UNIT-2

OPERATION ON RANDOM VARIABLE-EXPECTATIONS

Expected Value of a Random Variable

The expectation operation extracts a few parameters of a random variable and provides a summary description of the random variable in terms of these parameters.
It is far easier to estimate these parameters from data than to estimate the distribution or density function of the random variable.
Moments are some important parameters obtained through the expectation operation.

Expected value or mean of a random variable

The expected value of a random variableis defined by

X

$$EX = \int_{-\infty}^{\infty} x f_X(x)dx$$

Provided exists.

$$\int_{-\infty}^{\infty} x f_X(x)dx$$

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Is also called the mean or statistical average of the random variable and is denoted by $p_X(x_i), i = 1, 2, \ldots, N,$

$f_X(x)$

Note that, for a discrete RV with the probability mass function (pmf) the pdf is given by

$$f_X(x) = \sum_{i=1}^{N} p_X(x_i)\delta(x - x_i)$$

$$\therefore \mu_X = E[X] = \int_{-\infty}^{\infty} x \sum_{i=1}^{N} p_X(x_i)\delta(x - x_i)dx$$

$$= \sum_{i=1}^{N} p_X(x_i) \int_{-\infty}^{\infty} x\delta(x - x_i)dx$$

$$= \sum_{i=1}^{N} x_i p_X(x_i) \qquad \because \int_{-\infty}^{\infty} x\delta(x - x_i)dx$$

Thus for a discrete random variable with

$X \qquad p_X(x_i), i = 1, 2, \ldots, N,$

$$\boxed{\mu_X = \sum_{i=1}^{N} x_i p_X(x_i)}$$

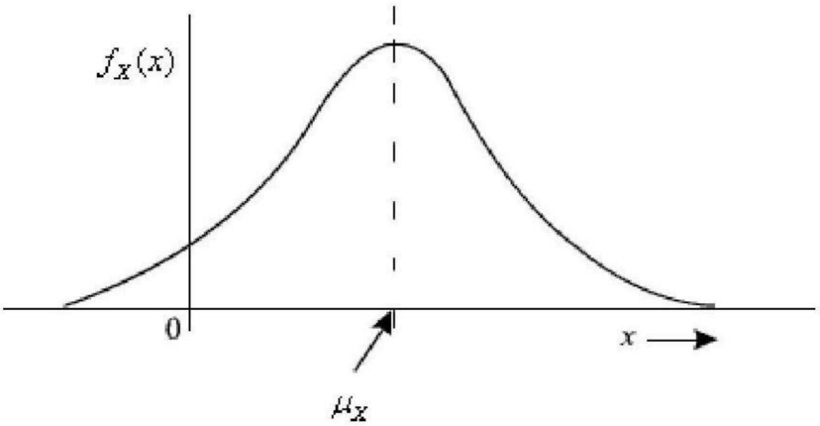

Figure1 Mean of a random variable

Example 1

Suppose is a random variable defined by the pdf

X

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\mu_X = \int_{-\infty}^{\infty} x f_X(x)dx = \int_a^b x\frac{1}{b-a}dx = \frac{a+b}{2}$$

Example 2

Consider the random variable X with the pmf as tabulated below

Value of the random variable x	0	1	2	3
pX(x)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$

Then

$$\mu_X = \sum_{i=1}^{N} x_i p_X(x_i) = 0\times\frac{1}{8}+1\times\frac{1}{8}+2\times\frac{1}{4}+3\times\frac{1}{2} = \frac{17}{8}$$

Example 3 Let X be a continuous random variable with

Then

$$f_X(x) = \frac{\alpha}{\pi(x^2+\alpha^2)} \qquad -\infty < x < \infty, \alpha > 0$$

=

$$EX = \int_{-\infty}^{\infty} x f_X(x)dx$$

$$\frac{\alpha}{\pi}\int_0^{\infty}\frac{2x}{x^2+\alpha^2}dx$$

$$\frac{\alpha}{\pi}\ln\left(1+x^2\right)\Big]_0^\infty$$

=

Hence EX does not exist. This density function is known as the Cauchy density function.

Expected value of a function of a random variable

Suppose $Y = g(X)$ is a real-valued function of a random variable X as discussed in the last class.

Then,

$$EY = Eg(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

We shall illustrate the above result in the special case $g(X)$ when $y = g(x)$ is one-to-one and monotonically increasing function of x In this case,

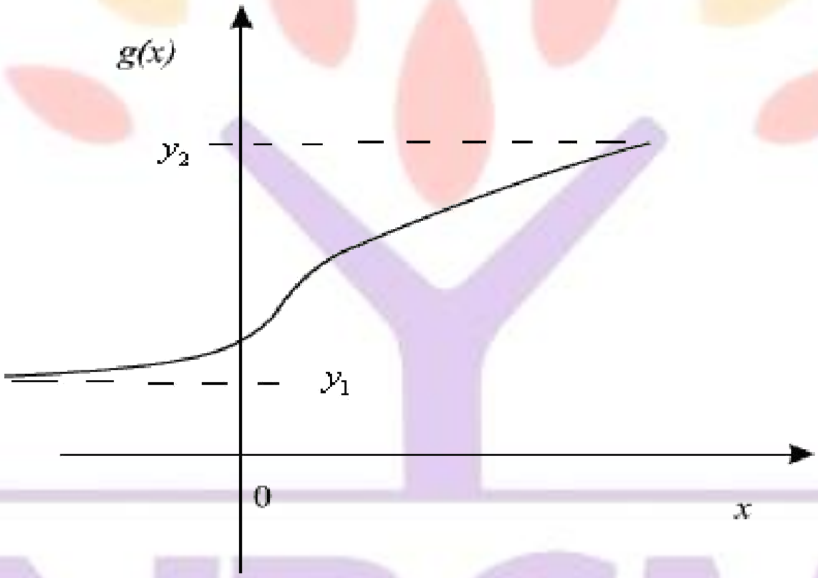

Figure 2

$$f_Y(y) = \frac{f_X(x)}{g'(x)}\Bigg]_{x=g^{-1}(y)}$$

$$EY = \int_{-\infty}^{\infty} y f_Y(y)dy$$

$$= \int_{y_1}^{y_2} y \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} dy$$

where $y_1 = g(-\infty)$ and $y_2 = g(\infty)$.

Substituting $x = g^{-1}(y)$ so that $y = g(x)$ and $dy = g'(x)dx$, we get

$$EY = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

The following important properties of the expectation operation can be immediately derived:

If c is a constant, $Ec = c$

Clearly $Ec = \int_{-\infty}^{\infty} c f_X(x)dx = c\int_{-\infty}^{\infty} f_X(x)dx = c$

If $g_1(X)$ and $g_2(X)$ are two functions of the random variable X and c_1 and c_2 are constants,

$$E[c_1 g_1(X) + c_2 g_2(X)] = c_1 Eg_1(X) + c_2 Eg_2(X)$$

$$E[c_1 g_1(X) + c_2 g_2(X)] = \int_{-\infty}^{\infty} c_1[g_1(x) + c_2 g_2(x)] f_X(x)dx$$

$$= \int_{-\infty}^{\infty} c_1 g_1(x) f_X(x)dx + \int_{-\infty}^{\infty} c_2 g_2(x) f_X(x)dx$$

$$= c_1 \int_{-\infty}^{\infty} g_1(x) f_X(x)dx + c_2 \int_{-\infty}^{\infty} g_2(x) f_X(x)dx$$

$$= c_1 Eg_1(X) + c_2 Eg_2(X)$$

The above property means that E is a linear operator.

MOMENTS ABOUT THE ORIGIN:

Mean-square valu

$$EX^2 = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

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MOMENTS ABOUT THE MEAN

Variance

Second central moment is called as variance

For a random variable X with the pdf $f_X(x)$ and mean $\mu_X,$ the variance of X is denoted by and σ_X^2

defined as $\sigma_X^2 = E(X-\mu_X)^2 = \int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)dx$

Thus for a discrete random variable X with $p_X(x_i), i = 1,2,\ldots,N,$

$$\sigma_X^2 = \sum_{i=1}^{N}(x_i-\mu_X)^2 p_X(x_i)$$

The standard deviation of X is defined as

$$\sigma_X = \sqrt{E(X-\mu_X)^2}$$

Example 4

Find the variance of the random variable in the above example

$$\begin{aligned}\sigma_X^2 &= E(X-\mu_X)^2\\ &= \int_a^b (x-\frac{a+b}{2})^2\frac{1}{b-a}dx\\ &= \frac{1}{b-a}[\int_a^b x^2dx - 2\times\frac{a+b}{2}\int_a^b xdx + \left(\frac{a+b}{2}\right)^2\int_a^b dx\\ &= \frac{(b-a)^2}{12}\end{aligned}$$

Example 5

Find the variance of the random variable discussed in above example. As already computed

$$\mu_X = \frac{17}{8}$$

$$\begin{aligned}\sigma_X^2 &= E(X-\mu_X)^2 \\ &= (0-\frac{17}{8})^2\times\frac{1}{8}+(1-\frac{17}{8})^2\times\frac{1}{8}+(2-\frac{17}{8})^2\times\frac{1}{4}+(3-\frac{17}{8})^2\times\frac{1}{2} \\ &= \frac{71}{64}\end{aligned}$$

For example, consider two random variables X_1 and X_2 with pmf as shown below. Note that each of X_1 and X_2 has zero mean. The variances are given by $\sigma_{X_1}^2=\frac{1}{2}$ and $\sigma_{X_2}^2=\frac{5}{3}$ implying that X_2 has more spread about the mean.

Properties of variance

(1) $\sigma_X^2 = EX^2-\mu_X^2$

$$\begin{aligned}\sigma_X^2 &= E(X-\mu_X)^2 \\ &= E(X^2-2\mu_X X+\mu_X^2) \\ &= EX^2-2\mu_X EX+E\mu_X^2 \\ &= EX^2-2\mu_X^2+\mu_X^2 \\ &= EX^2-\mu_X^2\end{aligned}$$

$$\therefore \sigma_X{}^2 = EX^2-\mu_X{}^2$$

If $Y=cX+b$, where c and b are constants, then $\sigma_Y^2=c^2\sigma_X^2$

$$\begin{aligned}\sigma_Y^2 &= E(cX+b-c\mu_X-b)^2 \\ &= Ec^2(X-\mu_X)^2 \\ &= c^2\sigma_X^2\end{aligned}$$

If c is a constant, $\operatorname{var}(c)=0.$

nth moment of a random variable

We can define the nth moment and the nth central- moment of a random variable X by the following relations nth-order moment $EX^n = \int_{-\infty}^{\infty} x^n f_X(x)dx \quad n = 1, 2, ..$

nth-order central moment $E(X-\mu_X)^n = \int_{-\infty}^{\infty} (x-\mu_X)^n f_X(x)dx \; n = 1, 2, ...$

Note that

$\mu_X = EX$ EX^2

The mean is the first moment and the mean-square value is the second moment
The first central moment is 0 and the variance is the second central moment

$\sigma_X^2 = E(X-\mu_X)^2$

SKEWNESS

The third central moment measures lack of symmetry of the pdf of a random variable

is called the coefficient of skewness and if the pdf is symmetric this coefficient will be zero.
The fourth central moment measures flatness or peakedness of the pdf of a random

variable. Is called kurtosis. If the peak of the pdf is sharper, then the random variable has a higher kurtosis.

$\frac{E(X-\mu_X)^3}{\sigma_X^3}$

Characteristic function

$\frac{E(X-\mu_X)^4}{\sigma_X^4}$

Consider a random variable with probability density function The characteristic function of denoted by $\phi_X(\omega)$, is defined as

Note the following:

$f_X(x)$

X

is a complex quantity, representing the Fourier transform of and traditionally using instead of This implies that the properties of the Fourier transform applies to the characteristic function.

$$\phi_X(\omega) = Ee^{j\omega X} = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)dx$$

where $j = \sqrt{-1}$

We can get
from by the inverse transform

$\phi_X(\omega)$, $f_X(x)$

$e^{j\omega X}$ $e^{-j\omega X}$

$f_X(x)$ $\phi_X(\omega)$,

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$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x} d\omega$$

Example 1

Consider the random variable X with pdf $f_X(x)$ given by

$$f_X(x) = \frac{1}{b-a} \qquad a \le x \le b$$

= 0 otherwise. The characteristics function is given by

$$\phi_X(\omega) = \frac{1}{j\omega(b-a)}\left(e^{j\omega b} - e^{j\omega a}\right)$$

Solution:

$$\phi_X(\omega) = \int_a^b \frac{1}{b-a} e^{j\omega x} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{j\omega x}}{j\omega}\right]_a^b$$

$$= \frac{1}{j\omega(b-a)}\left(e^{j\omega b} - e^{j\omega a}\right)$$

Example 2

The characteristic function of the random variable X with

$$f_X(x) = \lambda e^{-\lambda x} \quad \lambda > 0, x > 0 \text{ is}$$

$$\phi_X(\omega) = \int_0^\infty e^{j\omega x} \lambda e^{-\lambda x}$$

$$= \lambda \int_0^\infty e^{-(\lambda - j\omega)x} dx$$

$$= \frac{\lambda}{\lambda - j\omega}$$

Characteristic function of a discrete random variable

Suppose X is a random variable taking values from the discrete set $R_X = \{x_1, x_2, \ldots\}$ with corresponding probability mass function $p_X(x_i)$ for the value x_i.

Then,

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$$\phi_X(\omega) = Ee^{j\omega X} = \sum_{X_i \in R_X} p_X(x_i)e^{j\omega x_i}$$

$$\phi_X(\omega) = Ee^{j\omega X} = \sum_{X_i \in R_X} p_X(x_i)e^{j\omega x_i}$$

If Rx is the set of integers, we can write

In this case $\phi_X(\omega)$, can be interpreted as the discrete-time Fourier transform with substituting $e^{j\omega X}$ in the original discrete-time Fourier transform $e^{-j\omega X}$. The inverse relation is

$$p_X(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega k}\phi_X(\omega)d\omega$$

$p_X(k) = p(1-p)^k, \quad k = 0,1,\ldots,$ is given by

$$\phi_X(\omega) = \sum_{k=0}^{\infty} e^{j\omega k}p(1-p)^k$$
$$= p\sum_{k=0}^{\infty} e^{j\omega k}(1-p)^k$$
$$= \frac{p}{1-(1-p)e^{j\omega}}$$

Moments and the characteristic function

Given the characteristics function $\phi_X(\omega)$, the nth moment is given by

$$EX^n = \frac{1}{j^n}\frac{d^n}{d\omega^n}\phi_X(\omega)\bigg|_{\omega=0}$$

To prove this consider the power series expansion of $e^{j\omega X}$

$$e^{j\omega X} = 1 + j\omega X + \frac{(j\omega)^2X^2}{2!} + \ldots\ldots + \frac{(j\omega)^nX^n}{n!} + \ldots$$

Taking expectation of both sides and assuming $EX, EX^2, \ldots, EX^n$ to exist, we get

$$\phi_X(\omega) = 1 + j\omega EX + \frac{(j\omega)^2EX^2}{2!} + \ldots\ldots + \frac{(j\omega)^nEX^n}{n!} + \ldots\ldots$$

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Taking the first derivative of $\phi_X(\omega)$, with respect to ω at $\omega = 0$ we get

$$\left.\frac{d\phi_X(\omega)}{d\omega}\right|_{\omega=0} = jEX$$

Similarly, taking the nth derivative of $\phi_X(\omega)$, with respect to ω at $\omega = 0$ we get

$$\left.\frac{d^n\phi_X(\omega)}{d\omega^n}\right|_{\omega=0} = j^n EX^n$$

Thus ,

$$EX = \frac{1}{j}\left.\frac{d\phi_X(\omega)}{d\omega}\right|_{\omega=0}$$

and generally

$$EX^n = \frac{1}{j^n}\left.\frac{d^n\phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$$

TRANSFORMATION OF A RANDOM VARIABLE

Description:

Suppose we are given a random variable X with density fX(x). We apply a function g to produce a random variable Y = g(X). We can think of X as the input to a black box,and Y the output.

MULTIPLE RANDOM VARIABLES

Multiple Random Variables

In many applications we have to deal with more than two random variables. For example, in the navigation problem, the position of a space craft is represented by three random variables denoting the x, y and z coordinates. The noise affecting the R, G, B channels of colour video may be represented by three random variables. In such situations, it is convenient to define the vector-valued random variables where each component of the vector is a random variable.

In this lecture, we extend the concepts of joint random variables to the case of multiple random variables. A generalized analysis will be presented for random variables defined on the same sample space.

Jointly Distributed Random Variables

We may define two or more random variables on the same sample space. Let and be two real random variables defined on the same probability space The mapping such that for is called a joint random variable.

X Y

$(S, \mathbb{F}, P)$. $S \rightarrow \mathbb{R}^2$

$s \in S,\ (X(s), Y(s)) \in \mathbb{R}^2$

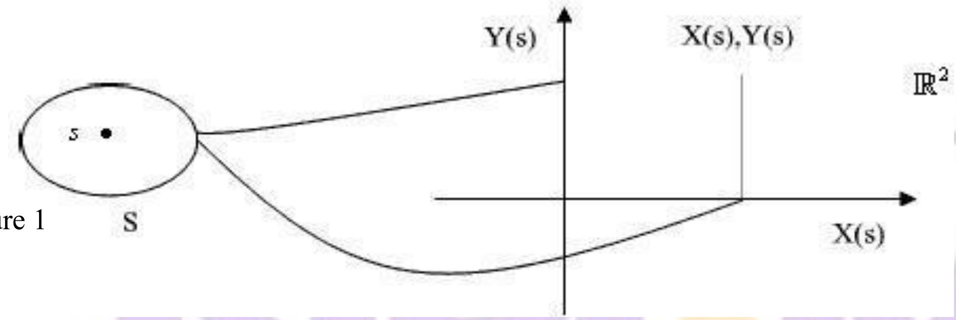

Figure 1

Joint Probability Distribution Function

Recall the definition of the distribution of a single random variable. The event was used to define the probability distribution function . Given , we can find the

$\{X \le x\}$

$F_X(x)$ $F_X(x)$

X

Y $\{X \le x, Y \le y\} = \{X \le x\} \cap \{Y \le y\}$

$P(\{X \le x, Y \le y\})\ \forall (x,y) \in \mathbb{R}^2$

probability of any event involving the random variable. Similarly, for two random variables X Y and , the event is considered as the representative event.

$F_{X,Y}(x,y)$

The probability is called the joint distribution function or the joint cumulative distribution function (CDF) of the random variables and and denoted by

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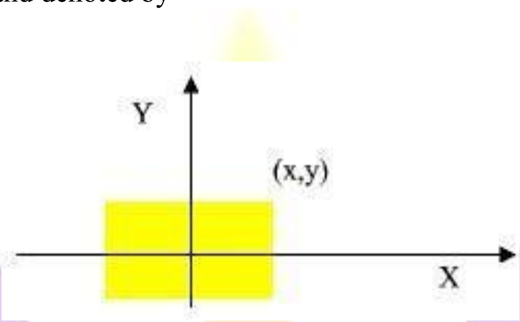

Figure 2

Properties of JPDF

satisfies the following properties:

1) $F_{X,Y}(x,y)$

2) $F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$ if $x_1 \le x_2$ and $y_1 \le y_2$

If $x_1 < x_2$ and $y_1 < y_2$,

3) $\{X \le x_1, Y \le y_1\} \subseteq \{X \le x_2, Y \le y_2\}$

Note that $\therefore P\{X \le x_1, Y \le y_1\} \le P\{X \le x_2, Y \le y_2\}$

$\therefore F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$

4) $F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = 0$

5) is right continuous in both the variables.

$\{X \le -\infty, Y \le y\} \subseteq \{X \le -\infty\}$

6)

$F_{X,Y}(\infty,\infty) = 1$

$F_{X,Y}(x,y)$

If $x_1 < x_2$ and $y_1 < y_2$

$$P(\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$$

$$F_{X,Y}(x,y), \ -\infty < x < \infty, -\infty < y < \infty$$

Given ,we have a complete description of the random variables X and Y.

7) $F_X(x) = F_{XY}(x,+\infty)$

To prove this

$$\{X \le x\} = \{X \le x\} \cap \{Y \le +\infty\}$$

$$\therefore F_X(x) = P(\{X \le x\}) = P(\{X \le x, Y \le \infty\}) = F_{X,Y}(x,+\infty)$$

Similarly .

$F_Y(y) = F_{XY}(\infty, y)$

Given , each of is called a marginal Distribution function or marginal cumulative distribution function (CDF).

$F_{X,Y}(x,y), \ -\infty < x < \infty, -\infty < y < \infty$ $\quad F_X(x)$ and $F_Y(y)$

Jointly Distributed Discrete Random Variables

If and are two discrete random variables defined on the same probability space such that takes values from the countable subset and takes values from the countable subset .Then the joint random variable can take values from the countable subset in . The joint random variable is completely specified by their joint probability mass function

X Y

(S,F,P) X R_X Y

Given , we can determine other probabilities involving the random variables and

R_Y (X,Y)

Remark $R_X \times R_Y$ (X,Y)

- $$p_{X,Y}(x,y) = P\{s \mid X(s) = x, Y(s) = y\}, \ \forall (x,y) \in R_X \times R_Y$$
- $p_{X,Y}(x,y)$ X

Y

$$p_{X,Y}(x,y) = 0 \text{ for } (x,y) \notin R_X \times R_Y$$

$$\sum_{(x,y) \in} \sum_{R_X \times R_Y} p_{X,Y}(x,y) = 1$$

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$$\sum_{(x,y)\in R_X \times R_Y}\sum p_{X,Y}(x,y) = P\left(\bigcup_{(x,y)\in R_X\times R_Y}\{x,y\}\right)$$
$$= P(R_X \times R_Y)$$
$$= P\{s \mid (X(s),Y(s)) \in (R_X \times R_Y)\}$$
$$= P(S) = 1$$

This is because

Marginal Probability Mass Functions: The probability mass functions $p_X(x)$ and $p_Y(y)$ are obtained from the joint probability mass function as follows

$$p_X(x) = P(\{X = x\} \cup R_Y)$$
$$= \sum_{y\in R_Y} p_{X,Y}(x,y)$$

and similarly

$$p_Y(y) = \sum_{x\in R_X} p_{X,Y}(x,y)$$

These probability mass functions $p_X(x)$ and $p_Y(y)$ obtained from the joint probability mass functions are called marginal probability mass functions .

Example 4 Consider the random variables X and Y with the joint probability mass function as tabulated in Table 1. The marginal probabilities $p_X(x)$ and $p_Y(y)$ are as shown in the last column and the last row respectively.

X \ Y	0	1	2	$p_Y(y)$
0	0.25	0.1	0.15	0.5
1	0.14	0.35	0.01	0.5
$p_X(x)$	0.39	0.45	0.16	

Table 1

Joint Probability Density Function

If X and Y are two continuous random variables and their joint distribution function is continuous in both x and y, then we can define joint probability density function $f_{X,Y}(x,y)$ by

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

provided it exists.

Clearly $$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$$

Properties of Joint Probability Density Function

$f_{X,Y}(x,y)$ is always a non-negative quantity. That is,

$$f_{X,Y}(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$$

- $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

The probability of any Borel set can be obtained by

$$P(B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

Marginal density functions

The marginal density functions $f_X(x)$ and $f_Y(y)$ of two joint RVs X and Y are given by the derivatives of the corresponding marginal distribution functions. Thus

$$\begin{aligned} f_X(x) &= \tfrac{d}{dx}F_X(x) \\ &= \tfrac{d}{dx}F_X(x,\infty) \\ &= \tfrac{d}{dx}\int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f_{X,Y}(u,y)\,dy\right)du \\ &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \end{aligned}$$

$$\therefore \quad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Thus $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

and similarly $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Example 5 The joint density function $f_{X,Y}(x,y)$ of the random variables in Example 3 is

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$
$$= \frac{\partial^2}{\partial x \partial y}[(1-e^{-2x})(1-e^{-y})] \qquad x \geq 0, y \geq 0$$
$$= 2e^{-2x}e^{-y} \qquad x \geq 0, y \geq 0$$

X Y

Example 6 The joint pdf of two random variables and are given by $f_{X,Y}(x,y) = cxy \quad 0 \leq x \leq 2, \; 0 \leq y \leq 2$

$$= 0 \quad \text{otherwise}$$

Find c.
Find $F_{X,Y}(x,y)$.
Find $f_X(x)$ and $f_Y(y)$
What is the probability $P(0 < X \leq 1, 0 < Y \leq 1)$?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = c\int_0^2\int_0^2 xydydx$$
$$= c\int_0^2 xdx \int_0^2 ydy$$
$$= 4c$$
$$\therefore 4c = 1$$
$$\Rightarrow c = \frac{1}{4}$$

$$F_{X,Y}(x,y) = \frac{1}{4}\int_0^y\int_0^x uvdudv$$
$$= \frac{x^2y^2}{16} \qquad 0 \leq x \leq 2, \; 0 \leq y \leq 2$$

$$f_X(x) = \int_0^2 \frac{xy}{4}dy \qquad 0 \leq y \leq 2$$
$$= \frac{x}{2}$$
$$\therefore f_X(x) = \frac{x}{2} \qquad 0 \leq y \leq 2$$

Similarly

$$f_Y(y) = \frac{y}{2} \qquad 0 \leq y \leq 2$$

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$P(0 < X \le 1, 0 < Y \le 1)$

$$= F_{X,Y}(1,1) + F_{X,Y}(0,0) - F_{X,Y}(0,1) - F_{X,Y}(1,0)$$
$$= \frac{1}{16} + 0 - 0 - 0$$
$$= \frac{1}{16}$$

Conditional Distributions

We discussed the conditional CDF and conditional PDF of a random variable conditioned on some events defined in terms of the same random variable. We observed that

$$F_X(x/B) = \frac{P(\{X \le x\} \cap B)}{P(B)} \qquad P(B) \ne 0$$

and

$$f_X(x/B) = \frac{d}{dx} F_X(x/B)$$

We can define these quantities for two random variables. We start with the conditional probability mass functions for two random variables.

Conditional Probability Density Functions

Suppose X and Y are two discrete jointly random variable with the joint PMF $p_{X,Y}(x,y)$. The conditional PMF of Y given $X = x$ is denoted by $p_{Y/X}(y/x)$ and defined as

$$p_{Y/X}(y/x) = P(\{Y = y\}/\{X = x\})$$
$$= \frac{P(\{X = x\} \cap \{Y = y\})}{P\{X = x\}}$$
$$= \frac{p_{X,Y}(x,y)}{p_X(x)} \quad \text{provided } p_X(x) \ne 0$$

Similarly we can define the conditional probability mass function

Thus,

$$p_{Y/X}(y/x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \quad \text{provided } p_X(x) \ne 0$$

$$p_{X/Y}(x/y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \quad \text{provided } p_Y(y) \ne 0$$

Conditional Probability Distribution Function

Consider two continuous jointly random variables X and Y with the joint probability distribution function $F_{X,Y}(x,y)$. We are interested to find the conditional distribution function of one of the random variables on the condition of a particular value of the other random variable.

We cannot define the conditional distribution function of the random variable Y on the condition of the event $\{X = x\}$ by the relation

$$F_{Y/X}(y/x) = P(Y \le y / X = x)$$

$$= \frac{P(Y \le y, X = x)}{P(X = x)}$$

as $P(X = x) = 0$ in the above expression. The conditional distribution function is defined in the limiting sense as follows:

$$F_{Y/X}(y/x) = lim_{\Delta x \to 0} P(Y \le y / x < X \le x + \Delta x)$$

$$= lim_{\Delta x \to 0} \frac{P(Y \le y, x < X \le x + \Delta x)}{P(x < X \le x + \Delta x)}$$

$$= lim_{\Delta x \to 0} \frac{\int_{\infty}^{y} f_{X,Y}(x,u) \Delta x du}{f_X(x) \Delta x}$$

$$= \frac{\int_{\infty}^{y} f_{X,Y}(x,u) du}{f_X(x)}$$

$$\therefore F_{Y/X}(y/x) = \frac{\int_{\infty}^{y} f_{X,Y}(x,u) du}{f_X(x)}$$

Conditional Probability Density Function

$f_{Y/X}(y / X = x) = f_{Y/X}(y/x)$ is called the conditional probability density function of Y given X

Let us define the conditional distribution function .

The conditional density is defined in the limiting sense as follows

$$f_{Y/X}(y/X=x)=\lim_{\Delta y\to 0}(F_{Y/X}(y+\Delta y/X=x)-F_{Y/X}(y/X=x))/\Delta y$$

$$\therefore f_{Y/X}(y/X=x)=\lim_{\Delta y\to 0,\Delta x\to 0}(F_{Y/X}(y+\Delta y/x<X\leq x+\Delta x)-F_{Y/X}(y/x<X\leq x+\Delta x))/\Delta y$$

$$(X=x)=\lim_{\Delta x\to 0}(x<X\leq x+\Delta x)$$

Because,

The right hand side of the highlighted equation is

$$\lim_{\Delta y\to 0,\Delta x\to 0}(F_{Y/X}(y+\Delta y/x<X<x+\Delta x)-F_{Y/X}(y/x<X<x+\Delta x))/\Delta y$$
$$=\lim_{\Delta y\to 0,\Delta x\to 0}(P(y<Y\leq y+\Delta y/x<X\leq x+\Delta x))/\Delta y$$
$$=\lim_{\Delta y\to 0,\Delta x\to 0}(P(y<Y\leq y+\Delta y,x<X\leq x+\Delta x))/P(x<X\leq x+\Delta x)\Delta y$$
$$=\lim_{\Delta y\to 0,\Delta x\to 0}f_{X,Y}(x,y)\Delta x\Delta y/f_X(x)\Delta x\Delta y$$
$$=f_{X,Y}(x,y)/f_X(x)$$

$$\therefore f_{Y/X}(y/x)=f_{X,Y}(x,y)/f_X(x)$$

Similarly we have

$$\therefore f_{X/Y}(x/y)=f_{X,Y}(x,y)/f_Y(y)$$

Two random variables are statistically independent if for all $(x,y)\in\mathbb{R}^2$,

$$f_{Y/X}(y/x)=f_Y(y)$$
or equivalently
$$f_{X,Y}(x,y)=f_X(x)f_Y(y)$$

•

Example 2 X and Y are two jointly random variables with the joint pdf given by

$$f_{X,Y}(x,y)=k \text{ for } 0\leq x\leq 1$$
$$= 0 \text{ otherwise}$$

find, (a) k

(b) $f_X(x)$ and $f_Y(y)$

$f_{X/Y}(x/y)$

Solution:

your roots to success...

$$\int\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = 1$$

Since

We get

$$k\, x\frac{1}{2}x1x1 = 1$$
$$\Rightarrow k = 2$$

$$\therefore f_{X,Y}(x,y) = 2 \quad for\ 0 \le x \le 1 \text{ as } y \le x$$
$$= 0 \text{ otherwise}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = 2\int_0^x dy = 2x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = 2\int_y^1 dx = 2(1-y)$$

Independent Random Variables (or) Statistical Independence

Let X and Y be two random variables characterized by the joint distribution function

Thus, $F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$

and the corresponding joint density function

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Then X and Y are independent if $\forall (x,y) \in \mathbb{R}^2$, $\{X \le x\}$ and $\{Y \le y\}$ are independent events.

your roots to success...

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$
$$= P\{X \le x\} P\{Y \le y\}$$
$$= F_X(x) F_Y(y)$$
$$\therefore f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$
$$= \frac{dF_X(x)}{dx} \frac{dF_Y(y)}{dy}$$
$$= f_X(x) f_Y(y)$$
$$\therefore f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_{Y/X}(y) = f_Y(y)$$

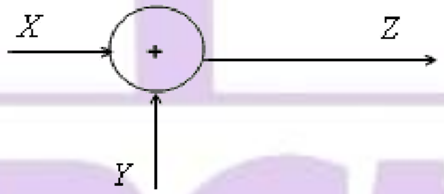
and equivalently

Sum of Two Random Variables

We are often interested in finding out the probability density function of a function of two or more RVs. Following are a few examples.

The received signal by a communication receiver is given by $Z = X + Y$

where Z is received signal which is the superposition of the message signal X and the noise Y.



The frequently applied operations on communication signals like modulation, demodulation, correlation etc. involve multiplication of two signals in the form Z = XY.

We have to know about the probability distribution of Z in any analysis of Z. More formally, given two random variables X and Y with joint probability density function $f_{X,Y}(x,y)$ and a function $Z = g(X,Y)$, we have to find $f_Z(z)$.
In this lecture, we shall address this problem.

Probability Density of the Function of Two Random Variabl

We consider the transformation $g : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Consider the event $\{Z \le z\}$ corresponding to each z. We can find a variable subset $D_Z \subseteq \mathbb{R}^2$ such that $D_Z = \{(x,y) \mid g(x,y) \le z\}$.

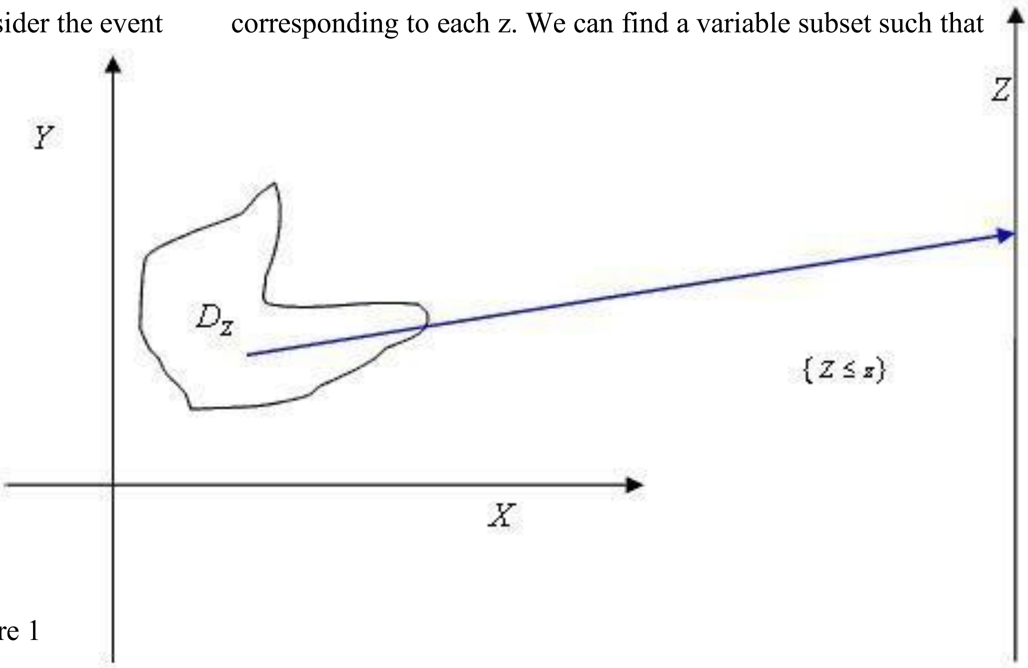


Figure 1

$$\begin{aligned}\therefore F_Z(z) &= P(\{Z \le z\}) \\ &= P\{(x,y) \mid (x,y) \in D_z\} \\ &= \iint_{(x,y)\in D_z} f_{X,Y}(x,y)\,dydx\end{aligned}$$

and $f_Z(z) = \dfrac{dF_Z(z)}{dz}$

Probability density function of Z = X + Y . Consider Figure 2

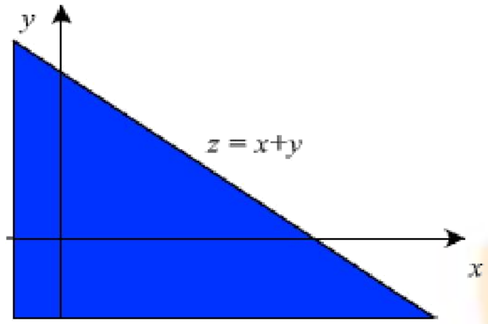


Figure 2

We have
$Z \le z$

$\Rightarrow X + Y \le z$

Therefore, D_Z is the colored region in the Figure 2.

$$\therefore F_Z(z) = \iint_{(x,y)\in D_z} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\right]dx$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z} f_{X,Y}(x,u-x)\,du\right]dx \quad \text{substituting } y = u - x$$
$$= \int_{-\infty}^{z}\left[\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx\right]du \quad \text{interchanging the order of integration}$$
$$\therefore f_Z(z) = \frac{d}{dz}\int_{-\infty}^{z}\left[\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx\right]du$$
$$= \int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx$$

If X and Y are independent

$$f_{X,Y}(x, z-x) = f_X(x) f_Y(z-x)$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

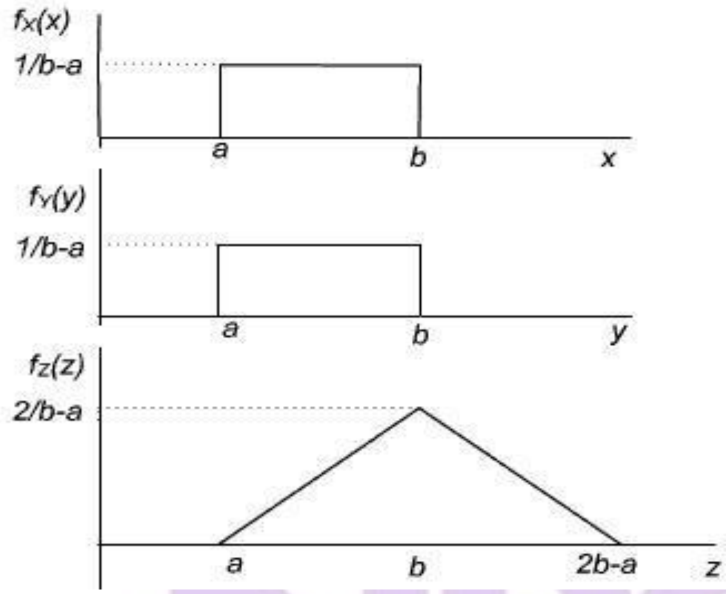
$$= f_X(z) * f_Y(z)$$

Where * is the convolution operation.

Example 1

Suppose X and Y are independent random variables and each uniformly distributed over (a, b).
And are as shown in the figure below.

$f_X(x)$ $f_Y(y)$



The PDF of is a triangular probability density function as shown in the figure.
$Z = X + Y$

Central Limit Theorem

Consider independent random variables .The mean and variance of each of the random variables are assumed to be known. Suppose and
. Form a random variable
$X_1, X_2, \ldots, X_n$
$E(X_i) = \mu_{X_i}$

$\operatorname{var}(X_i) = \sigma^2_{X_i}$

The mean and variance of are given by

$Y_n = X_1 + X_2 + \ldots + X_n$

Y_n

$$EY_n = \mu_{Y_n} = \mu_{X_1} + \mu_{X_2} + \ldots + \mu_{X_n}$$

$$\operatorname{var}(Y_n) = \sigma^2_{Y_n} = E\{\sum_{i=1}^{n}(X_i - \mu_{X_i})\}^2$$

$$= \sum_{i=1}^{n} E(X_i - \mu_{X_i})^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} E(X_i - \mu_i)(X_j - \mu_j)$$

$$= \sigma^2_{X_1} + \sigma^2_{X_2} + \ldots + \sigma^2_{X_n}$$

$$\because X_i \text{ and } X_j \text{ are independent for } i \neq j.$$

Y_n

and

Thus we can determine the mean and the variance of . Y_n

Can we guess about the probability distribution of ?

The central limit theorem (CLT) provides an answer to this question.

$$\left\{Y_n = \sum_{i=1}^{n} X_i\right\}$$

$Y \sim N(\mu_Y, \sigma_Y^2)$ $n \to \infty$

$X_1, X_2, \ldots, X_n$

The CLT states that under very general conditions converges in distribution to as. The conditions are:

The random variables are independent and identically distributed.

The random variables are independent with same mean and variance, but not identically distributed.

The random variables are independent with different means and same variance and not identically distributed.

The random variables are independent with different means and each variance being neither too small nor too large.

$X_1, X_2, \ldots, X_n$

We shall consider the first condition only. In this case, the central-limit theorem can be stated as follows:

Proof of the Central Limit Theorem:

We give a less rigorous proof of the theorem with the help of the characteristic function.

Further we consider each of to have zero mean. Thus,

Clearly,

$X_1, X_2, \ldots, X_n$ $Y_n = (X_1 + X_2 + \ldots + X_n)/\sqrt{n}.$

$$\mu_{Y_n} = 0,$$

$$\sigma^2_{Y_n} = \sigma^2_X,$$

$$E(Y_n^3) = E(X^3)/\sqrt{n} \text{ and so on.}$$

Y_n

$$\phi_{Y_n}(\omega) = E\left(e^{j\omega Y_n}\right) = E\left(e^{\left(j\omega \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)}\right)$$

$n \to \infty$ ϕ_{Y_n}

We will show that as $e^{j\omega Y_n}$ the characteristic function is of the form of the characteristic function of a Gaussian random variable.

Expanding $e^{j\omega Y_n}$ in power series $= 1 + j\omega Y_n + \frac{(j\omega)^2}{2!} Y_n^2 + \frac{(j\omega)^3}{3!} Y_n^3 + \ldots$

Assume all the moments of Y_n to be finite. Then

$$\phi_{Y_n}(\omega) = E\left(e^{j\omega Y_n}\right) = 1 + j\omega \mu_{Y_n} + \frac{(j\omega)^2}{2!} E(Y_n^2) + \frac{(j\omega)^2}{3!} E(Y_n^3) + \ldots$$

Substituting $\mu_{Y_n} = 0$ and $E(Y_n^2) = \sigma_{Y_n}^2 = \sigma_X^2$, we get

$$\phi_{Y_n}(\omega) = 1 - (\omega^2/2!)\sigma_X^2 + R(\omega, n)$$

where $R(\omega,n)$ is the average of terms involving ω^3 and higher powers of ω.

Note also that each term in $R(\omega,n)$ involves a ratio of a higher moment and a power of n and therefore,

$$\lim_{n\to\infty} R(\omega,n) = 0$$

which is the characteristic function of a Gaussian random variable with 0 mean and variance σ_X^2.

$$\therefore \lim_{n\to\infty} \phi_{Y_n}(\omega) \simeq 1 - \frac{\omega^2}{2!}\sigma_X^2 \simeq e^{-\frac{\sigma_X^2\omega^2}{2}}$$

$$Y_n \xrightarrow{d} N(0, \sigma_X^2)$$

OPERATIONS ON MULTIPLE RANDOM VARIABLES

Expected Values of Functions of Random Variables

If $Y = g(X)$ is a function of a continuous random variable $X,$ then

If $Y = g(X)$ is a function of a discrete random variable X, then $EY = Eg(X) = \sum_{x \in R_X} g(x) p_X(x)$

Suppose $Z = g(X,Y)$ is a function of continuous random variables X and Y then the expected value of Z is given by

$$EZ = Eg(X,y) = \int_{-\infty}^{\infty} z f_Z(z) dz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

Thus EZ can be computed without explicitly determining $f_Z(z)$.

We can establish the above result as follows.

Suppose $Z = g(X,Y)$ has n roots $(x_i, y_i),\ i = 1,2,..,n$ at $Z = z$. Then

$$\{z < Z \le z + \Delta z\} = \bigcup_{i=1}^{n} \{(x_i, y_i) \in \Delta D_i\}$$

Where ΔD_i Is the differential region containing (x_i, y_i). The mapping is illustrated in Figure 1 for $n = 3$.

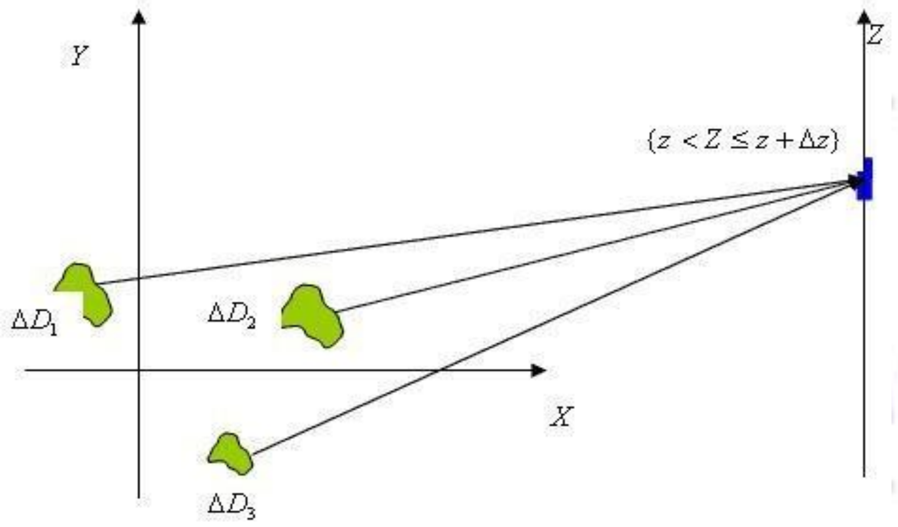


Figure 1

your roots to success...

Note that

$$P(\{z < Z \le z + \Delta z\}) = f_Z(z)\Delta z = \sum_{(x_i, y_i) \in D_i} f_{X,Y}(x_i, y_i)\Delta x_i \Delta y_i$$

$$\therefore z f_Z(z)\Delta z = \sum_{(x_i, y_i) \in \Delta D_i} z f_{X,Y}(x_i, y_i)\Delta x_i \Delta y_i$$

$$= \sum_{(x_i, y_i) \in \Delta D_i} g(x_i, y_i) f_{X,Y}(x_i, y_i)\Delta x_i \Delta y_i$$

As is varied over the entire axis, the corresponding (non-overlapping) differential regions in plane cover the entire plane.

Z Z

$X - Y$

Thus,

$$\therefore \int_{-\infty}^{\infty} z f_Z(z)dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)dxdy$$

If is a function of discrete random variables , we can similarly show that

$$\boxed{Eg(X, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)dxdy}$$

Example 1 The joint pdf of two random variables is given by

$Z = g(X, Y)$ X and Y

$$EZ = Eg(X, Y) = \sum_{x, y \in R_X \times R_Y} \sum g(x, y) p_{X,Y}(x, y)$$

X and Y

$$f_{X,Y}(x, y) = \frac{1}{4}xy \quad 0 \le x \le 2,\ 0 \le y \le 2$$
$$= 0 \quad \text{otherwise}$$

$$g(X,Y) = X^2Y$$

Find the joint expectation of

$$\begin{aligned} Eg(X,Y) &= EX^2Y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy \\ &= \int_0^2\int_0^2 x^2y\frac{1}{4}xydxdy \\ &= \frac{1}{4}\int_0^2 x^3dx\int_0^2 y^2dy \\ &= \frac{1}{4}\times\frac{2^4}{4}\times\frac{2^3}{3} \\ &= \frac{8}{3} \end{aligned}$$

$Z = aX + bY$, where a and b are constants, then

Example 2 If $EZ = aEX + bEY$

Proof:

$$\begin{aligned} EZ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by) f_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} axf_{X,Y}(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} byf_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} ax\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx + \int_{-\infty}^{\infty} by\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy \\ &= a\int_{-\infty}^{\infty} xf_X(x)dx + b\int_{-\infty}^{\infty} yf_Y(y)dy \\ &= aEX + bEY \end{aligned}$$

Thus, expectation is a linear operator.

Example 3

Consider the discrete random variables discussed in Example 4 in lecture 18.The joint probability mass function of the random variables are tabulated in Table . Find the joint expectation of X and Y .

$$g(X,Y) = XY$$

your roots to success...

X \ Y	0	1	2	$p_Y(y)$
0	0.25	0.1	0.15	0.5
1	0.14	0.35	0.01	0.5
$p_X(x)$	0.39	0.45	0.16	

$$\text{Clearly, } EXY = \sum_{x,y \in \mathbb{R}_X \times \mathbb{R}_Y} \sum g(x,y) p_{X,Y}(x,y)$$
$$= 1\times1\times0.35 + 1\times2\times0.01$$
$$= 0.37$$

Remark

$$E[a_1 g_1(X,Y) + a_2 g_2(X,Y)] = a_1 E g_1(X,Y) + a_2 E g_2(X,Y)$$

We have earlier shown that expectation is a linear operator. We can generally write

$$E(XY + 5\log_e XY) = EXY + 5E\log_e XY$$

Thus

If X and Y are independent random variables and $g(X,Y) = g_1(X)g_2(Y)$,then

$$Eg(X,Y) = Eg_1(X)g_2(Y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)g_2(Y) f_{X,Y}(x,y)dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)g_2(Y) f_X(x) f_Y(y)dxdy$$
$$= \int_{-\infty}^{\infty} g_1(X) f_X(x)dx \int_{-\infty}^{\infty} g_2(Y) f_Y(y)dy$$
$$= Eg_1(X)Eg_2(Y)$$

Joint Moments of Random Variables

Just like the moments of a random variable provide a summary description of the random variable, so also the joint moments provide summary description of two random variables. For two continuous random variables X and Y, the joint moment of order $m+n$ is defined as

$$E(X^m Y^n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n f_{X,Y}(x,y)dxdy$$

And the joint central moment of order $m+n$ is defined as

your roots to success...

$$E(X-\mu_X)^m(Y-\mu_Y)^n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)^m(y-\mu_Y)^n f_{X,Y}(x,y)\,dx\,dy$$

$\mu_X = EX \qquad \mu_Y = EY$

where and

X and Y $\qquad m \qquad n$

Remark

If are discrete random variables, the joint expectation of order and is defined as

$$E(X^mY^n) = \sum_{(x,y)\in R_{X,Y}}\sum x^m y^n p_{X,Y}(x,y)$$

$$E(X-\mu_X)^m(Y-\mu_Y)^n = \sum_{(x,y)\in R_{X,Y}}\sum (x-\mu_X)^m(y-\mu_Y)^n p_{X,Y}(x,y)$$

If and , we have the second-order moment of the random variables given by

$m=1 \qquad n=1$

X and Y

$$E(XY) = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy & \text{if } X \text{ and } Y \text{ are continuous} \\ \sum_{(x,y)\in R_{X,Y}}\sum xy\, p_{X,Y}(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \end{cases}$$

If are independent,

Covariance of two random variables

The covariance of two random variables is defined as X and Y $\quad E(XY) = EXEY$

Cov(X, Y) is also denoted as . Expanding the right-hand side, we get X and Y

$$Cov(X,Y) = E(X-\mu_X)(Y-\mu_Y)$$

$\sigma_{X,Y}$

The ratio is called the correlation coefficient.

$$\begin{aligned} Cov(X,Y) &= E(X-\mu_X)(Y-\mu_Y) \\ &= E(XY - \mu_Y X - \mu_X Y + \mu_X\mu_Y) \\ &= EXY - \mu_Y EX - \mu_X EY + \mu_X\mu_Y \\ &= EXY - \mu_X\mu_Y \end{aligned}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

your roots to success...

$\rho_{X,Y} > 0$ X and Y

$\rho_{X,Y} < 0$ X and Y

$\rho_{X,Y} = 0$ X and Y

$$|\rho(X,Y)| \leq 1.$$

X and Y $E^2(XY) \leq EX^2 EY^2$

If then are called positively correlated. If then are called negatively correlated If then are uncorrelated.

We will also show that To establish the relation, we prove the following result:

$Z = aX + Y$

For two random variables

Proof:

$$E(aX+Y)^2 \geq 0$$

Consider the random variable

$$\Rightarrow a^2 EX^2 + EY^2 + 2aEXY \geq 0$$

Non-negativity of the left-hand side implies that its minimum also must be nonnegative. For the minimum value,

so the corresponding minimum is

$$\frac{dEZ^2}{da} = 0 \Rightarrow a = -\frac{EXY}{EX^2}$$

$$\frac{E^2XY}{EX^2} + EY^2 - 2\frac{E^2XY}{EX^2}$$

Since the minimum is nonnegative,

$$= EY^2 - \frac{E^2XY}{EX^2}$$

Now

$$EY^2 - \frac{E^2XY}{EX^2} \geq 0$$

$$\Rightarrow E^2XY \leq EX^2 EY^2$$

$$\Rightarrow |EXY| \leq \sqrt{EX^2}\sqrt{EY^2}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$
$$= \frac{E(X-\mu_X)(Y-\mu_Y)}{\sqrt{E(X-\mu_X)^2 E(Y-\mu_Y)^2}}$$
$$\therefore |\rho(X,Y)| = \frac{|E(X-\mu_X)(Y-\mu_Y)|}{\sqrt{E(X-\mu_X)^2 E(Y-\mu_Y)^2}}$$
$$\leq \frac{\sqrt{E(X-\mu_X)^2}\sqrt{E(Y-\mu_Y)^2}}{\sqrt{E(X-\mu_X)^2 E(Y-\mu_Y)^2}}$$
$$= 1$$

$$|\rho(X,Y)| \leq 1$$

Thus

Uncorrelated random variables

Two random variables X and Y are called uncorrelated if

$$Cov(X,Y) = 0$$
which also means
$$E(XY) = \mu_X \mu_Y$$

Recall that if X and Y are independent random variables, then $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

$$EXY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy \quad \text{assuming } X \text{ and } Y \text{ are continuous}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)dxdy$$
$$= \int_{-\infty}^{\infty} x f_X(x)dx \int_{-\infty}^{\infty} y f_Y(y)dy$$

then
$$= EXEY$$

Thus two independent random variables are always uncorrelated.

Note that independence implies uncorrelated. But uncorrelated generally does not imply independence (except for jointly Gaussian random variables).

Joint Characteristic Functions of Two Random Variables

The joint characteristic function of two random variables X and Y is defined by

$$\phi_{X,Y}(\omega_1, \omega_2) = Ee^{j\omega_1 x + j\omega_2 y}$$

If X and Y are jointly continuous random variables, then

$$\phi_{X,Y}(\omega_1,\omega_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)e^{j\omega_1 x+j\omega_2 y}dydx$$

$\phi_{X,Y}(\omega_1,\omega_2)$

$e^{j\omega_1 x+j\omega_2 y}$

$e^{-(j\omega_1 x+j\omega_2 y)}$.

Note that $f_{X,Y}(x,y)$ is same as the two-dimensional Fourier transform with the basis function instead of

is related to the joint characteristic function by the Fourier inversion formula

$$f_{X,Y}(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_{X,Y}(\omega_1,\omega_2)e^{-j\omega_1 x-j\omega_2 y}d\,\omega_1 d\,\omega_2$$

If X and Y are discrete random variables, we can define the joint characteristic function in terms of the joint probability mass function as follows:

$$\phi_{X,Y}(\omega_1,\omega_2)=\sum_{(x,y)\in R_X\times R_Y}\sum p_{X,Y}(x,y)e^{j\omega_1 x+j\omega_2 y}$$

Properties of the Joint Characteristic Function

The joint characteristic function has properties similar to the properties of the chacteristic function of a single random variable. We can easily establish the following properties:

1.
2.
If and are independent random variables, then

$$\phi_X(\omega)=\phi_{X,Y}(\omega,0)$$
$$\phi_Y(\omega)=\phi_{X,Y}(0,\omega)$$

X Y

We have,

$$\begin{aligned}\phi_{X,Y}(\omega_1,\omega_2)&=Ee^{j\omega_1 X+j\omega_2 Y}\\&=E(e^{j\omega_1 X}e^{j\omega_2 Y})\\&=Ee^{j\omega_1 X}Ee^{j\omega_2 Y}\\&=\phi_X(\omega_1)\phi_Y(\omega_2)\end{aligned}$$

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$$\phi_{X,Y}(\omega_1, \omega_2) = Ee^{j\omega_1 X + j\omega_2 Y}$$

$$= E\left(1 + j\omega_1 X + j\omega_2 Y + \frac{j^2(\omega_1 X + j\omega_2 Y)^2}{2} + \ldots\ldots\ldots\ldots\right)$$

$$= 1 + j\omega_1 EX + j\omega_2 EY + \frac{j^2 \omega_1{}^2 EX^2}{2} + \frac{j^2 \omega_2{}^2 EY^2}{2} + \omega_1 \omega_2 EXY + \ldots\ldots$$

Hence,

$$\phi_{X,Y}(0,0) = 1$$

$$EX = \frac{1}{j}\frac{\partial}{\partial \omega_1}\phi_{X,Y}(\omega_1, \omega_2)\Big|_{\omega_1 = 0}$$

$$EY = \frac{1}{j}\frac{\partial}{\partial \omega_2}\phi_{X,Y}(\omega_1, \omega_2)\Big|_{\omega_2 = 0}$$

$$EXY = \frac{1}{j^2}\frac{\partial^2 \phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1 \partial \omega_2}\Big|_{\omega_1 = 0, \omega_2 = 0}$$

In general, the order joint moment is given by

$(m+n)th$

$$EX^m Y^n = \frac{1}{j^{m+n}}\frac{\partial^m \partial^n \phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1^m \partial \omega_2^n}\Big|_{\omega_1 = 0, \omega_2 = 0}$$

Example 2 The joint characteristic function of the jointly Gaussian random variables and with the joint pdf

X

Y

Let us recall the characteristic function of a Gaussian random variable

$$f_{X,Y}(x,y) = \frac{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

$$X \sim N(\mu_X, \sigma_X^2)$$

$$\phi_X(\omega) = Ee^{j\omega X}$$
$$= \frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} .e^{j\omega x}dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2-2(\mu_X-\sigma_X^2 j\omega)x+(\mu_X-\sigma_X^2 j\omega)^2-(\mu_X-\sigma_X^2 j\omega)^2+\mu_X^2}{\sigma_X^2}} dx$$
$$= e^{\frac{1}{2}\frac{(-\sigma_X^4\omega^2+2\mu_X\sigma_X^2 j\omega)}{\sigma_X^2}} \underbrace{\frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_X-\sigma_X^2 j\omega}{\sigma_X}\right)^2} dx}_{\text{Area under a Gaussian}}$$
$$= e^{\mu_X j\omega-\sigma_X^2\omega^2/2}\times 1$$
$$= e^{\mu_X j\omega-\sigma_X^{4\,2}\omega^2/2}$$

X Y

If and is jointly Gaussian,

$$f_{X,Y}(x,y) = \frac{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2-2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

we can similarly show that

$$\phi_{X,Y}(\omega_1,\omega_2) = Ee^{j(X\omega_1+Y\omega_2)}$$
$$= e^{j\mu_X\omega_1+j\mu_Y\omega_2-\frac{1}{2}(\sigma_X^2\omega_1^2+2\rho_{X,Y}\sigma_X\sigma_Y\omega_1\omega_2+\sigma_Y^2\omega_2^2)}$$

We can use the joint characteristic functions to simplify the probabilistic analysis as illustrated on next page:

Jointly Gaussian Random Variables

Many practically occurring random variables are modeled as jointly Gaussian random variables. For example, noise samples at different instants in the communication system are modeled as jointly Gaussian random variables.

Two random variables are called jointly Gaussian if their joint probability density

X and Y

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}-2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}, \quad -\infty < x < \infty,\ -\infty < y < \infty$$

The joint pdf is determined by 5 parameters

means μ_X and μ_Y

variances σ_X^2 and σ_Y^2

correlation coefficient $\rho_{X,Y}$

We denote the jointly Gaussian random variables X and Y with these parameters as $(X,Y) \sim N(\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho_{X,Y})$

The joint pdf has a bell shape centered at (μ_X,μ_Y) as shown in the Figure 1 below. The variances σ_X^2 and σ_Y^2 determine the spread of the pdf surface and $\rho_{X,Y}$ determines the orientation of the surface in the $X-Y$ plane.

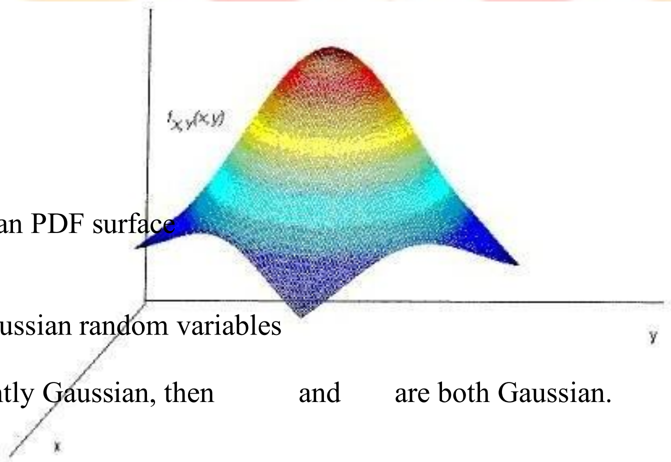

Figure 1 Jointly Gaussian PDF surface

Properties of jointly Gaussian random variables

If X and Y are jointly Gaussian, then X and Y are both Gaussian.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

We have

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} dy$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{\rho_{X,Y}^2(x-\mu_X)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} dy$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2\sigma_Y^2(1-\rho_{X,Y}^2)}\left[(y-\mu_Y - \frac{\rho_{X,Y}\sigma_Y(x-\mu_X)}{\sigma_X}\right]^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

Similarly

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}$$

The converse of the above result is not true. If each of X and Y is Gaussian, X and Y are not necessarily jointly Gaussian. Suppose

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}(1+\sin x \sin y)$$

$f_{X,Y}(x,y)$ in this example is non-Gaussian and qualifies to be a joint pdf. Because,

And

$$f_{X,Y}(x,y) \geq 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}(1+\sin x\sin y)dydx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}dydx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}\sin x\sin y\,dydx$$

$$=1+\frac{1}{2\pi\sigma_X\sigma_Y}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}}\sin x\,dx\underbrace{\int_{-\infty}^{\infty}e^{-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^2}}\sin y\,dy}_{\text{integration of an odd function}}$$

$$=1+0$$

$$=1$$

The marginal density $f_X(x)$ is given by

$$f_X(x)=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}(1+\sin x\sin y)dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}dy+\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}\underbrace{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}\sin x\sin y\,dy}_{\text{integration of an odd function}}$$

$$=\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}+0$$

$$=\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

Similarly, $f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}$

Thus X and Y are both Gaussian, but not jointly Gaussian.

If X and Y are jointly Gaussian, then for any constants a and b, the random variable Z given by $Z=aX+bY$ is Gaussian with mean $\mu_Z=a\mu_X+b\mu_Y$ and variance

$$\sigma_Z^2=a^2\sigma_X^2+b^2\sigma_Y^2+2ab\sigma_X\sigma_Y\rho_{X,Y}$$

Two jointly Gaussian RVs X and Y are independent if and only if X and Y are uncorrelated $(\rho_{X,Y}=0)$. Observe that if X and Y are uncorrelated, then

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}$$
$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{(x-\mu_X)^2}{\sigma_X^2}} \frac{1}{\sqrt{2\pi}\sigma_Y} e^{\frac{(y-\mu_Y)^2}{\sigma_Y^2}}$$
$$= f_X(x) f_Y(y)$$

Example 1 Suppose X and Y are two jointly-Gaussian 0-mean random variables with variances of 1 and 4 respectively and a covariance of 1. Find the joint PDF $f_{X,Y}(x,y)$

We have

$$\mu_X = \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4 \text{ and } \operatorname{cov}(X,Y) = 1.$$
$$\therefore \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{1}{1\times 2} = \frac{1}{2}$$

and

$$f_{X,Y}(x,y) = \frac{1}{2\pi\times 1\times 2\sqrt{1-\frac{1}{4}}} e^{-\frac{1}{2\times\frac{3}{4}}\left[\frac{x^2}{1}-2\times\frac{1}{2}\times\frac{xy}{1\times 2}+\frac{y^2}{4}\right]}$$
$$= \frac{1}{2\sqrt{3}\pi} e^{-\frac{2}{3}\left[x^2-\frac{xy}{2}+\frac{y^2}{4}\right]}$$

Example 2 Linear transformation of two random variables

Suppose $Z = aX + bY$. then

$$\phi_Z(\omega) = Ee^{j\omega Z} = Ee^{j(aX+bY)\omega} = \phi_{X,Y}(a\omega, b\omega)$$

If X and Y are jointly Gaussian, then

$$\phi_Z(\omega) = \phi_{X,Y}(a\omega, b\omega)$$
$$= e^{j(\mu_X+\mu_Y)\omega-\frac{1}{2}(a^2\sigma_X^2+2\rho_{X,Y}ab\sigma_X\sigma_Y+b^2\sigma_Y^2)\omega^2}$$

Which is the characteristic function of a Gaussian random variable with mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + 2\rho_{X,Y}\sigma_X\sigma_Y + \sigma_Y^2$

thus the linear transformation of two Gaussian random variables is a Gaussian random variable.

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Example 3 If Z = X + Y and X and Y are independent, then

$$\phi_Z(\omega) = \phi_{X,Y}(\omega, \omega)$$
$$= \phi_X(\omega)\phi_Y(\omega)$$

Using the property of the Fourier transform, we get

Hence proved. $f_Z(z) = f_X(z) * f_Y(z)$

Univariate transformations

When working on the probability density function (pdf) of a random variable X, one is often led to create a new variable Y defined as a function f(X) of the original variable X. For example, if X~N(μ, ²), then the new variable:

Y = f(X) = (X - μ)/
Is N (0, 1).

It is also often the case that the quantity of interest is a function of another (random) quantity whose distribution is known. Here are a few examples:
*Scaling: from degrees to radians, miles to kilometers, light-years to parsecs, degrees
Celsius to degrees Fahrenheit, linear to logarithmic scale, to the distribution of the variance
Laws of physics: what is the distribution of the kinetic energy of the molecules of a gas if the distribution of the speed of the molecules is known ?

So the general question is:
If Y = h(X),
And if f(x) is the pdf of X, Then what is the pdf g(y) of Y?
TRANSFORMATION OF A MULTIPLE RANDOM VARIABLES

χ^2

Multivariate transformations
The problem extends naturally to the case when several variables Yj are defined from several variables Xi through a transformation y = h(x).
Here are some examples:

Dividing by a constant: Y = X/m
Multiplying by a constant and adding a constant: Y = mX + b
Dividing by a constant and subtracting a constant: Y = X/m - b
θ

Suppose the vector of random variables has the joint distribution . Set for some square matrix and vector . If then has the joint distribution

Indeed, suppose (this is the notation for "the is the distribution density of ") and . For any domain of the space we can θ

θ θ

write

θ

We make the change of variables in the last integral.

(Line

ar transformation of random variables)

The linear transformation is distributed as . The was defined in the section (Definition of normal variable).

For two independent standard normal variables (s.n.v.) and the combination

is distributed as .

A product of normal variables is not a normal variable. See the section on the chi - squared distribution.

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