NARSIMHA REDDY ENGINEERING COLLEGE
UGC AUTONOMOUS INSTITUTION
Maisammaguda (V), Kompally - 500100, Secunderabad, Telangana State, India

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DYNAMICS OF MACHINERY

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Unit-V

Free Vibrations – concept checklist

You should be able to:

1. Understand simple harmonic motion (amplitude, period, frequency, phase)
2. Identify # DOF (and hence # vibration modes) for a system
3. Understand (qualitatively) meaning of ‘natural frequency’ and ‘Vibration mode’ of a system
4. Calculate natural frequency of a 1DOF system (linear and nonlinear)

5. Write the EOM for simple spring-mass systems by inspection

6. Understand natural frequency, damped natural frequency, and ‘Damping factor’ for a dissipative 1DOF vibrating system
7. Know formulas for nat freq, damped nat freq and ‘damping factor’ for spring-mass system in terms of k,m,c
8. Understand underdamped, critically damped, and over damped motion of a dissipative 1DOF vibrating system
9. Be able to determine damping factor from a measured free vibration response
10. Be able to predict motion of a freely vibrating 1DOF system given its initial velocity and position, and apply this to design-type problems

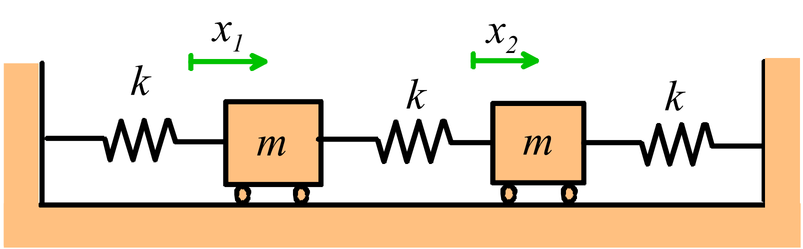
Number of DOF (and vibration modes)

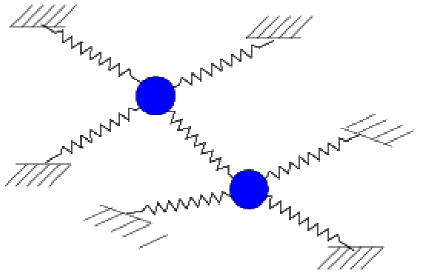
If masses are particles:

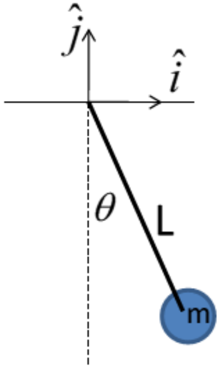
Expected # vibration modes = # of masses x # of directions masses can move independently

If masses are rigid bodies (can rotate, and have inertia)

Expected # vibration modes = # of masses x (# of directions masses can move + # possible axes of rotation)

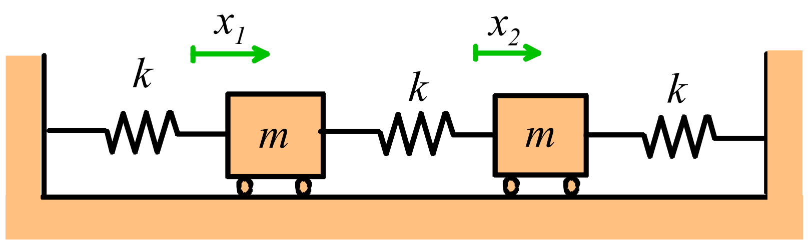


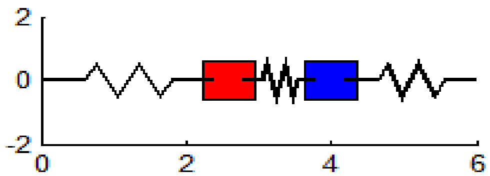


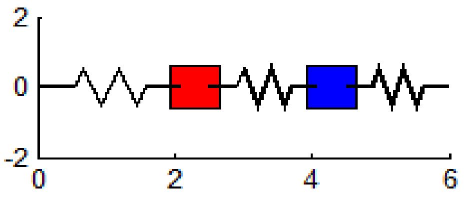


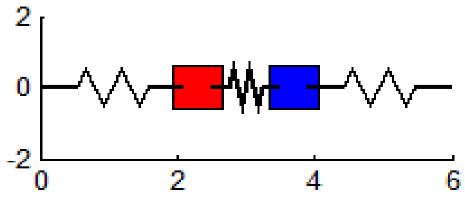
Vibration modes and natural frequencies

• A system usually has the same # natural freqs as degrees of freedom
•Vibration modes: special initial deflections that cause entire system to vibrate harmonically
•Natural Frequencies are the corresponding vibration frequencies

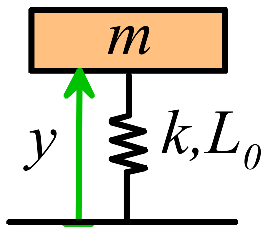


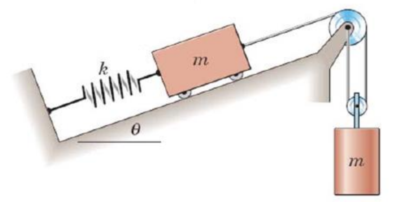






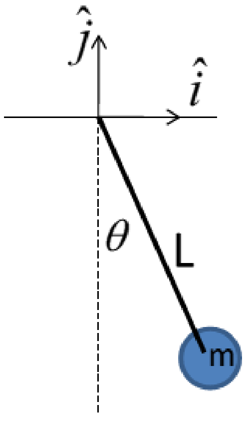
Calculating nat freqs for 1DOF systems – the basics





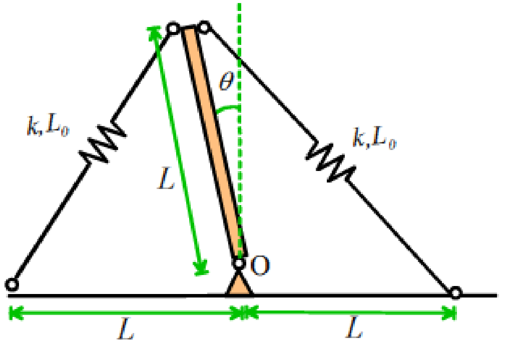
EOM for small vibration of any 1DOF undamped system has form

$$\frac{d^2 y}{dt^2} + \omega_n^2 y = C$$

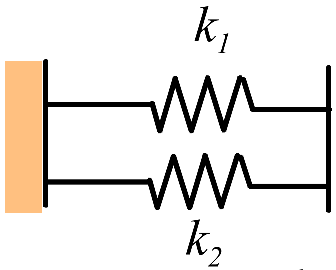


ω_n is the natural frequency

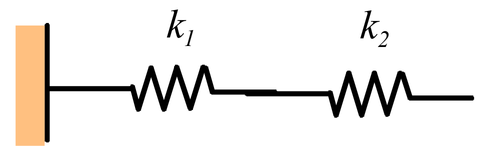
1. Get EOM (F=ma or energy)
2. Linearize (sometimes)
3. Arrange in standard form
4. Read off nat freq.



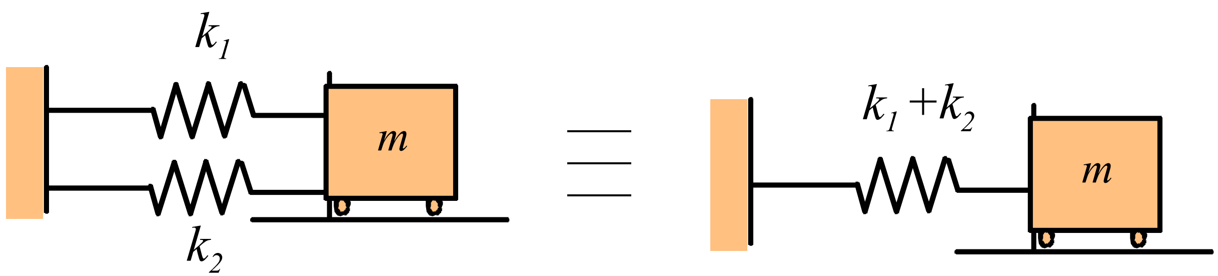
Useful shortcut for combining springs



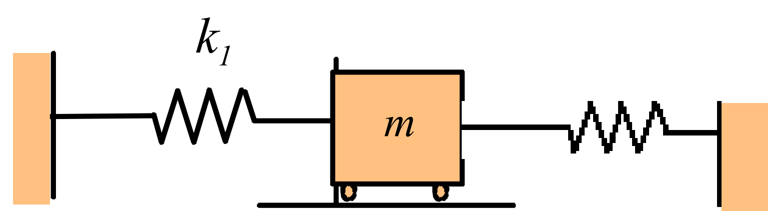
Parallel: stiffness $k = k_1 + k_2$



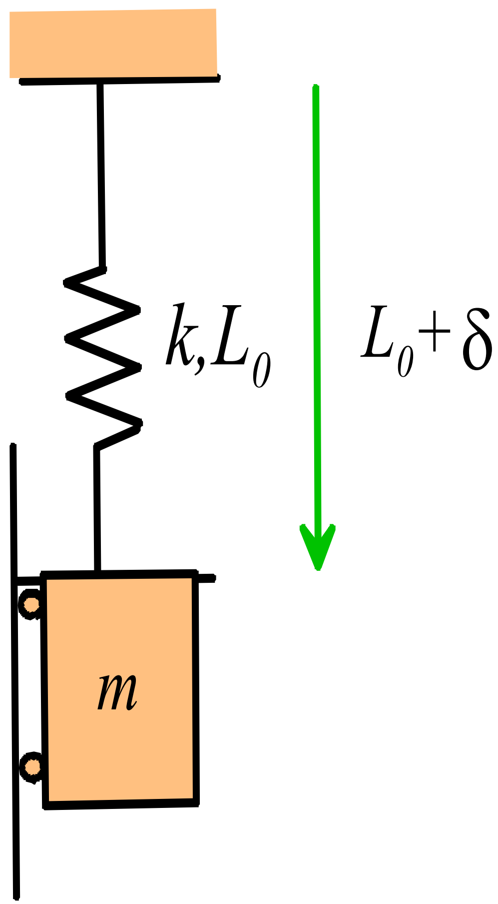
Series: stiffness $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$



Are these in series on parallel?



A useful relation



Suppose that static deflection δ (caused by earths gravity) of a system can be measured.

Then natural frequency is

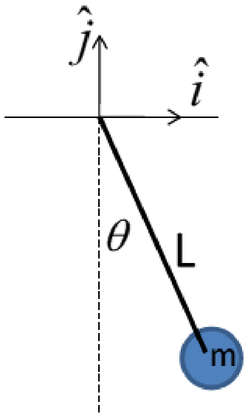
$$\omega_n = \sqrt{\frac{g}{\delta}}$$

Prove this!

Linearizing EOM

Sometimes EOM has form

$$\frac{d^2 y}{dt^2} + f(y) = C$$



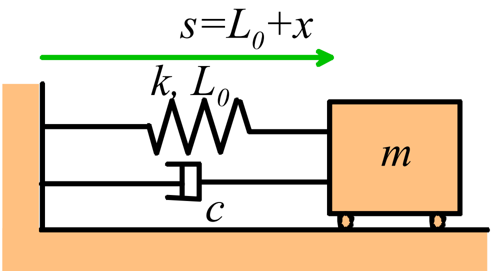
We cant solve this in general…
Instead, assume y is small

$$m\frac{d^2 y}{dt^2} + f(0) + \left.\frac{df}{dy}\right|_{y=0} y + \ldots = C$$

$$\frac{d^2 y}{dt^2} + \frac{1}{m}\left.\frac{df}{dy}\right|_{y=0} y = \frac{C - f(0)}{m}$$

There are short-cuts to doing the Taylor expansion

Writing down EOM for spring-mass systems

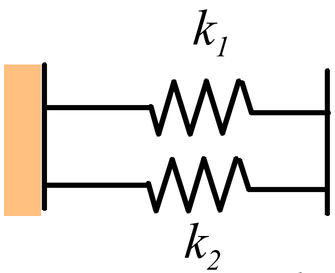


Commit this to memory! (or be able to derive it…)

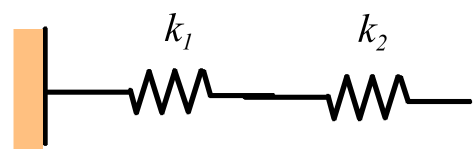
$$\mathbf{F}=m\mathbf{a} \Rightarrow \frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=0$$

$$\frac{d^2x}{dt^2}+2\zeta\omega_n\frac{dx}{dt}+\omega_n^2x=0 \qquad \omega_n=\sqrt{\frac{k}{m}} \qquad \zeta=\frac{c}{2\sqrt{km}}$$

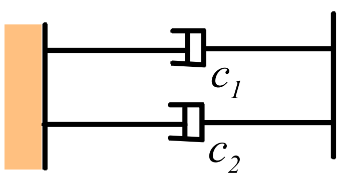
x(t) is the ‘dynamic variable’ (deflection from static equilibrium)



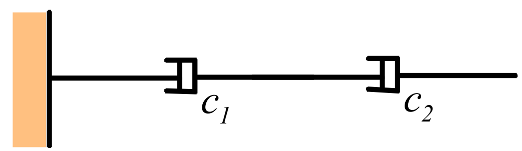
Parallel: stiffness $k=k_1+k_2$



Series: stiffness $\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$

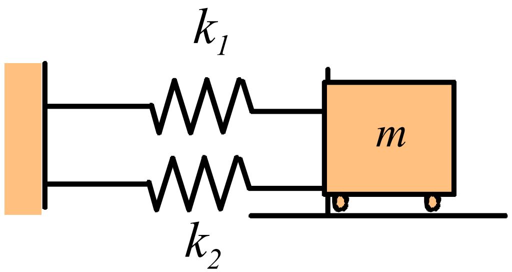


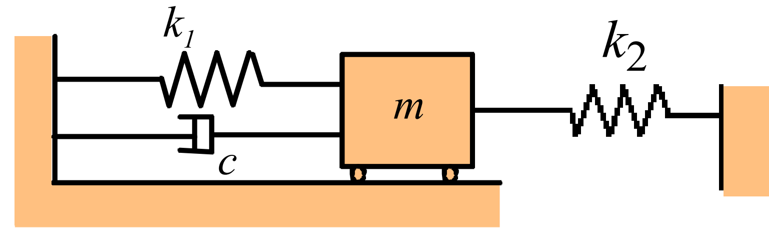
Parallel: coefficient $c=c_1+c_2$

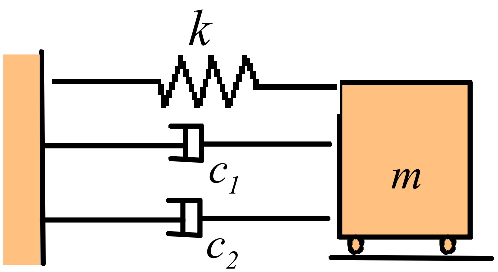


Parallel: coefficient $\frac{1}{c}=\frac{1}{c_1}+\frac{1}{c_2}$

Examples – write down EOM for





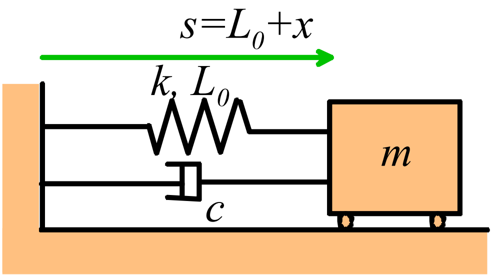


If in doubt – do F=ma, and arrange in 'standard form'

$$\mathbf{F} = m\mathbf{a} \Rightarrow \frac{d^2 y}{dt^2} + A\frac{dy}{dt} + By = C$$

$$\frac{d^2 x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{B} \qquad \zeta = \frac{A}{2\omega_n}$$

Solution to EOM for damped vibrations



$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\sqrt{km}}$$

Initial conditions: $x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$

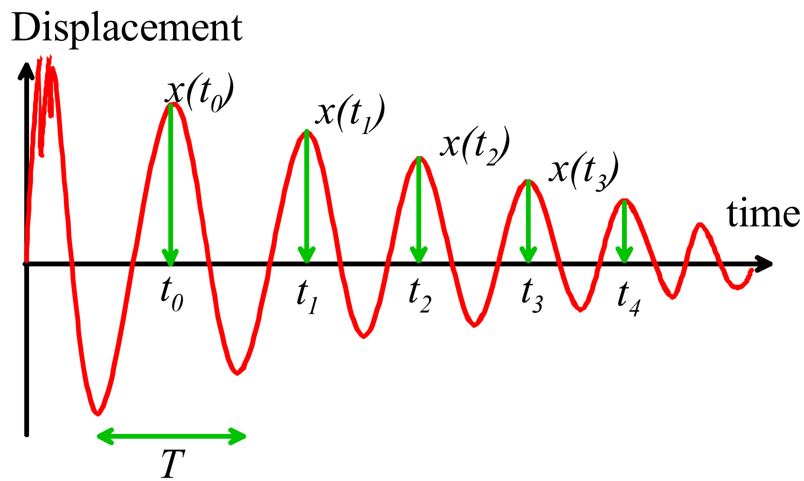
Underdamped: $\varsigma < 1$ $\quad x(t) = \exp(-\varsigma\omega_n t)\left\{ x_0 \cos\omega_d t + \frac{v_0 + \varsigma\omega_n x_0}{\omega_d} \sin\omega_d t \right\}$

Critically damped: $\varsigma = 1$ $\quad x(t) = \left\{ x_0 + \left[v_0 + \omega_n x_0 \right] t \right\} \exp(-\omega_n t)$

Overdamped: $\varsigma > 1$ $\quad x(t) = \exp(-\varsigma\omega_n t)\left\{ \frac{v_0 + (\varsigma\omega_n + \omega_d)x_0}{2\omega_d} \exp(\omega_d t) - \frac{v_0 + (\varsigma\omega_n - \omega_d)x_0}{2\omega_d} \exp(-\omega_d t) \right\}$

Critically damped gives fastest return to equilibrium

Calculating natural frequency and damping factor from a measured vibration response



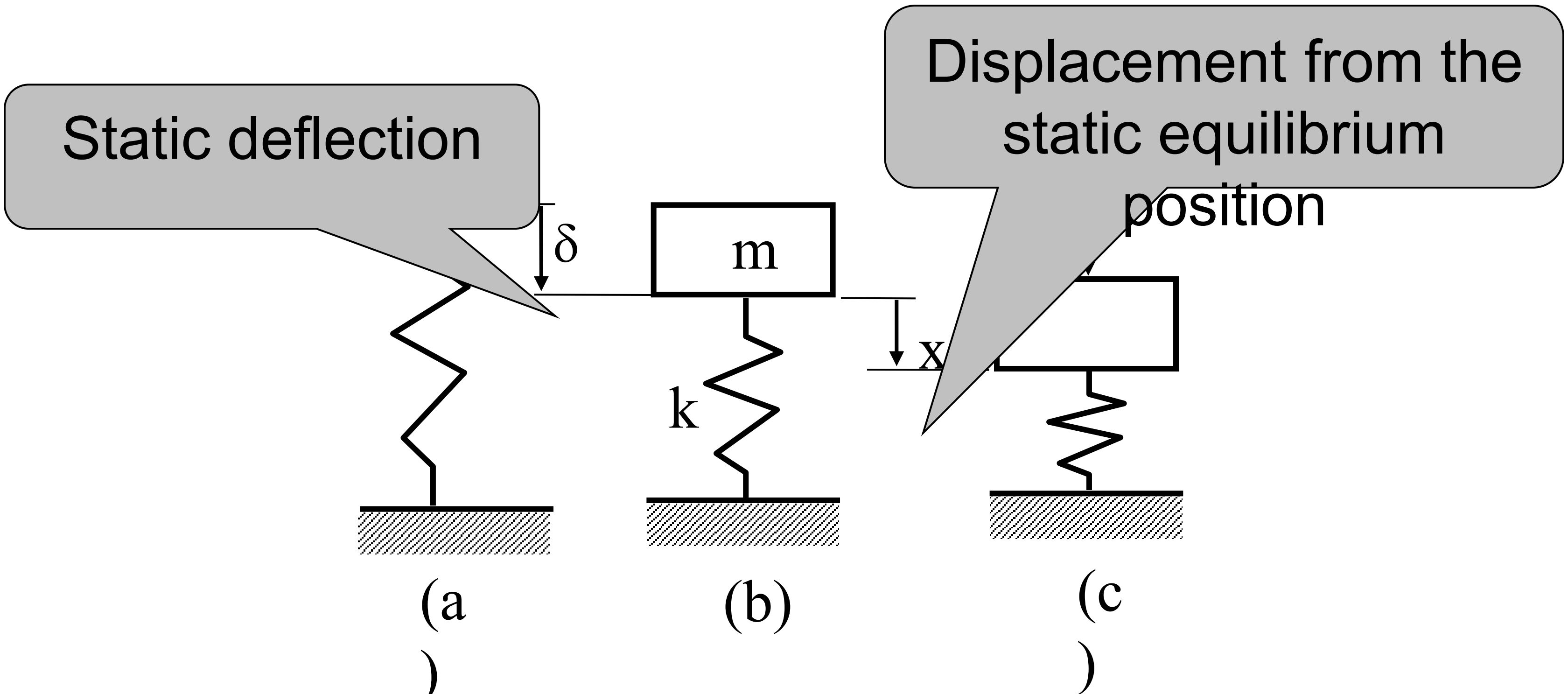
Measure log decrement: $\delta = \frac{1}{n}\log\left(\frac{x(t_0)}{x(t_n)}\right)$

Measure period: T

Then

$$\varsigma = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \qquad \omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T}$$

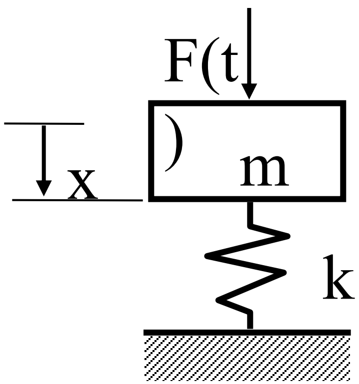
Single degree of freedom (SDOF) spring-mass system



(a) Free length of the spring; (b) spring-mass system in static equilibrium; (c) deflection from static equilibrium position

Equation of motion

$$m\ddot{x} + kx = F(t)$$



Terminology:

- t is the independent variable
- x=x(t) is the dependent variable
- This is a differential equation because it involves a derivative of x
- It is an ordinary differential equation because x is a function of t only
- This is a linear equation: if F is multiplied by 2, x will be multiplied by two also.
- This is a non-homogeneous equation since the right hand side is not zero.
- The equation is homogeneous when F=0

Free vibrations

Homogeneous equation:

$$m\ddot{x} + kx = 0$$

Assume that

$$x = A\sin\omega t + B\cos\omega t$$

$$\left(k - m\omega^2\right)\left[A\sin\omega t + B\cos\omega t\right] = 0$$

$\left[A\sin\omega t + B\cos\omega t\right] = 0 \longrightarrow$ A=B=0 No motion

$\left(k - m\omega^2\right) = 0 \longrightarrow \omega = \sqrt{\frac{k}{m}}$

Natural frequency

$$x = A\sin\omega t + B\cos\omega t$$

$$\omega = \sqrt{\frac{k}{m}}$$

Periodic functions with period 2π

$$\omega T = 2\pi$$

T= period in seconds

f = 1/T = frequency in Hertz (or cycles per second)

$\omega T = \frac{2\pi}{T} = 2\pi f$ = circular frequency in rad/s

Initial conditions

General solution to the homogeneous equation of motion

$$x = A\sin\omega t + B\cos\omega t$$

Determine the constants A and B

$$x(0) = x_o$$

$$\dot{x}(0) = v_o$$

$$x = \frac{v_o}{\omega}\sin\omega t + x_o\cos\omega t$$

Conservation of energy

Equation of motion

$$m\ddot{x} + kx = 0$$

multiply through by $\dot{x}$

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

Kinetic energy of the mass

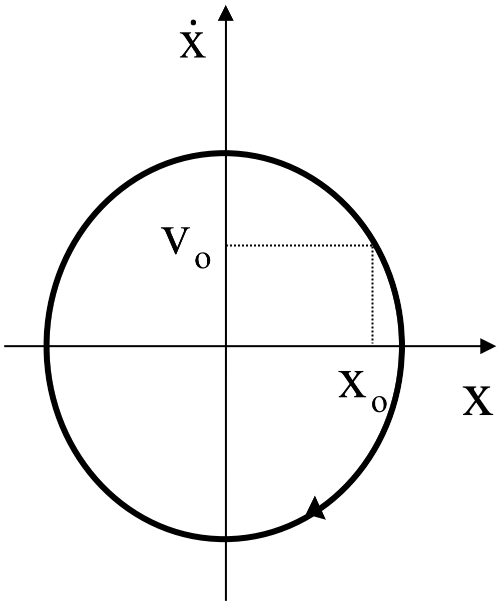
Energy stored in the spring

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}_o^2 + \frac{1}{2}kx_o^2$$

$$\frac{\dot{x}^2}{a^2} + \frac{x^2}{b^2} = 1$$

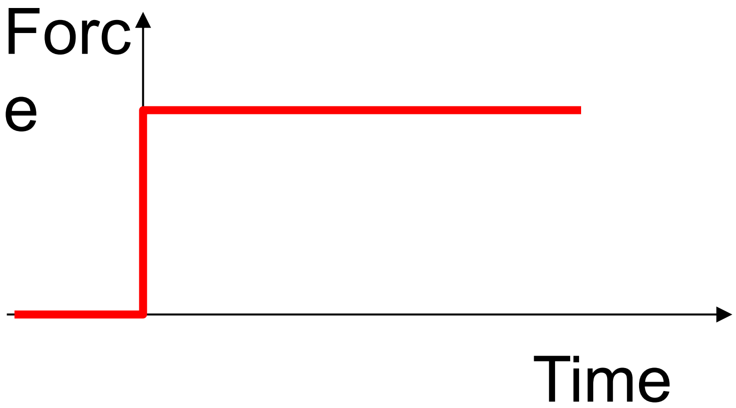
Ellipse

Phase diagram



1. Energy is conserved
2. The motion is periodic

Response of a SDOF system to a step load



F=0 for t<0

F=constant for t>0

Initial conditions: $x(0) = 0$, $\dot{x}(0) = 0$

$$m\ddot{x} + kx = F(t)$$

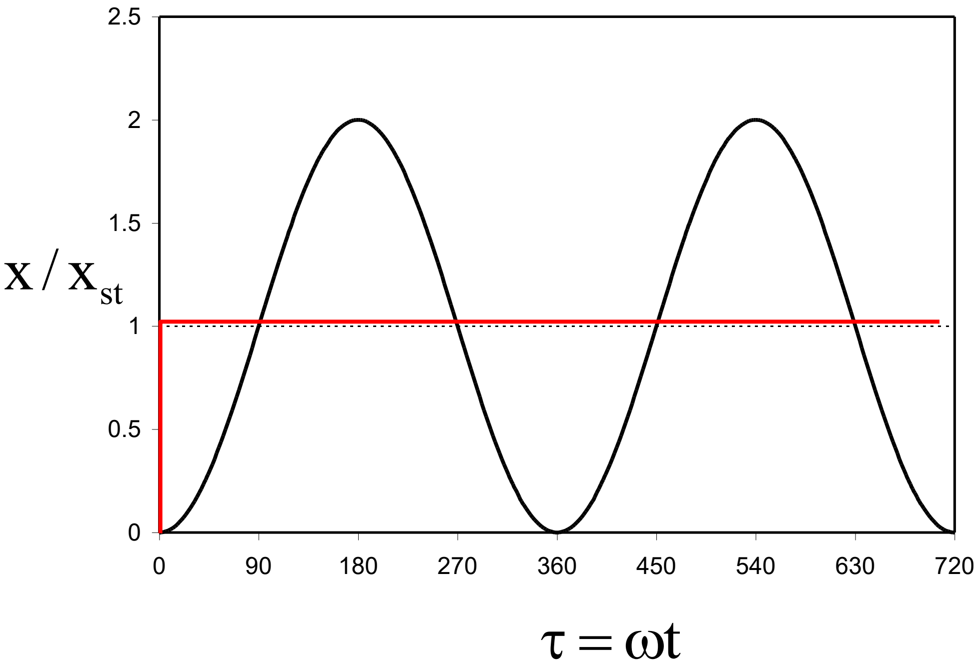
Static solution:
x=F/k

- This is called a particular solution to the equation of motion
- It satisfies the equation of motion for t>0
- It does not satisfy the initial conditions

Add the general solution to the homogeneous equation to this particular solution: $x = \frac{F}{k} + A\sin\omega t + B\cos\omega t$

Using the initial conditions to solve for the constants A and B

$$x = \frac{F}{k}\left(1 - \cos\omega t\right)$$



- Dynamic solution oscillates about the static solution
- maximum displacement is twice the static deflection
- the minimum will be zero.

Steady state forced vibration of SDOF system

$$F(t) = \tilde{F} \sin \Omega t$$

Amplitude

Forcing frequency

Assume a solution in the form $x = \tilde{X} \sin \Omega t$

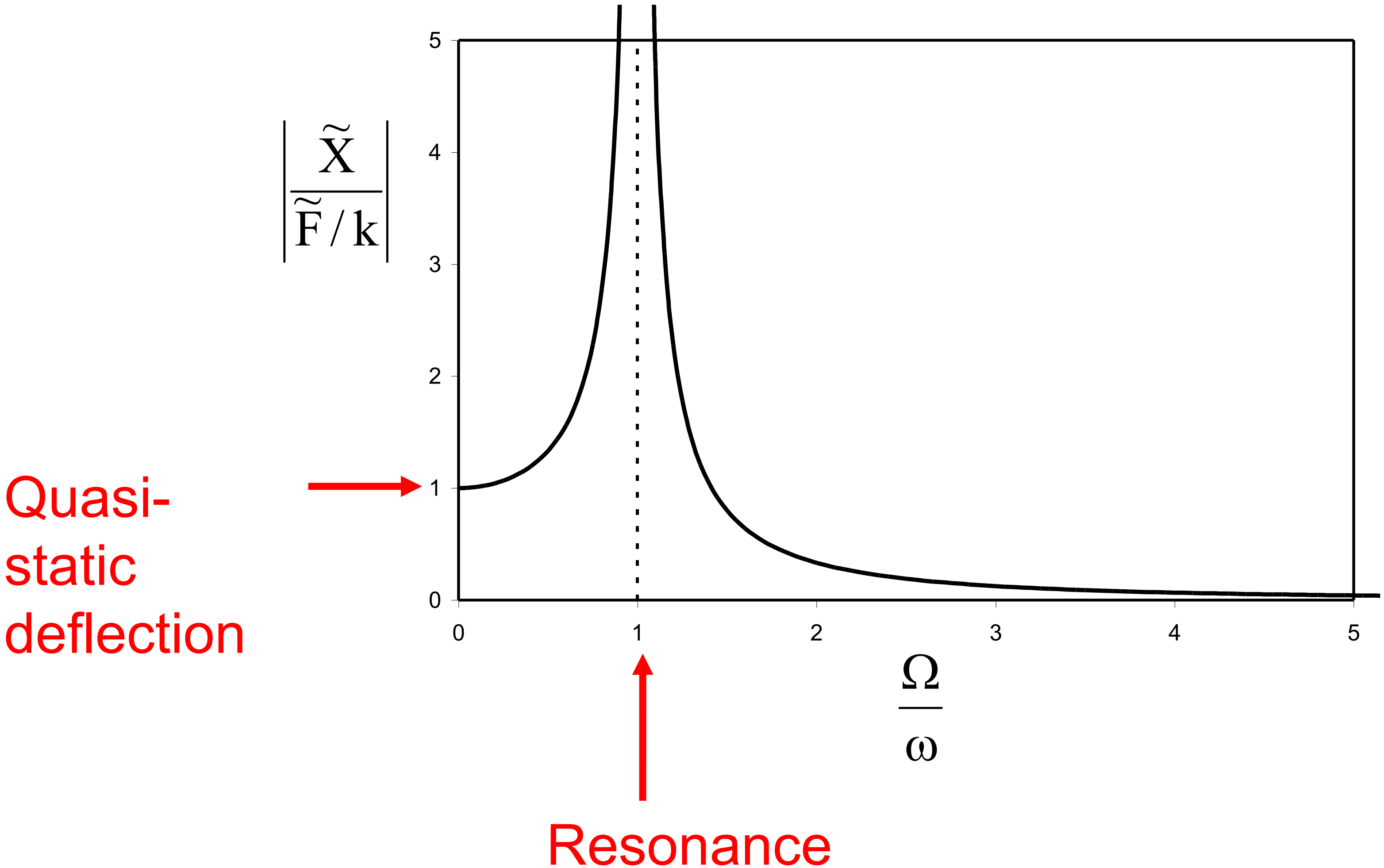
Substitute into the equation of motion $m\ddot{x} + kx = F(t)$

$$\left(k - m\Omega^2\right)\tilde{X} \sin \Omega t = \tilde{F} \sin \Omega t$$

$$\tilde{X} = \frac{\tilde{F}}{\left(k - m\Omega^2\right)} = \frac{\tilde{F}/k}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}$$

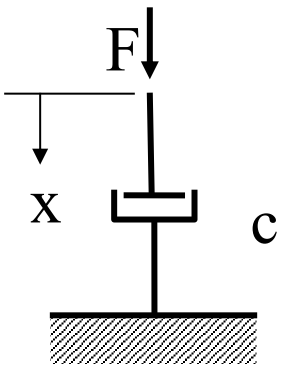
Quasi-static deflection

Denominator goes to zero



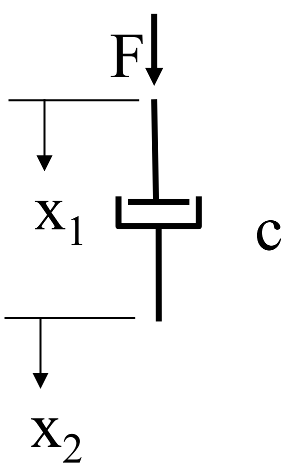
Quasi-static deflection
Resonance
$\left|\frac{\tilde{X}}{\tilde{F}/k}\right|$
$\frac{\Omega}{\omega}$
0
1
2
3
4
5

Viscous damping



$$F = c\dot{x}$$

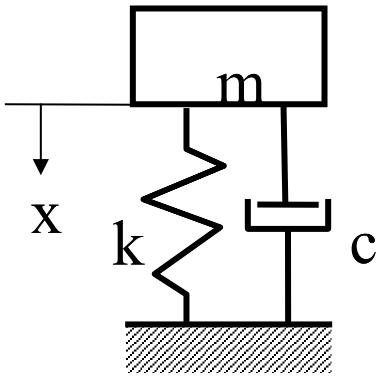
C = damping coefficient



$$F = c\left(\dot{x}_1 - \dot{x}_2\right)$$

Lord Rayleigh: Theory of sound 1877

Single degree of freedom model with damping



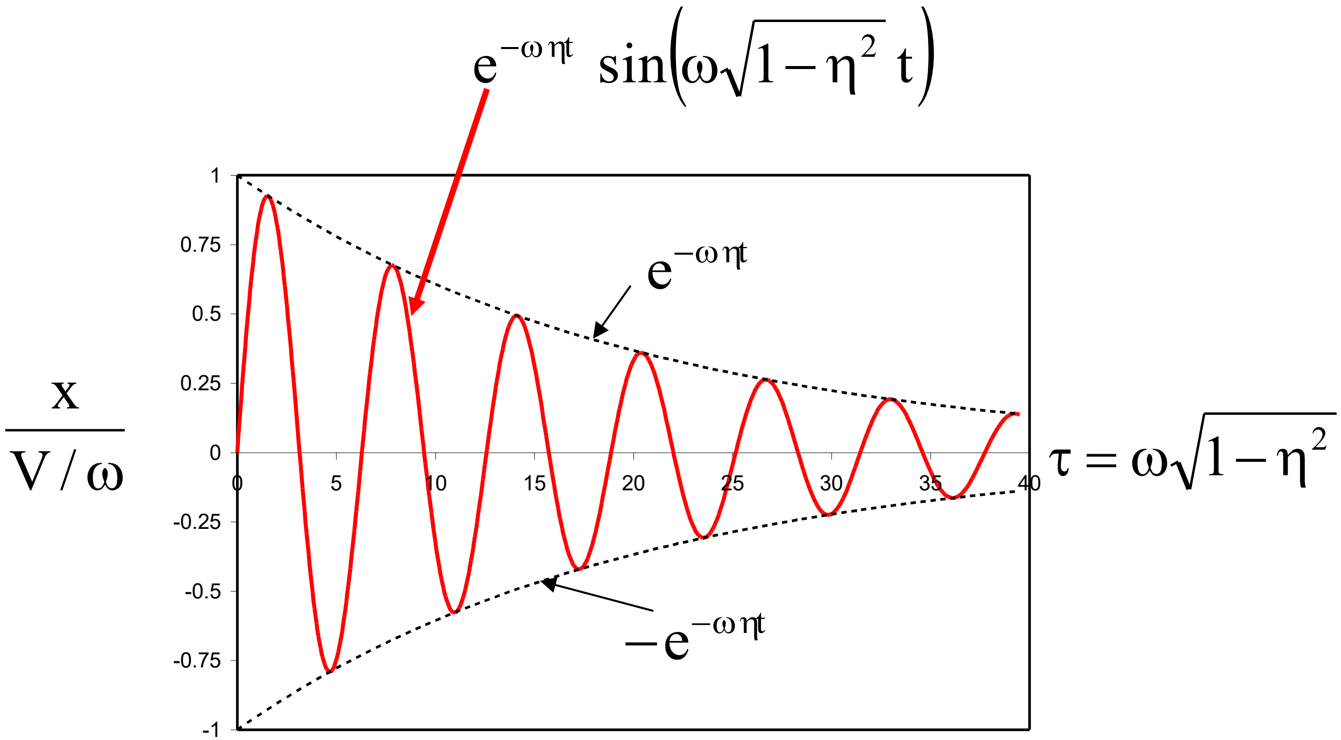
$$m\ddot{x} + c\dot{x} + kx = 0$$

Define $c/m = 2\omega\eta$ ← Damping ratio

$$\ddot{x} + 2\omega\eta\,\dot{x} + \omega^2 x = 0$$

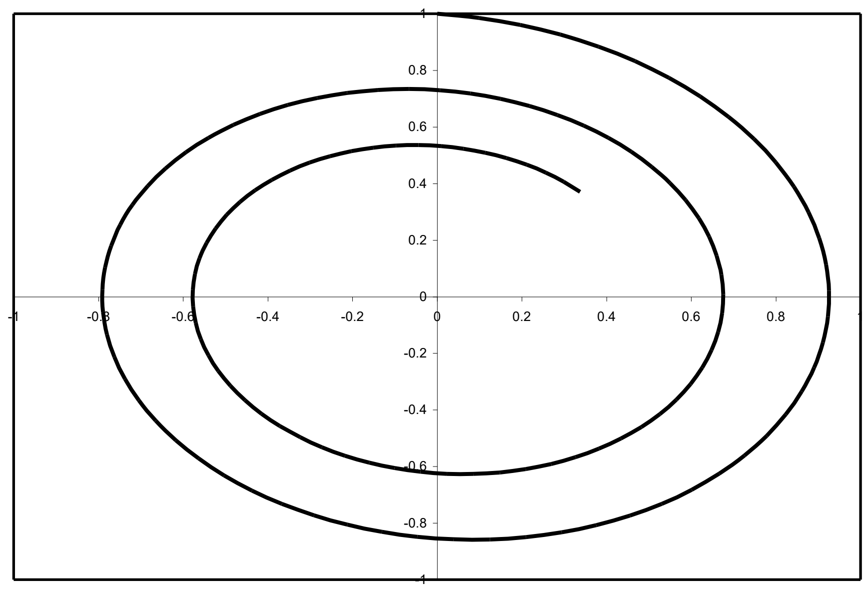
$$x = e^{-\omega\eta t}\left[A\sin\left(\omega\sqrt{1-\eta^2}\,t\right) + B\cos\left(\omega\sqrt{1-\eta^2}\,t\right)\right]$$

$x(0) = 0 \qquad \dot{x}(0) = V \longrightarrow \quad x = \dfrac{V}{\omega} e^{-\omega\eta t} \sin\left(\omega\sqrt{1-\eta^2}\,t\right)$



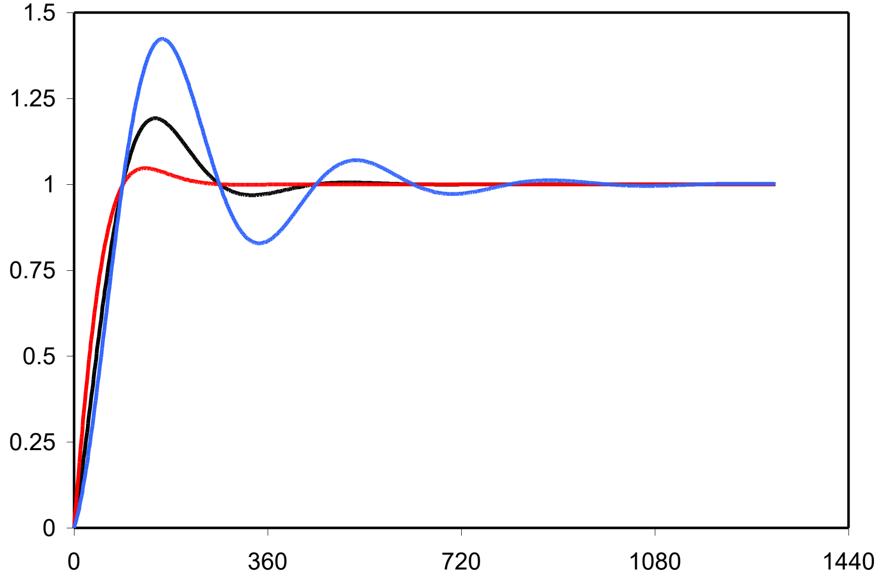
Exponential decay

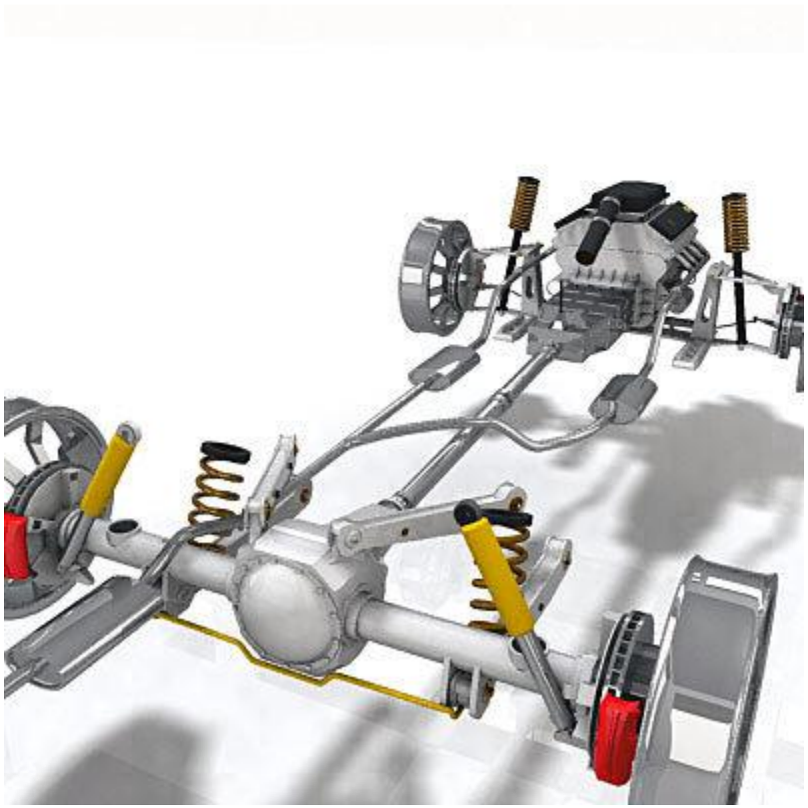
Phase diagram



Response of SDOF system with damping to a step load

$\eta =$ 0.005 (blue line), 0.01 (black line), 0.02 (red line))






Generic Picture
Gmax
Gmax