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UNIT-II

Random Variables and Probability Distributions

Concept of Random Variable:

A random variable X is a real function of the events of a given sample space S. Thus for a given experiment defined by a sample space S with events s, the random variable is a function of s. It is denoted by X(s). A random variable X can be considered to be a function that maps all events of the sample space into points on the real axis.

For example, an experiment consists of tossing two coins. Let the random variable be a function X chosen as the number of heads shown. So X maps the real numbers of the event "showing no head" as zero, the event "any one is head" as one and "both heads" as two. Therefore, the random variable is X = $\{0,1,2\}$. The elements of the random variable X are $x_1 = 0,\ x_2 = 1,\ and\ x_3 = 2$.

Types of Random Variable:

Random variables are classified into:

1. ***Discrete Random Variable*** **(DRV):** A random variable X which is defined on the discrete sample space is called discrete random variable.

   For example, consider a discrete sample space $S = \{1,2,3,4\}$. Let us define X = S^2 be a random variable. Then discrete values of S map to discrete values of X as $\{1,4,9,16\}$. The probabilities of the random variable x are equal to the probabilities of set S because of the one-to-one mapping of the discrete points.

   Let X be a discrete random variable with integer events $X = \{x_1, x_2, ..., x_n\}$. The probability of X at any event is a function of x_i and is given by

   $P(X = x_i) = p(x_i)$, $i = 1,2,3,........$

This function is called probability mass function and is abbreviated as p.m.f. (pmf).

Properties of probability mass function:

Consider a discrete random variable X in a sample space with infinite number of possible outcomes, that is, $X = \{x_1, x_2, ...\}$. If the probability of X, $p(x_i)$, $i = 1, 2, 3, ...$ satisfies the following properties then the function $p(x)$ is called probability mass function.

(i) $p(x_i) \geq 0, \forall i$ (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

2. ***Continuous Random Variable*** **(CRV):** A random variable X which is defined on the continuous sample space is called continuous random variable.

   Temperature, time, height and weight over a period of time etc. are examples of CRV.

   The probability density function of a random variable X is defined as the variable X falls in the infinitesimal interval $\left[x - \frac{dx}{2}, x + \frac{dx}{2}\right]$ such that

$$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right) = f(x)dx,$$

$$i.e.\ f(x) = \underset{\Delta t \to 0}{Lt} \frac{P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right)}{dx}$$

   Where $f(x)$ is called the probability density function of a random variable X and the continuous curve y = f(x) is called probability density curve.

***Properties of probability density function*:**

The continuous curve y =f(x) satisfies the following properties, then the function f(x) is called probability density function of a random variable and is abbreviated as p.d.f. (pdf).

(i) $f(x) \geq 0, \forall x,$ (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

3. ***Mixed Random Variable***: A random variable which is defined on both DSS and CSS partially, then the random variable is said to be a mixed random variable.

Probability distribution function:

Let X be a random variable. Then the probability distribution function associated with X is defined as the probability that the outcomes of an experiment will be one of the outcomes for which $X(s) \leq x, x \in R$. That is, the function $F(x)$ is defined by

$F(x) = P(X \leq x) = P\{s : X(s) \leq x\}, -\infty < x < \infty$ is called the distribution function of X. Sometimes it is also known as Cumulative Distribution function and is abbreviated as CDF.

Properties of cdf:

1 . If F is the distribution function of a random variable S and $a < b$, then

(i) $P(a < X \le b) = F(b) - F(a)$

(ii) $P(a \le X \le b) = P(X = a) + [F(b) - F(a)]$

(iii) $P(a < X < b) = [F(b) - F(a)] - P(X = b)$

(iv) $P(a \le X \le b) = [F(b) - F(a)] = P(X = b) + P(X = a)$

2. All distribution functions are monotonically increasing and lie between 0 and 1. That is, if F is the distribution function of the random variable X, then

(i) $0 \le F(x) \le 1$ i.e. F is bounded.

(ii) $F(x) < F(y)$ when $x < y$.

(ii) $F(-\infty) = 0$ and $F(\infty) = 0$.

Evaluation of mean and variance:

1. **A random variable 'X' has the following probability functions:**

x	0	1	2	3	4	5	6	7
P(x)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2$+K

(i) Determine 'K' (ii) Evaluate $P(X < 6), (PX \ge 6) \& Plo < x < 5$

(iii) Mean (iv) Variance

Sol: $Since \sum_{x=0}^{7} P(x) = 1$

K = 1/10 = 0.1

P(X<6)=P(X=0)+P(x=1)+……………..+P(x=5)

=0.81

P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)

$Mean\ \mu = \sum_{i=0}^{7} P_i x_i$

= 3.66 (K=1/10)

Variance $= \sum_{i=0}^{7} P_i x_i^2 - \mu^2$

=3.4044

2. **The probability density f(x) of a continuous random variable is given by f(x) = C.e$^{-|x|}$, - $-\infty < x < \infty$. S.T C = ½ and find (i) mean (ii) variance of the distribution. Also find $P(0 \le X \le 4)$.**

Sol: We have $\int_{-\infty}^{\infty} f(x)dx = 1$

$\Rightarrow C = 1/2$

(i) Mean $\mu = \int_{-\infty}^{\infty} xf(x)dx. = \frac{1}{2}\int_{-\infty}^{\infty} x.e^{-|x|}dx = \infty$

= 0 [integrand is odd]

(ii) $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

=2

(iii) $P(0 \le X \le 4)$=0.49

$\int_{0}^{4} f(x)dx = 0.49$

Example 1

A coin is tossed three times. Let Xbe the number of heads.
Find the possible values of X.
Solution
Sample space

$$S = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

Possible values of X

Outcome	X
TTT	0
TTH	1
THT	1
HTT	1
THH	2
HTH	2
HHT	2
HHH	3

Hence,

$$X = \{0,1,2,3\}$$

Prob : A random variable Xhas the following probability distribution.

X	0	1	2	3	4
P(X)	0.1	0.2	0.3	0.2	0.2

Find:

1. **Verify that it is a valid probability distribution.**
2. **Mean of X.**
3. **Variance of X.**
4. **Standard deviation.**

Sol :

Now,

$$\sum P(X) = 0.1 + 0.2 + 0.3 + 0.2 + 0.2$$
$$= 1.0$$

Hence,

$$\boxed{\sum P(X) = 1}$$

Therefore, **it is a valid probability distribution.**

Step 2: Calculate the Mean $E(X)$

The mean (expected value) is

$$E(X) = \sum XP(X)$$

Construct the following table.

X	$P(X)$	$XP(X)$
0	0.1	0.0
1	0.2	0.2
2	0.3	0.6
3	0.2	0.6
4	0.2	0.8

Now,

$$E(X) = 0 + 0.2 + 0.6 + 0.6 + 0.8$$
$$= 2.2$$

Therefore,

$$\boxed{\text{Mean } = E(X) = 2.2}$$

Step 3: Calculate $E(X^2)$

Construct another column.

X	$P(X)$	X^2	$X^2P(X)$
0	0.1	0	0.0
1	0.2	1	0.2
2	0.3	4	1.2
3	0.2	9	1.8
4	0.2	16	3.2

Now,

$$E(X^2) = 0 + 0.2 + 1.2 + 1.8 + 3.2$$
$$= 6.4$$

Thus,

$$\boxed{E(X^2) = 6.4}$$

Step 4: Calculate the Variance

$$Var(X) = E(X^2) - [E(X)]^2$$
$$Var(X) = (1.4)^2 = 1.96$$
$$\sigma$$

μ-σ+σVar(X) ≈ 1.96
Using the formula,

$$Var(X) = E(X^2) - [E(X)]^2$$

Substitute the values,

$$Var(X) = 6.4 - (2.2)^2$$
$$= 6.4 - 4.84$$
$$= 1.56$$

Therefore,

$$\boxed{Var(X) = 1.56}$$

Step 5: Calculate the Standard Deviation
The standard deviation is

$$\sigma = \sqrt{Var(X)}$$
$$= \sqrt{1.56}$$
$$\approx 1.249$$

Therefore,

$$\boxed{\sigma \approx 1.25}$$

2. Discrete Probability Distribution
A probability distribution gives the probability associated with each value of a discrete random variable.
Example 2
A fair die is rolled.
Let Xbe the number appearing on the die.
Find the probability distribution.
Solution
Since the die is fair

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

Check

$$\sum P(X) = 1$$

Hence, it is a valid probability distribution.

Example 3 : A random variable has the following distribution.

X	0	1	2	3
P(X)	0.2	0.3	0.4	0.1

Find Mean, Variance .

Solution :

Mean

$$E(X) = \sum XP(X)$$
$$= (0)(0.2) + (1)(0.3) + (2)(0.4) + (3)(0.1)$$
$$= 0 + 0.3 + 0.8 + 0.3$$
$$= 1.4$$

Mean = 1.4

Variance

$$Var(X) = E(X^2) - [E(X)]^2$$
$$Var(X) = (1.4)^2 = 1.96$$

First calculate

$$E(X^2) = \sum X^2 P(X)$$
$$= (0)^2(0.2) + (1)^2(0.3) + (2)^2(0.4) + (3)^2(0.1)$$
$$= 0 + 0.3 + 1.6 + 0.9$$
$$= 2.8$$

Now,

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= 2.8 - (1.4)^2$$
$$= 2.8 - 1.96$$
$$= 0.84$$

Continuous Probability Distribution

For a continuous random variable,

- Probability density function (PDF):

$$f(x) \geq 0$$

- Total area under the curve

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

Example 4: The probability density function is

$$\boldsymbol{f(x) = 2x, 0 < x < 1}$$

Find (i) Verify it is a PDF (ii) Find $P(X < 0.5)$.

Solution :

Step 1

$$\int_0^1 2x\, dx$$
$$= x^2 \,|_0^1$$
$$= 1$$

Hence it is a valid PDF.

Step 2

$$P(X < 0.5)$$
$$= \int_0^{0.5} 2x \, dx$$
$$= x^2 \,|_0^{0.5}$$
$$= (0.5)^2$$
$$= 0.25$$

Answer

$$P(X < 0.5) = 0.25$$

Example 5 : If $\boldsymbol{f(x) = 2x, 0 < x < 1}$ Find the mean and Variance .

Solution:

Mean : $E(X) = \int_0^1 x f(x) \, dx$

$$= \int_0^1 2\,x^2 dx$$
$$= 2\left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{2}{3}$$

Variance : $but \quad E(X) = \frac{2}{3}$

Now,

$$E(X^2) = \int_0^1 x^2 \, (2x) \, dx$$
$$= \int_0^1 2\,x^3 dx$$
$$= 2\left[\frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{2}$$

Therefore,

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$$
$$= \frac{1}{2} - \frac{4}{9}$$
$$= \frac{1}{18}$$

Answer

$$Var(X) = \frac{1}{18}$$

Prob : continuous random variable has

$$\boldsymbol{f(x) = kx^3, 0 < x < 2.}$$

Calculate:

1. $\boldsymbol{k}$
2. **Mean**
3. **Variance**
4. $\boldsymbol{P(X < 1)}$

Sol **:**

Since $f(x)$is a probability density function,

$$\int_0^2 f(x)\,dx = 1$$

Substitute $f(x) = kx^3$,

$$\int_0^2 k\,x^3\,dx = 1$$

Take $k$outside the integral,

$$k\int_0^2 x^3\;dx = 1$$

Evaluate the integral,

$$k\left[\frac{x^4}{4}\right]_0^2 = 1$$
$$k\left(\frac{2^4}{4}\right) = 1$$
$$k\left(\frac{16}{4}\right) = 1$$
$$4k = 1$$
$$\boxed{k = \frac{1}{4}}$$

Step 2: Find the Mean $E(X)$
The mean of a continuous random variable is

$$E(X) = \int_0^2 x\,f(x)\,dx$$

Substitute $f(x) = \frac{1}{4}x^3$,

$$E(X) = \int_0^2 x\left(\frac{1}{4}x^3\right)\, dx$$
$$= \frac{1}{4}\int_0^2 x^4 \, dx$$

Evaluate the integral,

$$= \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2$$
$$= \frac{1}{4}\left(\frac{32}{5}\right)$$
$$= \frac{8}{5}$$

Therefore, $\boxed{E(X) = \frac{8}{5} = 1.6}$

Step 3: Find $E(X^2)$ $E(X^2) = \int_0^2 x^2 f(x)\, dx$

Substitute $f(x) = \frac{1}{4}x^3$,

$$E(X^2) = \frac{1}{4}\int_0^2 x^5 \, dx$$
$$= \frac{1}{4}\left[\frac{x^6}{6}\right]_0^2$$
$$= \frac{1}{4}\left(\frac{64}{6}\right)$$
$$= \frac{16}{6}$$
$$= \frac{8}{3}$$

Thus, $\boxed{E(X^2) = \frac{8}{3}}$

Step 4: Find the Variance

$$Var(X) = E(X^2) - [E(X)]^2$$
$$Var(X) = (1.4)^2 = 1.96$$

Use the formula

$$Var(X) = E(X^2) - [E(X)]^2$$

Substitute the values,

$$Var(X) = \frac{8}{3} - \left(\frac{8}{5}\right)^2$$
$$= \frac{8}{3} - \frac{64}{25}$$

Take the LCM of 3and 25, which is 75,

$$= \frac{200}{75} - \frac{192}{75}$$
$$= \frac{8}{75}$$

Therefore, $\boxed{Var(X) = \frac{8}{75}}$ Or $\boxed{Var(X) \approx 0.1067}$

Step 5: Find $P(X < 1)$ $P(X < 1) = \int_0^1 f(x)\,dx$

Substitute the value of $f(x)$,

$$= \int_0^1 \frac{1}{4}x^3\,dx$$
$$= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{4}\left(\frac{1}{4}\right)$$
$$= \frac{1}{16}$$

Hence,

$$\boxed{P(X < 1) = \frac{1}{16} = 0.0625}$$

Problem :

A continuous random variable Xhas the probability density function

$$\boldsymbol{f(x) = k(1 - x^2), -1 < x < 1}$$

And $\boldsymbol{f(x) = 0}$**,otherwise.**

Find:

1. **The value of k**
2. **Mean of X**
3. **Variance of X**
4. $\boldsymbol{P(-0.5 < X < 0.5)}$

Sol: Step 1: Find the Value of k

Since $f(x)$is a probability density function,

$$\int_{-1}^{1} f(x)\,dx = 1$$

Substitute $f(x) = k(1 - x^2)$:

$$\int_{-1}^{1} k(1 - x^2)\,dx = 1$$

Take $k$outside the integral:

$$k\int_{-1}^{1} (1 - x^2)\,dx = 1$$

Now,

$$\int (1 - x^2)\, dx = x - \frac{x^3}{3}$$

Therefore,

$$k\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1$$

Substitute the limits:
At $x = 1$,

$$1 - \frac{1}{3} = \frac{2}{3}$$

At $x = -1$,

$$-1 - \left(\frac{-1}{3}\right) = -1 + \frac{1}{3} = -\frac{2}{3}$$

Hence,

$$k\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) = 1$$

$$k\left(\frac{4}{3}\right) = 1$$

$$k = \frac{3}{4}$$

Therefore,

$$\boxed{k = \frac{3}{4}}$$

Step 2: Find the Mean
The mean is

$$E(X) = \int_{-1}^{1} x f(x)\, dx$$

Substitute the PDF:

$$E(X) = \frac{3}{4}\int_{-1}^{1} x(1 - x^2)\, dx$$

$$= \frac{3}{4}\int_{-1}^{1} (x - x^3)\, dx$$

The function $x - x^3$is an odd function.
The integral of an odd function over $[-a, a]$is zero.
Therefore,

$$\boxed{E(X) = 0}$$

Step 3: Find $E(X^2)$

$$E(X^2) = \int_{-1}^{1} x^2 f(x)\, dx$$

Substitute the PDF:

$$= \frac{3}{4}\int_{-1}^{1} x^2 (1 - x^2)\, dx$$
$$= \frac{3}{4}\int_{-1}^{1} (x^2 - x^4)\, dx$$

Integrate:

$$= \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1}$$

At $x = 1$,

$$\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

At $x = -1$,

$$-\frac{1}{3} + \frac{1}{5} = -\frac{2}{15}$$

Therefore,

$$E(X^2) = \frac{3}{4}\left(\frac{2}{15} - \left(-\frac{2}{15}\right)\right)$$

$$= \frac{3}{4} \times \frac{4}{15}$$
$$= \frac{12}{60}$$
$$= \frac{1}{5}$$

Hence,

$$\boxed{E(X^2) = \frac{1}{5}}$$

Step 4: Find the Variance
The variance is

$$Var(X) = E(X^2) - [E(X)]^2$$

Substitute the values:

$$Var(X) = \frac{1}{5} - (0)^2$$
$$\boxed{Var(X) = \frac{1}{5}}$$

or $$\boxed{Var(X) = 0.2}$$

Step 5: Find $P(-0.5 < X < 0.5)$

$$P(-0.5 < X < 0.5) = \int_{-0.5}^{0.5} f(x)\,dx$$

Substitute the PDF:

$$= \frac{3}{4}\int_{-0.5}^{0.5} (1 - x^2)\,dx$$

Integrate:

$$= \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-0.5}^{0.5}$$

At $x = 0.5$,

$$0.5 - \frac{(0.5)^3}{3} = \frac{1}{2} - \frac{1/8}{3} = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$$

At $x = -0.5$,

$$-\frac{1}{2} + \frac{1}{24} = -\frac{11}{24}$$

Therefore,

$$P = \frac{3}{4}\left(\frac{11}{24} - \left(-\frac{11}{24}\right)\right)$$
$$= \frac{3}{4} \times \frac{22}{24}$$
$$= \frac{3}{4} \times \frac{11}{12}$$

$$= \frac{33}{48}$$
$$= \frac{11}{16}$$

Thus

$$\boxed{P(-0.5 < X < 0.5) = \frac{11}{16} = 0.6875}$$

Discrete Probability Distributions :

Binomial distribution: Binomial distribution was discovered by James Bernoulli in the year 1700 and it is a discrete probability distribution.

Where a trial or an experiment results in only two ways say ‘success’ or ‘failure’.

Some of the situations are:

(1) Tossing a coin – head or tail

(2) Birth of a baby – girl or boy

(3) Auditing a bill – contains an error or not.

Definition: A random variable X is said to be Binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x)=p(x)=n_{C_x}\,p^x q^{n-x},\ x=0,1,2,.....,n$$
$$=0,\ otherwise$$

$where \quad q=1-p, \quad p+q=1.$ **Here n, p are called parameters.**

Example: (1) The number of defective bolts in a box containing ‘n’ bolts.

(2) The number of post-graduates in a group of ‘n’ men.

Conditions: (1) The trials are repeated under identical conditions for a fixed number of times, say ‘n’ times.

(2) There are only two possible outcomes, for example success or failure for each trial.

(3) The probability of success in each trial remains constant and does not change from trail to trail.

(4) The trails are independent i.e. the probability of an event in any trail is not affected by the results of any other trail.

Mean and variance of Binomial distribution:

$$mean=E(X)=\sum xp(x)$$
$$=\sum_{x=0}^{n} x\,n_{C_x}\,p^x q^{n-x}=\sum_{x=1}^{n} x\frac{n}{x}\,n-1_{C_{x-i}}\,p^x q^{n-x}=np\sum_{x=1}^{n} n-1_{C_{x-1}}\,p^{x-1}q^{n-x}=np(q+p)^{n-1}=np$$

$$\mathrm{var}\,iance=E(X^2)-[E(X)]^2$$
$$=\sum[x(x-1)+x]p(x)=\sum x(x-1)p(x)+\sum xp(x)$$
$$=\sum x(x-1)\frac{n(n-1)}{x(x-1)}n-2_{C_{x-2}}\,p^x q^{n-x}+np=n(n-1)p^2\sum_{x=2}^{n} n-2_{C_{x-2}}\,p^{x-2}q^{n-x}+np$$
$$=n(n-1)p^2(q+p)^{n-2}+np=n(n-1)p^2+np=npq$$

Prob 1 : Ten coins are thrown simultaneously. Find the probability of getting at least 7 heads.

Sol: p = probability of getting a head = ½

q = probability of getting a head = ½

The p.d.f. of binomial distribution is

$$P(X = x) = n_{C_x} p^x q^{n-x},\ x = 0,1,2,.....,10$$

Given n = 10, $P(X = x) = 10_{C_x} p^x q^{10-x}$

Probability of getting at least 7 heads is given by

$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) = 0.1719.$

Prob 2: In 256 set of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails.

Sol : **Given**

- Number of tosses in each set: $n = 12$
- Probability of getting a head: $p = \frac{1}{2}$
- Probability of getting a tail: $q = \frac{1}{2}$
- Number of sets: $N = 256$

We need to find the expected number of cases in which there are **8 heads and 4 tails**.

Step 1: Use the Binomial Distribution

The probability of getting exactly $r$heads in $n$tosses is

$$P(X = r) = \binom{n}{r} p^r q^{n-r}$$

Substitute the given values: $P(X = 8) = \binom{12}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^4$

Since $\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{12}$

And $\binom{12}{8} = \frac{12!}{8!\,4!} = 495$

Also $2^{12} = 4096$

Therefore, $P(X = 8) = \frac{495}{4096}$

Step 2: Find the Expected Number of Cases

Expected number of cases

$$= N \times P(X = 8)$$

$$= 256 \times \frac{495}{4096}$$

Since $\frac{256}{4096} = \frac{1}{16}$

$$= \frac{495}{16} = 30.9375 \approx 31$$

The expected number of cases in which **8 heads and 4 tails** occur is 31

Hence, out of 256 sets of 12 tosses, approximately 31 sets are expected to contain exactly 8 heads and 4 tails.

Poisson distribution:

Poisson distribution due to French mathematician Denis Poisson in 1837 is a discrete probability distribution.

It is a rare distribution of rare events i.e. the events whose probability of occurrence is very small but the number of trails which would lead to the occurrence of the event, are very large.

As $n \to \infty,\ p \to 0$ B.D. tends to P.D.

Definition: **A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability mass function is given by**

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0,1,2,....$$
$$= 0,\ otherwise.$$

Here $\lambda > 0$, is called parameter of the distribution.

$e = 2.71828.$

Example:

(1) The number defective bulbs manufactured by a company.

(2) The number of telephone calls per minute at a switch board.

Conditions:

(1) The variable or number of occurrences is a discrete variable.

(2) The occurrences are rare.

(3) The number of trails 'n' is large.

(4) The probability of success (p) is very small.

(5) np = λ is finite.

Mean and variance of Poisson distribution:

$$mean = E(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda}.e^{\lambda} = \lambda.$$

$$\text{var}\,iance = E(X^2) - [E(X)]^2$$
$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-1)!} + \lambda - \lambda^2 = \lambda^2 e^{-\lambda}.e^{\lambda} + \lambda - \lambda^2 = \lambda.$$

***Example*: A car hire firm has two cars which it hires out day by day, The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand (ii) on which demand is refused.**

Solution: Given mean, λ = 1.5

We have $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

(i) P(no demand) = p(0) = 0.2231

Number of days in a year there is no demand of car = 365 (0.2231) = 81 days.

(ii) P(demand refused) = p(x>2) = 1- [p(0) +p(1) +p(2)] = 0.1913

Number of days in a year when some demand is refused = 365 (0.1913) = 70 days.

Ex 1 : A machine produces an average of 4 defective items per day. Assuming a Poisson distribution, find the mean, variance, and standard deviation.

Sol : **Given** machine produces an average of 4 defective items per day **i.e ,** $\lambda = 4$

then Mean $\mu = \lambda = 4$

Variance $\sigma^2 = \lambda = 4$

Standard Deviation $\sigma = \sqrt{4} = 2$ **.**

Ex 2 : If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals

i. **Exactly 3**
ii. **More than 2 individuals**
iii. **None**
iv . **More than one individual suffers bad reaction .**

Sol : Given

- Number of individuals is $n = 2000$
- Probability of bad reaction is $p = 0.001$

Since $n$is large and $p$is small, use the Poisson approximation.

Parameter,

$\lambda = np = 2000 \times 0.001 = 2$

Poisson formula: $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Here, $\lambda = 2, e^{-2} = 0.1353$

(i) Probability of exactly 3 individuals

$$P(X=3)=\frac{e^{-2}2^3}{3!}=\frac{0.1353\times 8}{6}=0.1804$$

Answer : $\boxed{P(X=3)=0.1804}$

(ii) Probability of more than 2 individuals

$$P(X>2)=1-[P(0)+P(1)+P(2)]$$

Calculate, $P(0)=0.1353,$ $P(1)=0.1353\times 2=0.2707$

$$P(2)=\frac{0.1353\times 4}{2}=0.2707$$

Therefore, $P(X>2)=1-(0.1353+0.2707+0.2707)$

$$=1-0.6767$$

$$=0.3233$$

Answer : $\boxed{P(X>2)=0.3233}$

(iii) Probability that none suffers bad reaction

$$P(X=0)=e^{-2}$$

$$=0.1353$$

Answer : $\boxed{P(X=0)=0.1353}$

(iv) Probability that more than one suffers bad reaction

$$P(X>1)=1-[P(0)+P(1)]$$

$$=1-(0.1353+0.2707)$$

$$=1-0.4060$$

$$=0.5940$$

Answer : $\boxed{P(X>1)=0.5940}$

Ex 3 : A die is tossed until 6 appears. Find the probability that it must be cast more than 5 times.

Sol : Probability of getting 6 is $p=\frac{1}{6}$

Probability of not getting 6 is $q=\frac{5}{6}$

"More than 5 times" means the first five throws are not 6.

Hence $P(X>5)=q^5=\left(\frac{5}{6}\right)^5=\frac{3125}{7776}=0.4019$

Answer $\boxed{P(X>5)=0.4019}$