

UNIT-2

MATHEMATICAL EXPECTATION

Mean of a Random Variable:

Definition *:

Let X be a random variable with a probability distribution $f(x)$. The mean (or expected value) of X is denoted by μ_X (or $E(X)$) and is defined by:

$$E(X)=\mu_X=\begin{cases} \sum_{all\ x} x f(x); & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x) dx; & \text{if } X \text{ is continuous} \end{cases}$$

Example *: (Reading Assignment)

Example: (Example A)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the expected number of defective computers purchased³⁶

Solution:

Let X = the number of defective computers purchased.
In Example A, we found that the probability distribution of X is:

x	0	1	2
$F(x)=p(X=x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

or:

$$f(x) = P(X = x) = \begin{cases} \frac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}; & x = 0, 1, 2 \\ 0; & \text{otherwise} \end{cases}$$

The expected value of the number of defective computers purchased is the mean (or the expected value) of X , which is:

$$\begin{aligned} E(X) &= \mu_X = \sum_{x=0}^2 x f(x) \\ &= (0) f(0) + (1) f(1) + (2) f(2) \\ &= (0) \frac{10}{28} + (1) \frac{15}{28} + (2) \frac{3}{28} \\ &= \frac{15}{28} + \frac{6}{28} = \frac{21}{28} = 0.75 \quad (\text{computers}) \end{aligned}$$

Example B:

Let X be a continuous random variable that represents the life (in hours) of a certain electronic device. The pdf of X is given by:

$$f(x) = \begin{cases} \frac{20,000}{x^3} & ; x > 100 \\ 0 & ; \text{elsewhere} \end{cases}$$

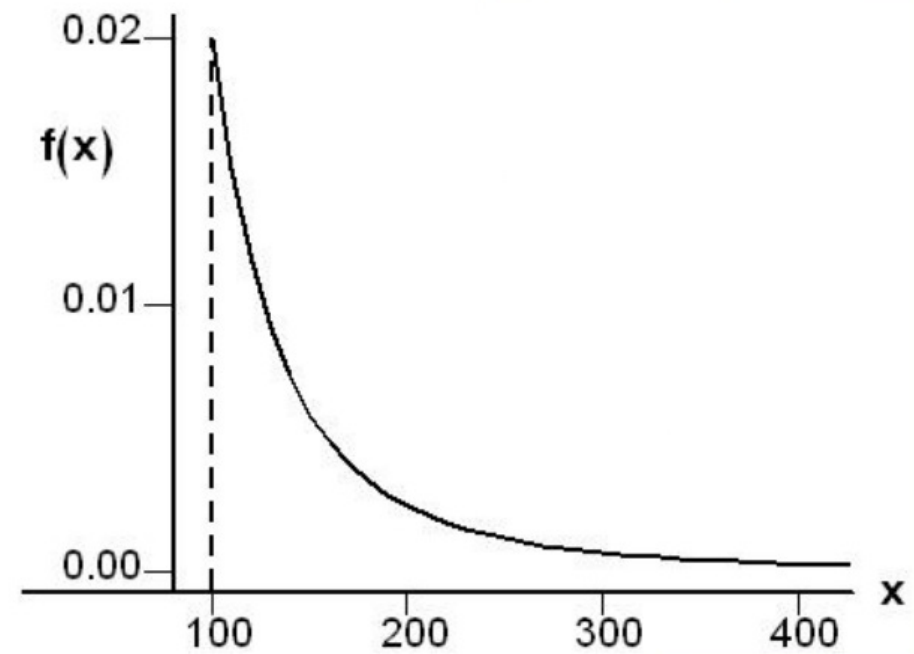
Find the expected Number of this type of devices.

Solution:

$$\begin{aligned} E(X) &= \mu_X = \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{100}^{\infty} x \frac{20000}{x^3} dx \\ &= 20000 \int_{100}^{\infty} \frac{1}{x^2} dx \end{aligned}$$

$$= 20000 \left[-\frac{1}{x} \right]_{x=100}^{x=\infty}$$

$$= -20000 \left[0 - \frac{1}{100} \right] = 200 \quad (\text{hours})$$



Therefore, we expect that this type of electronic devices to last, on Average of 200 hours

Theorem *:

Let X be a random variable with a probability distribution $f(x)$, and let $g(X)$ be a function of the random variable X . The mean (or expected value) of the random variable $g(X)$ is denoted by $\mu_{g(X)}$ (or $E[g(X)]$) and is defined by:

$$E[g(X)] = \mu_{g(X)} = \begin{cases} \sum_{\text{all } x} g(x) f(x); & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x) dx; & \text{if } X \text{ is continuous} \end{cases}$$

Example:

Let X be a discrete random variable with the following probability distribution

x	0	1	2
$F(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Find MEAN WHERE $g(X) = (X - 1)^2$.

Solution:

$$g(X) = (X - 1)^2$$

$$E[g(X)] = \mu_{g(X)} = \sum_{x=0}^2 g(x) f(x) = \sum_{x=0}^2 (x - 1)^2 f(x)$$

$$= (0 - 1)^2 f(0) + (1 - 1)^2 f(1) + (2 - 1)^2 f(2)$$

$$= (-1)^2 \frac{10}{28} + (0)^2 \frac{15}{28} + (1)^2 \frac{3}{28}$$

$$= \frac{10}{28} + 0 + \frac{3}{28} = \frac{10}{28}$$

Example:

In Example B, find $E\left(\frac{1}{X}\right)$

Solution:

$$f(x) = \begin{cases} \frac{20,000}{x^3} & ; x > 100 \\ 0 & ; \text{elsewhere} \end{cases}$$

$$g(X) = \frac{1}{X}$$

$$E\left(\frac{1}{X}\right) = E[g(X)] = \mu_{g(X)} = \int_{-\infty}^{\infty} g(x) f(x) dx = \int_{-\infty}^{\infty} \frac{1}{x} f(x) dx$$

$$= \int_{100}^{\infty} \frac{1}{x} \frac{20000}{x^3} dx = 20000 \int_{100}^{\infty} \frac{1}{x^4} dx = \frac{20000}{-3} \left[\frac{1}{x^3} \right]_{x=100}^{x=\infty}$$

$$= \frac{-20000}{3} \left[0 - \frac{1}{1000000} \right] = 0.0067$$

Variance (of a Random Variable):

The most important measure of variability of a random variable X is called the variance of X and is denoted by $\text{Var}(X)$ or σ_X^2

Definition B:

Let X be a random variable with a probability distribution $f(x)$ and mean μ . The variance of X is defined by:

$$\text{Var}(X) = \sigma_X^2 = E[(X - \mu)^2] = \begin{cases} \sum_{\text{all } x} (x - \mu)^2 f(x); & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx; & \text{if } X \text{ is continuous} \end{cases}$$

Definition:

The positive square root of the variance of X , $\sigma_X = \sqrt{\sigma_X^2}$, is called the standard deviation of X .

Note:

$\text{Var}(X) = E[g(X)]$, where $g(X) = (X - \mu)^2$

Theorem :

The variance of the random variable X is given by:

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$

where $E(X^2) = \begin{cases} \sum_{\text{all } x} x^2 f(x); & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^2 f(x) dx; & \text{if } X \text{ is continuous} \end{cases}$

Example 4.9:

Let X be a discrete random variable with the following probability distribution

x	0	1	2	3
F(x)	0.15	0.38	0.10	0.01

Find $\text{Var}(X) = \sigma_X^2$.

Solution:

$$\begin{aligned}\mu &= \sum_{x=0}^3 x f(x) = (0) f(0) + (1) f(1) + (2) f(2) + (3) f(3) \\ &= (0) (0.51) + (1) (0.38) + (2) (0.10) + (3) (0.01) = 0.61\end{aligned}$$

1. First method:

$$\begin{aligned}\text{Var}(X) &= \sigma_X^2 = \sum_{x=0}^3 (x - \mu)^2 f(x) \\ &= \sum_{x=0}^3 (x - 0.61)^2 f(x) \\ &= (0 - 0.61)^2 f(0) + (1 - 0.61)^2 f(1) + (2 - 0.61)^2 f(2) + (3 - 0.61)^2 f(3) \\ &= (-0.61)^2 (0.51) + (0.39)^2 (0.38) + (1.39)^2 (0.10) + (2.39)^2 (0.01) \\ &= 0.4979\end{aligned}$$

2. Second method:

$$\begin{aligned}\text{Var}(X) &= \sigma_X^2 = E(X^2) - \mu^2 \\ E(X^2) &= \sum_{x=0}^3 x^2 f(x) = (0^2) f(0) + (1^2) f(1) + (2^2) f(2) + (3^2) f(3) \\ &= (0) (0.51) + (1) (0.38) + (4) (0.10) + (9) (0.01) = 0.87 \\ \text{Var}(X) &= \sigma_X^2 = E(X^2) - \mu^2 = 0.87 - (0.61)^2 = 0.4979\end{aligned}$$

Example *0:

Let X be a continuous random variable with the following pdf:

$$f(x) = \begin{cases} 2(x-1) & ; 1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

Find the mean and the variance of X .

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_1^2 x [2(x-1)] dx = 2 \int_1^2 x(x-1) dx = 5/3$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_1^2 x^2 [2(x-1)] dx = 2 \int_1^2 x^2 (x-1) dx = 17/6$$

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2 = 17/6 - (5/3)^2 = 1/8$$

B Means and Variances of Linear Combinations of Random Variables:

If $X_1, X_2, \dots, X_n$ are n random variables and $a_1, a_2, \dots, a_n$ are constants, then the random variable :

$$Y = \sum_{i=1}^n a_i X_i = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

is called a linear combination of the random variables $X_1, X_2, \dots, X_n$.

Theorem 4.5:

If X is a random variable with mean $\mu = E(X)$, and if a and b are constants, then:

$$E(aX \pm b) = a E(X) \pm b$$

$$\Leftrightarrow$$

$$\mu_{aX \pm b} = a \mu_X \pm b$$

Corollary 1: $E(b) = b$ (a=0 in Theorem 4.5)

$E(aX) = a E(X)$ (b=0 in Theorem 4.5)

Example *6:

Let X be a random variable with the following probability density function:

$$f(x) = \begin{cases} \frac{1}{3}x^2 & ; -1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

Find $E(4X+3)$.

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^2 x \left[\frac{1}{3} x^2 \right] dx = \frac{1}{3} \int_{-1}^2 x^3 dx = \frac{1}{3} \left[\frac{1}{4} x^4 \right]_{x=-1}^{x=2} = 5/4$$

$$E(4X+3) = 4 E(X) + 3 = 4(5/4) + 3 = 8$$

Another solution:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \quad ; g(X) = 4X+3$$

$$E(4X+3) = \int_{-1}^2 (4x+3) f(x) dx = \int_{-1}^2 (4x+3) \left[\frac{1}{3} x^2 \right] dx = 8$$

Theorem:

If $X_1, X_2, \dots, X_n$ are n random variables and $a_1, a_2, \dots, a_n$ are constants, then:

$$E(a_1X_1 + a_2X_2 + \dots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \dots + a_nE(X_n)$$

$$\Leftrightarrow$$

$$E\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E(X_i)$$

Corollary:

If X , and Y are random variables, then:

$$E(X \pm Y) = E(X) \pm E(Y)$$

Theorem 4.9:

If X is a random variable with variance $\text{Var}(X) = \sigma_X^2$ and if a and b are constants, then:

$$\text{Var}(aX \pm b) = a^2 \text{Var}(X)$$

$$\Leftrightarrow$$

$$\sigma_{aX \pm b}^2 = a^2 \sigma_X^2$$

Theorem:

If $X_1, X_2, \dots, X_n$ are n independent random variables and $a_1, a_2, \dots, a_n$ are constants, then:

$$\text{Var}(a_1X_1 + a_2X_2 + \dots + a_nX_n)$$

$$= \text{Var}(a_1^2X_1^2) + \text{Var}(a_2^2X_2^2) + \dots + \text{Var}(a_n^2X_n^2)$$

$$\Leftrightarrow$$

$$\text{Var}\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 \text{Var}(X_i)$$

$$\Leftrightarrow$$

$$\sigma_{a_1X_1 + a_2X_2 + \dots + a_nX_n}^2 = a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2 + \dots + a_n^2 \sigma_{X_n}^2$$

Corollary:

If X , and Y are independent random variables, then:

- $\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
- $\text{Var}(aX - bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
- $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y)$

Example:

Let X , and Y be two independent random variables such that $E(X)=2$, $\text{Var}(X)=4$, $E(Y)=7$, and $\text{Var}(Y)=1$. Find:

1. $E(3X+7)$ and $\text{Var}(3X+7)$
2. $E(5X+2Y-2)$ and $\text{Var}(5X+2Y-2)$.

Solution:

1. $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$
 $\text{Var}(3X+7) = (3)^2 \text{Var}(X) = (3)^2 (4) = 36$
2. $E(5X+2Y-2) = 5E(X) + 2E(Y) - 2 = (5)(2) + (2)(7) - 2 = 22$
 $\text{Var}(5X+2Y-2) = \text{Var}(5X+2Y) = 5^2 \text{Var}(X) + 2^2 \text{Var}(Y)$
 $= (25)(4) + (4)(1) = 104$

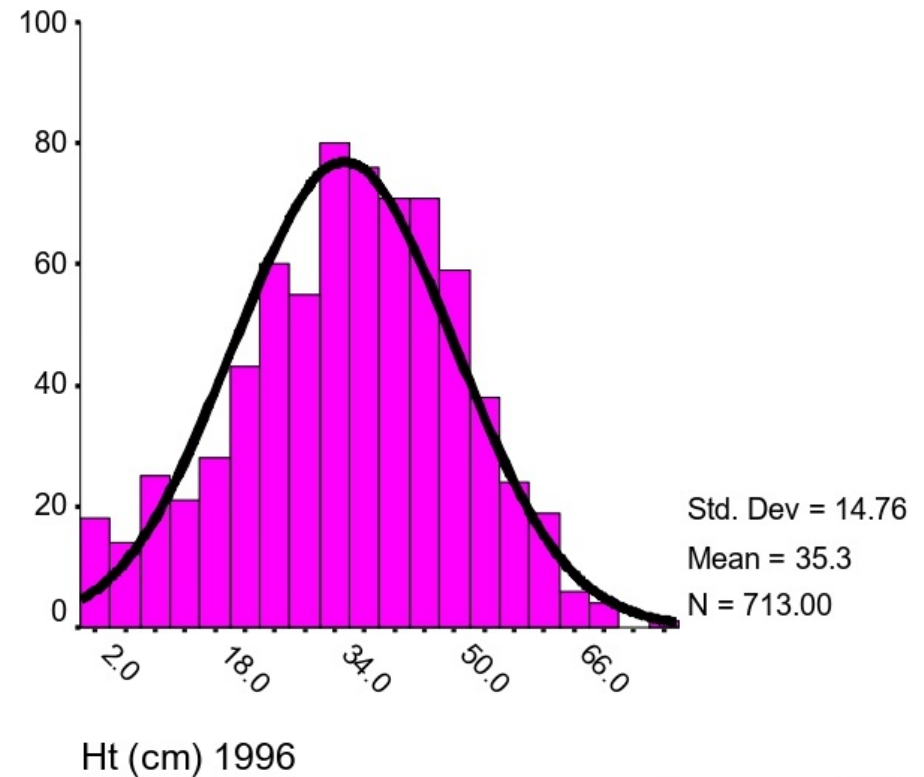
The Binomial, Poisson, and Normal Distributions

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Probability distributions

- u We use probability distributions because they work -they fit lots of data in real



Height (cm) of *Hypericum cumulicola* at Archbold Biological Station

Random variable

- The mathematical rule (or function) that assigns a given numerical value to each possible outcome of an experiment in the sample space of interest.

Random variables

- ▮ Discrete random variables
- ▮ Continuous random variables

The Binomial Distribution

Bernoulli Random Variables

- Imagine a simple trial with only two possible outcomes
 - Success (S)
 - Failure (F)
- Examples
 - Toss of a coin (heads or tails)
 - Sex of a newborn (male or female)
 - Survival of an organism in a region (live or die)



Jacob Bernoulli (1654 - 1705)

The Binomial Distribution

Overview

- u Suppose that the probability of success is p
- u What is the probability of failure?
 - u $q = 1 - p$
- u Examples
 - u Toss of a coin ($S = \text{head}$): $p = 0.5 \Rightarrow q = 0.5$
 - u Roll of a die ($S = 1$): $p = 0.1667 \Rightarrow q = 0.8333$
 - u Fertility of a chicken egg ($S = \text{fertile}$): $p = 0.8 \Rightarrow q = 0.2$

The Binomial Distribution

Overview

- Imagine that a trial is repeated n times
- Examples
 - A coin is tossed 5 times
 - A die is rolled 25 times
 - 50 chicken eggs are examined
- Assume p remains constant from trial to trial and that the trials are statistically independent of each other

The Binomial Distribution

Overview

- What is the probability of obtaining x successes in n trials?
- Example
 - What is the probability of obtaining 2 heads from a coin that was tossed 5 times?

$$P(HHTTT) = (1/2)^5 = 1/32$$

The Binomial Distribution

Overview

- But there are more possibilities:

HHTTT HTHTT HTTHT HTTTH
THHTT THTHT THTTH
TTHHT TTHTH
TTTHH

$$P(2 \text{ heads}) = 10 \times 1/32 = 10/32$$

The Binomial Distribution

Overview

- u In general, if trials result in a series of success and failures,

FFSFFFFSFSFSFFFFFSF...

Then the probability of x successes in that order is

$$\begin{aligned} P(x) &= q \cdot q \cdot p \cdot q \cdot \dots \\ &= p^x \cdot q^{n-x} \end{aligned}$$

The Binomial Distribution

Overview

- However, if order is not important, then

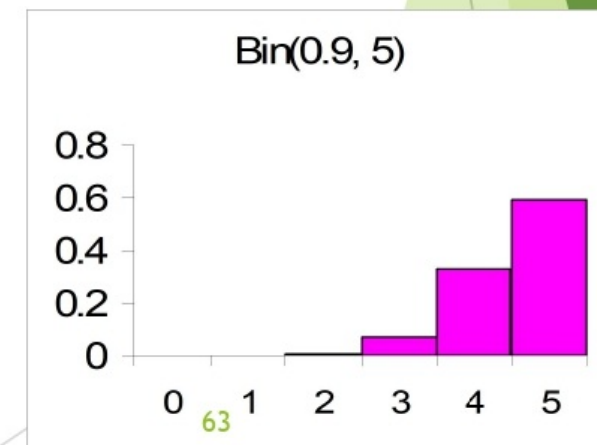
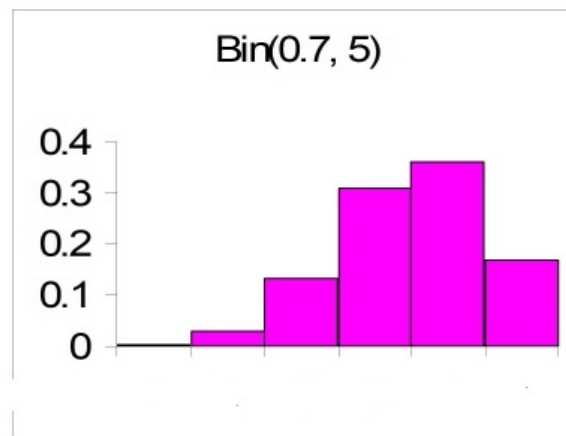
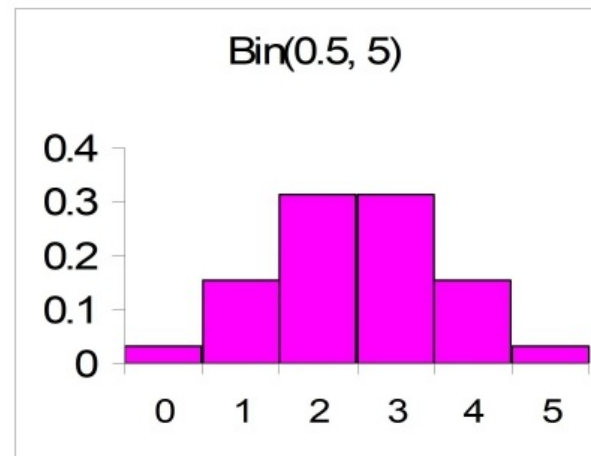
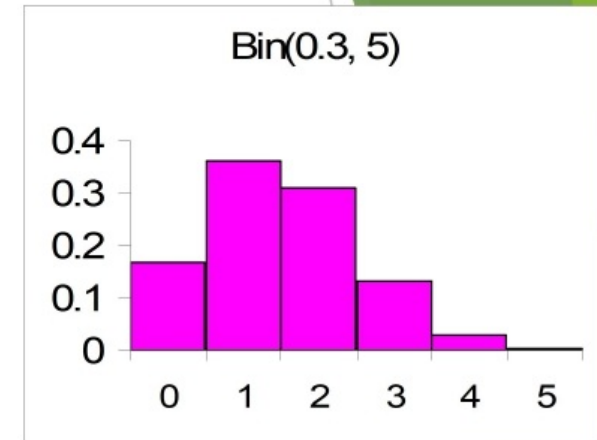
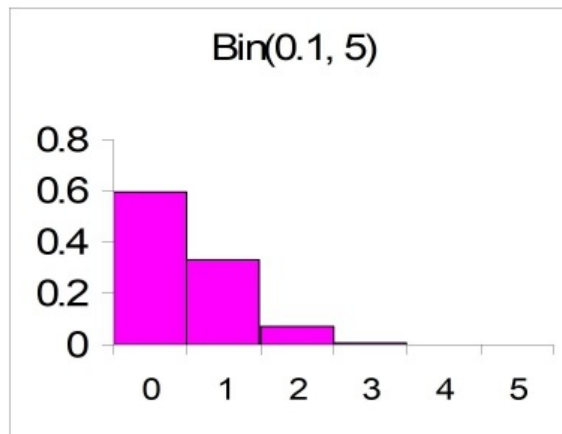
$$P(x) = \frac{n!}{x!(n-x)!} p^x \cdot q^{n-x}$$

where $\frac{n!}{x!(n-x)!}$ is the number of ways to obtain x successes

in n trials, and $i! = i \cdot (i-1) \cdot (i-2) \cdot \dots \cdot 2 \cdot 1$

The Binomial Distribution

Overview



The Poisson Distribution

Overview

- u When there is a large number of trials, but a small probability of success, binomial calculation becomes impractical
 - u Example: Number of deaths from horse kicks in the Army in different years
- u The mean number of successes from n trials is $\mu = np$
 - u Example: 64 deaths in 20 years from thousands of soldiers



Simeon D. Poisson (1781-1840)

The Poisson Distribution

Overview

- If we substitute μ/n for p , and let n tend to infinity, the binomial distribution becomes the Poisson distribution:

$$P(x) = \frac{e^{-\mu} \mu^x}{x!}$$

The Poisson Distribution

Overview

- u Poisson distribution is applied where random events in space or time are expected to occur
- u Deviation from Poisson distribution may indicate some degree of non-randomness in the events under study
- u Investigation of cause may be of interest

The Poisson Distribution

Emission of α -particles

- υ Rutherford, Geiger, and Bateman (1910) counted the number of α -particles emitted by a film of polonium in 2608 successive intervals of one-eighth of a minute
 - υ What is n ?
 - υ What is p ?
- υ Do their data follow a Poisson distribution?

The Poisson Distribution

Emission of α -particles

- Calculation of μ :

μ = No. of particles per interval

$$= 10097/2608$$

$$= 3.87$$

- Expected values:

$$2608 \times P(x) = 2608 \times \frac{e^{-3.87} (3.87)^x}{x!}$$

No. α -particles	Observed
0	57
1	203
2	383
3	525
4	532
5	408
6	273
7	139
8	45
9	27
10	10
11	4
12	0
13	1
14	1
Over 14	0
Total	2608

The Poisson Distribution

Emission of α -particles

No. α -particles	Observed	Expected
0	57	54
1	203	210
2	383	407
3	525	525
4	532	508
5	408	394
6	273	254
7	139	140
8	45	68
9	27	29
10	10	11
11	4	4
12	0	1
13	1	1
14	1	1
Over 14	0	0
Total	2608	2680

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Emission of α -particles

Random events

Regular events

Clumped events

The Poisson Distribution

