}(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$$

Thus the autocorrelation of the output process $\{Y(t)\}$ depends on the time-lag τ, i.e.,

Thus $$EY(t)Y(t+\tau) = R_Y(\tau)$$

The above analysis indicates that for an LTI system with WSS input

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

the output is WSS and
The input and output are jointly WSS.

The average power of the output process is given by $\{Y(t)\}$

$$P_Y = R_Y(0)$$
$$= R_X(0) * h(0) * h(0)$$

Power spectrum of the output process

Using the property of Fourier transform, we get the power spectral density of the output process given by

Also note that

$$S_Y(\omega) = S_X(\omega) H(\omega) H^*(\omega)$$
$$= S_X(\omega) |H(\omega)|^2$$

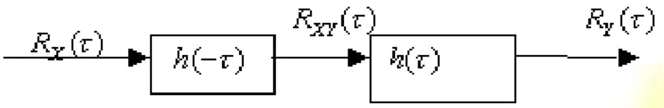
Taking the Fourier transform of we get the cross power spectral density given by

$$R_{XY}(\tau) = h(-\tau) * R_X(\tau)$$
$$and \; R_{YX}(\tau) = h(\tau) * R_X(\tau)$$

$R_{XY}(\tau)$

$S_{XY}(\omega)$

$$S_{XY}(\omega) = H^*(\omega) S_X(\omega)$$
and
$$S_{YX}(\omega) = H(\omega) S_X(\omega)$$



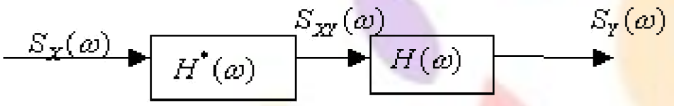


Figure 4

Example 3

A random voltage modeled by a white noise process $\{X(t)\}$ with power spectral density $\frac{N_0}{2}$ is input to an RC network shown in the Figure 7.
Find (a) output PSD $S_Y(\omega)$
output auto correlation function $R_Y(\tau)$
average output power $EY^2(t)$

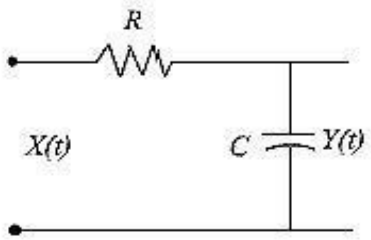


Figure 7

The frequency response of the system is given by

$$H(\omega) = \frac{\frac{1}{jC\omega}}{R + \frac{1}{jC\omega}}$$

$$= \frac{1}{jRC\omega + 1}$$

Therefore,

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$$

$$= \frac{1}{R^2C^2\omega^2 + 1} S_X(\omega)$$

$$= \frac{1}{R^2C^2\omega^2 + 1} \frac{N_0}{2}$$

(a)

Taking the inverse Fourier transform

$$R_Y(\tau) = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}$$

Average output power

$$EY^2(t) = R_Y(0) = \frac{N_0}{4RC}$$

Rice's representation or quadrature representation of a WSS process

An arbitrary zero-mean WSS process $\{X(t)\}$ can be represented in terms of the slowly varying components $X_c(t)$ and $X_s(t)$ as follows:

(1)

$$X(t) = X_c(t)\cos\omega_0 t - X_s(t)\sin\omega_0 t$$

where ω_0 is a center frequency arbitrary chosen in the band $\{(\omega_0 - \frac{B}{2} \le |\omega| \le \omega_0 + \frac{B}{2})\}$. $X_c(t)$ and $X_s(t)$ are respectively called the in-phase and the quadrature-phase components of $X(t)$.

Let us choose a dual process $\{Y(t)\}$ such that

$$X(t) + jY(t) = \left(X_c(t) + jX_s(t)\right)e^{j\omega_0 t}$$
$$= \underbrace{\left(X_c(t)\cos\omega_0 t - X_s(t)\sin\omega_0 t\right)}_{X(t)} + j\underbrace{\left(X_c(t)\sin\omega_0 t + X_s(t)\cos\omega_0 t\right)}_{Y(t)}$$

then ,

$$X_c(t) = X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t$$

(2)

and

$$X_s(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$$

(3)

For such a representation, we require the processes $\{X_c(t)\}$ and $\{X_s(t)\}$ to be WSS. Note that

As is zero mean, we require that

$$EX(t) = \cos\omega_0 t EX_c(t) - \sin\omega_0 t EX_s(t)$$

And $\{X(t)\}$

$$EX_c(t) = 0$$

Again

$$EX_s(t) = 0$$

$$EX_c(t) = \cos\omega_0 t\; EX(t) + \sin\omega_0 t\; EY(t)$$
$$EX_s(t) = \cos\omega_0 t EX(t) - \sin\omega_0 t EY(t)$$

As each of $EX_c(t)$, $EX_s(t)$ and $EX(t)$ is zero-mean, we require that $EY(t) = 0$.

Also

$$R_{X_c}(t+\tau, t) = E[X(t+\tau)\cos\omega_0(t+\tau) + Y(t+\tau)\sin\omega_0(t+\tau)][X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t]$$
$$= R_X(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t + R_Y(\tau)\sin\omega_0(t+\tau)\sin\omega_0 t + R_{XY}(\tau)\cos\omega_0(t+\tau)\sin\omega_0 t$$
$$+ R_{YX}(\tau)\sin\omega_0(t+\tau)\cos\omega_0 t$$

and

$$R_{X_s}(t+\tau, t) = R_X(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t + R_Y(\tau)\sin\omega_0(t+\tau)\sin\omega_0 t$$
$$- R_{XY}(\tau)\cos\omega_0(t+\tau)\sin\omega_0 t - R_{YX}(\tau)\sin\omega_0(t+\tau)\cos\omega_0 t$$

and

$$R_{X_cX_s}(t+\tau,t) = R_X(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t - R_Y(\tau)\sin\omega_0(t+\tau)\sin\omega_0 t$$
$$- R_{XY}(\tau)\cos\omega_0(t+\tau)\sin\omega_0 t + R_{YX}(\tau)\sin\omega_0(t+\tau)\cos\omega_0 t$$

Thus, $R_{X_c}(t+\tau,t)$, $R_{X_s}(t+\tau,t)$ and $R_{X_cX_s}(t+\tau,t)$ will be independent of t if and only if

and

$$R_{X_cX_s}(\tau) = R_X(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t - R_Y(\tau)\sin\omega_0(t+\tau)\sin\omega_0 t$$
$$- R_{XY}(\tau)\cos\omega_0(t+\tau)\sin\omega_0 t + R_{YX}(\tau)\sin\omega_0(t+\tau)\cos\omega_0 t$$
$$= R_X(\tau)[\cos\omega_0(t+\tau)\cos\omega_0 t - \sin\omega_0(t+\tau)\sin\omega_0 t]$$
$$- R_{XY}(\tau)[\cos\omega_0(t+\tau)\sin\omega_0 t - \sin\omega_0(t+\tau)\cos\omega_0 t]$$
$$= R_X(\tau)\cos\omega_0\tau - R_{XY}(\tau)\sin(-\omega_0\tau)$$
$$= R_X(\tau)\cos\omega_0\tau - R_{YX}(\tau)\sin\omega_0\tau$$

$\{Y(t)\}$

How to find $\{Y(t)\}$ satisfying the above two conditions? $\{X(t)\}$

For this, consider to be the Hilbert transform of , i.e.

$$Y(t) = \int_{-\infty}^{\infty} X(s)h(t-s)ds$$

$$h(t) = \frac{1}{\pi t}$$

Where and the integral is defined in the mean-square sense. See the illustration in Figure 2.

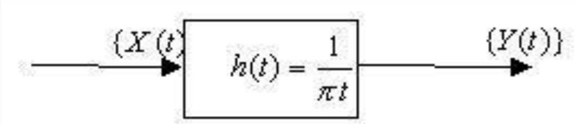


Figure 2

The frequency response of the Hilbert transform is given by

$H(\omega)$

$$H(\omega) = \begin{cases} -j, & \text{if } \omega>0 \\ j, & \text{if } \omega<0 \\ 0, & \text{if } \omega=0 \end{cases}$$

$$\therefore H(\omega) = -j\,\text{sgn}(\omega)$$
$$\text{and } |H(\omega)|^2 = 1$$
$$\therefore S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = S_X(\omega)$$

and

$$S_{XY}(\omega) = H(\omega) S_{XX}(\omega) = \begin{cases} jS_{XX}(\omega), & \text{for } \omega > 0 \\ -jS_{XX}(\omega), & \text{for } \omega < 0 \end{cases}$$

$$S_{YX}(\omega) = H^*(\omega) S_{XX}(\omega) = \begin{cases} -jS_{XX}(\omega), & \text{for } \omega>0 \\ jS_{XX}(\omega), & \text{for } \omega<0 \end{cases}$$

The Hilbert transform of Y(t) satisfies the following spectral relations

$$S_Y(\omega) = S_X(\omega)$$
and
$$S_{XY}(\omega) = -S_{YX}(\omega)$$

From the above two relations, we get

$$R_X(\tau) = R_Y(\tau)$$
and
$$R_{XY}(\tau) = -R_{YX}(\tau)$$

The Hilbert transform of $X(t)$ is generally denoted as $\hat{X}(t)$. Therefore, from (2) and (3) we establish

and

$$X_c(t) = X(t)\cos\omega_0 t + \hat{X}(t)\sin\omega_0 t,$$
$$X_s(t) = X(t)\cos\omega_0 t - \hat{X}(t)\sin\omega_0 t$$

$$X(t) = X_c(t)\cos\omega_0 t - X_S(t)\sin\omega_0 t$$

The realization for the in phase and the quadrature phase components is shown in Figure 3 below.

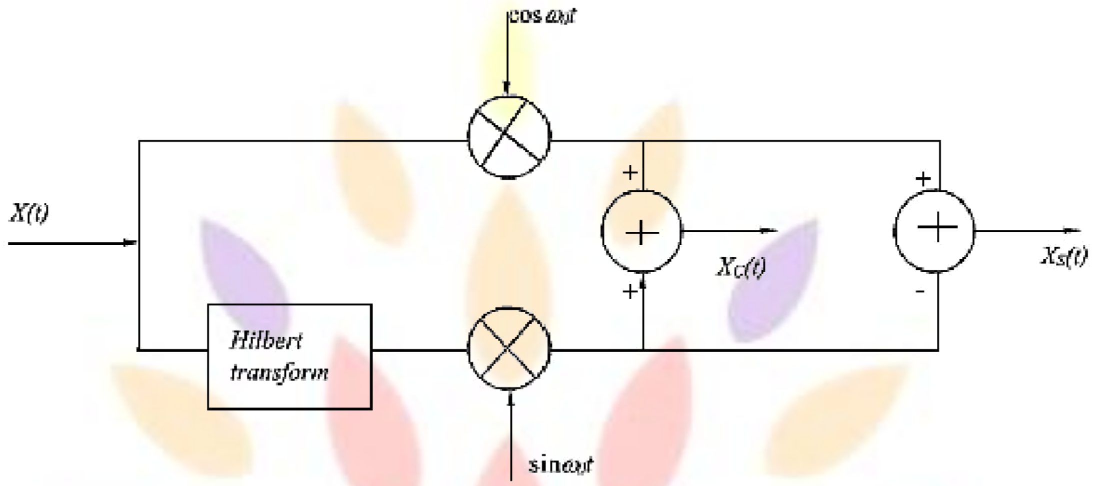


Figure 3

From the above analysis, we can summarize the following expressions for the autocorrelation functions

$$R_{X_c}(\tau) = R_{X_s}(\tau)$$

Where

$$= R_X(\tau)\cos\omega_0\tau + R_{YX}(\tau)\sin\omega_0\tau$$

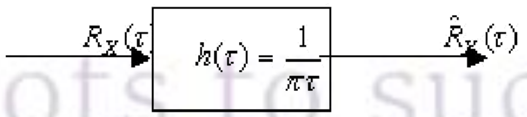
$$= R_X(\tau)\cos\omega_0\tau + h(t) * R_X(\tau)\sin\omega_0\tau \qquad \because R_{YX}(\tau) = h(\tau) * R_X(\tau)$$

$$= R_X(\tau)\cos\omega_0\tau + \hat{R}_X(\tau)\sin\omega_0\tau$$

See the illustration in Figure 4

$$\hat{R}_X(\tau) = \text{Hilbert transform of } R_X(\tau)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi s} R_X(\tau - s)ds$$



The variances $\sigma^2_{X_c}$ and $\sigma^2_{X_s}$ are given by

$$\sigma^2_{X_c} = \sigma^2_{X_s} = R_X(0).$$

Taking the Fourier transform of $R_{X_c}(\tau)$ and $R_{X_s}(\tau)$, we get

$$S_{X_c}(\omega) = S_{X_s}(\omega) = \begin{cases} S_X(\omega-\omega_0)+S_X(\omega+\omega_0) & |\omega| \le B \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

$$\begin{aligned} R_{X_cX_s}(\tau) &= R_X(\tau)\sin\omega_0\tau - R_{YX}(\tau)\cos\omega_0\tau \\ &= R_X(\tau)\sin\omega_0\tau - \hat{R}_X(\tau)\cos\omega_0\tau \end{aligned}$$

and

$$S_{X_cX_s}(\omega) = \begin{cases} j[S_X(\omega+\omega_0)-S_X(\omega-\omega_0)] & |\omega| \le B \\ 0 & \text{otherwise} \end{cases}$$

Notice that the cross power spectral density $S_{X_cX_s}(\omega)$ is purely imaginary. Particularly, if $S_X(\omega)$ is locally symmetric about ω_0

Implying that

$$S_{X_cX_s}(\omega) = 0$$

Consequently, the zero-mean processes $\{X_c(t)\}$ and $\{X_s(t)\}$ are also uncorrelated

$$R_{X_cX_s}(\tau) = 0$$