Department of Civil Engineering

Study Material

UNIT–V: Thin Cylinders and Thick Cylinders

Introduction

Pressure vessels are containers used to store liquids or gases under pressure.

Examples:

- Boilers
- LPG cylinders
- Air compressors
- Water tanks
- Gas storage tanks
- Pipelines

Depending on wall thickness, cylinders are classified into:

1. Thin Cylinders
2. Thick Cylinders

PART–A: THIN CYLINDERS

1. Thin Cylindrical Shell

Definition

A cylinder is called **thin** when

Dt>20\boxed{\frac{D}{t}>20}tD >20

or

t<D20\boxed{t<\frac{D}{20}}t<20D

Where:

- DDD = Internal diameter
- ttt = Thickness

Assumptions

1. Thickness is small compared to diameter.
2. Stress is uniformly distributed across thickness.
3. Material is homogeneous and isotropic.
4. Internal pressure is uniformly distributed.

2. Types of Stresses in Thin Cylinders

Two principal stresses are developed:

1. Hoop (Circumferential) Stress
2. Longitudinal (Axial) Stress

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3. Hoop Stress (Circumferential Stress)

Definition

Stress acting around the circumference of the cylinder.

It tends to split the cylinder into two halves along its length.

Derivation

Considering equilibrium of half the cylinder,

Internal pressure force

P×D×LP \times D \times LP×D×L

Resisting force

2tLσh2tL\sigma_h2tLσh

Equating,

PDL=2tLσhPDL=2tL\sigma_hPDL=2tLσh

Hence,

σh=PD2t\boxed{\sigma_h=\frac{PD}{2t}}σh =2tPD

Where:

- PPP = Internal pressure
- DDD = Diameter
- ttt = Thickness

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4. Longitudinal Stress

Definition

Stress acting along the axis of the cylinder.

It tends to split the cylinder across its cross-section.

Derivation

Pressure force

P(πD24)P\left(\frac{\pi D^2}{4}\right)P(4πD2)

Resisting force

πDtσl\pi Dt\sigma_lπDtσl

Equating,

PπD24=πDtσlP\frac{\pi D^2}{4}=\pi Dt\sigma_lP4πD2 =πDtσl

Therefore,

σl=PD4t\boxed{\sigma_l=\frac{PD}{4t}}σl =4tPD

Important Relation

σh=2σl\boxed{\sigma_h=2\sigma_l}σh =2σl

Thus, the hoop stress is twice the longitudinal stress.

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5. Hoop Strain

Using Hooke's Law,

ϵh=1E(σh−μσl)\boxed{\epsilon_h=\frac{1}{E}\left(\sigma_h-\mu\sigma_l\right)}ϵh =E1 (σh −μσl)

Where:

- EEE = Young's Modulus
- μ\muμ = Poisson's Ratio

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6. Longitudinal Strain

ϵl=1E(σl−μσh)\boxed{\epsilon_l=\frac{1}{E}\left(\sigma_l-\mu\sigma_h\right)}ϵl =E1 (σl −μσh)

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7. Volumetric Strain

The volumetric strain is

ϵv=2ϵh+ϵl\boxed{\epsilon_v=2\epsilon_h+\epsilon_l}ϵv =2ϵh +ϵl

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8. Change in Diameter

ΔD=ϵhD\boxed{\Delta D=\epsilon_h D}ΔD=ϵh D

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9. Change in Length

ΔL=ϵlL\boxed{\Delta L=\epsilon_l L}ΔL=ϵl L

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10. Change in Volume

ΔV=ϵvV\boxed{\Delta V=\epsilon_v V}ΔV=ϵv V

Where:

- VVV = Original volume

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11. Thin Spherical Shell

Definition

A pressure vessel having spherical shape and small thickness.

Condition

Dt>20\boxed{\frac{D}{t}>20}tD >20

Stress in Thin Sphere

Only one stress develops because the stress is equal in all directions.

σ=PD4t\boxed{\sigma=\frac{PD}{4t}}σ=4tPD

Comparison:

- Thin Cylinder:
  - σh=PD2t\sigma_h=\frac{PD}{2t}σh =2tPD
- Thin Sphere:
  - σ=PD4t\sigma=\frac{PD}{4t}σ=4tPD

Thus, for the same pressure and dimensions, the stress in a thin sphere is **half the hoop stress in a thin cylinder**.

PART–B: THICK CYLINDERS

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12. Introduction

A cylinder is called **thick** when

Dt<20\boxed{\frac{D}{t}<20}tD <20

or

t>D20\boxed{t>\frac{D}{20}}t>20D

Examples:

- Gun barrels
- Hydraulic presses
- High-pressure vessels
- Hydraulic cylinders

In thick cylinders, stress is **not uniform** across the thickness, so thin-cylinder formulas cannot be used.

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13. Lame's Theory

Developed by **Gabriel Lamé**.

Used for:

- Thick cylinders
- High internal pressure

Assumptions:

1. Material is homogeneous and isotropic.
2. Stresses vary only in the radial direction.
3. Material obeys Hooke's Law.

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14. Types of Stresses

(a) Radial Stress (σr\sigma_rσr)

Acts perpendicular to the wall.

- Maximum at inner surface.
- Minimum at outer surface.

(b) Hoop Stress (σh\sigma_hσh)

Acts tangentially.

- Maximum at inner surface.
- Decreases toward the outer surface.

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15. Lame's Formulae

The general equations are

Radial Stress

σ r=A−Br2\boxed{\sigma_r=A-\frac{B}{r^2}} σ r =A−r2B

Hoop Stress

σ h=A+Br2\boxed{\sigma_h=A+\frac{B}{r^2}} σ h =A+r2B

Where:

- A,BA, BA,B = Constants determined from boundary conditions
- rrr = Radius at the point considered

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16. Boundary Conditions

For a cylinder with:

- Internal radius = rir_iri
- External radius = ror_oro
- Internal pressure = PiP_iPi
- External pressure = PoP_oPo

At r=rir=r_ir=ri ,

σ r=−Pi\sigma_r=-P_i σ r =−Pi

At r=ror=r_or=ro ,

σ r=−Po\sigma_r=-P_o σ r =−Po

These conditions are used to determine the constants AAA and BBB.

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17. Stress Distribution

Radial Stress

- Maximum compressive value at the inner surface.
- Decreases toward the outer surface.
- Equal to external pressure at the outer surface.

Hoop Stress

- Maximum at the inner surface.
- Decreases continuously toward the outer surface.
- Always greater than radial stress.

18. Design of Thick Cylinders

The design is based on the **maximum allowable hoop stress**.

Procedure:

1. Determine the maximum hoop stress using Lame's equations.
2. Compare it with the allowable stress.
3. Select suitable inner and outer radii (or thickness).

19. Compound Cylinders

Definition

A compound cylinder is made by fitting one cylinder over another using **shrink fitting**.

Examples:

- Large gun barrels
- High-pressure hydraulic cylinders

Advantages

- Higher pressure capacity.
- Better stress distribution.
- Increased safety.
- Longer service life.

20. Shrinkage (Interference Fit)

The outer cylinder is heated and fitted over the inner cylinder.

After cooling:

- The outer cylinder contracts.
- A compressive pressure develops at the common surface.
- This improves the load-carrying capacity.

21. Necessary Difference of Radii for Shrinkage

The required radial interference is

δ =u1+u2\boxed{\delta=u_1+u_2} δ =u1 +u2

Where:

- u1u_1u1 = Expansion of the inner cylinder
- u2u_2u2 = Contraction of the outer cylinder

This interference ensures proper shrink fit and develops the desired contact pressure.

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22. Comparison of Thin and Thick Cylinders

Thin Cylinder	Thick Cylinder
D/t>20D/t > 20D/t>20	D/t<20D/t < 20D/t<20
Stress is uniform across thickness	Stress varies across thickness
Simple formulas are used	Lamé's equations are used
Radial stress is negligible	Radial stress is significant
Used for low pressure	Used for high pressure

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23. Important Formula Sheet

Quantity	Formula
Thin Cylinder Condition	D/t>20D/t>20D/t>20
Thick Cylinder Condition	D/t<20D/t<20D/t<20
Hoop Stress (Thin Cylinder)	σh=PD2t\sigma_h=\frac{PD}{2t}σh =2tPD
Longitudinal Stress	σl=PD4t\sigma_l=\frac{PD}{4t}σl =4tPD
Hoop Strain	ϵh=σh−μσlE\epsilon_h=\frac{\sigma_h-\mu\sigma_l}{E}ϵh =Eσh −μσl
Longitudinal Strain	ϵl=σl−μσhE\epsilon_l=\frac{\sigma_l-\mu\sigma_h}{E}ϵl =Eσl −μσh
Volumetric Strain	ϵv=2ϵh+ϵl\epsilon_v=2\epsilon_h+\epsilon_lϵv =2ϵh +ϵl
Change in Diameter	ΔD=ϵhD\Delta D=\epsilon_hDΔD=ϵh D
Change in Length	ΔL=ϵlL\Delta L=\epsilon_lLΔL=ϵl L
Change in Volume	ΔV=ϵvV\Delta V=\epsilon_vVΔV=ϵv V
Thin Sphere Stress	σ=PD4t\sigma=\frac{PD}{4t}σ=4tPD
Lamé Radial Stress	σr=A−Br2\sigma_r=A-\frac{B}{r^2}σr =A−r2B
Lamé Hoop Stress	σh=A+Br2\sigma_h=A+\frac{B}{r^2}σh =A+r2B

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24. Frequently Asked University Exam Questions (2 Marks)

1. Define a thin cylinder.
2. Define a thick cylinder.
3. State the condition for a thin cylinder.
4. Define hoop stress.
5. Define longitudinal stress.
6. State Lamé's equations.
7. What is a compound cylinder?
8. What is shrink fitting?
9. Why is radial stress neglected in thin cylinders?

10. State the condition for a thin spherical shell.

Curvature (R)

The radius of the curved elastic line.

Relationship

1R=d2ydx2\boxed{\frac{1}{R}=\frac{d^2y}{dx^2}}R1 =dx2d2y

For small deflections,

1R=MEI\boxed{\frac{1}{R}=\frac{M}{EI}}R1 =EIM

Where:

- MMM = Bending moment
- EEE = Young's modulus
- III = Moment of inertia

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2. Differential Equation of Elastic Curve

From bending theory,

EId2ydx2=M\boxed{EI\frac{d^2y}{dx^2}=M}EIdx2d2y =M

This is called the **Differential Equation of the Elastic Curve**.

Where:

- EEE = Young's Modulus
- III = Moment of Inertia
- MMM = Bending Moment
- yyy = Deflection

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3. Methods of Finding Slope and Deflection

There are four commonly used methods:

1. Double Integration Method
2. Macaulay's Method
3. Moment Area Method
4. Conjugate Beam Method

PART–A: DOUBLE INTEGRATION METHOD

Principle

Using

$$EI\frac{d^2y}{dx^2}=M(x)$$

Integrate twice.

First Integration

Gives

$$EI\frac{dy}{dx}=\int Mdx+C_1$$

Slope equation.

Second Integration

$$EIy=\int\left(\frac{dy}{dx}\right)dx+C_2$$

Deflection equation.

Procedure

1. Find support reactions.
2. Write bending moment equation.
3. Integrate once.
4. Integrate again.
5. Apply boundary conditions.
6. Calculate slope and deflection.

Standard Results (Double Integration)

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1. Cantilever with Point Load at Free End

Load = P

Length = L

Maximum Slope

$$\boxed{\theta=\frac{PL^2}{2EI}}$$

Maximum Deflection

$$\boxed{y=\frac{PL^3}{3EI}}$$

Occurs at free end.

2. Cantilever with UDL

Maximum Slope

θ =wL36EI\boxed{\theta=\frac{wL^3}{6EI}} θ =6EIwL3

Maximum Deflection

y=wL48EI\boxed{y=\frac{wL^4}{8EI}}y=8EIwL4

3. Cantilever with UVL

Maximum Deflection

y=wL430EI\boxed{y=\frac{wL^4}{30EI}}y=30EIwL4

4. Simply Supported Beam with Central Point Load

Maximum Slope

θ =PL216EI\boxed{\theta=\frac{PL^2}{16EI}} θ =16EIPL2

Maximum Deflection

y=PL348EI\boxed{y=\frac{PL^3}{48EI}}y=48EIPL3

Occurs at center.

5. Simply Supported Beam with UDL

Maximum Slope

θ =wL324EI\boxed{\theta=\frac{wL^3}{24EI}} θ =24EIwL3

Maximum Deflection

y=5wL4384EI\boxed{y=\frac{5wL^4}{384EI}}y=384EI5wL4

Occurs at center.

PART–B: MACAULAY'S METHOD

Introduction

Macaulay's Method is a modified Double Integration Method used when beams have **multiple point loads at different locations**.

Macaulay Bracket

The bracket is written as

(x−a) (x−a) (x−a)

Rule:

- If x<ax<ax<a, the term is taken as zero.
- If x>ax>ax>a, use the actual value.

Procedure

1. Calculate reactions.
2. Write a single bending moment equation using Macaulay brackets.
3. Integrate twice.
4. Apply boundary conditions.
5. Find slope and deflection.

Advantages

- Simple for multiple concentrated loads.
- Only one equation is required.
- Eliminates separate equations for each region.

PART–C: MOMENT AREA METHOD

Developed by **Otto Mohr**.

Uses the bending moment diagram instead of integration.

Mohr's First Theorem

Statement

The **change in slope** between any two points on the elastic curve equals the **area of the M/EI diagram** between those points.

Mathematically,

θB− θA= ∫ MEIdx\boxed{\theta_B-\theta_A=\int\frac{M}{EI}dx} θB − θA = ∫ EIM dx

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Mohr's Second Theorem

Statement

The **deflection** of one point relative to the tangent drawn at another point equals the **moment of the M/EI area** between the two points about the point where deflection is required.

Mathematically,

δ = ∫ MxEIdx\boxed{\delta=\int\frac{Mx}{EI}dx} δ = ∫ EIMx dx

Procedure

1. Draw Bending Moment Diagram.
2. Draw M/EI diagram.
3. Calculate area.
4. Apply Mohr's First Theorem for slope.
5. Apply Mohr's Second Theorem for deflection.

Advantages

- Easy for simple loading cases.
- No integration required.
- Saves calculation time.

Applications of Moment Area Method

- Cantilever beams
- Simply supported beams
- Point load
- UDL
- Couple
- UVL

PART–D: CONJUGATE BEAM METHOD

Introduction

The Conjugate Beam Method is another method to determine:

- Slope
- Deflection

using an imaginary beam called the **Conjugate Beam**.

Concept

A fictitious beam is constructed having

Load on conjugate beam

MEI\boxed{\frac{M}{EI}}EIM

Where:

- M = Bending Moment of real beam

Basic Principle

Real Beam	Conjugate Beam
Slope	Shear Force
Deflection	Bending Moment

Thus,

θ =Vc\boxed{\theta=V_c} θ =Vc   y=Mc\boxed{y=M_c} y=Mc

Where:

- VcV_cVc = Shear in conjugate beam
- McM_cMc = Moment in conjugate beam

Difference Between Real Beam and Conjugate Beam

Real Beam	Conjugate Beam
Actual beam	Imaginary beam
Carries actual loads	Carries M/EI loading
Slope is calculated	Shear gives slope
Deflection is calculated	Moment gives deflection

Support Conditions

Real Beam	Conjugate Beam
Fixed End	Free End
Free End	Fixed End
Pin Support	Pin Support
Roller Support	Roller Support

Procedure

1. Draw real beam.
2. Calculate Bending Moment Diagram.
3. Draw M/EI loading.

4. Convert supports.
5. Analyze conjugate beam.
6. Shear = slope.
7. Moment = deflection.

Applications

- Cantilever beams
- Simply supported beams
- Overhanging beams
- Determinate beams

Comparison of Methods

Method	Best Used For
Double Integration	Simple loading
Macaulay	Multiple point loads
Moment Area	Manual calculations
Conjugate Beam	Long beams & complicated supports

Important Formula Sheet

Quantity	Formula
Differential Equation	EId2ydx2=MEI\frac{d^2y}{dx^2}=MEIdx2d2y =M
Slope	θ=dydx\theta=\frac{dy}{dx} θ=dxdy
Radius of Curvature	1R=MEI\frac{1}{R}=\frac{M}{EI}R1 =EIM
Mohr's First Theorem	θ=∫MEIdx\theta=\int\frac{M}{EI}dx θ=∫EIM dx
Mohr's Second Theorem	δ=∫MxEIdx\delta=\int\frac{Mx}{EI}dx δ=∫EIMx dx

Standard Deflection Formulas

Beam	Maximum Deflection
Cantilever + Point Load	PL33EI\frac{PL^3}{3EI}3EIPL3
Cantilever + UDL	wL48EI\frac{wL^4}{8EI}8EIwL4
Cantilever + UVL	wL430EI\frac{wL^4}{30EI}30EIwL4
Simply Supported + Center Point Load	PL348EI\frac{PL^3}{48EI}48EIPL3
Simply Supported + UDL	5wL4384EI\frac{5wL^4}{384EI}384EI5wL4

Standard Slope Formulas

Beam	Maximum Slope
Cantilever + Point Load	PL22EI\frac{PL^2}{2EI}2EIPL2
Cantilever + UDL	wL36EI\frac{wL^3}{6EI}6EIwL3
Simply Supported + Center Point Load	PL216EI\frac{PL^2}{16EI}16EIPL2
Simply Supported + UDL	wL324EI\frac{wL^3}{24EI}24EIwL3

Frequently Asked University Exam Questions (2 Marks)

1. Define slope and deflection.
2. What is the elastic curve?
3. State the differential equation of the elastic curve.
4. What is the radius of curvature?
5. State Mohr's First Theorem.
6. State Mohr's Second Theorem.
7. What is Macaulay's Method?
8. What is a conjugate beam?
9. State the relation between slope and shear in the conjugate beam.
10. State the relation between deflection and bending moment in the conjugate beam.

Frequently Asked University Exam Questions (5–10 Marks)

1. Derive the differential equation of the elastic curve.
2. Determine the slope and deflection of a cantilever carrying a point load using the Double Integration Method.
3. Determine the slope and deflection of a simply supported beam carrying a UDL using the Double Integration Method.
4. Explain Macaulay's Method with a suitable example.
5. State and explain Mohr's First and Second Theorems.
6. Determine the slope and deflection of a beam using the Moment Area Method.
7. Explain the Conjugate Beam Method with support transformations.
8. Compare the Double Integration, Macaulay, Moment Area, and Conjugate Beam methods.
9. Solve numerical problems on determinate beams using the Conjugate Beam Method.
10. Find the slope and deflection of beams subjected to point loads, UDLs, UVLs, and couples.