Department of Civil Engineering

Study Material

UNIT–IV: Deflection of Beams

Introduction

When a beam is subjected to loads, it bends. Due to bending:

- The beam rotates at different sections.
- The beam deflects from its original position.

The study of **slope** and **deflection** is important to ensure that structures remain safe, functional, and comfortable.

1. Important Terms

(a) Elastic Curve

The shape assumed by the neutral axis of a beam after bending is called the **Elastic Curve**.

(b) Deflection (y)

Definition

The perpendicular displacement of any point on the beam from its original position.

Unit

- mm
- m

(c) Slope (θ)

Definition

The angle between the tangent to the elastic curve and the original axis of the beam.

Formula

θ =dydx\boxed{\theta=\frac{dy}{dx}} θ =dxdy

Unit

- Radians

(d) Radius of Curvature (R)

The radius of the curved elastic line.

Relationship

1R=d2ydx2\boxed{\frac{1}{R}=\frac{d^2y}{dx^2}}R1 =dx2d2y

For small deflections,

1R=MEI\boxed{\frac{1}{R}=\frac{M}{EI}}R1 =EIM

Where:

- MMM = Bending moment
- EEE = Young's modulus
- III = Moment of inertia

2. Differential Equation of Elastic Curve

From bending theory,

EId2ydx2=M\boxed{EI\frac{d^2y}{dx^2}=M}EIdx2d2y =M

This is called the **Differential Equation of the Elastic Curve**.

Where:

- EEE = Young's Modulus
- III = Moment of Inertia
- MMM = Bending Moment
- yyy = Deflection

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3. Methods of Finding Slope and Deflection

There are four commonly used methods:

1. Double Integration Method
2. Macaulay's Method
3. Moment Area Method
4. Conjugate Beam Method

PART–A: DOUBLE INTEGRATION METHOD

Principle

Using

EId2ydx2=M(x)EI\frac{d^2y}{dx^2}=M(x)EIdx2d2y =M(x)

Integrate twice.

First Integration

Gives

EIdydx=∫Mdx+C1EI\frac{dy}{dx}=\int Mdx+C_1EIdxdy =∫Mdx+C1

Slope equation.

Second Integration

EIy=∫(dydx)dx+C2EIy=\int\left(\frac{dy}{dx}\right)dx+C_2EIy=∫(dxdy)dx+C2

Deflection equation.

Procedure

1. Find support reactions.
2. Write bending moment equation.
3. Integrate once.
4. Integrate again.
5. Apply boundary conditions.
6. Calculate slope and deflection.

Standard Results (Double Integration)

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1. Cantilever with Point Load at Free End

Load = P

Length = L

Maximum Slope

θ=PL22EI\boxed{\theta=\frac{PL^2}{2EI}} θ=2EIPL2

Maximum Deflection

y=PL33EI\boxed{y=\frac{PL^3}{3EI}} y=3EIPL3

Occurs at free end.

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2. Cantilever with UDL

Maximum Slope

$$\boxed{\theta=\frac{wL^3}{6EI}}$$

Maximum Deflection

$$\boxed{y=\frac{wL^4}{8EI}}$$

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3. Cantilever with UVL

Maximum Deflection

$$\boxed{y=\frac{wL^4}{30EI}}$$

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4. Simply Supported Beam with Central Point Load

Maximum Slope

$$\boxed{\theta=\frac{PL^2}{16EI}}$$

Maximum Deflection

$$\boxed{y=\frac{PL^3}{48EI}}$$

Occurs at center.

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5. Simply Supported Beam with UDL

Maximum Slope

$$\boxed{\theta=\frac{wL^3}{24EI}}$$

Maximum Deflection

$$\boxed{y=\frac{5wL^4}{384EI}}$$

Occurs at center.

PART–B: MACAULAY'S METHOD

Introduction

Macaulay's Method is a modified Double Integration Method used when beams have **multiple point loads at different locations**.

Macaulay Bracket

The bracket is written as

(x−a) (x−a) (x−a)

Rule:

- If x<ax<ax<a, the term is taken as zero.
- If x>ax>ax>a, use the actual value.

Procedure

1. Calculate reactions.
2. Write a single bending moment equation using Macaulay brackets.
3. Integrate twice.
4. Apply boundary conditions.
5. Find slope and deflection.

Advantages

- Simple for multiple concentrated loads.
- Only one equation is required.
- Eliminates separate equations for each region.

PART–C: MOMENT AREA METHOD

Developed by **Otto Mohr**.

Uses the bending moment diagram instead of integration.

Mohr's First Theorem

Statement

The **change in slope** between any two points on the elastic curve equals the **area of the M/EI diagram** between those points.

Mathematically,

$$\boxed{\theta_B-\theta_A=\int\frac{M}{EI}dx}$$

Mohr's Second Theorem

Statement

The **deflection** of one point relative to the tangent drawn at another point equals the **moment of the M/EI area** between the two points about the point where deflection is required.

Mathematically,

$$\delta = \int \frac{Mx}{EI} dx \quad \boxed{\delta=\int\frac{Mx}{EI}dx} \quad \delta = \int \frac{Mx}{EI} dx$$

Procedure

1. Draw Bending Moment Diagram.
2. Draw M/EI diagram.
3. Calculate area.
4. Apply Mohr's First Theorem for slope.
5. Apply Mohr's Second Theorem for deflection.

Advantages

- Easy for simple loading cases.
- No integration required.
- Saves calculation time.

Applications of Moment Area Method

- Cantilever beams
- Simply supported beams
- Point load
- UDL
- Couple
- UVL

PART–D: CONJUGATE BEAM METHOD

Introduction

The Conjugate Beam Method is another method to determine:

- Slope
- Deflection

using an imaginary beam called the **Conjugate Beam**.

Concept

A fictitious beam is constructed having

Load on conjugate beam

MEI\boxed{\frac{M}{EI}}EIM

Where:

- M = Bending Moment of real beam

Basic Principle

Real Beam	Conjugate Beam
Slope	Shear Force
Deflection	Bending Moment

Thus,

θ =Vc\boxed{\theta=V_c} θ =Vc   y=Mc\boxed{y=M_c} y=Mc

Where:

- VcV_cVc = Shear in conjugate beam
- McM_cMc = Moment in conjugate beam

Difference Between Real Beam and Conjugate Beam

Real Beam	Conjugate Beam
Actual beam	Imaginary beam
Carries actual loads	Carries M/EI loading
Slope is calculated	Shear gives slope
Deflection is calculated	Moment gives deflection

Support Conditions

Real Beam	Conjugate Beam
Fixed End	Free End
Free End	Fixed End
Pin Support	Pin Support
Roller Support	Roller Support

Procedure

1. Draw real beam.
2. Calculate Bending Moment Diagram.
3. Draw M/EI loading.
4. Convert supports.
5. Analyze conjugate beam.
6. Shear = slope.
7. Moment = deflection.

Applications

- Cantilever beams
- Simply supported beams
- Overhanging beams
- Determinate beams

Comparison of Methods

Method	Best Used For
Double Integration	Simple loading
Macaulay	Multiple point loads
Moment Area	Manual calculations
Conjugate Beam	Long beams & complicated supports

Important Formula Sheet

Quantity	Formula
Differential Equation	EId2ydx2=MEI\frac{d^2y}{dx^2}=MEIdx2d2y =M
Slope	θ=dydx\theta=\frac{dy}{dx} θ=dxdy
Radius of Curvature	1R=MEI\frac{1}{R}=\frac{M}{EI}R1 =EIM
Mohr's First Theorem	θ=∫MEIdx\theta=\int\frac{M}{EI}dx θ=∫EIM dx
Mohr's Second Theorem	δ=∫MxEIdx\delta=\int\frac{Mx}{EI}dx δ=∫EIMx dx

Standard Deflection Formulas

Beam	Maximum Deflection
Cantilever + Point Load	PL33EI\frac{PL^3}{3EI}3EIPL3
Cantilever + UDL	wL48EI\frac{wL^4}{8EI}8EIwL4
Cantilever + UVL	wL430EI\frac{wL^4}{30EI}30EIwL4
Simply Supported + Center Point Load	PL348EI\frac{PL^3}{48EI}48EIPL3
Simply Supported + UDL	5wL4384EI\frac{5wL^4}{384EI}384EI5wL4

Standard Slope Formulas

Beam	Maximum Slope
Cantilever + Point Load	PL22EI\frac{PL^2}{2EI}2EIPL2
Cantilever + UDL	wL36EI\frac{wL^3}{6EI}6EIwL3
Simply Supported + Center Point Load	PL216EI\frac{PL^2}{16EI}16EIPL2
Simply Supported + UDL	wL324EI\frac{wL^3}{24EI}24EIwL3

Frequently Asked University Exam Questions (2 Marks)

1. Define slope and deflection.
2. What is the elastic curve?

3. State the differential equation of the elastic curve.
4. What is the radius of curvature?
5. State Mohr's First Theorem.
6. State Mohr's Second Theorem.
7. What is Macaulay's Method?
8. What is a conjugate beam?
9. State the relation between slope and shear in the conjugate beam.
10. State the relation between deflection and bending moment in the conjugate beam.

Frequently Asked University Exam Questions (5–10 Marks)

1. Derive the differential equation of the elastic curve.
2. Determine the slope and deflection of a cantilever carrying a point load using the Double Integration Method.
3. Determine the slope and deflection of a simply supported beam carrying a UDL using the Double Integration Method.
4. Explain Macaulay's Method with a suitable example.
5. State and explain Mohr's First and Second Theorems.
6. Determine the slope and deflection of a beam using the Moment Area Method.
7. Explain the Conjugate Beam Method with support transformations.
8. Compare the Double Integration, Macaulay, Moment Area, and Conjugate Beam methods.
9. Solve numerical problems on determinate beams using the Conjugate Beam Method.
10. Find the slope and deflection of beams subjected to point loads, UDLs, UVLs, and couples.