UNIT-3 Stochastic Processes-Temporal Characteristics

- The Stochastic process Concept
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- Distribution and Density Functions
- Statistical Independence
- Concept of Stationarity: First-Order Stationary Processes, Second-Order and Wide-Sense Stationarity, Nth-Order and Strict-Sense Stationarity
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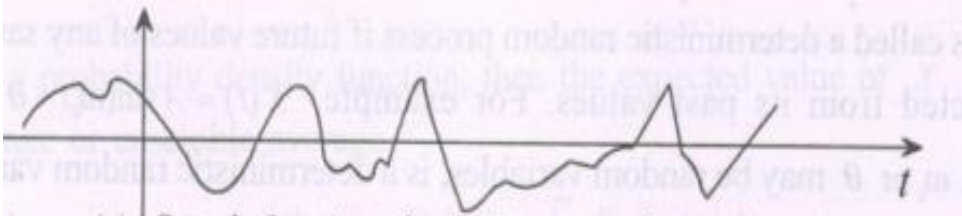
Stochastic process Concept:

The random processes are also called as stochastic processes which deal with randomly varying time wave forms such as any message signals and noise. They are described statistically since the complete knowledge about their origin is not known. So statistical measures are used. Probability distribution and probability density functions give the complete statistical characteristics of random signals. A random process is a function of both sample space and time variables. And can be represented as {X x(s,t)}.

Deterministic and Non-deterministic processes: In general a random process may be deterministic or non deterministic. A process is called as deterministic random process if future values of any sample function can be predicted from its past values. For example, X(t) = A sin (ω_0t+Θ), where the parameters A, ω_0 and Θ may be random variables, is deterministic random process because the future values of the sample function can be detected from its known shape. If future values of a sample function cannot be detected from observed past values, the process is called non-deterministic process.

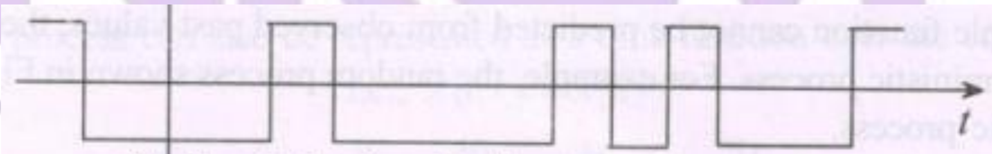
Classification of random process: Random processes are mainly classified into four types based on the time and random variable X as follows.

1. Continuous Random Process: A random process is said to be continuous if both the random variable X and time t are continuous. The below figure shows a continuous random process. The fluctuations of noise voltage in any network is a continuous random process.



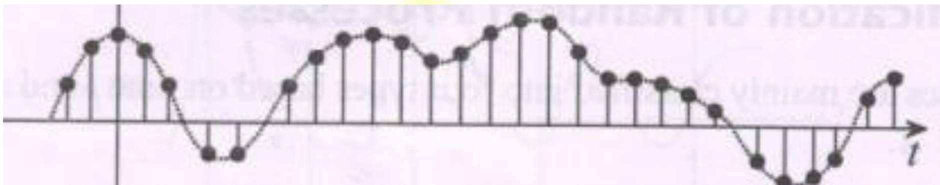
(a) Sample function of a continuous random process

2. Discrete Random Process: In discrete random process, the random variable X has only discrete values while time, t is continuous. The below figure shows a discrete random process. A digital encoded signal has only two discrete values a positive level and a negative level but time is continuous. So it is a discrete random process.



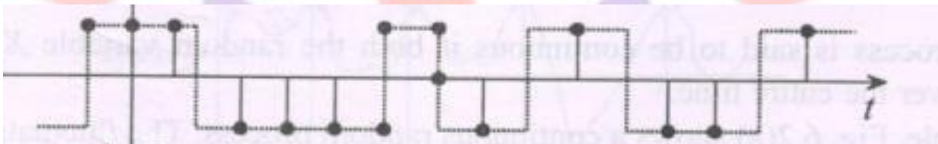
(b) Sample function of a discrete random process

3. Continuous Random Sequence: A random process for which the random variable X is continuous but t has discrete values is called continuous random sequence. A continuous random signal is defined only at discrete (sample) time intervals. It is also called as a discrete time random process and can be represented as a set of random variables {X(t)} for samples t_k, k=0, 1, 2,….

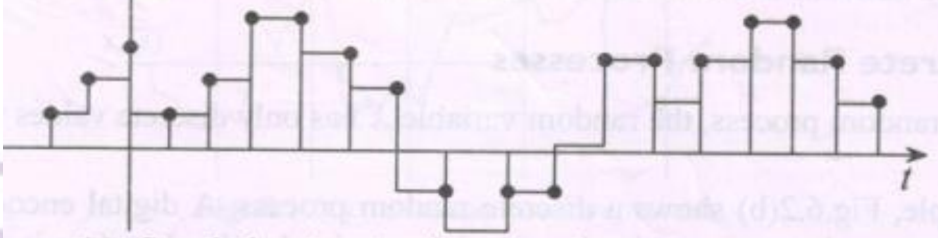


(c) Sample function of a continuous random sequence

4. Discrete Random Sequence: In discrete random sequence both random variable X and time t are discrete. It can be obtained by sampling and quantizing a random signal. This is called the random process and is mostly used in digital signal processing applications. The amplitude of the sequence can be quantized into two levels or multi levels as shown in below figure s (d) and (e).



(d) Sample function of a discrete random sequence



(e) Sample function of a discrete random sequence (multi levels)

Joint distribution functions of random process: Consider a random process X(t). For a single random variable at time t_1, $X_1=X(t_1)$, The cumulative distribution function is defined as $F_X(x_1;t_1) = P\{(X(t_1)\ x_1\}$ where x_1 is any real number. The function $F_X(x_1;t_1)$ is known as the first order distribution function of X(t). For two random variables at time instants t_1 and t_2 $X(t_1) = X_1$ and $X(t_2) = X_2$, the joint distribution is called the second order joint distribution function of the random process X(t) and is given by

$F_X(x_1, x_2 ; t_1, t_2) = P\{(X(t_1)\ x_1, X(t_2)\ x_2\}$. In general for N random variables at N time intervals $X(t_i) = X_i$ i=1,2,…N, the N^{th} order joint distribution function of X(t) is defined as

$F_X(x_1, x_2 \ldots\ldots x_N ; t_1, t_2, \ldots\ldots t_N) = P\{(X(t_1)\ x_1, X(t_2)\ x_2, \ldots X(t_N)\ x_N\}$.

Joint density functions of random process:: Joint density functions of a random process can be obtained from the derivatives of the distribution functions.

1. First order density function:

$$f_X(x_1;t_1) = \frac{dF_X(x1;t1)}{dx_1}$$

2. second order density function

$$f_X(x_1, x_2 ; t_1, t_2) = \frac{\partial^2 F_X(x1,x2 ; t1,t2)}{\partial x_1 \partial x_2}$$

3. Nth order density function:

$$f_X(x_1, x_2 \ldots\ldots x_N ; t_1, t_2 \ldots\ldots t_N) = \frac{\partial^2 F_X(x1,x2 \ldots\ldots xN ; t1,t2,\ldots\ldots tN)}{\partial x_1 \partial x_2 \ldots\ldots \partial x_N}$$

Independent random processes: Consider a random process X(t). Let $X(t_i) = x_i$, i= 1,2,…N be N Random variables defined at time constants $t_1,t_2, \ldots t_N$ with density functions $f_X(x_1;t_1), f_X(x_2;t_2), \ldots f_X(x_N ; t_N)$. If the random process X(t) is statistically independent, then the N^{th} order joint density function is equal to the product of individual joint functions of X(t) i.e.

$f_X(x_1, x_2 \ldots\ldots x_N ; t_1, t_2 \ldots\ldots t_N) = f_X(x_1;t_1)\, f_X(x_2;t_2).\, f_X(x_N ; t_N)$. Similarly the joint distribution will be the product of the individual distribution functions.

Statistical properties of Random Processes: The following are the statistical properties of random processes.

1. **Mean:** The mean value of a random process X(t) is equal to the expected value of the random process i.e

$$\bar{X}(\bar{t}) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;t)dx$$

2. **Autocorrelation:** Consider random process X(t). Let X_1 and X_2 be two random variables defined at times t_1 and t_2 respectively with joint density function $f_X(x_1, x_2 ; t_1, t_2)$. The correlation of X_1 and X_2, $E[X_1 X_2] = E[X(t_1) X(t_2)]$ is called the autocorrelation function of the random process X(t) defined as
$R_{XX}(t_1,t_2) = E[X_1 X_2] = E[X(t_1) X(t_2)]$ or

$$R_{XX}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_x(x1,x2 ;\ t1,t2)\, dx_1 dx_2$$

3. **Cross correlation:** Consider two random processes X(t) and Y(t) defined with random variables X and Y at time instants t_1 and t_2 respectively. The joint density function is $f_{xy}(x,y ; t_1,t_2)$.Then the correlation of X and Y, $E[XY] = E[X(t_1) Y(t_2)]$ is called the cross correlation function of the random processes X(t) and Y(t) which is defined as
$R_{XY}(t_1,t_2) = E[X Y] = E[X(t_1) Y(t_2)]$ or

$$R_{XY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{x,y}(x,y ;\ t1,t2)\, dxdy$$

Stationary Processes: A random process is said to be stationary if all its statistical properties such as mean, moments, variances etc… do not change with time. The stationarity which depends on the density functions has different levels or orders.

1. **First order stationary process:** A random process is said to be stationary to order one or first order stationary if its first order density function does not change with time or shift in time value. If X(t) is a first order stationary process then

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta t) \text{ for any time } t_1.$$

Where Δt is shift in time value. Therefore the condition for a process to be a first order stationary random process is that its mean value must be constant at any time instant. i.e.

$$E[X(t)] = \bar{X} = constant$$

2. **Second order stationary process:** A random process is said to be stationary to order two or second order stationary if its second order joint density function does not change with time or shift in time value i.e. $f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2;t_1+\Delta t, t_2+\Delta t)$ for all t_1,t_2 and Δt. It is a function of time difference (t_2, t_1) and not absolute time t. Note that a second order stationary process is also a first order stationary process. The condition for a process to be a second order stationary is that its autocorrelation should depend only on time differences and not on absolute time. i.e. If

$R_{XX}(t_1,t_2) = E[X(t_1) X(t_2)]$ is autocorrelation function and $\tau = t_2 - t_1$ then

$R_{XX}(t_1,t_1+\tau) = E[X(t_1) X(t_1+\tau)] = R_{XX}(\tau)$. $R_{XX}(\tau)$ should be independent of time t.

3. **Wide sense stationary (WSS) process:** If a random process X(t) is a second order stationary process, then it is called a wide sense stationary (WSS) or a weak sense stationary process. However the converse is not true.

The conditions for a wide sense stationary process are

1. $E[X(t)] = \bar{X} = constant$
2. $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ is independent of absolute time t.

Joint wide sense stationary process: Consider two random processes X(t) and Y(t). If they are jointly WSS, then the cross correlation function of X(t) and Y(t) is a function of time difference $\tau = t_2 - t_1$ only and not absolute time. i.e. $R_{XY}(t_1,t_2) = E[X(t_1) Y(t_2)]$. If $\tau = t_2 - t_1$ then $R_{XY}(t,t+\tau) = E[X(t) Y(t+\tau)] = R_{XY}(\tau)$.

Therefore the conditions for a process to be joint wide sense stationary are

1. $E[X(t)] = \bar{X} = constant.$
2. $E[Y(t)] = \bar{Y} = constant$
3. $E[X(t) Y(t+\tau)] = R_{XY}(\tau)$ is independent of time t.

Strict sense stationary (SSS) processes: A random process X(t) is said to be strict Sense stationary if its Nth order joint density function does not change with time or shift in time value. i.e.

$f_X(x_1, x_2 \ldots\ldots x_N ; t_1, t_2, \ldots\ldots t_N) = f_X(x_1, x_2 \ldots\ldots x_N ; t_1+\Delta t, t_2+\Delta t, \ldots t_N+\Delta t)$ for all $t_1, t_2 \ldots t_N$ and Δt.

A process that is stationary to all orders n=1,2,. . . N is called strict sense stationary process. Note that SSS process is also a WSS process. But the reverse is not true.

Time Average Function: Consider a random process X(t). Let x(t) be a sample function which exists for all time at a fixed value in the given sample space S. The average value of x(t) taken over all times is called the time average of x(t). It is also called mean value of x(t). It can be expressed as

$$\bar{x} = A[X(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

Time autocorrelation function: Consider a random process X(t). The time average of the product X(t) and X(t+ τ) is called time average autocorrelation function of x(t) and is denoted

$$R_{XX}(\tau) = E[X(t)\,X(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Time mean square function: If τ = 0, the time average of x (t) is called time mean square value of x(t) defined as

$$A[X^2(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt \;.$$

Time cross correlation function: Let X(t) and Y(t) be two random processes with sample functions x(t) and y(t) respectively. The time average of the product of x(t) y(t+ τ) is called time cross correlation function of x(t) and y(t). Denoted as

$$R_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt$$

Ergodic Theorem and Ergodic Process: The Ergodic theorem states that for any random process X(t), all time averages of sample functions of x(t) are equal to the corresponding statistical or ensemble averages of X(t). Random processes that satisfy the Ergodic theorem are called Ergodic processes.

Joint Ergodic Process: Let X(t) and Y(t) be two random processes with sample functions x(t) and y(t) respectively. The two random processes are said to be jointly Ergodic if they are individually Ergodic and their time cross correlation functions are equal to their respective statistical cross correlation functions. i.e.

1. $\bar{x} = \bar{X}$, $\bar{y} = \bar{Y}$ 2. $\mathcal{R}_{xx}(\tau) = R_{XX}(\tau)$, $\mathcal{R}_{xy}(\tau) = R_{XY}(\tau)$

and $\mathcal{R}_{yy}(\tau) = R_{YY}(\tau)$.

Mean Ergodic Random Process: A random process X(t) is said to be mean Ergodic if time average of any sample function x(t) is equal to its statistical average, ¯ which is constant and the probability of all other sample functions is equal to one. i.e.

$$E[X(t)] = \bar{X} = A[X(t)] = \bar{x}$$

with probability one for all x(t).

Autocorrelation Ergodic Process: A stationary random process X(t) is said to be Autocorrelation Ergodic if and only if the time autocorrelation function of any sample function x(t) is equal to the statistical autocorrelation function of X(t). i.e. $A[x(t)\, x(t+\tau)] = E[X(t)\, X(t+\tau)]$ or $\mathcal{R}_{xx}(\tau) = R_{XX}(\tau)$.

Cross Correlation Ergodic Process: Two stationary random processes X(t) and Y(t) are said to be cross correlation Ergodic if and only if its time cross correlation function of sample functions x(t) and y(t) is equal to the statistical cross correlation function of X(t) and Y(t). i.e. $A[x(t)\, y(t+\tau)] = E[X(t)\, Y(t+\tau)]$ or $\mathcal{R}_{xy}(\tau) = R_{XY}(\tau)$.

Properties of Autocorrelation function: Consider that a random process X(t) is at least WSS and is a function of time difference $\tau = t_2 - t_1$. Then the following are the properties of the autocorrelation function of X(t).

1. Mean square value of X(t) is $E[X^2(t)] = R_{XX}(0)$. It is equal to the power (average) of the process, X(t).
   Proof: We know that for X(t), $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$. If $\tau = 0$, then $R_{XX}(0) = E[X(t)\, X(t)] = E[X^2(t)]$ hence proved.
2. Autocorrelation function is maximum at the origin i.e.

   $$|R_{XX}(\tau)| \le R_{XX}(0).$$

   Proof: Consider two random variables $X(t_1)$ and $X(t_2)$ of X(t) defined at time intervals t_1 and t_2 respectively. Consider a positive quantity $[X(t_1) \pm X(t_2)]^2 \ge 0$
   Taking Expectation on both sides, we get $E\{[X(t_1) \pm X(t_2)]^2\} \ge 0$ $E[X^2(t_1) + X^2(t_2) \pm 2X(t_1)\, X(t_2)] \ge 0$
   $E[X^2(t_1)] + E[X^2(t_2)\, 2E[X(t_1)\, X(t_2)]\, 0$
   $R_{XX}(0) + R_{XX}(0) \pm 2\, R_{XX}(t_1,t_2) \ge 0$ [Since $E[X^2(t)] = R_{XX}(0)$]
   Given X(t) is WSS and $\tau = t_2 - t_1$. Therefore
   $2\, R_{XX}(0) \pm 2\, R_{XX}(\tau) \ge 0$ $R_{XX}(0) \pm R_{XX}(\tau) \ge 0$
   or

   $$|R_{XX}(\tau)| \le R_{XX}(0).$$

3. $R_{XX}(\tau)$ is an even function of τ i.e. $R_{XX}(-\tau) = R_{XX}(\tau)$.
   Proof: We know that $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$
   Let $\tau = -\tau$ then
   $R_{XX}(-\tau) = E[X(t)\, X(t-\tau)]$
   Let $u=t-\tau$ or $t= u+\tau$
   Therefore $R_{XX}(-\tau) = E[X(u+\tau)\, X(u)] = E[X(u)\, X(u+\tau)]$

   $R_{XX}(-\tau) = R_{XX}(\tau)$ hence proved.

4. If a random process X(t) has a non zero mean value, $E[X(t)] \neq 0$ and Ergodic with no periodic components, then
   $$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2.$$

Proof: Consider a random variable X(t)with random variables $X(t_1)$ and $X(t_2)$. Given the mean value is

$$E[X(t)] = \bar{X} \neq 0$$

We know that
$R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[X(t_1)\, X(t_2)]$. Since the process has no periodic components, as $|\tau|$ tends to ∞, the random variable becomes independent, i.e.

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = E[X(t_1)\, X(t_2)] = E[X(t_1)]\, E[\, X(t_2)]$$

Since X(t) is Ergodic $E[X(t1)] = E[(X(t2)] = \bar{X}$ so it satisfies above property.

5. If X(t) is periodic then its autocorrelation function is also periodic.
   Proof: Consider a Random process X(t) which is periodic with period T_0 Then
   $X(t) = X(t \pm T_0)$ or
   $X(t+\tau) = X(t+\tau \pm T_0)$. Now we have $R_{XX}(\tau) = E[X(t)X(t+\tau)]$ then
   $R_{XX}(\tau \pm T_0) = E[X(t)X(t+\tau \pm T_0)]$
   Given X(t) is WSS, $R_{XX}(\tau \pm T_0) = E[X(t)X(t+\tau)]$ $R_{XX}(\tau \pm T_0) = R_{XX}(\tau)$
   Therefore $R_{XX}(\tau)$ is periodic hence proved.

6. The autocorrelation function of random process $R_{XX}(\tau)$ cannot have any arbitrary shape.
   Proof:
   The autocorrelation function $R_{XX}(\tau)$ is an even function of and has maximum value at the origin.
   Hence the autocorrelation function cannot have an arbitrary shape hence proved.

7. If a random process X(t) with zero mean has the DC component A as Y(t) =A + X(t), Then $R_{YY}(\tau) = A^2 + R_{XX}(\tau)$.

   Proof: Given a random process Y(t) =A + X(t).

   We know that $R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = E[(A + X(t))(A + X(t+\tau))]$

   $= E[(A^2 + AX(t) + AX(t+\tau) + X(t)X(t+\tau)]$

   $= E[(A^2] + AE[X(t)] + E[AX(t+\tau)] + E[X(t)X(t+\tau)]$

   $= A^2 + 0 + 0 + R_{XX}(\tau)$.

   Therefore $R_{YY}(\tau) = A^2 + R_{XX}(\tau)$ hence proved.

8. If a random process Z(t) is sum of two random processes X(t) and Y(t) i.e, Z(t) = X(t) + Y(t). Then $R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$ Proof: Given Z(t) = X(t) + Y(t).

   We know that $R_{ZZ}(\tau) = E[Z(t)Z(t+\tau)]$

   $= E[(X(t)+Y(t))(X(t+\tau)Y(t+\tau))]$

   $= E[(X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau))]$

   $= E[(X(t)X(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau))]$

   Therefore $R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$ hence proved.

Properties of Cross Correlation Function: Consider two random processes X(t) and Y(t) are at least jointly WSS. And the cross correlation function is a function of the time difference $\tau = t_2 - t_1$. Then the following are the properties of cross correlation function.

i) $R_{XY}(\tau) = R_{YX}(-\tau)$ is a Symmetrical property.

Proof: We know that $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$

also $R_{YX}(\tau) = E[Y(t)X(t+\tau)]$

Let $\tau = -\tau$ then

$R_{YX}(-\tau) = E[Y(t)X(t-\tau)]$

Let $u = t-\tau$ or $t = u+\tau$. then

$R_{YX}(-\tau) = E[Y(u+\tau)X(u)] = E[X(u)Y(u+\tau)]$

Therefore $R_{YX}(-\tau) = R_{XY}(\tau)$ hence proved.

(ii) $|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)}$

We have

$$|R_{XY}(\tau)|^2 = |EX(t+\tau)Y(t)|^2$$

$$\le EX^2(t+\tau)EY^2(t) \quad \text{using Cauch-Schwarts Inequality}$$

$$= R_X(0)R_Y(0)$$

$$\therefore |R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)}$$

Further,

$$\sqrt{R_X(0)R_Y(0)} \le \frac{1}{2}\left(R_X(0)+R_Y(0)\right) \quad \because \text{ Geometric mean} \le \text{Arithmatic mean}$$

$$\therefore |R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)} \le \frac{1}{2}\left(R_X(0)+R_Y(0)\right)$$

Hence the absolute value of the cross correlation function is always less than or equal to the geometrical mean of the autocorrelation functions.

(iii) If $X(t)$ and Y (t) are uncorrelated, then $R_{XY}(\tau) = EX(t+\tau)EY(t) = \mu_X\mu_Y$

(iv) If $X(t)$ and Y (t) is orthogonal process, $R_{XY}(\tau) = EX\left(t+\tau\right)Y\left(t\right) = 0$

Example Suppose $X(t) = A\cos(w_0 t + \phi_1)$ and $Y(t) = A\sin(w_0 t + \phi_2)$ where A and w_0 are constants and ϕ_1 and ϕ_2 are independent random variables each uniformly distributed between 0 and 2π. Then

$$\begin{aligned} R_{XY}(t_1,t_2) &= EX(t_1)X(t_2) \\ &= EA\cos(w_0 t_1 + \phi_1)A\sin(w_0 t + \phi_2) \\ &= EA\cos(w_0 t_1 + \phi_1)EA\sin(w_0 t + \phi_2) \\ &= 0 \times 0 = 0 \end{aligned}$$

$\because \phi_1$ and ϕ_2 are independent

Therefore, random processes $\{X(t), t \in \Gamma\}$ and $\{Y(t), t \in \Gamma\}$ are orthogonal.

Covariance functions for random processes:

Auto Covariance function:

Consider two random processes X(t) and X(t+ τ) at two time intervals t and t+ τ. The auto covariance function can be expressed as

$C_{XX}(t, t+\tau) = E[(X(t)-E[X(t)])((X(t+\tau) - E[X(t+\tau)])]$ or $C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[(X(t)]E[X(t+\tau)]$

Properties:

1. If X(t) is WSS, then $C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$ since $E[X(t)]=E[X(t+\tau)]= \bar{X}$
2. $C_{XX}(0) = R_{XX}(0) - \bar{X}^2 = E[X^2(t)] - \{E[X(t)]\}^2 = \sigma_X^2$

Therefore at τ = 0, the auto covariance function becomes the Variance of the random process.

3. The autocorrelation coefficient of the random process, X(t) is defined as

$$\rho_{XX}(t, t+\tau) = \frac{C_{XX}(t,t+\tau)}{\sqrt{C_{XX}(t,t)C_{XX}(t+\tau,t+\tau)}} \text{ if } \tau = 0,$$

$$\rho_{XX}(0) = \frac{C_{XX}(t,t)}{C_{XX}(t,t)} = 1. \text{ Also } |\rho_{XX}(t, t+\tau)| \leq 1.$$

Cross Covariance Function:

If two random processes X(t) and Y(t) have random variables X(t) and Y(t+ τ), then the cross covariance function can be defined as

$C_{XY}(t, t+\tau) = E[(X(t)-E[X(t)])((Y(t+\tau) - E[Y(t+\tau)])]$ or $C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E[(X(t)E[Y(t+\tau)]$.

Properties:

1. If X(t) and Y(t) are jointly WSS, then
$$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$$
2. If X(t) and Y(t) are Uncorrelated then $C_{XY}(t, t+\tau) = 0$.
3. The cross correlation coefficient of random processes X(t) and Y(t) is defined as

$$\rho_{XY}(t, t+\tau) = \frac{C_{XY}(t,t+\tau)}{\sqrt{C_{XX}(t,t)C_{YY}(t+\tau,t+\tau)}} \text{ if } \tau = 0,$$

$$\rho_{XY}(0) = \frac{C_{XY}(0)}{\sqrt{C_{XX}(0)C_{YY}(0)}} = \frac{C_{XY}(0)}{\sigma_X\sigma_Y}.$$

Response of Linear time-invariant system to WSS input

In many applications, physical systems are modeled as linear time invariant (LTI) systems. The dynamic behavior of an LTI system to deterministic inputs is described by linear differential equations. We are familiar with time and transform domain (such as Laplace transform and Fourier transform) techniques to solve these equations. In this lecture, we develop the technique to analyze the response of an LTI system to WSS random process.

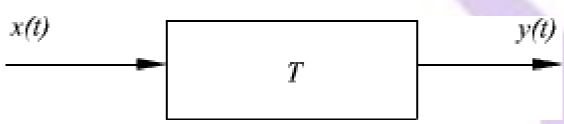
The purpose of this study is two-folds:

(1) Analysis of the response of a system
(2) Finding an LTI system that can optionally estimate an unobserved random process from an observed process. The observed random process is statistically related to the unobserved random process. For example, we may have to find LTI system (also called a filter) to estimate the signal from the noisy observations.

Basics of Linear Time Invariant Systems:

A system is modeled by a transformation T that maps an input signal $x(t)$ to an output signal y(t). We can thus write,

$$y(t) = T\left[x(t)\right]$$



Linear system
The system is called linear if superposition applies: **the weighted sum of inputs results in the weighted sum of the corresponding outputs**. Thus for a linear system

$$T\left[a_1x_1(t)+a_2x_2(t)\right]=a_1T\left[x_1(t)\right]+a_2T\left[x_2(t)\right]$$

Example : Consider the output of a linear differentiator, given by

$$y(t) = \frac{d\,x(t)}{dx}$$

Then, $\frac{d}{dt}\left(a_1x_1(t) + a_2x_2(t) \right)$

$$= a_1\frac{d}{dt}x_1(t) + a_2\frac{d}{dt}x_2(t)$$

Hence the linear differentiator is a linear system.

Linear time-invariant system

Consider a linear system with $y(t) = Tx(t)$. The system is called time-invariant if

$$Tx(t-t_0) = y(t-t_0) \quad \forall \quad t_0$$

It is easy to check that that the differentiator in the above example is a linear time-invariant system.

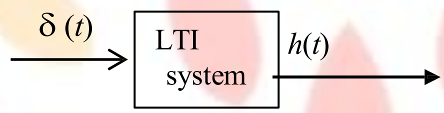
Causal system

The system is called causal if the output of the system at $t = t_0$ depends only on the present and past values of input. Thus for a causal system

$$y(t_0) = T(x(t), t \le t_0)$$

Response of a linear time-invariant system to deterministic input

A linear system can be characterised by its impulse response $h(t) = T\delta(t)$ where $\delta(t)$ is the *Dirac delta function.*



Recall that any function $x(t)$ can be represented in terms of the Dirac delta function as follows

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-s)\; ds$$

If $x(t)$ is input to the linear system $y(t) = Tx(t)$, then

$$y(t) = T\int_{-\infty}^{\infty} x(s)\,\delta(t-s)\; ds$$

$$= \int_{-\infty}^{\infty} x(s)\; T\delta(t-s)\; ds \quad [\text{ Using the linearity property }]$$

$$= \int_{-\infty}^{\infty} x(s)\; h(t,s)\; ds$$

Where $h(t,s) = T\delta(t-s)$ is the response at time t due to the shifted impulse $\delta(t-s)$.

If the system is time invariant,

$$h(t,s) = h(t-s)$$

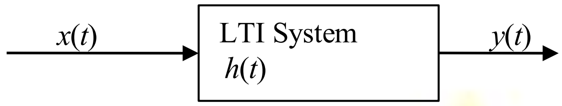
Therefore for a linear time invariant system,

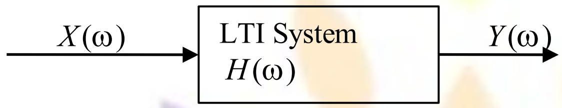
$$y(t) = \int_{-\infty}^{\infty} x(s)\; h(t-s)\; ds = x(t) * h(t)$$

where * denotes the convolution operation.

Thus for a LTI System,

$$\boxed{y(t) = x(t) * h(t) = h(t) * x(t)}$$





Taking the Fourier transform, we get

$$Y(\omega) = H(\omega)\, X(\omega)$$

where $H(\omega) = FT\, h(t) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$ is thefrequency response of the system

Response of an LTI System to WSS input

Consider an LTI system with impulse response *h*(*t*). Suppose $\{X(t)\}$ is a WSS process input to the system. The output $\{Y(t)\}$ of the system is given by

$$Y(t) = \int_{-\infty}^{\infty} h(s)\, X(t-s)\, ds = \int_{-\infty}^{\infty} h(t-s)\, X(s)\, ds$$

where we have assumed that the integrals exist in the mean square (m.s.) sense.

Mean and autocorrelation of the output process $\{Y(t)\}$

The mean of the output process is given by,

$$\begin{aligned} EY(t) &= E\int_{-\infty}^{\infty} h(s)X(t-s)\,ds \\ &= \int_{-\infty}^{\infty} h(s)EX(t-s)\,ds \\ &= \int_{-\infty}^{\infty} h(s)\mu_X\,ds \\ &= \mu_X \int_{-\infty}^{\infty} h(s)\,ds \\ &= \mu_X H(0) \end{aligned}$$

where $H(0)$ is the frequency response $H(\omega)$ at 0 frequency ($\omega = 0$) given by

$$H(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt\Big|_{\omega=0} = \int_{-\infty}^{\infty} h(t)dt$$

Therefore, the mean of the output process $\{Y(t)\}$ is a constant

Cross correlation function of input and output of LTI system:

The Cross correlation of the input *X(t)* and the out put *Y*(*t*) is given by

$$E\left(X(t+\tau)Y(t)\right) = E\ X(t+\tau)\int_{-\infty}^{\infty} h(s)\,X(t-s)\,ds$$

$$= \int_{-\infty}^{\infty} h(s)\quad E\ X(t+\tau)\ X(t-s)\ ds$$

$$= \int_{-\infty}^{\infty} h(s)\quad R_X(\tau+s)\,ds$$

$$= \int_{-\infty}^{\infty} h(-u)\quad R_X(\tau-u)\,du \qquad [\text{ Put } s = -u\ ]$$

$$= h(-\tau) * R_X(\tau)$$

$$\therefore \quad R_{XY}(\tau) = h(-\tau) * R_X(\tau)$$

$$also\ \ R_{YX}(\tau) = R_{XY}(-\tau) = h(\tau) * R_X(-\tau)$$

$$= h(\tau) * R_X(\tau)$$

Thus we see that $R_{XY}(\tau)$ is a function of lag τ only. Therefore, $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

The autocorrelation function of the output process Y(t) is given by,

$$\therefore \quad E\left(Y(t+\tau)Y(t)\right) = E\int_{-\infty}^{\infty} h(s)\quad X(t+\tau-s)\,ds\,Y(t)$$

$$= \int_{-\infty}^{\infty} h(s)\ E\ X(t+\tau-s)\ Y(t)\ ds$$

∞

UNIT 3
STOCHASTIC PROCESSES-TEMPORAL CHARACTERISTICS

Random Processes

In practical problems, we deal with time varying waveforms whose value at a time is random in nature. For example, the speech waveform recorded by a microphone, the signal received by communication receiver or the daily record of stock-market data represents random variables that change with time. How do we characterize such data? Such data are characterized as random or stochastic processes. This lecture covers the fundamentals of random processes.

Recall that a random variable maps each sample point in the sample space to a point in the real line. A random process maps each sample point to a waveform.

Consider a probability space $\{S,\mathbb{F},P\}$. A random process can be defined on $\{S,\mathbb{F},P\}$ as an indexed family of random variables $\{X(s,t),\ s\in S, t\in\Gamma\}$ where Γ is an index set, which may be discrete or continuous, usually denoting time. Thus a random process is a function of the sample point s and index variable t and may be written as $X(t,s)$.

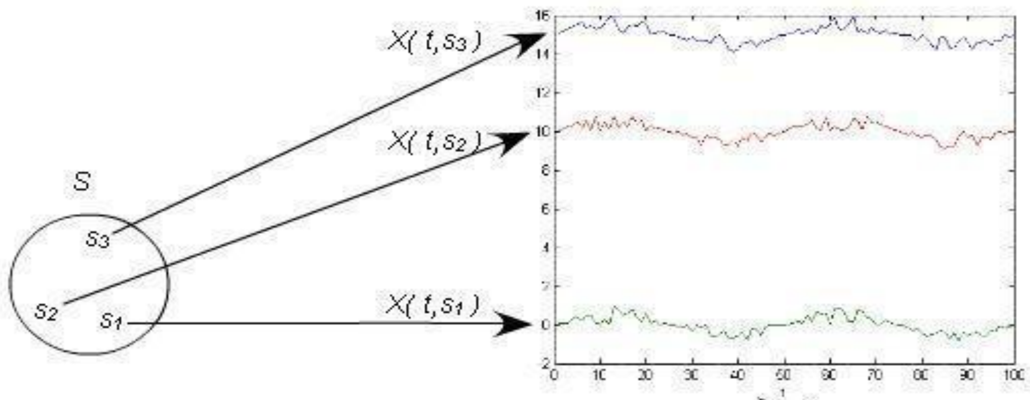


Figure : Random Process

Example 1 Consider a sinusoidal signal $X(t) = A\cos\omega t$ where A is a binary random variable with probability mass functions $p_A(1) = p$ and $p_A(-1) = 1-p$.

$\{X(t),\ t \in \Gamma\}$

= $X_1(t) = \cos \omega t$ $X_2(t) = -\cos \omega t.$ t_0 $X(t_0)$

Clearly, is a random process with two possible realizations and . At a particular time is a random variable with two values $\cos \omega t_0$ and $-\cos \omega t_0$.

Classification of a Random Process

Continuous-time vs. Discrete-time process

Γ $\{X(t),\ t \in \Gamma\}$

If the index set is continuous, is called a continuous-time process.
If the index set is a countable set, is called a discrete-time process. Such a random process can be represented as Γ and called a random. Sometimes sequence $\{X[n],\ n \in Z\}$
the notation is used to describe a random sequence indexed by the set of positive integers. $\{X_n,\ n \geq 0\}$

We can define a discrete-time random process on discrete points of time. Particularly, we can get a discrete-time random process by sampling a continuous-time process at a uniform interval such that
The discrete-time random process is more important in practical implementations. Advanced statistical signal processing techniques have been developed to process this type of signals.

Continuous-state vs. Discrete-state process T $X[n] = X(nT).$

The value of a random process is at any time can be described from its probabilistic model.

The state is the value taken by at a time , and the set of all such states is called the state space. A random process is discrete-state if the state-space is finite or countable. It also means that the corresponding sample space is also finite or countable. Otherwise , the random process is called continuous state.

Firtst order and nth order Probability density function and Distribution functions t

As we have observed above that at a specific time is a random variable and can be described by its probability distribution function This distribution function is called the first-order probability distribution function. $X(t)$ t

We can similarly define the first-order probability density function

$X(t)$ t

$$F_{X(t)}(x) = P(X(t) \leq x).$$

$$f_{X(t)}(x) = \frac{dF_{X(t)}(x)}{dx}.$$

$\{X(t), t \in \Gamma\}$

t n $X(t_1), X(t_2), \ldots, X(t_n)$ n

To describe , we have to use joint distribution function of the random variables at all possible values of . For any positive integer , represents jointly distributed random variables. Thus a random process can thus be described by
specifying the joint distribution function .

$$F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n) = P(X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n), \forall n \ge 1 \text{ and } \forall t_n \in \Gamma$$

or th the $n-th\ order$ joint probability density function

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n)$$

Mean of the random process

$\{X(t), t \in \Gamma\}$

Note that

If is a discrete-state random process, then it can be also specified by the collection of joint probability mass function

Moments of a random process

We defined the moments of a random variable and joint moments of random variables. We can define all the possible moments and joint moments of a random process .

Particularly, following moments are important.

$\mu_X(t) =$ $t = E(X(t))$

The autocovariance function of the random process at time is defined by

$$R_X(t_1,t_2) = autocorrelation\ function\ \textbf{of the process at times}\ t_1, t_2 = E(X(t_1)X(t_2))$$

These moments give partial information about the process.

$$R_X(t_1,t_2) = R_X(t_2,t_1) \textbf{ and}$$

$$R_X(t,t) = EX^2(t) = \text{sec}ond\ moment\ or\ mean\ square\ value\ at\ \textbf{time}\ t$$

$C_X(t_1,t_2)$ t_1 and t_2

$$C_X(t_1,t_2) = E(X(t_1) - \mu_X(t_1))(X(t_2) - \mu X(t_2))$$
$$= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

Clearly

$$C_X(t,t) = E(X(t) - \mu_X(t))^2 = \text{variance of the process at time } t$$

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)CX(t_2,t_2)}}$$

The ratio is called the correlation coefficient.

The autocorrelation function and the autocovariance functions are widely used to characterize a class of random process called the wide-sense stationary process.

We can also define higher-order moments like

$R_X(t_1,t_2,t_3) = E(X(t_1),X(t_2),X(t_3))$ = Triple correlation function at t_1,t_2,t_3 etc.

The above definitions are easily extended to a random sequence . $\{X[n], n \in \mathbb{Z}\}$

Cross-*covariance function* of the processses at times t_1,t_2

$$C_{XY}(t_1,t_2) = E(X(t_1)-\mu_X(t_1))(Y(t_2)-\mu_Y(t_2))$$
$$= R_{XY}(t_1,t_2)-\mu_X(t_1)\mu_Y(t_2)$$

Cross-correlation coefficient

$$\rho_{XY}(t_1,t_2) = \frac{C_{XY}(t_1,t_2)}{\sqrt{C_X(t_1,t_1)C_Y(t_2,t_2)}}$$

On the basis of the above definitions, we can study the degree of dependence between two random processes

This also implies that for such two process

Orthogonal processes: Two random processes and are called orthogonal if

Stationary Random Process

$R_{XY}(t_1,t_2) = \mu_X(t_1)\mu_Y(t_2)$

$\{X(t), t \in \Gamma\}$

$\{Y(t), t \in \Gamma\}$

$R_{XY}(t_1,t_2) = 0 \ \forall\, t_1,t_2 \in \Gamma$

The concept of stationarity plays an important role in solving practical problems involving random processes. Just like time-invariance is an important characteristic of many deterministic systems, stationarity describes certain time-invariant property of a class of random processes. Stationarity also leads to frequency-domain description of a random process.

Strict-sense Stationary Process

$\{X(t)\}$

$\{X(t)\}$

A random process is called strict-sense stationary (SSS) if its probability structure is invariant with time. In terms of the joint distribution function, is called SSS if

$$F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n) = F_{X(t_1+t_0),X(t_2+t_0),\ldots,X(t_n+t_0)}(x_1, x_2, \ldots, x_n)$$

$\forall n \in N,\ \forall t_0 \in \Gamma$ and for all choices of sample points $t_1, t_2, \ldots, t_n \in \Gamma$.

Thus, the joint distribution functions of any set of random variables $X(t_1), X(t_2), \ldots X(t_n)$ does not depend on the placement of the origin of the time axis. This requirement is a very strict. Less strict form of stationarity may be defined.

Particularly,

If then is called order stationary.

$$F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2 \ldots x_n) = F_{X(t_1+t_0),X(t_2+t_0),\ldots,X(t_n+t_0)}(x_1, x_2 \ldots x_n) \text{ for } n = 1, 2, .., k, \quad \{X(t)\}$$

Is called kth order stationary does not depend on the placement of the origin of the time axis. This requirement is a very strict. Less strict form of stationary may be defined.

If $\{X(t)\}$ is stationary up to order 1 kth

Let us assume $\{X(t)\}$. Then

$$F_{X(t_1)}(x_1) = F_{X(t_1+t_0)}(x_1),\ \forall t_0 \in T$$

As a consequence

$$t_0 = -t_1.$$

If $F_{X(t_1)}(x_1) = F_{X(0)}(x_1)$ which is independent of time.

is stationary up to order 2 Put

$$EX(t_1) = EX(0) = \mu_X(0) = \text{constant}$$

$\{X(t)\}$

$$t_0 = -t_2$$

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(t_1-t_2),X(0)}(x_1, x_2)$$

This implies that the second-order distribution depends only on the time-lag $t_1 - t_2$.

$R_X(t_1,t_2) = E(X(t_1)X(t_2))$ As a consequence, for such a process

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(0),X(t_1-t_2)}(x_1,x_2)\,dx_1 dx_2$$
$$= R_X(t_1 - t_2)$$

Similarly,

$$C_X(t_1,t_2) = C_X(t_1 - t_2)$$

Therefore, the autocorrelation function of a SSS process depends only on the time lag

$t_1 - t_2$ We can also define the joint stationary of two random processes. Two processes

And are called jointly strict-sense stationary if their joint probability distributions of any order is invariant under the translation of time. A complex random process is called SSS if and are jointly SSS.

Example 1 A random process is SSS. This is because

$\{Z(t) = X(t) + jY(t)\}$ $\{X(t)\}$ $\{Y(t)\}$

Wide-sense stationary process

It is very difficult to test whether a process is SSS or not. A subclass of the SSS process called the wide sense stationary process is extremely important from practical point of view.

$\forall n,$

$$F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1,x_2,\ldots,x_n) = F_{X(t_1)}(x_1)F_{X(t_2)}(x_2)\ldots F_{X(t_n)}(x_n)$$
$$= F_X(x_1)F_X(x_2)\ldots F_X(x_n)$$
$$= F_{X(t_1+t_0)}(x_1)F_{X(t_2+t_0)}(x_2)\ldots F_{X(t_n+t_0)}(x_n)$$
$$= F_{X(t_1+t_0),X(t_2+t_0),\ldots,X(t_n+t_0)}(x_1,x_2,\ldots,x_n)$$

A random process is called wide sense stationary process (WSS) if

$EX(t) = \mu_X = \text{constant}$

and

$EX(t_1)X(t_2) = R_X(t_1 - t_2)$ is a function of time lag $t_1 - t_2$.

$\{X(t)\}$

Remark

For a WSS process

$EX^2(t) = R_X(0) = \text{constant}$

$\text{var}(X(t)) = EX^2(t) - (EX(t))^2 = \text{constant}$

$C_X(t_1, t_2) = EX(t_1)X(t_2) - EX(t_1)EX(t_2)$

$= R_X(t_2 - t_1) - \mu_X^2$

$\therefore C_X(t_1, t_2)$ is a function of the lag $(t_2 - t_1)$.

An SSS process is always WSS, but the converse is not always true.

Example 3 Sinusoid with random phase

Consider the random process $\{X(t)\}$ given by

$X(t) = A\cos(\omega_0 t + \phi)$

where A and w_0 are constants and ϕ are unifirmly distributed between 0 and 2π.

This is the model of the carrier wave (sinusoid of fixed frequency) used to analyse the noise performance of many receivers.

Note that

$$f_\Phi(\phi) = \begin{cases} \frac{1}{2\pi} & 0 \le \phi \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

By applying the rule for the transformation of a random variable, we get

$$f_{X(t)}(x) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - x^2}} & -A \le x \le A \\ 0 & \text{otherwise} \end{cases}$$

$t.$ $\{X(t)\}$

Which is independent ofHence is first-order stationary.

Note that

$$
\begin{aligned}
EX(t) &= EA\cos(\omega_0 t+\phi)\\
&= \int_0^{2\pi} A\cos(\omega_0 t+\phi)\frac{1}{2\pi}d\phi\\
&= 0 \text{ which is a constant}
\end{aligned}
$$

and

$$
\begin{aligned}
R_X(t_1,t_2) &= EX(t_1)X(t_2)\\
&= EA\cos(\omega_0 t_1+\phi)A\cos(\omega_0 t_2+\phi)\\
&= \frac{A^2}{2}E[\cos(\omega_0 t_1+\phi+\omega_0 t_2+\phi)+\cos(\omega_0 t_1+\phi-\omega_0 t_2-\phi)]\\
&= \frac{A^2}{2}E[\cos(\omega_0(t_1+t_2)+2\phi)+\cos(\omega_0(t_1-t_2))]\\
&= \frac{A^2}{2}\cos(\omega_0(t_1-t_2)) \text{ which is a function of the lag } t_1-t_2.
\end{aligned}
$$

$\{X(t)\}$

Hence is wide-sense stationary

Properties of Autocorrelation Function of a Real WSS Random Process Autocorrelation of a deterministic signal Consider a deterministic signal such that

$x(t)$

Such signals are called power signals. For a power signalthe autocorrelation function is defined as

$$0 < \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)dt < \infty$$

$x(t)$

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt$$

$R_x(\tau)$

$$R_x(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

$x(t)$

Measures the similarity between a signal and its time-shifted version.

$R_x(0)$

Particularly, is the mean-square value. If is a voltage waveform across a 1 ohm resistance, then $R_x(0)$ is the average power delivered to the resistance. In this sense, represents the average power of the signal.

Example 1 Suppose $x(t) = A\cos \omega t.$ $x(t)$ τ

The autocorrelation function of at lag is given by

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} A\cos \omega(t+\tau) A\cos \omega t\,dt$$

$$= \lim_{T \to \infty} \frac{A^2}{4T} \int_{-T}^{T} [\cos(2\omega t+\tau) + \cos \omega\tau]\,dt$$

$$= \frac{A^2 \cos \omega\tau}{2}$$

We see that of the above periodic signal is also periodic and its maximum occurs when

$R_x(\tau)$

The power of the signal is

$\tau = 0, \pm\frac{2\pi}{\omega}, \pm\frac{4\pi}{\omega},$ etc. $R_x(0) = \frac{A^2}{2}.$

The autocorrelation of the deterministic signal gives us insight into the properties of the autocorrelation function of a WSS process. We shall discuss these properties next.

Properties of the autocorrelation function of a real WSS process

Consider a real WSS process Since the autocorrelation function of such a process is a function of the lag we can redefine a one-parameter autocorrelation function as

$\{X(t)\}.$ $R_X(t_1, t_2)$

If is a complex WSS process, then

$\tau = t_1 - t_2,$

$$R_X(\tau) = EX(t+\tau)X(t)$$

Where is the complex conjugate of For a discrete random sequence, we can define the autocorrelation sequence similarly.

$$R_X(\tau) = EX(t+\tau)X^*(t)$$

$X(t).$

The autocorrelation function is an important function charactering a WSS random process. It possesses some general properties. We briefly describe them below.

$R_X(0) = EX^2(t)$

Is the mean-square value of the process? Thus,

$R_X(0) = EX^2(t) \geq 0.$

Remark If $X(t)$ is a voltage signal applied across a 1 ohm resistance, and then is the ensemble average power delivered to the resistance.

For a real WSS process is an even function of the time Thus,

$X(t), R_X(\tau)$ $\tau.$

$R_X(-\tau) = R_X(\tau).$

Because,

$$\begin{aligned} R_X(-\tau) &= EX(t-\tau)X(t) \\ &= EX(t)X(t-\tau) \\ &= EX(t_1+\tau)X(t_1) \quad (\text{Substituting } t_1 = t-\tau) \\ &= R_X(\tau) \end{aligned}$$

Remark For a complex process

This follows from the Schwartz inequality

$R_X(-\tau) = R_X^*(\tau)$

We have $|R_X(\tau)| \leq R_X(0).$

$$|<X(t), X(t+\tau)>|^2 \leq \|X(t)\|^2 \|X(t+\tau)\|^2$$

is a positive semi-definite function in the sense that for any positive integer and real

Proof

$$\begin{aligned} R_X^2(\tau) &= \{EX(t)X(t+\tau)\}^2 \\ &\leq EX^2(t)EX^2(t+\tau) \\ &= R_X(0)R_X(0) \\ \therefore |R_X(\tau)| &\leq R_X(0) \end{aligned}$$

$R_X(\tau)$ n

a_j, a_j $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j R_X(t_i - t_j) \geq 0$

Define the random variable

$$Y = \sum_{j=1}^{n} a_i X(t_i)$$

Then we have

$$0 \le EY^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j EX(t_i)X(t_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j R_X(t_i - t_j)$$

It can be shown that the sufficient condition for a function $R_X(\tau)$ to be the autocorrelation function of a real WSS process $\{X(t)\}$ is that $R_X(\tau)$ be real, even and positive semidefinite.

If $X(t)$ is MS periodic, then $R_X(\tau)$. Is also periodic with the same period.

Proof: Note that a real WSS random process $\{X(t)\}$ is called mean-square periodic (MS periodic) with a period T_p if for every $t \in \Gamma$

$$E(X(t+T_p) - X(t))^2 = 0$$

$$\Rightarrow EX^2(t+T_p) + EX^2(t) - 2EX(t+T_p)X(t) = 0$$

$$\Rightarrow R_X(0) + R_X(0) - 2R_X(T_p) = 0$$

$$\Rightarrow R_X(T_p) = R_X(0)$$

Again

$$(E(X(t+\tau+T_p) - X(t+\tau))X(t))^2 \le \quad E(X(t+\tau+T_p) - X(t+\tau))^2 EX^2(t)$$

(By applying Cauchy Schwartz inequality)

$$\Rightarrow (R_X(\tau+T_p) - R_X(\tau))^2 \le 2(R_X(0) - R_X(T_p))R_X(0)$$

$$\Rightarrow (R_X(\tau+T_p) - R_X(\tau))^2 \le 0 \qquad \because R_X(0) = R_X(T_p)$$

$$\therefore R_X(\tau+T_p) = R_X(\tau)$$

Cross correlation function of jointly WSS processes

If $\{X(t)\}$ and $\{Y(t)\}$ are two real jointly WSS random processes, their cross-correlation functions are independent of t and depends on the time-lag. We can write the cross-correlation function

$$R_{XY}(\tau) = EX(t+\tau)Y(t)$$

The cross correlation function satisfies the following properties:

$$\begin{aligned} R_{XY}(\tau) &= EX(t+\tau)Y(t) \\ &= EY(t)X(t+\tau) \\ &= R_{YX}(-\tau) \end{aligned}$$

$$(ii)\ |R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$$

We Have

$$\begin{aligned} |R_{XY}(\tau)|^2 &= |EX(t+\tau)Y(t)|^2 \\ &\leq EX^2(t+\tau)EY^2(t) \quad \text{using Cauchy-Schwartz Inequality} \\ &= R_X(0)R_Y(0) \end{aligned}$$

$$\therefore |R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$$

Further,

$$\sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}\left(R_X(0)+R_Y(0)\right) \quad \because \text{Geometric mean} \leq \text{Arithmatic mean}$$

If $X(t)$ and Y (t) are uncorrelated, $R_{XY}(\tau) = EX(t+\tau)EY(t) = \mu_X\mu_Y$

If X (t) and Y (t) are orthogonal processes, $R_{XY}(\tau) = EX(t+\tau)Y(t) = 0$

Example 2

Consider a random process $Z(t)$ which is sum of two real jointly WSS random processes.

We have $X(t)$ and $Y(t)$.

$$\begin{aligned} Z(t) &= X(t)+Y(t) \\ R_Z(\tau) &= EZ(t+\tau)Z(t) \\ &= E[X(t+\tau)+Y(t+\tau)][X(t)+Y(t)] \\ &= R_X(\tau)+R_Y(\tau)+R_{XY}(\tau)+R_{YX}(\tau) \end{aligned}$$

If $X(t)$ and $Y(t)$ are orthogonal processes,then $R_{XY}(\tau) = R_{YX}(\tau) = 0$

$$\therefore R_Z(\tau) = R_X(\tau)+R_Y(\tau)$$

Example 3

Suppose

$$Z_1(t) = X(t)\cos(\omega_0 t + \Phi) \ and$$
$$Z_2(t) = X(t)\sin(\omega_0 t + \Phi)$$

Where X (t) is a WSS process and $\Phi \sim U[0\ 2\pi]$

$$R_{Z_1Z_2}(\tau) = E\left[X_1(t)X_2(t-\tau)\right] = \frac{1}{2\pi}\int_0^{2\pi} x_1(t)x_2(t-\tau)d\phi$$
$$= E\left[X(t)X(t-\tau)\right]E\left[\cos(\omega_0 t + \Phi)\sin(\omega_0 t - \omega_0\tau + \Phi)\right]$$
$$= \frac{1}{2}R_X(\tau)\left\{\underline{E\left[\sin(2\omega_0 t - \omega_0\tau + 2\Phi\right]} - E\left[\sin(\omega_{0c}\tau)\right]\right\}$$
$$= -\frac{1}{2}R_X(\tau)\sin(\omega_0\tau)$$

Time averages and Ergodicity

Often we are interested in finding the various ensemble averages of a random process $\{X(t)\}$ by means of the corresponding time averages determined from single realization of the random process. For example we can compute the time-mean of a single realization of the random process by the formula

$$\langle\mu_x\rangle_T = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt$$

which is constant for the selected realization. Note that represents the dc value of $x(t)$.

Another important average used in electrical engineering is the rms value given by

$$\langle x_{rms}\rangle_T = \lim_{T\to\infty}\sqrt{\frac{1}{2T}\int_{-T}^{T} x^2(t)dt}$$

Time averages of a random process

The time-average of a function $g(X(t))$ of a continuous random process $\{X(t)\}$ is defined by

$$\langle g(X(t))\rangle_T = \frac{1}{2T}\int_{-T}^{T} g(X(t))dt$$

where the integral is defined in the mean-square sense.

$g(X_n)$ $\{X_n\}$

$$\langle g(X_n)\rangle_N = \frac{1}{2N+1}\sum_{i=-N}^{N} g(X_i)$$

Similarly, the time-average of a function of a continuous random process is defined by

The above definitions are in contrast to the corresponding ensemble average defined by

$$Eg(X(t)) = \int_{-\infty}^{\infty} g(x) f_{X(t)}(x)dx \qquad \text{for continuous case}$$

$$= \sum_{i \in R_{X(t)}} g(x_i) p_{X(t)}(x_i) \qquad \text{for discrete case}$$

The following time averages are of particular interest

Time-averaged me

Time-averaged autocorrelation function

$$\langle \mu_X \rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt \qquad \text{(continuous case)}$$

$$\langle \mu_X \rangle_N = \frac{1}{2N+1}\sum_{i=-N}^{N} X_i \qquad \text{(discrete case)}$$

Note that, and are functions of random variables and are governed by respective probability distributions. However, determination of these distribution functions is difficult and we shall discuss the behaviour of these averages in terms of their mean and variances. We shall further assume that the random processes and are WSS.

Mean and Variance of the Time Averages

Let us consider the simplest case of the time averaged mean of a discrete-time WSS random process given by

$$\langle R_X(\tau)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt \qquad \text{(continuous case)}$$

$\{X(t)\}$

$$\langle R_X[m]\rangle_N = \frac{1}{2N+1}\sum_{i=-N}^{N} X_i X_{i+m} \qquad \text{(discrete case)}$$

$\langle g(X(t))\rangle_T$ $\langle g(X_n)\rangle_N$

$\{X_n\}$

$$\langle \mu_X \rangle_N = \frac{1}{2N+1}\sum_{i=-N}^{N} X_i$$

$\langle \mu_X \rangle_N$

$$E\langle \mu_X \rangle_N = E\,\frac{1}{2N+1}\sum_{i=-N}^{N} X_i$$
$$= \frac{1}{2N+1}\sum_{i=-N}^{N} EX_i$$
$$= \mu_X$$

and the variance

$$E\left(\langle \mu_X \rangle_N - \mu_X\right)^2 = E\left(\frac{1}{2N+1}\sum_{i=-N}^{N} X_i - \mu_X\right)^2$$
$$= E\left(\frac{1}{2N+1}\sum_{i=-N}^{N} (X_i - \mu_X)\right)^2$$
$$= \frac{1}{(2N+1)^2}\left[\sum_{i=-N}^{N} E(X_i-\mu_X)^2 + 2\sum_{i=-N}^{N}\sum_{j=-N,\, i\neq j}^{N} E(X_i-\mu_X)(X_j-\mu_X)\right]$$

If the samples $X_{-N}, X_{-N+1}, \ldots, X_1, X_2, \ldots, X_N$ are uncorrelated,

$$E\left(\langle \mu_X \rangle_N - \mu_X\right)^2 = E\left(\frac{1}{2N+1}\sum_{i=-N}^{N} X_i - \mu_X\right)^2$$
$$= \frac{1}{(2N+1)^2}\left[\sum_{i=-N}^{N} E(X_i-\mu_X)^2\right]$$
$$= \frac{\sigma_X^2}{2N+1}$$

We also observe that

$$\lim_{N\to\infty} E\left(\langle \mu_X \rangle_N - \mu_X\right)^2 = 0$$

From the above result, we conclude that

$$\langle \mu_X \rangle_N \xrightarrow{m.s.} \mu_X$$

Let us consider the time-averaged mean for the continuous case. We have

$$\langle \mu_X \rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$
$$\therefore E\langle \mu_X \rangle_T = \frac{1}{2T}\int_{-T}^{T} EX(t)\,dt$$
$$= \frac{1}{2T}\int_{-T}^{T} \mu_X\,dt = \mu_X$$

and the variance

$$
\begin{aligned}
E\left(\langle \mu_X \rangle_T - \mu_X\right)^2 &= E\left(\frac{1}{2T}\int_{-T}^{T} X(t)dt - \mu_X\right)^2 \\
&= E\left(\frac{1}{2T}\int_{-T}^{T} (X(t) - \mu_X)dt\right)^2 \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E(X(t_1) - \mu_X)(X(t_2) - \mu_X)dt_1 dt_2 \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t_1 - t_2)dt_1 dt_2
\end{aligned}
$$

The above double integral is evaluated on the square area bounded by $t_1 = \pm T$ and $t_2 = \pm T.$ We divide this square region into sum of trapezoidal strips parallel to $t_1 - t_2 = 0.$ (See Figure 1)Putting $t_1 - t_2 = \tau$ and noting that the differential area between $t_1 - t_2 = \tau$ and $t_1 - t_2 = \tau + d\tau$ is $(2T - |\tau|)d\tau$, the above double integral is converted to a single integral as follows:

$$
\begin{aligned}
E\left(\langle \mu_X \rangle_T - \mu_X\right)^2 &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t_1 - t_2)dt_1 dt_2 \\
&= \frac{1}{4T^2}\int_{-2T}^{2T} (2T - |\tau|)C_X(\tau)d\tau \\
&= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)C_X(\tau)d\tau
\end{aligned}
$$

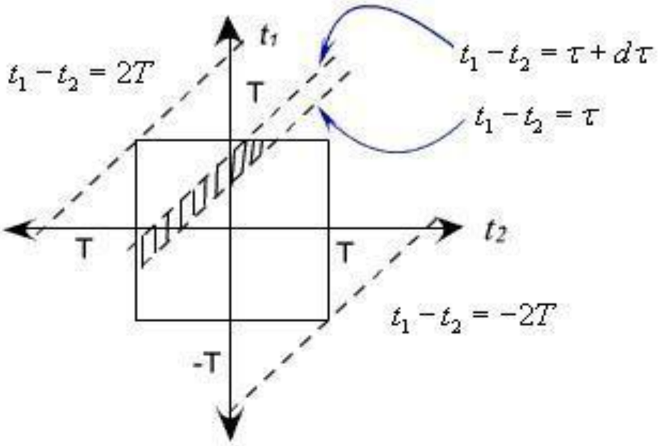


Figure 1

Ergodicity Principle

averages converge to the corres

If the time ponding ensemble averages in the probabilistic sense, then a time-average computed from a large realization can be used as the value for the corresponding ensemble average. Such a principle is the ergodicity principle to be discussed below:

Mean ergodic process $\{X(t)\}$ $\langle \mu_X \rangle_T \xrightarrow{m.s.} \mu_X \quad T \to \infty$

A WSS process $\{X(t)\}$ is said to be ergodic in mean, if as . Thus for a mean ergodic process ,

$$\lim_{T\to\infty} E\langle \mu_X \rangle_T = \mu_X$$

and

$$\lim_{T\to\infty} \operatorname{var}\langle \mu_X \rangle_T = 0$$

We have earlier shown that

and

$$E\langle \mu_X \rangle_T = \mu_X$$

therefore, the condition for ergodicity in mean is

$$\operatorname{var}\langle \mu_X \rangle_T = \frac{1}{2T}\int_{-2T}^{2T} C_X(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau$$

Further,

$$\boxed{\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T} C_X(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau = 0}$$

Therefore, a sufficient condition for mean ergodicity is

$$\frac{1}{2T}\int_{-2T}^{2T} C_X(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau \le \frac{1}{2T}\int_{-2T}^{2T} |C_X(\tau)|d\tau$$

$$\int_{-2T}^{2T} |C_X(\tau)| d\tau < \infty$$

$\{X(t)\}$

Example 1 Consider the random binary waveform discussed in Example 5 of lecture 32.The process has the auto-covariance function given by

$$C_X(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_p} & |\tau| \le T_p \\ 0 & \text{otherwise} \end{cases}$$

Here

$$\int_{-2T}^{2T} |C_X(\tau)| d\tau = 2\int_{0}^{2T} |C_X(\tau)| d\tau$$
$$= 2\int_{0}^{T_p} \left(1 - \frac{\tau}{T_p}\right) d\tau$$
$$= 2\left(T_p + \frac{T_p^3}{3T_p^2} - \frac{T_p^2}{T_p}\right)$$
$$= \frac{2T_p}{3}$$

Hence is mean ergodic.

$$\int_{-2T}^{2T} |C_X(\tau)| d\tau < \infty$$

Autocorrelation ergodicity

$\{X(t)\}$

We consider so that,

Then will be autocorrelation ergodic if is mean ergodic. Thuswill be autocorrelation ergodic if

$$\langle R_X(\tau) \rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt$$

$Z(t) = X(t)X(t+\tau)$ $\mu_Z = R_X(\tau)$

$\{X(t)\}$ $\{Z(t)\}$

$\{X(t)\}$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T} \left(1 - \frac{|\tau_1|}{2T}\right) C_Z(\tau_1)\, d\tau_1 = 0$$

where

$$C_Z(\tau_1) = EZ(t)Z(t-\tau_1) - EZ(t)EZ(t-\tau_1)$$
$$= EX(t)X(t-\tau)X(t-\tau)X(t-\tau-\tau_1) - R_X^2(\tau)$$

$C_Z(\tau_1)$

involves fourth order moment.

Simpler condition for autocorrelation ergodicity of a jointly Gaussian process can be found.

Consider the random–phased sinusoid given by
where are constants and is a random variable. We have earlier proved that this process is WSS with

$X(t) = A\cos(\omega_0 t + \phi)$ A and ω_0 $\phi \sim U[0,\ 2\pi]$

For any particular realization

$\mu_X = 0$ $R_X(\tau) = \frac{A^2}{2}\cos\omega_0\tau$

$$x(t) = A\cos(w_0 t + \phi_1),$$

and

$$\langle \mu_x \rangle_T = \frac{1}{2T}\int_{-T}^{T} A\cos(w_0 t + \phi_1)dt$$
$$= \frac{1}{Tw_0}A\sin(w_0 T)$$

We see that as and

For each realization, both the time-averaged mean and the time-averaged autocorrelation function converge to the corresponding ensemble averages. Thus the random-phased sinusoid is ergodic in both mean and autocorrelation

$$\langle R_x(\tau) \rangle_T = \frac{1}{2T}\int^{T} A\cos(w_0 t + \phi_1)A\cos(w_0(t+\tau) + \phi_1)dt$$
$T \to \infty$
$$= \frac{A^2}{4T}\int_{-T}^{T}[\cos w_0\tau + A\cos(w_0(2t+\tau) + 2\phi_1)]dt$$
$\langle \mu_x \rangle_T \to 0$
$$= \frac{A^2\cos w_0\tau}{2} + \frac{A^2\sin(w_0(2T+\tau))}{4w_0 T}$$

$$\langle R_x(\tau) \rangle_T \to \frac{A^2\cos w_0\tau}{2}$$