UNIT III

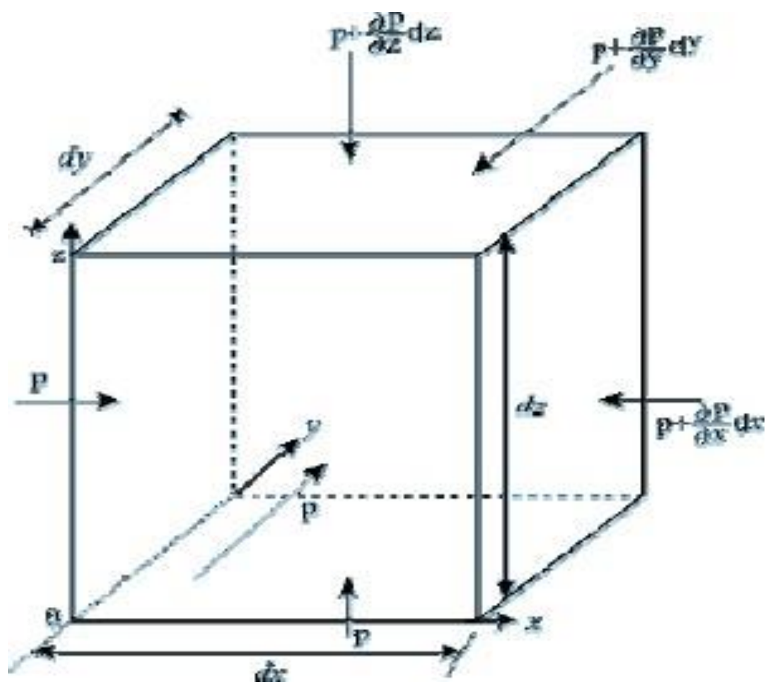
FLUID DYNAMICS

Euler and Navier Stokes Equation:

Euler's Equation: The Equation of Motion of an Ideal Fluid

Using the Newton's second law of motion the relationship between the velocity and pressure field for a flow of an inviscid fluid can be derived. The resulting equation, in its differential form, is known as Euler's Equation. The equation is first derived by the scientist Euler.

Derivation:
Let us consider an elementary parallelopiped of fluid element as a control mass system in a frame of rectangular cartesian coordinate axes as shown in Fig.. The external forces acting on a fluid element are the body forces and the surface forces.



A Fluid Element appropriate to a Cartesian Coordinate System

used for the derivation of Euler's Equation

Let Xx, Xy, Xz be the components of body forces acting per unit mass of the fluid element along the coordinate axes x, y and z respectively. The body forces arise due to external force fields like gravity, electromagnetic field, etc., and therefore, the detailed description of Xx, Xy and Xz are provided by the laws of physics describing the force fields. The surface forces for an inviscid fluid will be the pressure forces acting on different surfaces as shown in Fig. Therefore, the net forces acting on the fluid element along x, y and z directions can be written as

$$F_x = \rho X_x dx\, dy\, dz + p\, dy\, dz - \left(p + \frac{\partial p}{\partial x} dx\right) dy\, dz = \left(\rho X_x - \frac{\partial p}{\partial x}\right) dx\, dy\, dz$$

$$F_y = \rho X_y dx\, dy\, dz + p\, dx\, dz - \left(p + \frac{\partial p}{\partial y} dy\right) dx\, dz = \left(\rho X_y - \frac{\partial p}{\partial y}\right) dx\, dy\, dz$$

$$F_z = \rho X_z dx\, dy\, dz + p\, dy\, dx - \left(p + \frac{\partial p}{\partial z} dz\right) dx\, dz = \left(\rho X_z - \frac{\partial p}{\partial z}\right) dx\, dy\, dz$$

Since each component of the force can be expressed as the rate of change of momentum in the respective directions, we have

$$\frac{D}{Dt}(\rho\, dx\, dy\, dz\, u) = \left(\rho X_x - \frac{\partial p}{\partial x}\right) dxdydz$$

$$\frac{D}{Dt}(\rho\, dx\, dy\, dz\, v) = \left(\rho X_y - \frac{\partial p}{\partial y}\right) dxdydz$$

$$\frac{D}{Dt}(\rho\, dx\, dy\, dz\, w) = \left(\rho X_z - \frac{\partial p}{\partial z}\right) dxdydz$$

s the mass of a control mass system does not change with time, $\rho dxdydz$ is constant with time and can be taken common. Therefore we can write as

$$\frac{Du}{Dt} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Expanding the material accelerations in Eqs in terms of their respective temporal and convective components, we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} - u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{X}$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}.\nabla)\vec{V} = \vec{X} - \frac{1}{\rho}\nabla p$$

Derivation of Bernoulli's Equation for Inviscid and Viscous Flow Field

Bernoulli's Equation

Energy Equation of an ideal Flow along a Streamline

Euler's equation (the equation of motion of an inviscid fluid) along a stream line for a steady flow with gravity as the only body force can be written as

$$V\frac{dV}{ds} = -\frac{1}{\rho}\frac{dp}{ds} - g\frac{dz}{ds}$$

Application of a force through a distance ds along the streamline would physically imply work interaction. Therefore an equation for conservation of energy along a streamline can be obtained by integrating the above Eq. with respect to ds as

$$\int V\frac{dV}{ds}ds = -\int\frac{1}{\rho}\frac{dp}{ds}ds - \int g\frac{dz}{ds}ds$$

$$\text{or,} \qquad \frac{V^2}{2} + \int\frac{dp}{\rho} + gz = C$$

Where C is a constant along a streamline. In case of an incompressible flow, above Eq. can be written as

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = C$$

Pressure head + Velocity head + Potential head =Total head (total energy per unit weight).

Bernoulli's Equation with Head Loss

The derivation of mechanical energy equation for a real fluid depends much on the information about the frictional work done by a moving fluid element and is excluded from the scope of the book. However, in many practical situations, problems related to real fluids can be analysed with the help of a modified form of Bernoulli’s equation as

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f$$

where, hf represents the frictional work done (the work done against the fluid friction) per unit weight of a fluid element while moving from a station 1 to 2 along a streamline in the direction of flow. The term hf is usually referred to as head loss between 1 and 2, since it amounts to the loss in total mechanical energy per unit weight between points 1 and 2 on a streamline due to the effect of fluid friction or viscosity. It physically signifies that the difference in the total mechanical energy between stations 1 and 2 is dissipated into intermolecular or thermal energy and is expressed as loss of head hf in above Eq. The term head loss, is conventionally symbolized as hL instead of hf in dealing with practical problems. For an inviscid flow hL = 0, and the total mechanical energy is constant along a streamline.

Bernoulli's Equation In Irrotational Flow

$$\boxed{\frac{p}{\rho}+\frac{V^2}{2}+gz=C}$$

- This equation was obtained by integrating the Euler’s equation (the equation of motion) with respect to a displacement '**ds'** along a streamline. Thus, the value of C in the above equation is constant only along a streamline and should essentially vary from streamline to streamline.
- The equation can be used to define relation between flow variables at point B on the streamline and at point A, along the same streamline. So, in order to apply this equation, one should have knowledge of velocity field beforehand. This is one of the limitations of application of Bernoulli's equation.

Irrotationality of flow field

Under some special condition, the constant C becomes invariant from streamline to streamline and the Bernoulli’s equation is applicable with same value of C to the entire flow field. The typical condition is the irrotationality of flow field.

Momentum Equation in Integral Form:

Conservation of Momentum: Momentum Theorem

In Newtonian mechanics, the conservation of momentum is defined by Newton’s second law of motion.

Newton’s Second Law of Motion

- The rate of change of momentum of a body is proportional to the impressed action and takes place in the direction of the impressed action.
- If a force acts on the body ,linear momentum is implied.
- If a torque (moment) acts on the body,angular momentum is implied.

Reynolds Transport Theorem

A study of fluid flow by the Eulerian approach requires a mathematical modeling for a control volume either in differential or in integral form. Therefore the physical statements of the principle of conservation of mass, momentum and energy with reference to a control volume become necessary. This is done by invoking a theorem known as the Reynolds transport theorem which relates the control volume concept with that of a control mass system in terms of a general property of the system.

Statement of Reynolds Transport Theorem

The theorem states that "the time rate of increase of property N within a control mass system is equal to the time rate of increase of property N within the control volume plus the net rate of efflux of the property N across the control surface”.

Equation of Reynolds Transport Theorem

After deriving Reynolds Transport Theorem according to the above statement we get

$$\left(\frac{dN}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho\, dV + \iint_{CS} \eta\rho \vec{V}\cdot d\vec{A}$$

In this equation
N - flow property which is transported
η - intensive value of the flow property
Application of the Reynolds Transport Theorem to Conservation of Mass and Momentum

Angular Momentum Equation in Integral Form:

Angular Momentum
The angular momentum or moment of momentum theorem is also derived from below Eq in consideration of the property N as the angular momentum and accordingly η as the angular momentum per unit mass. Thus,

$$\frac{d}{dt}(A_{CMS}) = \frac{\partial}{\partial t}\iiint_{CV} \rho(\vec{r}\times\vec{V}_r)\, dV + \iint_{CS} (\vec{r}\times\vec{V}_r)\rho\vec{V}_r\cdot d\vec{A}$$

where
Control mass system is the **angular momentum of the control mass system**. . It has to be noted that the origin for the angular momentum is the origin of the position vector

Laplace Equation:

Potential Flow Theory
Let us imagine a pathline of a fluid particle. Rate of spin of the particle is **ω**z . The flow in which this spin is zero throughout is known as irrotationalflow.

$$\nabla\times\vec{V} = 0$$

For irrotational flows,

Pathline of a Fluid Particle

Velocity Potential and Stream Function

Since for irrotational flows . $\nabla \times \vec{V} = C$.
the velocity for an irrotational flow, can be expressed as the gradient of a scalar function called

$$\vec{V} = \nabla \phi$$

the velocity potential, denoted by Φ
Combination of above eq'ns yields

$$\nabla^2 \phi = 0$$

Laplace equation

For irrotational flows

$$\vec{\omega} = 0$$

For two-dimensional case

$$\omega = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0$$
$$\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$-\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = C \qquad \left[u = \frac{\partial \psi}{\partial y}; v = -\frac{\partial \psi}{\partial x}\right]$$
$$-\left(+\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}\right) = 0$$
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$
$$\nabla^2 \psi = 0$$

which is again Laplace's equation.

From the above Eq. we see that **an inviscid, incompressible, irrotational flow is governed by Laplace's equation.**

A complicated flow pattern for an inviscid, incompressible, irrotational flow can be synthesized by adding together a number of elementary flows (provided they are also inviscid, incompressible and irrotational) **The Superposition Principle**

Stream Function

Let us consider a two-dimensional incompressible flow parallel to the x - y plane in a rectangular cartesian coordinate system. The flow field in this case is defined by

$$u = u(x, y, t)$$
$$v = v(x, y, t)$$
$$w = 0$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

If a function $\psi(x, y, t)$ is defined in the manner

$$u = \frac{\partial \psi}{\partial y}$$
$$v = -\frac{\partial \psi}{\partial x}$$

so that it automatically satisfies the equation of continuity , then the function is known as stream function.

Note that for a **steady flow, ψ is a function of two variables x and y only.**

Constancy of ψ on a Streamline

Since ψ is a point function, it has a value at every point in the flow field. Thus a change in the stream function ψ can be written as

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

The equation of a streamline is given by

$$\frac{u}{dx} = \frac{v}{dy} \qquad \text{or} \qquad udy - vdx = 0 \text{(since tangent dy/dx equals the velocity v/u)}$$

It follows that $d\psi = 0$ on a streamline.This implies the value of ψ is constant along a streamline.

Therefore, the equation of a streamline can be expressed in terms of stream function as

$\psi(x, y)$ = constant

Once the function ψ is known, streamline can be drawn by joining the same values of ψ in the flow field.

Stream function for an irrotational flow

In case of a two-dimensional irrotational flow

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \psi_{xx} + \psi_{yy} = 0$$

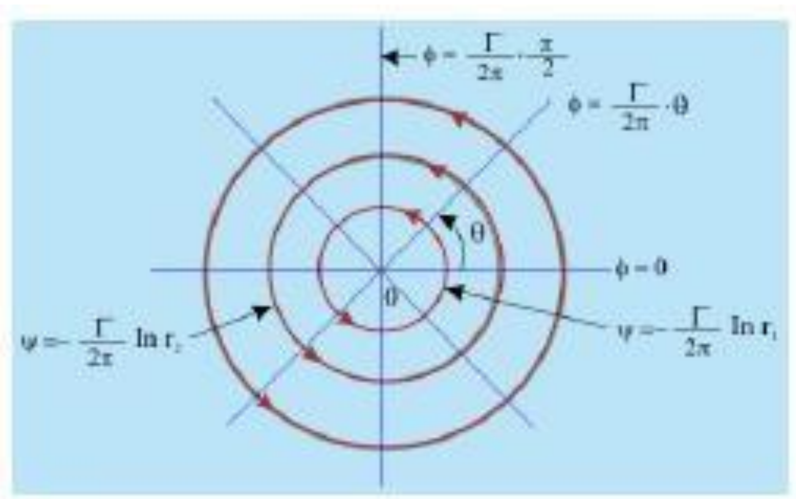
$$\Rightarrow \nabla^2 \psi = 0$$

Conclusion drawn: For an irrotational flow, stream function satisfies the Laplace's equation

Concept of Circulation in a Free Vortex Flow

Free Vortex Flow

- Fluid particles move in circles about a point.
- The only non-trivial velocity component is tangential.
- This tangential speed varies with radius r so that same circulation is maintained.
- Thus,all the **streamlines are concentric circles** about a given point where the velocity along each streamline is inversely proportional to the distance from the centre. This **flow is necessarily irrotational.**



Flownet for a vortex (free vortex)

Lift and Drag for Flow Past a Cylinder without Circulation

Pressure in the Cylinder Surface

Pressure becomes uniform at large distances from the cylinder (where the influence of doublet is small).

Let us imagine the pressure p0 is known as well as uniform velocity U_0 .

We can apply Bernoulli's equation between infinity and the points on the boundary of the cylinder.

Neglecting the variation of potential energy between the aforesaid point at infinity and any point on the surface of the cylinder, we can write

$$\frac{p_0}{\rho g}+\frac{U_0^2}{2g}=\frac{p_b}{\rho g}+\frac{U_b^2}{2g}$$

where the subscript b represents the surface on the cylinder.

Since fluid cannot penetrate the solid boundary, the velocity **Ub should be only in the transverse direction** , or in other words, only vθ component of velocity is present on the streamline $\psi = 0$.

Thus at $r=\left(\frac{\Lambda}{U_0}\right)^{1/2}$

$$U_b = v_\theta\Big|_{r=\left(\frac{\Lambda}{U_0}\right)^{1/2}} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r=\left(\frac{\Lambda}{U_0}\right)^{1/2}} = -2U_0\sin\theta$$

$$p_b=\rho g\left[\frac{U_0^2}{2g}+\frac{p_0}{\rho g}-\frac{(2U_0\sin\theta)^2}{2g}\right]$$