

**NARSIMHA REDDY ENGINEERING COLLEGE
(NRCM)**

Maisammaguda, Telangana, Hyderabad-500100

B.Tech II Year

PROBABILITY AND STATISTICS

23MA302

UNIT-1

PROBABILITY

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NRCM

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Probability Review

- Events and Event spaces
- Random variables
- Joint probability distributions
 - Marginalization, conditioning, chain rule, Bayes Rule, law of total probability, etc.
- Structural properties
 - Independence, conditional independence
- Mean and Variance
- Examples

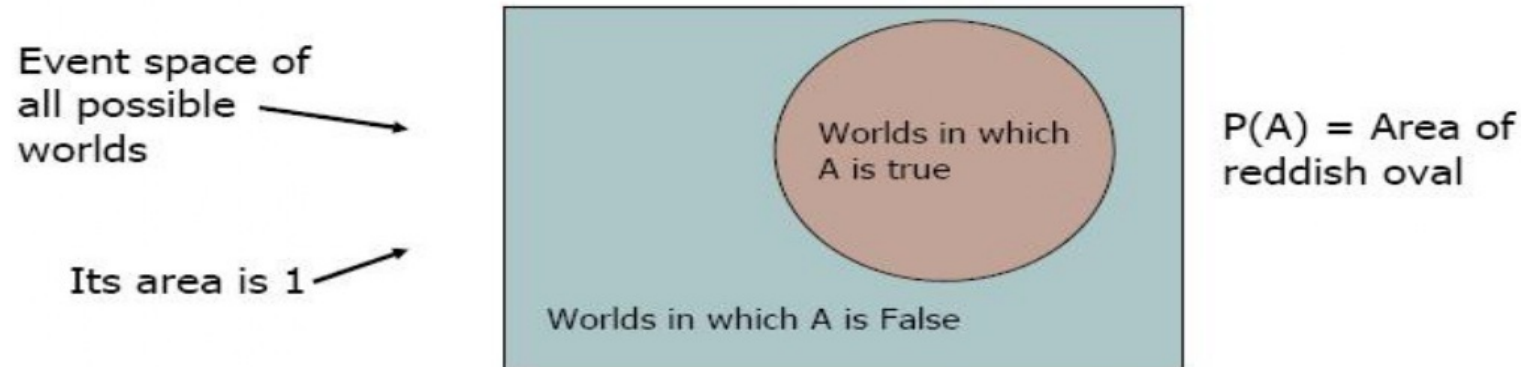
Sample space and Events

- Ω : Sample Space, result of an experiment
 - If you toss a coin twice $\Omega = \{HH, HT, TH, TT\}$
- Event: a subset of Ω
 - First toss is head = $\{HH, HT\}$
- $\mathcal{S}$: event space, a set of events:
 - Closed under finite union and complements
 - Entails other binary operation: union, diff, etc.
 - Contains the empty event and Ω

Probability Measure

- S : Defined over (E, S) so that
 - $P(\alpha) \geq 0$ for all α in S
 - $P(E) = 1$
 - If α, β are disjoint, then
 - $P(\alpha \cup \beta) = p(\alpha) + p(\beta)$
- We can deduce other axioms from the above ones
 - Ex: $P(\alpha \cup \beta)$ for non-disjoint event
$$P(\alpha \cup \beta) = p(\alpha) + p(\beta) - p(\alpha \cap \beta)$$

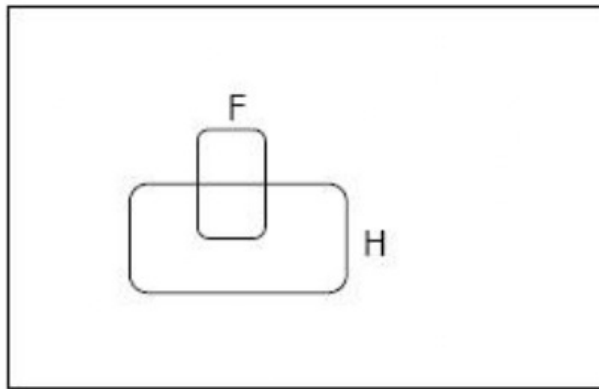
Visualization



- We can go on and define conditional probability, using the above visualization

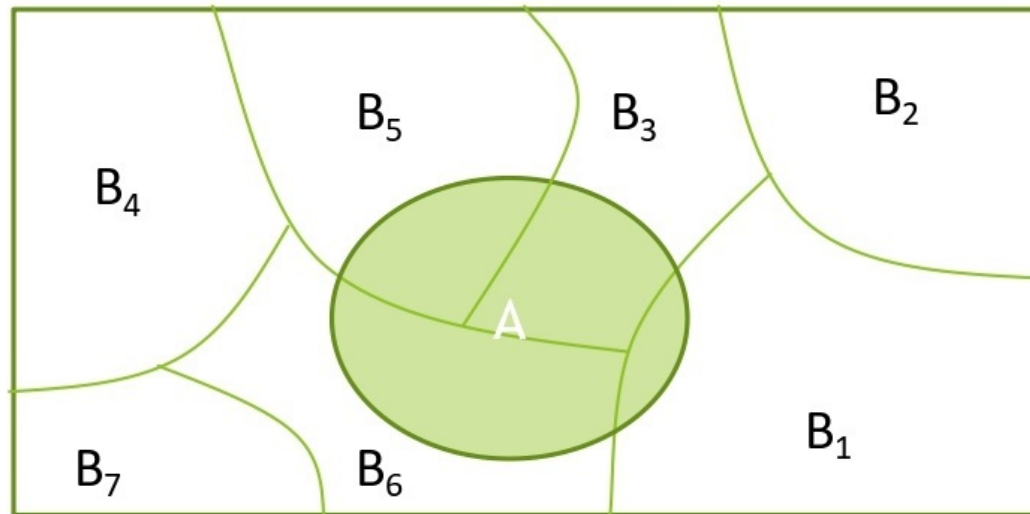
Conditional Probability

$P(F|H)$ = Fraction of worlds in which H is true that also have F true



$$p(f | h) = \frac{p(F \cap H)}{p(H)}$$

Rule of total probability



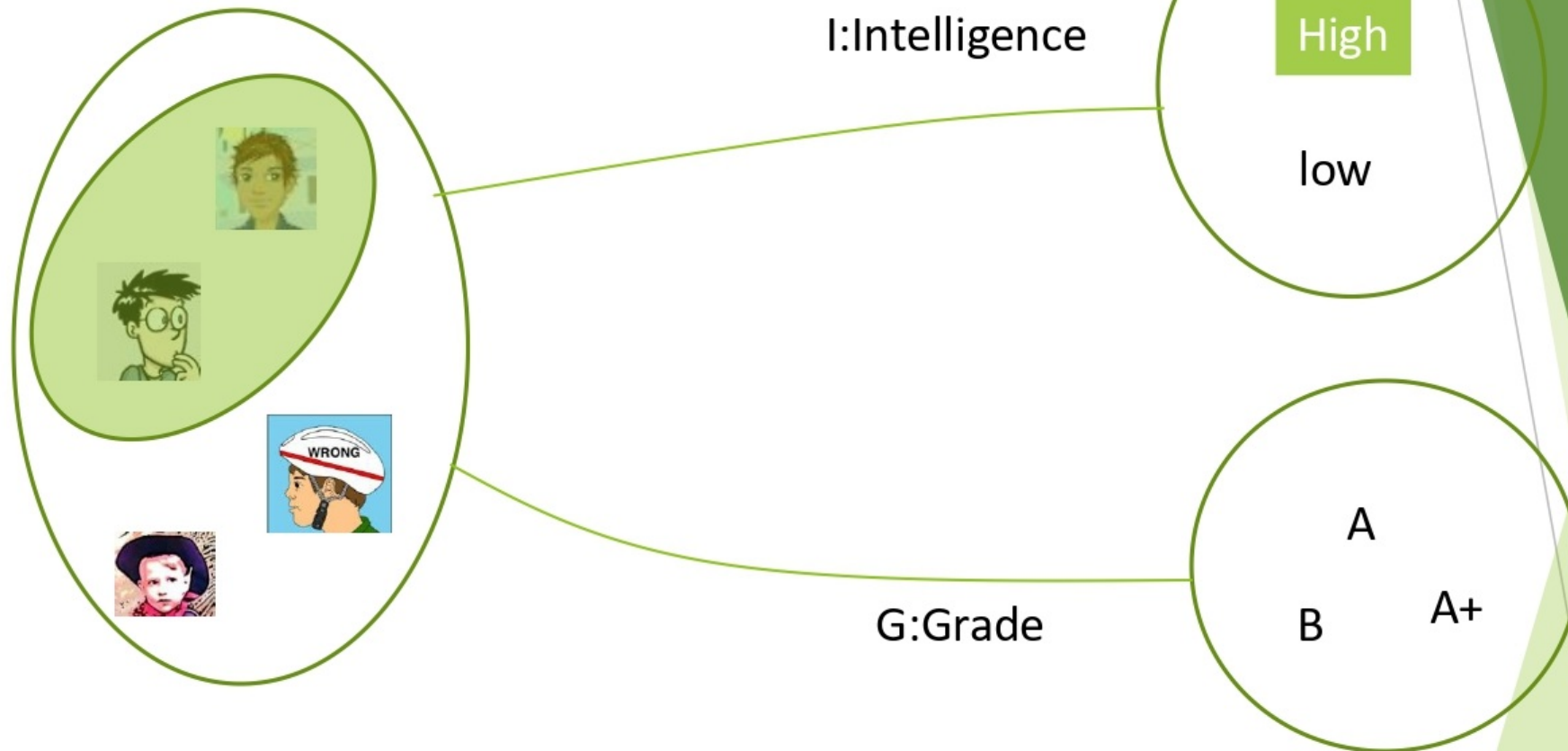
$$p(A) = \sum P(B_i)P(A|B_i)$$

From Events to Random

- Almost all the semester we will be dealing with RV
- Concise way of specifying attributes of outcomes
- Modeling students (Grade and Intelligence):
 - Ω = all possible students
 - What are events
 - Grade_A = all students with grade A
 - Grade_B = all students with grade B
 - Intelligence_High = ... with high intelligence
 - Very cumbersome
 - We need “functions” that maps from Ω to an attribute space.

Random Variables

Ω



$$P(I = \text{high}) = P(\{\text{all students whose intelligence is high}\})$$

Discrete Random Variables

- Random variables (RVs) which may take on only a **countable** number of **distinct** values
 - E.g. the total number of tails X you get if you flip 100 coins
- X is a RV with arity k if it can take on exactly one value out of $\{x_1, \dots, x_k\}$
 - E.g. the possible values that X can take on are 0, 1, 2, ..., 100

Probability of Discrete RV

- Probability mass function (pmf): $P(X = x_i)$
- Easy facts about pmf
 - $\sum_i P(X = x_i) = 1$
 - $P(X = x_i \cap X = x_j) = 0$ if $i \neq j$
 - $P(X = x_i \cup X = x_j) = P(X = x_i) + P(X = x_j)$ if $i \neq j$
 - $P(X = x_1 \cup X = x_2 \cup \dots \cup X = x_k) = 1$

Common Distributions

- Uniform $XU[1, \dots, N]$

- X takes values $1, 2, \dots, N$
- $P(X = i) = 1/N$
- E.g. picking balls of different colors from a box

- Binomial $X \sim \text{Bin}(n, p)$

- X takes values $0, 1, \dots, n$
-
- E.g. coin flips
$$p(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}$$

Continuous Random Variables

- Probability density function (pdf) instead of probability mass function (pmf)
- A pdf is any function $f(x)$ that describes the probability density in terms of the input variable x .

Probability of Continuous RV

- Properties of pdf

- $f(x) \geq 0, \forall x$

- $$\int_{-\infty}^{+\infty} f(x) = 1$$

- Actual probability can be obtained by taking the integral of pdf

- E.g. the probability of X being between 0 and 1 is

Cumulative Distribution Function

- $F_X(v) = P(X \leq v)$

- Discrete RVs

- $F_X(v) = \sum_{v_i} P(X = v_i)$

- Continuous RVs

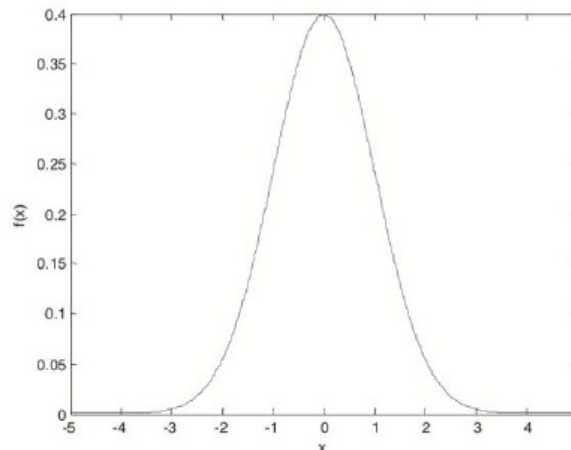
- $$F_X(v) = \int_{-\infty}^v f(x) dx$$

- $$\frac{d}{dx} F_x(x) = f(x)$$

Common Distributions

Normal $X \sim N(\mu, \sigma^2)$

- $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$
- E.g. the height of the entire population



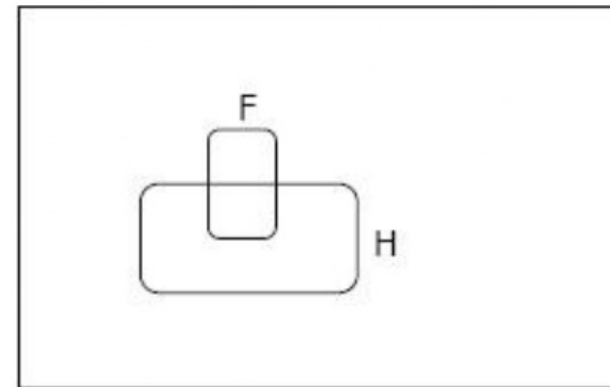
Conditional Probability

$$P(X = x | Y = y) = \frac{P(X = x \cap Y = y)}{P(Y = y)}$$

events

But we will always write it this way:

$$P(x | y) = \frac{p(x, y)}{p(y)}$$



Bayes Rule

- We know that $P(\text{rain}) = 0.5$
 - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(\text{rain} \mid \text{wet}) = \frac{P(\text{rain})P(\text{wet} \mid \text{rain})}{P(\text{wet})}$$

$$P(x \mid y) = \frac{P(x)P(y \mid x)}{P(y)}$$

- You can condition on more variables

$$P(x | y, z) = \frac{P(x | z)P(y | x, z)}{P(y | z)}$$

Independence

- X is independent of Y means that knowing Y does not change our belief about X.
 - $P(X|Y=y) = P(X)$
 - $P(X=x, Y=y) = P(X=x) P(Y=y)$
 - The above should hold for all x, y
 - It is symmetric and written as $X \perp Y$

Independence

- $X_1, \dots, X_n$ are independent if and only if

$$P(X_1 \in A_1, \dots, X_n \in A_n) = \prod_{i=1}^n P(X_i \in A_i)$$

- If $X_1, \dots, X_n$ are independent and identically distributed we say they are *iid* (or that they are a random sample) and we write

$$X_1, \dots, X_n \sim P$$

Mean and Variance

$$\mu = E(X)$$

- Mean (Expectation): $E(X) = \sum_{v_i} v_i P(X = v_i)$
 - Discrete RVs:

$$E(g(X)) = \sum_{v_i} g(v_i) P(X = v_i)$$

- Continuous RVs: $E(X) = \int_{-\infty}^{+\infty} x f(x) dx$

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f(x) dx$$

Mean and Variance

$$\text{Var}(X) = E((X - \mu)^2)$$

u Variance:

$$\text{Var}(X) = E(X^2) - \mu^2$$

u Discrete RVs:

$$V(X) = \sum_{v_i} (v_i - \mu)^2 P(X = v_i)$$

u Continuous RVs:

$$V(X) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx$$

u Covariance:

$$\text{Cov}(X, Y) = E((X - \mu_x)(Y - \mu_y)) = E(XY) - \mu_x \mu_y$$

Mean and Variance

u Correlation:

$$\rho(X,Y) = Cov(X,Y) / \sigma_x \sigma_y$$

$$-1 \leq \rho(X,Y) \leq 1$$

Properties

Mean

- $E(X + Y) = E(X) + E(Y)$
- $E(aX) = aE(X)$
- If X and Y are independent, $E(XY) = E(X) \cdot E(Y)$

Variance

- $V(aX + b) = a^2V(X)$
- If X and Y are independent,

$$V(X + Y) = V(X) + V(Y)$$

Properties

- The conditional expectation of Y given X when the value of $X = x$ is:

$$E(Y | X = x) = \int y * p(y | x) dy$$

- The Law of Total Expectation or Law of Iterated Expectation:

$$E(Y) = E[E(Y | X)] = \int E(Y | X = x) p_X(x) dx$$

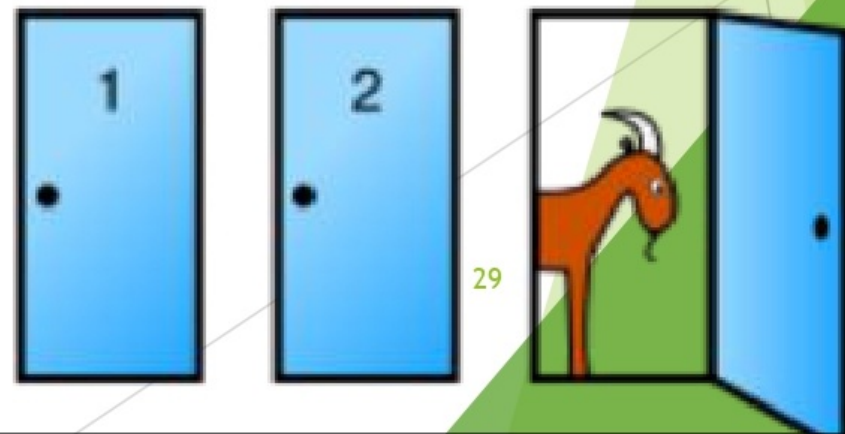
- u The law of Total Variance:

$$\text{Var}(Y) = \text{Var}[E(Y | X)] + E[\text{Var}(Y | X)]$$

Examples

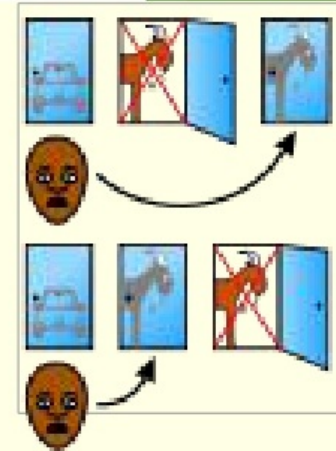
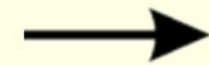
Monty Hall Problem

- u You're given the choice of three doors: Behind one door is a car; behind the others, goats.
- u You pick a door, say No. 1
- u The host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.
- u Do you want to pick door No. 2 instead?

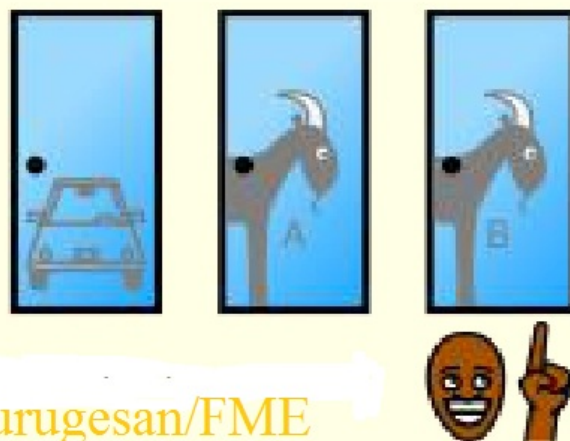




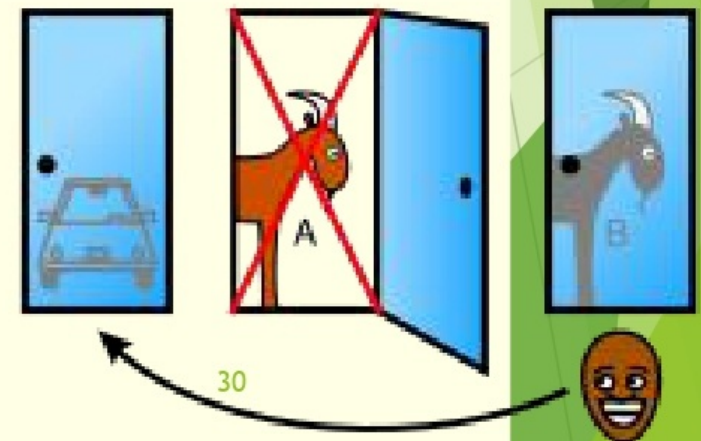
*Host reveals
Goat A
or
Host reveals
Goat B*



*Host must
reveal Goat B*



*Host must
reveal Goat A*



Monty Hall Problem: Bayes Rule

C_i

• the car is behind door i , $i = 1, 2, 3$

• $P(C_i) = 1/3$

• H_{ij} : the host opens door j after you pick door i

•
$$P(H_{ij} | C_k) = \begin{cases} 0 & i = j \\ 0 & j = k \\ 1/2 & i = k \\ 1 & i \neq k, j \neq k \end{cases}$$

Monty Hall Problem: Bayes Rule cont.

WLOG, $i=1, j=3$

$$P(C_1 | H_{13}) = \frac{P(H_{13} | C_1) P(C_1)}{P(H_{13})}$$

$$P(H_{13} | C_1) P(C_1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

Monty Hall Problem: Bayes Rule cont.

$$\begin{aligned}P(H_{13}) &= P(H_{13}, C_1) + P(H_{13}, C_2) + P(H_{13}, C_3) \\&= P(H_{13} | C_1) P(C_1) + P(H_{13} | C_2) P(C_2) \\&= \frac{1}{6} + 1 \cdot \frac{1}{3} \\&= \frac{1}{2}\end{aligned}$$

$$P(C_1 | H_{13}) = \frac{1/6}{1/2} = \frac{1}{3}$$

Monty Hall Problem: Bayes Rule cont.

- $P(C_1 | H_{13}) = \frac{1/6}{1/2} = \frac{1}{3}$
- $P(C_2 | H_{13}) = 1 - \frac{1}{3} = \frac{2}{3} > P(C_1 | H_{13})$
- *You should switch!*