UNIT-III

Continuous Random Variables:

Probability Density Function:

A continuous random variable X is identified with a function called as Probability Density Function denoted by $f(x)$ *and given by*

$$f(x) = P\left[x - \frac{dx}{2} \le X \le x + \frac{dx}{2}\right]$$

Properties:

1. $f(x) \ge 0$

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

3. $P[E] = \int_E f(x)\, dx$

Ex: 1) $P[1 \le X \le 2] = \int_1^2 f(x)\, dx$

2) $P[X \le 5] = \int_{-\infty}^{5} f(x)\, dx$

3) $P[X \ge 5] = \int_5^{\infty} f(x)\, dx$

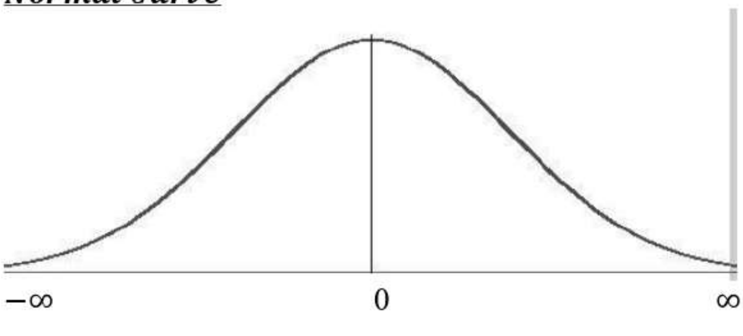
NORMAL DISTRIBUTION

Definition:

A Random variable X is said to follow the normal distribution if its probability density function is given by $\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}}$

Limits: $-\infty < X < \infty$; $-\infty < \mu < \infty$; $\sigma > 0$

Normal Curve

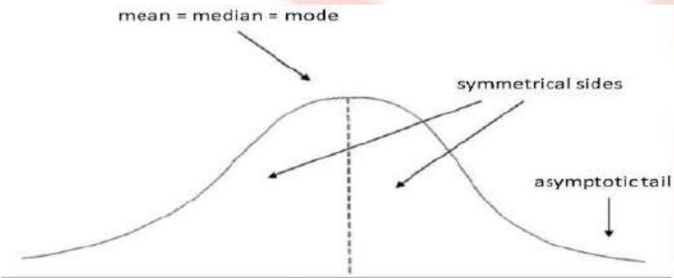


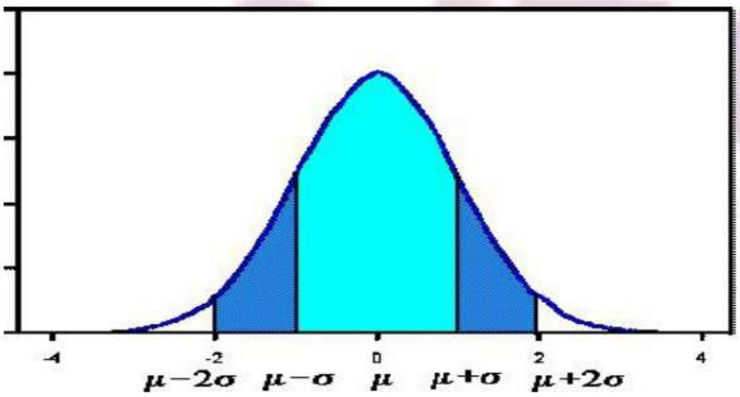
The probability can be evaluated with an integration given by

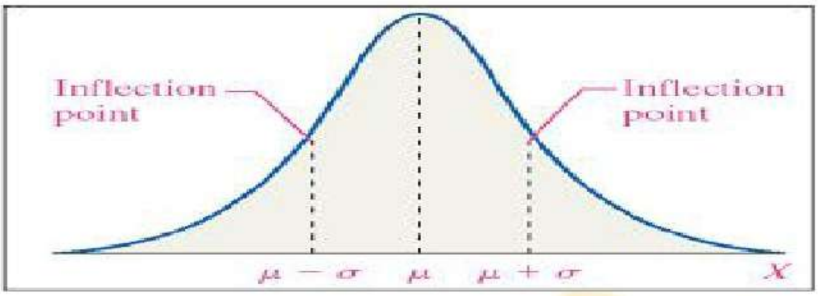
$$P[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} f(x)\ dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$

The computations for normal distributions are taken over the standard normal variate denoted by $\boxed{\ = \frac{X-\mu}{\ }}$







Examples:

1) *Heights and weights of persons.*
2) *Aggregate marks of students.*
3) *Lifetime of electric bulbs or motors, tires, etc*

Applications of Normal Distribution:

1) *Normal distribution can be used in sampling theory.*
2) *It is used in testing of hypothesis where it is assumed that the population is always normal.*
3) *This can also be used in statistical inference to draw conclusions about the population.*
4) *This can also be used in approximating binomial and possion distributions.*

Mean:

we know that $E[X] = \int_{-\infty}^{\infty} x.f(x)\, dx$

$= \int_{-\infty}^{\infty} x.\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$

$Let\ z = \frac{X-\mu}{\sigma} \Rightarrow \sigma z = X - \mu \ and\ X = \sigma z + \mu$

$then\ \sigma\, dz = dX$

$\therefore Mean = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu).e^{-\frac{z^2}{2}}.\sigma\, dz$

$= \frac{1}{\sqrt{2\pi}} [\int_{-\infty}^{\infty} \sigma z\, .\, e^{-\frac{z^2}{2}}.dz + \int_{-\infty}^{\infty} \mu\, .\, e^{-\frac{z^2}{2}}.dz]$

$= \frac{1}{\sqrt{2\pi}} [0 + 2\int_{0}^{\infty} \mu\, .\, e^{-\frac{z^2}{2}}.dz]$ $\quad since\ ze^{-\frac{z^2}{2}} is\ an\ odd\ \&\ e^{-\frac{z^2}{2}} is\ an\ even\ function$

$= \frac{\mu}{\sqrt{2\pi}} 2\sqrt{\frac{\pi}{2}}$ $\quad since\ \int_{0}^{\infty} e^{-\frac{z^2}{2}}.dz = \sqrt{\frac{\pi}{2}}$

$\therefore \boxed{Mean = \mu}$

Variance:

$Variance = E[X-\mu]^2$

$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $\quad since\ we\ have\ E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$

$$let\ \frac{x-\mu}{\sigma} = z \Rightarrow (x-\mu) = \sigma z$$

$$then\ dx = \sigma\, dz$$

$$Variance = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2\, e^{-\frac{z^2}{2}}.\sigma\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\, e^{-\frac{z^2}{2}} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} 2\int_{0}^{\infty} z^2\, e^{-\frac{z^2}{2}} dz \quad since\ z^2 e^{-\frac{z^2}{2}}\ is\ an\ even\ function$$

$$put\ \frac{z^2}{2} = t\ \Rightarrow z^2 = 2t$$

$$2z\, dz = 2dt\ then\ dz = \frac{dt}{z}$$

$$\Rightarrow dz = \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2t.e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{1/2}.e^{t}\ \ dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t}.t^{\frac{3}{2}-1}\ dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad since\ \Gamma(n) = \int_{0}^{\infty} e^{-t}t^{n-1}dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \quad since\ \Gamma(n) = (n-1)\Gamma(n-1)$$

$$= \frac{\sigma^2}{\sqrt{\pi}}.\sqrt{\pi}$$

$$= \sigma^2$$

$$\therefore\ \boxed{Variance = \sigma^2}$$

Mode:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} [e^{\frac{-(x-\mu)^2}{2\sigma^2}}.\frac{-2(x-\mu)}{2\sigma^2}]$$

$$f'(x) = -f(x).\frac{(x-\mu)}{\sigma^2}$$

$$\therefore f'(x) = 0 \Rightarrow (x-\mu) = 0$$

$$\Rightarrow \boxed{x = \mu}$$

$$f'(x) = -f(x).\frac{(x-\mu)}{\sigma^2}$$

$$f''(x) = -\left[f'(x).\frac{(x-\mu)}{\sigma^2} + f(x).\frac{1}{\sigma^2}\right]$$

$$at\ x = \mu\ , f''(x) = -\left[0 + \frac{f(x)}{\sigma^2}\right] = -\frac{f(x)}{\sigma^2} < 0$$

$\therefore \boxed{Mode = \mu}$

Median:

As per the definiton the Median M can be given by

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$Consider \int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} \dots\dots\dots(1)$$

$$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2} \dots\dots.(2)$$

$$\int_{-\infty}^{\mu} f(x)\,dx = \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

$$let\ \frac{x-\mu}{\sigma} = z \Rightarrow (x-\mu) = \sigma z$$

$$then\ dx = \sigma\,dz$$

$$if\ x = -\infty\ then\ z = -\infty\ and\ if\ x = \mu\ then\ z = 0$$

$$\int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} \sigma\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = \frac{1}{2}$$

$$\therefore \int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2} \dots\dots\dots(3)$$

$$sub\ (3)\ in\ (2)$$

$$\Rightarrow \frac{1}{2} + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

$$\int_{\mu}^{M} f(x)\,dx = 0$$

$$\Rightarrow M = \mu$$

$\therefore \boxed{Median = \mu}$

Moment generating function:

$M_X(t) = E[e^{tX}]$

$= \int_{-\infty}^{\infty} e^{tx} f(x)dx$

$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$

$Let\ z = \frac{x-\mu}{\sigma}\ then\ x = \sigma z + \mu$

$\Rightarrow dx = \sigma\, dz$

$\therefore\ M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)}. e^{-\frac{z^2}{2}} \sigma\ dz$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z}. e^{\mu t}. e^{-\frac{z^2}{2}}\ dz$

$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2-2t\sigma z]}\ dz$

$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2-2t\sigma z+t^2\sigma^2-t^2\sigma^2]}\ dz$

$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-t\sigma)^2-t^2\sigma^2]}\ dz$

$= \frac{e^{\mu t+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z-t\sigma]^2}\ dz$

$= \frac{e^{\mu t+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-\frac{v^2}{2}} dv\quad by\ taking\ v = z - t\sigma\ \ then\ dv = dz$

$= \frac{e^{\mu t+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} 2\frac{\sqrt{\pi}}{\sqrt{2}}$

$\therefore\ \boxed{M_X(t) = e^{\mu t+\frac{t^2\sigma^2}{2}}}$

Calculation of Probabilities:

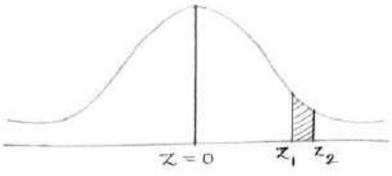
$case(i) : To\ evaluate\ the\ probability\ of\ [x_1 \le X \le x_2]$

$compute\ z_1 = \frac{x_1-\mu}{\sigma}\ \ and\ z_2 = \frac{x_2-\mu}{\sigma}$

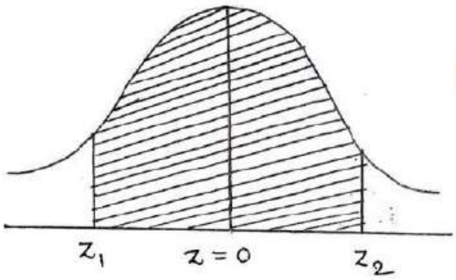
$Hence\ we\ have\ to\ evaluate\ p[z_1 \le z \le z_2]$

$(a)\ z_1\ and\ z_2\ are\ of\ same\ sign$

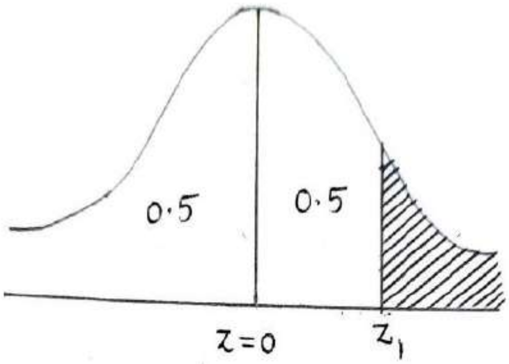
$Let\ us\ assume\ srictly\ that\ z_1\ and\ z_2\ are\ positive.$



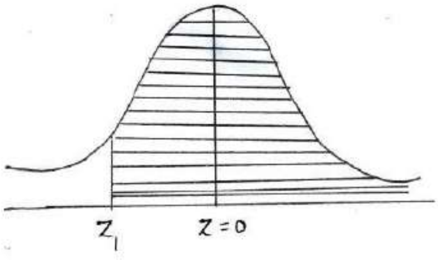
$p[z_1 \le z \le z_2] = A(z_2) - A(z_1)$
$A(z_1) - represents\ the\ area\ between\ 0\ and\ z_1$
$A(z_2) - represents\ the\ area\ between\ 0\ and\ z_2 (b)$
$z_1\ and\ z_2\ are\ of\ opposite\ sign$



$p[z_1 \le z \le z_2] = A(z_1) + A(z_2)$
$case(ii) : P[X > x_1]$
$(a)\, z_1 > 0\ , z_1 = \frac{x_1 - \mu}{\sigma}$



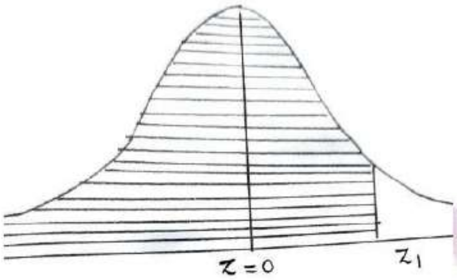
$p[z > z_1] = 0.5 - A(z_1)$
$(b)\, z_1 < 0\ ,$



$p[z > z_1] = 0.5 + A(z_1)$

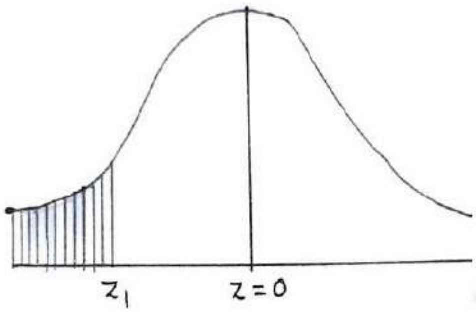
$case(iii) : P[X < x_1]$

$(a)\, z_1 > 0\ , z_1 = \frac{x_1 - \mu}{\sigma}$



$p[z < z_1] = 0.5 + A(z_1)$

$(b)\, z_1 < 0$



$p[z < z_1] = 0.5 - A(z_1)$

Problems:

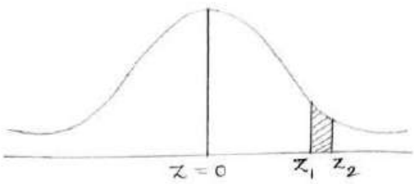
1) *Calculate the probability of* $[3.43 \leq X \leq 6.19]$ *for a normal distributed variate with* $mean = 1\, , S.D = 3.$

Sol: Given that $\mu = 1;\ \sigma = 3$

$x_1 = 3.43\, , x_2 = 6.19$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{3.43 - 1}{3} = \frac{2.43}{3} = 0.81$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{6.19 - 1}{3} = \frac{5.19}{3} = 1.73$$



To evaluate $p[0.81 \leq z \leq 1.73]$

$P(z_1 < Z < z_2) = A[z_2] - A[z_1]$

$= A[1.73] - A[0.81]$

$= 0.4582 - 0.2910$

$= 0.1672$

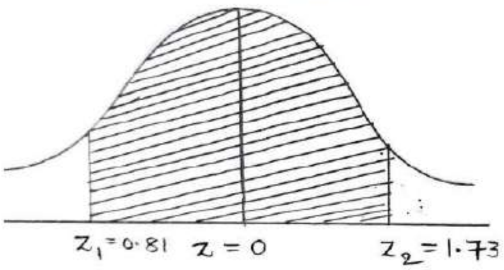
2) *Evaluate the probability between* $[-1.43 \leq X \leq 6.19]$,
$mean = 1\ , S.D = 3$

Sol: Given that $\mu = 1;\ \sigma = 3$

$x_1 = -1.43\ , x_2 = 6.19$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{-1.43 - 1}{3} = \frac{-2.43}{3} = -0.81$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{6.19 - 1}{3} = \frac{5.19}{3} = 1.73$$



To evaluate $p[-0.81 \leq z \leq 1.73]$

$P(z_1 < Z < z_2) = A[z_2] + A[z_1]$

$p[-0.81 \leq z \leq 1.73] = A[1.73] + A[-0.81]$

$= 0.4582 + 0.2910$

$= 0.7492$

3) $Mean = 30\ , S.D = 5$ *then evaluate*
i) $p[26 \leq X \leq 40]$ *ii)* $p[X \geq 45]$

i) Given that $\mu = 30;\ \sigma = 5$

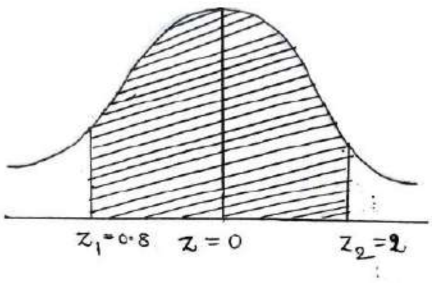
$x_1 = 26\ , x_2 = 40$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{26 - 30}{5} = \frac{-4}{5} = -0.8$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{40 - 30}{5} = \frac{10}{5} = 2$$

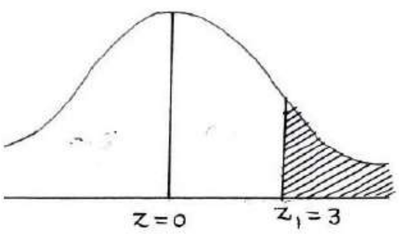
To evaluate $p[-0.8 \leq z \leq 2]$

$= A[2] + A[0.8]$



$= 0.4772 + 0.2881$
$= 0.7653$
$ii)$ *Given that* $\mu = 30;\ \sigma = 5$

$$z_1 = \frac{45 - 30}{5} = \frac{15}{5} = 3$$



$p[z \geq 3] = 0.5 - A(3)$

$= 0.5 - 0.4987$

$= 0.0013$

4) *In a normal distribution* 7% *of the items are under* 35 *and* 89% *of items under* 63. *Determine the mean and variance of the distribution.*

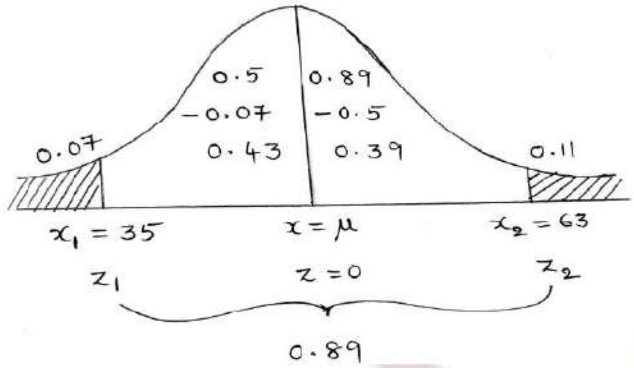
Sol: $7\% \rightarrow 35$
$89\% \rightarrow 63$
$x_1 = 35$ *and* $x_2 = 63$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{35 - \mu}{\sigma}$$

$$\therefore\ z_1 = \frac{35 - \mu}{\sigma} \ \ldots\ldots (1)$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{63 - \mu}{\sigma} \ \ldots\ldots (2)$$

$p[0 \leq z \leq z_1] = 0.43$
$z_1 = -1.48 \ldots\ldots (3)$
$p[0 \leq z \leq z_2] = 0.39$
$z_2 = 1.23 \ldots\ldots (4)$



$sub\ (3)\ in\ (1)$

$-1.48 = \dfrac{35-\mu}{\sigma}$

$35 - \mu = -1.48\sigma$

$\mu - 1.48\sigma = 35 \ldots\ldots(5)$

$sub\ (4)\ in\ (2)$

$1.23 = \dfrac{63-\mu}{\sigma}$

$63 - \mu = 1.23\sigma$

$\mu + 1.23\sigma = 63 \ \ldots\ldots\ldots(6)$

$solving\ (5)\ and\ (6)$

$\mu = 50.28\ , \sigma = 10.33$

$Mean = 50.28\ and\ variance(\sigma^2) = 106.708$

5) *In a normal distribution* 31% *of the items are under* 45 *and* 8% *of items are over* 64 . *Find the mean and variance.*

Sol: 31% *under* 45 *and* 8% *over* 64

$x_1 = 45\ , x_2 = 64$

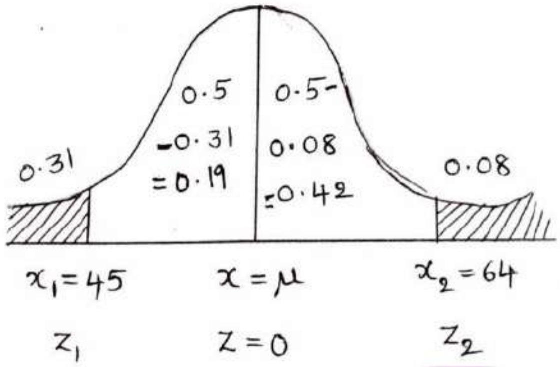
$z_1 = \dfrac{45-\mu}{\sigma} \ldots\ldots(1)\ and\ z_2 = \dfrac{64-\mu}{\sigma} \ldots\ldots(2)$

$p[0 \le z \le z_1] = 0.19$

$z_1 = -0.5 \ldots\ldots.(3)$

$p[0 \le z \le z_2] = 0.42$

$z_2 = 1.41 \ldots\ldots.(4)$



$sub\ (3)\ in\ (1)$

$$-0.5 = \frac{45-\mu}{\sigma}$$

$$45-\mu = -0.5\sigma$$

$$\mu - 0.5\sigma = 45 \ldots\ldots (5)$$

$sub\ (4)\ in\ (2)$

$$1.41 = \frac{64-\mu}{\sigma}$$

$$64-\mu = 1.41\sigma$$

$$\mu + 1.41\sigma = 64 \ \ldots\ldots..(6)$$

$solving\ (5)\ and\ (6)$

$$\mu = 49.9738\,, \sigma = 9.9476$$

$$Mean = 49.97\ and\ variance(\sigma^2) = 98.8036$$

$$\therefore X \sim N(49.97, 98.803)$$

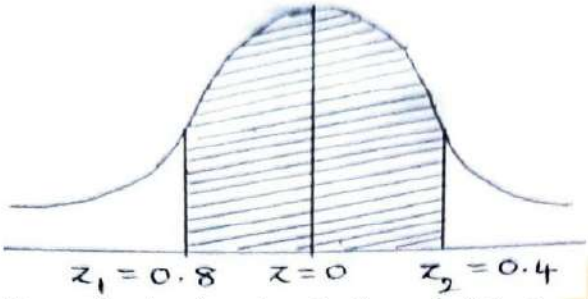
6) *In a sample of* 1000 *items the mean is* 14 *and standard deviation is* 2.5. *Assuming the normal distribution find i) how many score between* 12 *and* 15
ii) how many score above 18 *and iii) how many score below* 15

Sol: Given that $\mu = 14$ *and* $\sigma = 2.5$

i) $x_1 = 12$ *and* $x_2 = 15$

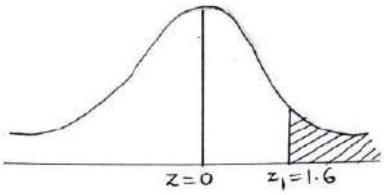
$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{12-14}{2.5} = \frac{-2}{2.5} = -0.8$$

$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{15-14}{2.5} = \frac{1}{2.5} = 0.4$$



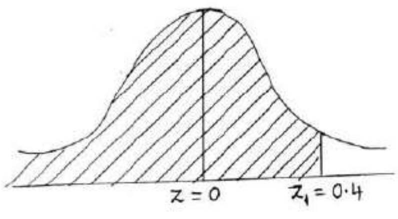
$To\ evaluate\ p(z_1 \le z \le z_2) = p(-0.8 \le z \le 0.4)$
$= A(z_1) + A(z_2)$
$= A(-0.8) + A(0.4)$
$= 0.2881 + 0.1554$
$\therefore p(z_1 \le z \le z_2) = 0.4435$
$\Rightarrow no.of\ items\ between\ 12\ and\ 15\ are\ 1000 \times 0.4435 = 443.5 \cong 444$
$ii)\ x_1 = 18$

$$z_1 = \frac{18-14}{2.5} = \frac{4}{2.5} = 1.6$$



$p(z \ge z_1) = 0.5 - Ar(1.6)$
$= 0.5 - 0.4452 = 0.0548$
$\Rightarrow no.of\ items\ above\ 18\ are\ 1000 \times 0.0548 = 54.8 \cong 55$
$iii)\ x_1 = 15$

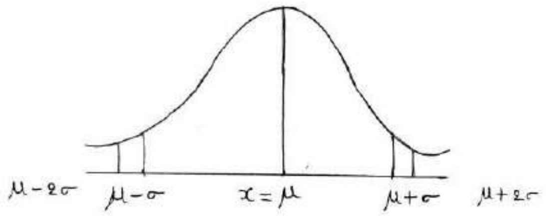
$$z_1 = \frac{15-14}{2.5} = \frac{1}{2.5} = 0.4$$



$p(z \le z_1) = 0.5 + Ar(0.4)$
$= 0.5 + 0.1554 = 0.6554$
$\Rightarrow no.of\ items\ below\ 15\ are\ 1000 \times 0.6554 = 655.4 \cong 655$

Features of chief characteristics:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



Chief Characteristics of Normal Distribution:

1) *The graph of the curve* $y = f(x)$ *is known as Normal Curve.*
2) *The curve is bell − shaped and is symmetrical about the line* $x = \mu$ *which has 2 tails left and right that extends upto* $-\infty$ *and* $+\infty$ *respectively.*
3) *The complete curve represents the total population in which it is equally divided on both sides.*
4) *The curve has only one peak that represents the value of mean which is also equal the median and the mode for the distribution. Because of the unique mode the distribution is called as Uni − model.*
5) *The points at which the shape of the curve changesbfrom concave to convex are called points of inflection denoted by* $\mu \pm \sigma$, $\mu \pm 2\sigma$.
6) *If* X_1 *and* X_2 *are 2 normal variates then their linear combination* $a_1X_1 + a_2X_2$ *is also a normal variate.*
7) *The probability can be evaluated with an integration given by*

$$P[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} F(x)\ dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\ dx$$

8) *The computations for normal distributions are taken over the standard normal variate denoted by* $\boxed{\quad = \frac{X-\mu}{\quad}}$

EXPONENTIAL DISTRIBUTION:

Definition: The continuous random variable X follows an exponential distribution if its probability density function is given by f(x)={ $\lambda e^{-\lambda x}, for\ x \geq 0, \lambda > 0$
$0,\ \ otherwise$

Mean: μ=E(X)= $\int_0^\infty x\, f(x)dx$
$= \int_0^\infty x\, \lambda e^{-\lambda x}dx$
$= \lambda \int_0^\infty x\, e^{-\lambda x}dx$......(1)

Put $\lambda x = t \Rightarrow x = t/\lambda$

$\Rightarrow dx = dt/\lambda$

(1)$\Rightarrow \mu = \lambda \int_0^\infty \frac{t}{\lambda} e^{-t} \frac{dt}{\lambda}$

$= \frac{1}{\lambda} \int_0^\infty t e^{-t}$ dt

$= \frac{1}{\lambda} \int_0^\infty t^{2-1} e^{-t}$ dt $= \frac{1}{\lambda}\ \gamma(2) = \frac{1}{\lambda}$

(since by definition, $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$ *and* γ(n)=(n-1)Γ(n-1)

$\Rightarrow \gamma(2) = (2-1)\ \gamma\ (2-1) = \gamma\ (1) = 1)$

$\therefore$ Mean$=\frac{1}{\lambda}$

<u>Variance</u>: V(X)$= \int_0^\infty x^2 f(x) dx - \mu^2$

$= \lambda \int_0^\infty x^2 e^{-\lambda x} dx - \frac{1}{\lambda^2}$ …….(1)

Put λx=t$\Rightarrow$x$=\frac{t}{\lambda}$

$\Rightarrow$ dx$=\frac{dt}{\lambda}$

(1) $= \lambda \int_0^\infty (\frac{t}{\lambda})^2 e^{-t} \frac{dt}{\lambda} - \frac{1}{\lambda^2}$

$= \frac{1}{\lambda^2} \int_0^\infty t^2 e^{-t}$ dt- $\frac{1}{\lambda^2}$

$= \frac{1}{\lambda^2} \int_0^\infty t^{3-1} e^{-t}$ dt- $\frac{1}{\lambda^2}$

$= \frac{1}{\lambda^2} \Gamma 3 - \frac{1}{\lambda^2}$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$ (since $\Gamma 3 = (3-1)\Gamma(3-1) = 2\Gamma 2 = 2$)

$= \frac{1}{\lambda^2}$

$\therefore$ variance v(x)$= \frac{1}{\lambda^2}$

<u>Moment Generating Function(M.G.F):</u>

$M_X^{(t)} = E(e^{tx}) = \int_0^\infty e^{tx}$ f(x) dx

$= \lambda \int_0^\infty e^{tx}\ e^{-\lambda x}$dx

$= \lambda \int_0^\infty e^{(t-\lambda)x}$ dx------(1)

Put $(t-\lambda)x$=-y

$\Rightarrow (t-\lambda)dx$=-dy

(1) $= \lambda \int_0^\infty e^{-y} \frac{-dy}{t-\lambda}$

$\frac{-\lambda}{t-\lambda} \int_0^\infty e^{-y}$ dy

$= \frac{\lambda}{t-\lambda}$ (-1)$= \frac{\lambda}{\lambda - t}$

$\therefore M_X^{(t)} = \frac{\lambda}{\lambda - t}$

Problem 1: The mileage which car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40,000 km. Find the probabilities that one of these tyres will last(i)atleast 20,000 km and (ii)atmost 30,000 km.

Sol:Let X denote the mileage obtained with the tyre.

Given mean=$\frac{1}{\lambda}$=40,000 $\Rightarrow \lambda = \frac{1}{40,000}$

Since the density function is given by $f(x) = \lambda e^{-\lambda x}, x \geq 0$

$\therefore$ f(x)=$\frac{1}{40,000} e^{\frac{-x}{40,000}}$

(i)P(X>20000)= $\int_{20000}^{\infty} \frac{1}{40,000} e^{\frac{-x}{40,000}}$ dx

=$\frac{-1}{40,000}$(40,000)$\left(e^{\frac{-x}{40,000}} \right)_{20000}^{\infty}$

=$e^{-0.5}$=0.606

(ii) P(X≤30000)= $\int_0^{30000} \frac{1}{40,000} e^{\frac{-x}{40,000}}$ dx

=$\frac{-1}{40,000}$(40,000)$\left(e^{\frac{-x}{40,000}} \right)_{0}^{30000}$

=$1 - e^{-0.75}$=0.5276

Problem 2:The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.(i)What is the probability that the repair time exceeds 2hours.

Solution: Given $\lambda = \frac{1}{2}$ where λ denotes the time to repair a machine.

Since the density function is given by $f(x) = \lambda e^{-\lambda x}, x \geq 0$

f(x)=$\frac{1}{2} e^{\frac{-x}{2}}, x \geq 0$

(i)p(x>2)=$\frac{1}{2}\int_2^{\infty} e^{\frac{-x}{2}} dx = -\left(e^{\frac{-x}{2}}\right)_2^{\infty}$ =e^{-1}=0.3679

Problem 3: The length of time a person speaks over phone follows exponentially distributed with mean 6 .What is the probability that the person will take for (1)more than 8 minutes (2)between 4 and 8 minutes.

Solution: Given $\lambda = \frac{1}{6}$ where λ denotes the length of time a person speaks over phone.

Since the density function is given by $f(x) = \lambda e^{-\lambda x}, x \geq 0$

f(x)=$\frac{1}{6} e^{\frac{-x}{6}}, x \geq 0$

(i) p(x>8)=$\frac{1}{6}\int_8^{\infty} e^{\frac{-x}{6}} dx = -\left(e^{\frac{-x}{6}}\right)_8^{\infty}$ =$e^{-\frac{4}{3}}$=0.2636

(ii) $p(4 \le x \le 8) = \frac{1}{6}\int_4^8 e^{\frac{-x}{6}}\,dx = -\left(e^{\frac{-x}{6}}\right)_4^8 = e^{\frac{-4}{6}} - e^{\frac{-8}{6}}$ =0.5134-0.2635=0.2499

Problem 4: If x is exponentially distributed with parameter λ, find the value of k if $\frac{p(x>k)}{p(x\le k)}$=a.

Solution: Given $\frac{p(x>k)}{p(x\le k)}$=a

$\Rightarrow \frac{p(x>k)}{1-p(x>k)}$=a

$\Rightarrow p(x>k)$=a[1- $p(x>k)$]

$\Rightarrow$(1+a) $p(x>k)$=a

$\Rightarrow p(x>k)=\frac{a}{1+a}$

$\Rightarrow \int_k^\infty f(x)\,dx=\frac{a}{1+a}$

$\Rightarrow \lambda\int_k^\infty e^{-\lambda x}\,dx=\frac{a}{1+a}$

$\Rightarrow \lambda\left(\frac{e^{-\lambda x}}{-\lambda}\right)_k^\infty=\frac{a}{1+a}$

$\Rightarrow e^{-\lambda k}=\frac{a}{1+a}$

$\Rightarrow -\lambda k=\log\left(\frac{a}{1+a}\right)$

$\Rightarrow \lambda k=\log\left(\frac{1+a}{a}\right)$

$\Rightarrow k=\frac{1}{\lambda}\log\left(\frac{1+a}{a}\right)$

Problem 5: The amount of time that a watch will run without having to be reset is a random variable having an exponentially distribution with mean 120 days. Find the probability that such a watch will (i)have to set less than 24 days and (ii)not have to be reset in atleast 180 days.

Solution: Let X denote the number of days the watch will run without reset. Then $f(x)=\lambda e^{-\lambda x}, x \ge 0$

Given $\frac{1}{\lambda}$=120$\Rightarrow\lambda=\frac{1}{120}$

(i)P(x<24)=1-p(x≥24)

$=1-\frac{1}{120}\int_{24}^\infty e^{\frac{-x}{120}}\,dx$

$=1-e^{-0.2}$=0.1813

(ii)p(x>180)=$\frac{1}{120}\int_{180}^\infty e^{\frac{-x}{120}}\,dx$=0.2231

Problem 6: If x=e^λ with p(x≤1)=p(x>1) find v(x)

Solution: $(x)=\lambda e^{-\lambda x}$

p(x≤1)=p(x>1)

$\Rightarrow$1- p(x>1)= p(x>1)

$\Rightarrow 2\, p(x>1)=1$

$\Rightarrow p(x>1)=\frac{1}{2}$

$\Rightarrow \lambda \int_1^{\infty} e^{-\lambda x}\,dx=\frac{1}{2}$

$\Rightarrow -(e^{-\lambda x})_1^{\infty}=\frac{1}{2}$

$\Rightarrow e^{-\lambda}=\frac{1}{2}$

$\Rightarrow -\lambda=\log(\frac{1}{2})$

$\Rightarrow \lambda=\log 2$

$\therefore$ variance$=\frac{1}{\lambda^2}=\frac{1}{(\log 2)^2}$

Problem 7:The daily consumption of milk in excess of 20,000 gallons is exponentially distributed with μ=3000.The city has a daily stack of 35,000 gallons. What is the probability that of 2 days selected at random ,the stock is sufficient for both days.

Solution: Given $\lambda=\frac{1}{3000}$

$\therefore p(x>35000)=e^{-35/3}$(for one day)

$\therefore P(\text{for 2 days})=(e^{-35/3})^2=e^{-70/3}$

GAMMA DISTRIBUTION

Definition: The Gamma function denoted by $\gamma(\alpha)$ is defined as

$\gamma(\alpha)=\int_0^{\infty} x^{\alpha-1} e^{-x}\,dx$; $\alpha>0$(a real number)

Note 1:

1. $\gamma(\alpha+1)=\alpha\,\gamma(\alpha)$ OR $\gamma(\alpha)=(\alpha-1)\,\gamma(\alpha-1)$;$\alpha>0$

2. $\int_0^{\infty} x^{\alpha-1} e^{-\lambda x}=\frac{\gamma(\alpha)}{\lambda^{\alpha}}$; $\lambda>0$

3. $\gamma(n)=(n-1)!;\ n=1,2,3,4,\ldots$

4. $\gamma(\frac{1}{2})=\sqrt{\pi}$

Definition: The probability density function is given by $f(x)=\begin{cases}\frac{\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}}{\gamma(\alpha)}, & x>0\\ 0, & othewise\end{cases}$

If $\alpha=1$ then $f(x)=\begin{cases}\lambda\, e^{-\lambda x}; & x>0\\ 0, & otherwise\end{cases}$

Note 2:Sum of α independent exponential random variables then we get a Gamma random variable with parameters (α,λ).

Problem 1: Show that for $\alpha,\lambda > 0$; $\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\gamma(\alpha)}dx=1$.

Solution: $\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\gamma(\alpha)}dx = \frac{\lambda^\alpha}{\gamma(\alpha)}\int_0^\infty x^{\alpha-1} e^{-\lambda x}dx$

$= \frac{\lambda^\alpha}{\gamma(\alpha)} \cdot \frac{\gamma(\alpha)}{\lambda^\alpha}$ (by note (2))

$=1$

<u>Mean:</u>

$E(x)=\int_0^\infty x\, f(x)dx$

$=\int_0^\infty x \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\gamma(\alpha)}dx$

$= \frac{\lambda^\alpha}{\gamma(\alpha)}\int_0^\infty x^{1+\alpha-1} e^{-\lambda x}dx$

$= \frac{\lambda^\alpha}{\gamma(\alpha)}\int_0^\infty x^{(\alpha+1)-1} e^{-\lambda x}dx$

$= \frac{\lambda^\alpha}{\gamma(\alpha)} \frac{\gamma(\alpha+1)}{\lambda^{\alpha+1}}$

$= \frac{1}{\lambda}\frac{\alpha\,\gamma(\alpha)}{\gamma(\alpha)}$ (by note (1))

$=\frac{\alpha}{\lambda}$

$\therefore$ **E(x)=** $\frac{\alpha}{\lambda}$

<u>Variance:</u>

$V(x)=E(X^2)-(E(X))^2$

$= \int_0^\infty x^2 f(x)dx - (\frac{\alpha}{\lambda})^2$

$=\int_0^\infty x^2 \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\gamma(\alpha)}dx - (\frac{\alpha}{\lambda})^2$

$=\int_0^\infty \frac{\lambda^\alpha x^{\alpha+1} e^{-\lambda x}}{\gamma(\alpha)}dx - (\frac{\alpha}{\lambda})^2$

$= \frac{\lambda^\alpha}{\gamma(\alpha)}\int_0^\infty x^{(\alpha+2)-1} e^{-\lambda x}dx - (\frac{\alpha}{\lambda})^2$

$$= \frac{\lambda^{\alpha}}{\gamma(\alpha)} \frac{\gamma(\alpha+2)}{\lambda^{\alpha+2}} - \left(\frac{\alpha}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2} \frac{(\alpha+1)\gamma(\alpha+1)}{\gamma(\alpha)} - \left(\frac{\alpha}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2} \frac{(\alpha+1)\alpha\gamma(\alpha)}{\gamma(\alpha)} - \left(\frac{\alpha}{\lambda}\right)^2$$

$$= \frac{(\alpha+1)\alpha}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2$$

$$= \frac{\alpha^2}{\lambda^2} + \frac{\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

$\therefore V(X) = \frac{\alpha}{\lambda^2}$

<u>M.G.F:</u>

$M_X^{(t)} = E(e^{tx})$

$$= \int_0^{\infty} e^{tx} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\gamma(\alpha)} dx$$

$$= \frac{\lambda^{\alpha}}{\gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x} dx \ldots\ldots(1)$$

$$Put\ y = (\lambda - t)x \Rightarrow dy = (\lambda - t)dx$$

$$= \frac{\lambda^{\alpha}}{\gamma(\alpha)} \int_0^{\infty} \left(\frac{y}{\lambda-t}\right)^{\alpha-1} e^{-y} \frac{dy}{\lambda-t}$$

$$= \frac{\lambda^{\alpha}}{\gamma(\alpha)} \frac{1}{(\lambda-t)^{\alpha-1+1}} \int_0^{\infty} (y)^{\alpha-1} e^{-y} dy$$

$$= \frac{\lambda^{\alpha}}{(\lambda-t)^{\alpha}} \frac{\gamma(\alpha)}{\gamma(\alpha)} = \frac{\lambda^{\alpha}}{(\lambda-t)^{\alpha}}$$

Problem 1: Find $\Gamma\frac{7}{2}$

Solution: $\Gamma\frac{7}{2} = (\frac{7}{2}-1)\Gamma(\frac{7}{2}-1)$ (since $\gamma(\alpha) = (\alpha-1)\,\gamma(\alpha-1)$)

$= \frac{5}{2}\Gamma\frac{5}{2}$

$= \frac{5}{2}(\frac{5}{2}-1)\Gamma(\frac{5}{2}-1)$

$= \frac{5}{2}\,\frac{3}{2}\,\Gamma\frac{3}{2}$

$= \frac{5}{2}\,\frac{3}{2}(\frac{3}{2}-1)\Gamma(\frac{3}{2}-1)$

$= \frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}$

$= \frac{15}{8}\sqrt{\pi}$

Problem 2:Find $\int_0^\infty x^6 e^{-5x}\ dx$

Solution: $\int_0^\infty x^{7-1} e^{-5x}\ dx$ which is of the form $\int_0^\infty x^{\alpha-1} e^{-\lambda x}\ dx= \frac{\Gamma\alpha}{\lambda^\alpha}$

$$\therefore \int_0^\infty x^6 e^{-5x}\ dx = \frac{\Gamma 7}{5^7}$$

Problem 3: In a certain city, the daily consumption of electric power is millions of kilowatt hours can be treated as a random variable having an Gamma distribution with parameters ($\frac{1}{2}$,3).If the power plant of this city has a daily capacity of 12 millions kilowatt hours, what is the probability that this power supply will be inadequate on any given day?

Solution: Let X represent the daily consumption of electric power in millions of kilowatt hours.

Then f(x)= $\frac{e^{-\lambda x}\lambda^{\alpha} x^{\alpha-1}}{\Gamma\alpha}$

$$=\frac{e^{-\frac{x}{2}}(\frac{1}{2})^3 x^2}{\Gamma 3}$$

$$=\frac{e^{-\frac{x}{2}}(\frac{1}{2})^3 x^2}{\Gamma 3}$$

$$\therefore P(X>12)= \frac{1}{2^4}\int_{12}^{\infty} x^2 e^{-\frac{x}{2}}\ dx \ldots(1)$$

Put $\frac{x}{2}$ =t

$\Rightarrow$x=2t

$\Rightarrow$dx=2 dt

For x=$\infty\Rightarrow$t=∞ and For x=12$\Rightarrow$t=6

$$P(X>12)=\frac{1}{16}\int_6^\infty (2t)^2 e^{-t}\ 2\,dt$$

$$=\frac{8}{16}\int_6^\infty t^2 e^{-t}\quad dt \qquad \text{(since } \int uv\,dx = uv_1 - u'v_2 + u''v_3 \text{)}$$

$$u = t^2 \quad , v = e^{-t}$$

$$=\frac{8}{16}\left(-t^2e^{-t} - 2t\,e^{-t} - 2e^{-t}\right)_6^\infty$$

(using Bernoulli's Rule of Integration)

$$=\frac{1}{2}\,[0-(-36e^{-6}-2\{6\ e^{-6}\ +e^{-6}\})]$$

$$= \frac{1}{2}[36e^{-6}+14e^{-6}]= 25\ e^{-6}$$

Problem 4:The daily consumption of milk in a city, in excess of 20,000 L is approximately distributed as a Gamma variate with the parameters α=2 and $\lambda=\frac{1}{10,000}$.The city has a daily stock of 30,000 L. what is the probability that the stock is insufficient on a particular day?

Solution: Given α=2 and $\lambda=\frac{1}{10,000}$

$$f(x)=\frac{e^{-\lambda x}\lambda^{\alpha}x^{\alpha-1}}{\Gamma\alpha} = \frac{e^{-\frac{x}{10,000}}\left(\frac{1}{10,000}\right)^{2}x}{\Gamma 2}$$

$$=x\left(\frac{1}{10,000}\right)^{2}e^{-\frac{x}{10,000}}$$

$$\therefore p(X>30,000)=\left(\frac{1}{10,000}\right)^{2}\int_{30000}^{\infty} x\, e^{-\frac{x}{10,000}}\, dx \text{.....(1)}$$

Put $\frac{x}{10,000}=t \Rightarrow x=(10000)t$

$\Rightarrow dx=(10000)dt$

For $x=\infty\Rightarrow t=\infty$ and For $x=30000\Rightarrow t=3$

$$\therefore p(X>30,000)=\left(\frac{1}{10,000}\right)^{2}(10000)^{2}\int_{3}^{\infty} t\, e^{t}\, dt$$

$$=\int_{3}^{\infty} t\, e^{t}\, dt$$

$$=(-te^{-t}-e^{-t})_{3}^{\infty}=4e^{-3}$$