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UNIT-III

Continuous Distributions and Sampling

Normal Distribution: It was first discovered by English Mathematician De-Moivre in 17333 and further refined by French Mathematician Laplace in 1744 and independently by Karl Friedrich Gauss. Normal distribution is also known as 'Gaussian distribution'.

Definition: A random variable X is said to have a Normal distribution, if its probability density function is given by

$$f(x;\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ ;\ -\infty<x<\infty, -\infty<\mu<\infty, \sigma>0$$

Where μ is mean and σ^2 is variance are called parameters.

Notation: $X \sim N(\mu, \sigma^2)$.

***Problem*: In a normal distribution, 7% of the items are under 35 and 89% under 63. Determine the mean and variance of the distribution.**

Solution: Given $P(X<35) = 0.07$ and $P(X<63) = 0.89$

Therefore, $P(X>63) = 1- P(X<63) = 1 – 0.89 = 0.11$

When $X = 35$, $Z = (X-\mu)/\sigma = (35-\mu)/\sigma = -z_1$ (say) ……(1)

When $X = 63$, $Z = (X-\mu)/\sigma = (63-\mu)/\sigma = -z_2$ (say) …….(2)

$P(0<Z<z_2) = 0.39 \Rightarrow z_2 = 1.23$ (from tables)

and $P(0<Z<z_1) = 0.43 \Rightarrow z_1 = 1.48$

from(1) we have $(35-\mu)/\sigma = -1.48$ ……(3)

from (2) we have $(63-\mu)/\sigma = 1.23$ ……(4)

(4) – (3) gives $\sigma = 10.332$

From equation (3), $\mu = 50.3$

Therefore, mean = 50.3 and variance = 106.75

Characteristics:

(1). The graph of the Normal distribution y = f(x) in the xy-plane is known as the normal curve.

(2). The curve is a bell shaped curve and symmetrical with respect to mean i.e., about the line x =μ and the two tails on the right and the left sides of the mean (μ) extends to infinity. The top of the bell is directly above the mean μ.

(3). Area under the normal curve represents the total population.

(4). Mean, median and mode of the distribution coincide at x = μ as the distribution is symmetrical. So normal curve is Uni modal (has only one maximum point).

(5). x-axis is an asymptote to the curve.

(6). Linear combination of independent normal variates is also a normal variate.

(7). The points of inflexion of the curve are at x = μ ± σ and the curve changes from concave to convex at x = μ + σ to x = μ - σ.

(8). The probability that the normal variate X with mean μ and standard deviation σ lies between x_1 and x_2 is given by

$$P(x_1 \le X \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \dots\dots\dots(1)$$

Since (1) depends on the two parameters μ and σ, we get different normal curves for different values of μ and σ and it is an impracticable task to plot all such normal curves. Instead, by putting z=(x-μ)/σ, the R.H.S. of equation (1) becomes independent of the two parameters μ and σ. Here z is known as the standard variable.

(9). Area under the normal curve is distributed as follows:

P(μ - σ<X< μ + σ) = 0.6826 ; P(μ - 2σ<X< μ + 2σ) = 0.9543; P(μ - 3σ<X< μ + 3σ) = 0.9973.

Prob : In a normal distribution 7% of the items are under 35 and 89% are under 63. Determine the mean and variance of the distribution .

Sol : Let the normal distribution have:

- Mean $= \mu$
- Standard deviation $= \sigma$

We are given: $P(X < 35) = 0.07$, $P(X < 63) = 0.89$

We use the standard normal variable $Z = \frac{X-\mu}{\sigma}$

$$z = \frac{(1.2) - (0)}{1} = 1.2$$

Step 1: Convert the given probabilities into Z-values

From the standard normal distribution table:

For $P(X < 35) = 0.07$

$$P(Z < z) = 0.07$$

The corresponding Z-value is $z_1 \approx -1.48$

Hence,

$$\frac{35 - \mu}{\sigma} = -1.48$$

$$\boxed{\mu - 35 = 1.48\sigma}$$

For $P(X < 63) = 0.89$

$$P(Z < z) = 0.89$$

The corresponding Z-value is $z_2 \approx 1.23$

Hence $\frac{63-\mu}{\sigma} = 1.23$ => $\boxed{63 - \mu = 1.23\sigma}$

Step 2: Solve for σ

Add the two equations: $(\mu - 35) + (63 - \mu) = 1.48\sigma + 1.23\sigma$

$$28 = 2.71\sigma$$

$$\sigma = \frac{28}{2.71}$$

$$\boxed{\sigma \approx 10.33}$$

Step 3: Find the Mean

Using $\mu - 35 = 1.48(10.33)$

$$\mu - 35 = 15.29$$

$$\boxed{\mu \approx 50.29}$$

Step 4: Find the Variance

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = (1.4)^2 = 1.96$$

$$\text{Variance} = \sigma^2 = (10.33)^2$$

$$\boxed{\sigma^2 \approx 106.7}$$

Prob : A population consists of five numbers 2,3,6,8,11. Consider all possible samples of size two which can be drawn with replacement from this population. Find

i. **The mean of the population.**
ii. **The standard deviation of the population.**
iii. **The mean of sampling distributions of means and**

The standard deviation of the sampling distributions of means.

Sol : **Solution**

Given Population 2,3,6,8,11

Population size, N=5

Sample size, n=2

Sampling is **with replacement**.

Hence, the total number of possible samples is $N^n = 5^2 = 25$.

(i) Mean of the Population

The population mean is $\mu = \sum X / N = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$

(ii) Standard Deviation of the Population

First compute the squared deviations from the mean $\sum(X-\mu)^2 = 16+9+0+4+25 = 54$

Population variance: $\sigma^2 = \frac{\sum(X-\mu)^2}{N} = 54/5 = 10.8$

Population standard deviation: $\sigma = \sqrt{10.8}$ $\boxed{\sigma \approx 3.29}$

(iii) Mean of the Sampling Distribution of Means

Step 1: List all possible samples (with replacement)

Sample	**Mean**
(2,2)	2.0
(2,3)	2.5
(2,6)	4.0
(2,8)	5.0
(2,11)	6.5
(3,2)	2.5

(3,3)	3.0
(3,6)	4.5
(3,8)	5.5
(3,11)	7.0
(6,2)	4.0
(6,3)	4.5
(6,6)	6.0
(6,8)	7.0
(6,11)	8.5
(8,2)	5.0
(8,3)	5.5
(8,6)	7.0
(8,8)	8.0
(8,11)	9.5
(11,2)	6.5
(11,3)	7.0
(11,6)	8.5
(11,8)	9.5
(11,11)	11.0

Step 2: Sum of sample means

$$\sum \bar{X} = 150$$

Therefore, $\bar{\bar{X}} = \frac{150}{25} = 6$

Mean of sampling distribution=6

Observation: The mean of the sampling distribution equals the population mean.

(iv) Standard Deviation of the Sampling Distribution of Means

For sampling **with replacement**, $\sigma_{\bar{X}} = \sigma / \sqrt{n}$

Substitute the values:

$$= \frac{3.29}{\sqrt{2}}$$

$$= \frac{3.29}{1.414}$$

$$= 2.324$$

$$\sigma_X \approx 2.32$$

PROBLEM: If the masses of 300 students are normally distributed with mean 68 kgs and standard deviation 3kgs how many students have masses

i. **Greater than 72kgs**

Less than or equal to 64 kgs.

SOL: Given:

$$\mu = 68, \sigma = 3, N = 300$$

For a normal distribution, first convert the given value into the standard normal variable (Z-score).

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{(1.2)-(0)}{1} = 1.2$$

$$\Phi(z) \approx 88.5\%$$

(i) Number of students having masses greater than 72 kg

Step 1: Calculate the Z-score

$$Z = \frac{X-\mu}{\sigma}$$

$$= \frac{72-68}{3}$$

$$= \frac{4}{3}$$

$$Z = 1.33$$

Step 2: Obtain the probability from the Standard Normal Table

From the Z-table, $P(0 < Z < 1.33) = 0.4082$

Therefore, $P(Z > 1.33) = 0.5 - 0.4082$

$$= 0.0918$$

Hence, $P(X > 72) = 0.0918$

Step 3: Find the Number of Students

$$\text{Number of students} = 300 \times 0.0918$$
$$= 27.54$$
$$\boxed{\approx 28 \text{ students}}$$

(ii) Number of students having masses less than or equal to 64 kg

Step 1: Calculate the Z-score

$$Z = \frac{64 - 68}{3}$$
$$= \frac{-4}{3}$$
$$= -1.33$$

Step 2: Obtain the Probability

From the Z-table,

$$P(0 < Z < 1.33) = 0.4082$$

Hence,

$$P(Z \leq -1.33) = 0.5 - 0.4082$$
$$= 0.0918$$

Therefore, $$P(X \leq 64) = 0.0918$$

Step 3: Find the Number of Students

$$300 \times 0.0918$$
$$= 27.54$$
$$\boxed{\approx 28 \text{ students}}$$

Prob : If X is a Normal variate with mean 30 and standard deviation 5. find P(26≤X≤40).

Sol:

Given Data

- Mean,

$$\mu = 30$$

- Standard deviation,

$$\sigma = 5$$

We need to find

$$P(26 \leq X \leq 40).$$

Step 2: Convert to Standard Normal Variable

The standard normal variable is

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{(1.2)-(0)}{1} = 1.2$$

$$\Phi(z) \approx 88.5\%$$

$$Z = \frac{X-\mu}{\sigma}$$

For $X = 26$

$$Z = \frac{26-30}{5}$$

$$= \frac{-4}{5}$$

$$= -0.8$$

For $X = 40$

$$Z = \frac{40-30}{5}$$

$$= \frac{10}{5}$$

$$= 2$$

Therefore,

$$P(26 \le X \le 40) = P(-0.8 \le Z \le 2).$$

Step 3: Use the Standard Normal Table
From the standard normal table,

$$P(0 < Z < 0.8) = 0.2881$$

and

$$P(0 < Z < 2.0) = 0.4772$$

Hence,

$$P(-0.8 \le Z \le 0) = 0.2881$$

Therefore,

$$P(-0.8 \le Z \le 2) = P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$$
$$= 0.2881 + 0.4772$$
$$= 0.7653$$

Prob : A population consists of the five numbers $\boxed{4, 6, 8, 10, 12}$

Consider all possible samples of size 2 drawn without replacement from this population.
Find:
The mean of the population.
The standard deviation of the population.
The mean of the sampling distribution of sample means.

The standard deviation of the sampling distribution of sample means.

Sol:

Population: $4, 6, 8, 10, 12$

Population size $N = 5$

Sample size $n = 2$

(i) Mean of the Population
The population mean is

$$\mu = \frac{\sum X}{N}$$
$$= \frac{4 + 6 + 8 + 10 + 12}{5}$$
$$= \frac{40}{5}$$
$$\boxed{\mu = 8}$$

(ii) Standard Deviation of the Population
The population variance is

$$\sigma^2 = \frac{\sum (X - \mu)^2}{N}$$

Calculation Table

X	$X - \mu$	$(X - \mu)^2$
4	-4	16
6	-2	4
8	0	0
10	2	4
12	4	16

$$\sum (X - \mu)^2 = 40$$

Therefore,

$$\sigma^2 = \frac{40}{5} = 8$$

Population standard deviation

$$\sigma = \sqrt{8}$$
$$\boxed{\sigma = 2.828}$$

(iii) Mean of the Sampling Distribution of Means
Step 1: List All Possible Samples
The number of samples is

$$^5C_2 = 10$$

Sample	Sample Mean
(4,6)	5
(4,8)	6
(4,10)	7
(4,12)	8
(6,8)	7
(6,10)	8
(6,12)	9
(8,10)	9
(8,12)	10
(10,12)	11

Step 2: Mean of Sample Means

$$\sum \bar{X} = 5 + 6 + 7 + 8 + 7 + 8 + 9 + 9 + 10 + 11 = 80$$

Number of sample means = 10

$$\mu_{\bar{X}} = \frac{80}{10} = 8$$

Therefore,

$$\boxed{\mu_{\bar{X}} = 8}$$

Observation: The mean of the sampling distribution is equal to the population mean.

(iv) Standard Deviation of the Sampling Distribution of Means
Using the formula,

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

Substitute

$$\sigma = 2.828, n = 2, N = 5$$

$$= \frac{2.828}{\sqrt{2}}\sqrt{\frac{3}{4}}$$

$$= \frac{2.828}{1.414} \times 0.866$$

$$= 2 \times 0.866$$

$$= 1.732$$

Therefore, $\boxed{\sigma_{\bar{X}} = 1.732}$

Verification from Sample Means

Sample Mean	$(\bar{X}-8)^2$
5	9
6	4
7	1
8	0
7	1
8	0
9	1
9	1
10	4
11	9

$$\sum(\bar{X}-8)^2 = 30$$

Variance of the sampling distribution

$$= \frac{30}{10} = 3$$

Standard deviation

$$= \sqrt{3}$$
$$= 1.732$$

Quantity	Answer
Population Mean	$\boxed{8}$
Population Standard Deviation	$\boxed{2.828}$
Mean of Sampling Distribution	$\boxed{8}$
Standard Deviation of Sampling Distribution	$\boxed{1.732}$