

UNIT-IV

LAMINATES

Plate stiffness and compliance

Assumptions

1. The laminate consists of perfectly bonded layers. There is no slip between the adjacent layers. In other words, it is equivalent to saying that the displacement components are continuous through the thickness.
2. Each lamina is considered to be a homogeneous layer such that its effective properties are known.
3. Each lamina is in a state of plane stress.
4. The individual lamina can be isotropic, orthotropic or transversely isotropic.
5. The laminate deforms according to the Kirchhoff - Love assumptions for bending and stretching of thin plates (as assumed in classical plate theory). The assumptions are:
 - a. The normals to the mid-plane remain straight and normal to the midplane even after deformation.
 - b. The normals to the mid-plane do not change their lengths.
6. The classical laminate theory is abbreviated as CLT. This theory is known as the classical laminated plate theory and abbreviated as CLPT.

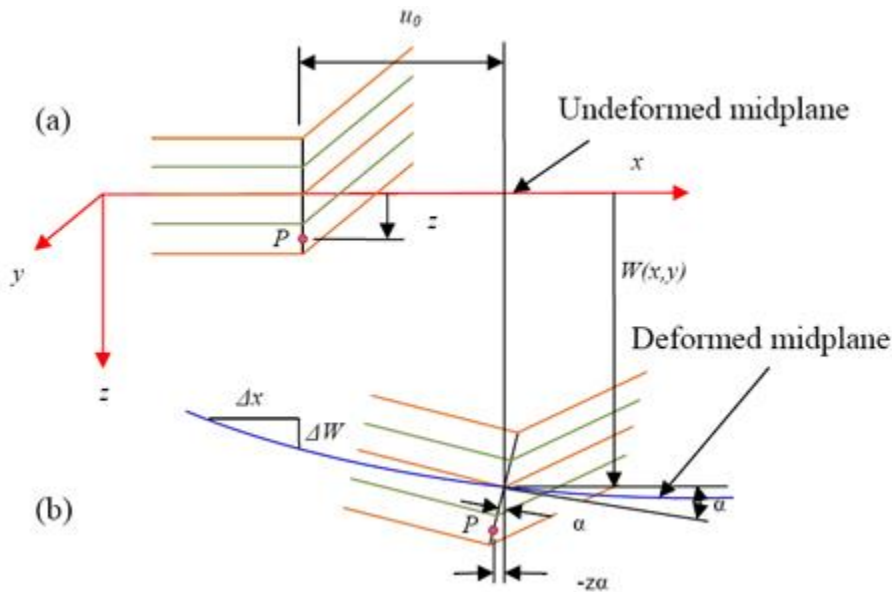


Fig: Plate deformation

$$u(x, y, z) = u_0(x, y) - z \tan \alpha = u_0(x, y) - z \alpha = u_0(x, y) - z \frac{\partial W}{\partial x}$$

Similarly, for the deformation in yz plane we can express the slope of the deformed mid-plane as. Thus, the displacement of a generic point along y axis can be given as

$$v(x, y, z) = v_0(x, y) - z \frac{\partial w}{\partial y}$$

Thus, the complete displacement field for a generic point in the laminate according to the classical laminate theory is given below:

$$\begin{aligned} u(x, y, z) &= u_0(x, y) - z \frac{\partial w}{\partial x} \\ v(x, y, z) &= v_0(x, y) - z \frac{\partial w}{\partial y} \\ w(x, y, z) &= w_0(x, y) \end{aligned}$$

From the first assumption of the Kirchhoff-Love theory that the normals remain straight and normal to mid-plane even after deformation, results into zero transverse shear strains. Thus,

$$\gamma_{xz} = \gamma_{yz} = 0$$

Using the definitions of small strain, we can write the above equation as

$$\begin{aligned} \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \end{aligned}$$

From the first of the above equation we can write

$$\frac{\partial u}{\partial z} = - \frac{\partial w}{\partial x}$$

Integrating this with respect to z , we get

$$u(x, y, z) = -z \frac{\partial w}{\partial x} + u_0(x, y)$$

Where $u_0(x, y)$ is a constant of integration which is function of x and y alone. Similarly, from the second of Equation (5.9), we can get

$$v(x, y, z) = -z \frac{\partial w}{\partial y} + v_0(x, y)$$

Strain Displacements Relations:

The strain displacement relations for infinitesimal strains using the displacement field can be given as

$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - Z \frac{\partial^2 w}{\partial x^2} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - Z \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2Z \frac{\partial^2 w}{\partial x \partial y}\end{aligned}$$

The above equation can be written as

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

or

$$\{\epsilon\}_{xy} = \{\epsilon^{(0)}\}_{xy} + z \{\kappa\}_{xy}$$

$$\{\epsilon^{(0)}\}_{xy} = \{\epsilon_{xx}^{(0)} \quad \epsilon_{yy}^{(0)} \quad \gamma_{xy}^{(0)}\}^T = \left\{ \frac{\partial u_0}{\partial x} \quad \frac{\partial v_0}{\partial y} \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right\}^T$$

$$\{\kappa\}_{xy} = \{\kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}\}^T = \left\{ -\frac{\partial^2 w}{\partial x^2} \quad -\frac{\partial^2 w}{\partial y^2} \quad -2 \frac{\partial^2 w}{\partial x \partial y} \right\}^T$$

The terms κ_{xx} and κ_{yy} are the bending moment curvatures and κ_{xy} is the twisting moment curvature.

Strain Displacements Relations:

The strain displacement relations for infinitesimal strains using the displacement field can be given as

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The above equation can be written as

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

or

$$\{\epsilon\}_{xy} = \{\epsilon^{(0)}\}_{xy} + z \{\kappa\}_{xy}$$

where $\{\epsilon^{(0)}\}_{xy} = \{\epsilon_{xx}^{(0)} \quad \epsilon_{yy}^{(0)} \quad \gamma_{xy}^{(0)}\}^T = \left\{ \frac{\partial u_0}{\partial x} \quad \frac{\partial v_0}{\partial y} \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right\}^T$ are the midplane strains

and $\{\kappa\}_{xy} = \{\kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}\}^T = \left\{ -\frac{\partial^2 w}{\partial x^2} \quad -\frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y} \right\}^T$ represents the midplane curvatures. The terms κ_{xx} and κ_{yy} are the bending moment curvatures and κ_{xy} is the twisting moment curvature.

State of Stress in a Laminate:

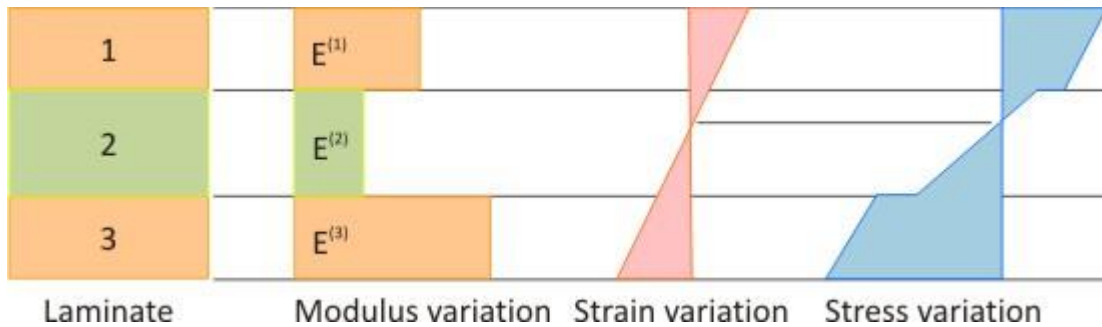
The stresses at any location can be calculated from the strains and lamina constitutive relations. It is assumed that the lamina properties are known. Hence, the constitutive equation for a k^{th} lamina is known, that is, the reduced stiffness matrices (in principal material directions and global directions) are known. Thus, the stresses in k^{th} lamina can be given as

$$\{\sigma\}_{xy}^k = [\bar{Q}]^k \{\epsilon\}_{xy}^k$$

Now, using Equation (5.13), we can write the stresses as

$$\{\sigma\}_{xy}^k = [\bar{Q}]^k \{\epsilon^{(0)}\}_{xy} + [\bar{Q}]^k z \{\kappa\}_{xy}$$

. In these equations, the strains are given at a z location where the stresses are required. It should be noted that the strains are continuous and vary linearly through the thickness. If we look at the stress distribution through the thickness it is clear that the stresses are not continuous through the thickness, because the stiffness is different for different laminae in thickness direction. In a lamina the stress varies linearly. The slope of this variation in a lamina depends upon its moduli. However, at the interface of two adjacent laminae there is a discontinuity in the stresses. The same thing is depicted in below Figure with three layers.



Elucidation of stress discontinuity at lamina interfaces in a laminate

Inplane Resultant Forces:

The inplane forces per unit length are defined as

$$N_{xx} = \int_{-H}^H \sigma_{xx} dz, \quad N_{yy} = \int_{-H}^H \sigma_{yy} dz, \quad N_{xy} = \int_{-H}^H \tau_{xy} dz \quad (5.16)$$

Or these can be written as

$$\{N\}_{xy} = \int_{-H}^H \{\sigma\}_{xy} dz$$

$$\{N\}_{xy} = \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} [\bar{Q}]^k \{\epsilon^{(0)}\}_{xy} dz + \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} [\bar{Q}]^k \{\kappa\}_{xy} z dz$$

Now recall that the midplane strains $\{\epsilon^{(0)}\}_{xy}$ and the curvatures $\{\kappa\}_{xy}$ are independent of z location. The reduced transformed stiffness matrix $[\bar{Q}]$ is function of thickness and constant over a given lamina thickness. Now we can replace the integration over the laminate thickness as sum of the integrations over individual lamina thicknesses. Thus, Equation can be written as

$$\{N\}_{xy} = \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} [\bar{Q}]^k \{\epsilon^{(0)}\}_{xy} dz + \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} [\bar{Q}]^k \{\kappa\}_{xy} z dz$$

Here, N_{Lay} is the total number of layers in the laminate. This equation can be written as

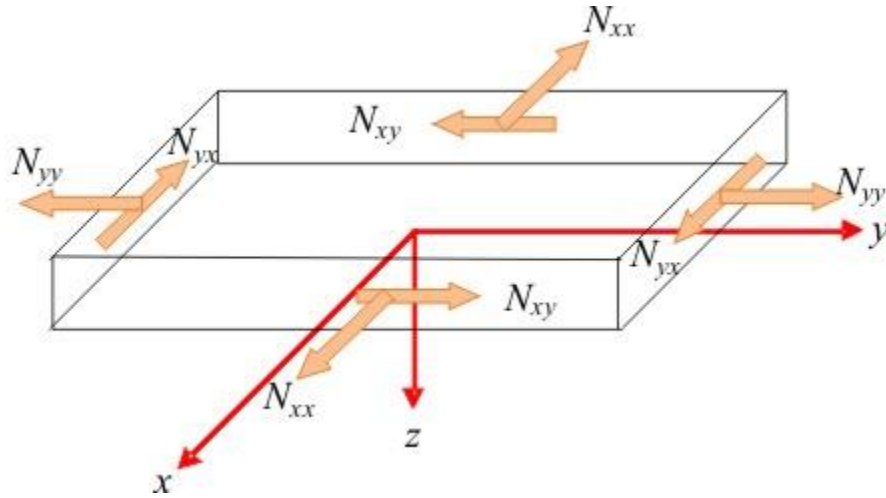
$$\{N\}_{xy} = [A]\{\epsilon^{(0)}\}_{xy} + [B]\{\kappa\}_{xy}$$

where

$$[A] = \sum_{k=1}^{N_{Lay}} [\bar{Q}]^k (z_k - z_{k-1}) \quad \text{and} \quad [B] = \frac{1}{2} \sum_{k=1}^{N_{Lay}} [\bar{Q}]^k (z_k^2 - z_{k-1}^2)$$

The matrix $[A]$ represents the in-plane stiffness, that is, it relates the in-plane forces with mid-plane strains and the matrix $[B]$ represents the bending stiffness coupling, that is, it relates the in-plane forces with mid-plane curvatures.

It should be noted that the matrices $[A]$ and $[B]$ are symmetric as the matrix $[\bar{Q}]$ is also symmetric for each lamina in the laminate. The resultant in-plane forces are shown.



In plane resultant forces per unit length on a laminate

Resultant Moments:

The resultant moments per unit length are defined as

$$M_{xx} = \int_{-H}^H \sigma_{xx} z dz, \quad M_{yy} = \int_{-H}^H \sigma_{yy} z dz, \quad M_{xy} = \int_{-H}^H \tau_{xy} z dz$$

Or these can be written as

$$\{M\}_{xy} = \int_{-H}^H \{\sigma\}_{xy} z dz$$

Now, using Equation (5.15) we can write,

$$\{M\}_{xy} = \int_{-H}^H [\bar{Q}]^k \{\epsilon^{(0)}\}_{xy} z dz + \int_{-H}^H [\bar{Q}]^k \{\kappa\}_{xy} z^2 dz$$

Now, with the same justification as given, we can write the above equation as

$$\{M\}_{xy} = \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} [\bar{Q}]^k \{\epsilon^{(0)}\}_{xy} z dz + \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} [\bar{Q}]^k \{\kappa\}_{xy} z^2 dz$$

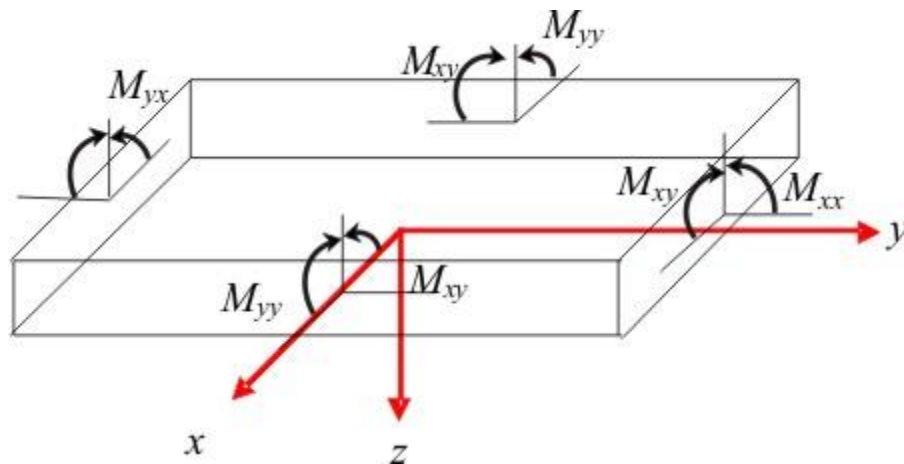
This can be written as

$$\{M\}_{xy} = [B]\{\epsilon^{(0)}\}_{xy} + [D]\{\kappa\}_{xy} \tag{5.26}$$

where

$$[D] = \frac{1}{3} \sum_{k=1}^{N_{Lay}} [\bar{Q}]^k (z_k^3 - z_{k-1}^3) \quad (5.27)$$

The matrix $[D]$ represents the bending stiffness, that is, it relates resultant moments with mid-plane curvatures. Again, the matrix $[D]$ is also symmetric. Further, it is important to note that the matrix $[B]$ relates the resultant moments with mid-plane curvatures as well.



Resultant moments per unit length on a laminate

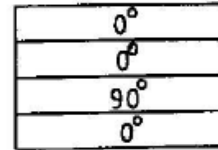
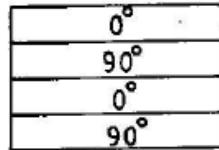
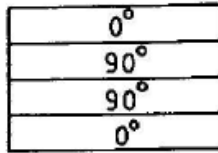
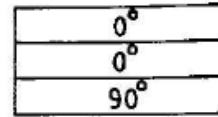
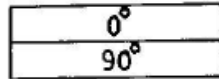
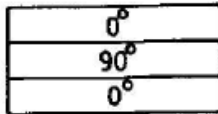
Types of Laminates:-

1. Based on Layer angle orientation

- Crossply Laminate
- Angle-ply Laminate

2. Based on layer orientation about midplane

- Symmetric Laminate
- Anti symmetric Laminate
- Un symmetric Laminate



a) Symmetric

b) Antisymmetric

c) Unsymmetric

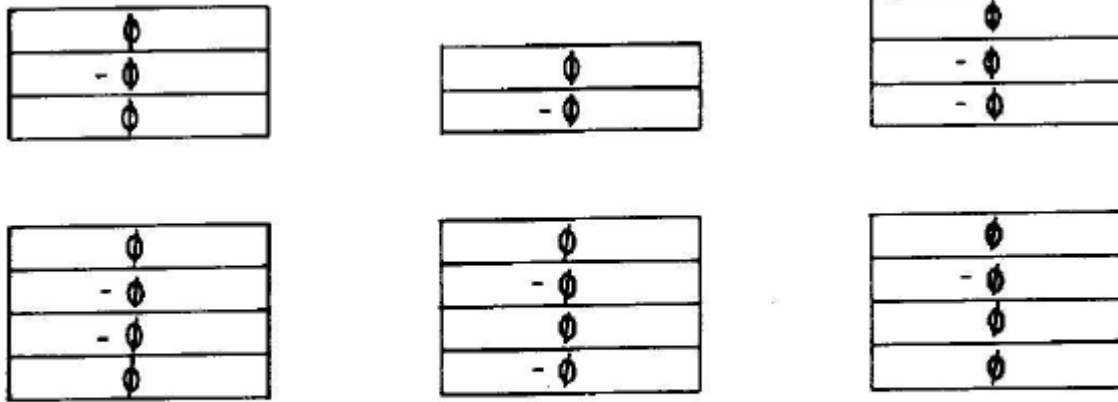
Cross-ply laminates

A laminate is called cross-ply laminate if all the plies used to fabricate the laminate are only 0° and 90°

For a cross ply laminate the terms $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$. This is because these terms involve the terms \bar{Q}_{16} and \bar{Q}_{26} which have the products of mn terms. This product is zero for any cross-ply. Thus, the terms \bar{Q}_{16} and \bar{Q}_{26} are identically zero for each ply.

Note: For a cross-ply following relations hold true. The readers should verify these relations from earlier lectures on planar constitutive relations.

$$\begin{aligned} \bar{Q}_{11}(0) &= \bar{Q}_{22}(90), & \bar{Q}_{22}(0) &= \bar{Q}_{11}(90) \\ \bar{Q}_{12}(0) &= \bar{Q}_{12}(90), & \bar{Q}_{66}(0) &= \bar{Q}_{66}(90) \end{aligned}$$



a) Symmetric

b) Antisymmetric

c) Unsymmetric

Angle ply laminates

Angle-Ply Laminates:

A laminate is called angle-ply laminate if it has plies of the same thickness and material and are oriented at $+\theta$ and $-\theta$.

For example $[45/-45/-30/30]$ is shown. For angle-ply laminates the terms $A_{16} = A_{26}$ are zero. This can be justified by that fact that \bar{Q}_{16} and \bar{Q}_{26} have the term mn . Due to this term \bar{Q}_{16} and \bar{Q}_{26} have opposite signs for layers with $+\theta$ and $-\theta$ fibre orientation. Since the thicknesses and materials of these layers are same, by the definition the terms $A_{16} = A_{26}$ are zero for the laminate.

Note: For angle-ply laminates the following relations are very useful in computing $[A]$, $[B]$ and $[D]$.

$$\begin{aligned} \bar{Q}_{11}(+\theta) &= \bar{Q}_{11}(-\theta), & \bar{Q}_{22}(+\theta) &= \bar{Q}_{22}(-\theta) \\ \bar{Q}_{12}(+\theta) &= \bar{Q}_{12}(-\theta), & \bar{Q}_{66}(+\theta) &= \bar{Q}_{66}(-\theta) \\ \bar{Q}_{16}(+\theta) &= -\bar{Q}_{16}(-\theta), & \bar{Q}_{26}(+\theta) &= -\bar{Q}_{26}(-\theta) \end{aligned}$$

Anti-symmetric Laminates:

A laminate is called anti-symmetric when the material and thickness of the plies are same above and below the mid-plane but the orientation of the plies at same distance above and below the mid-plane have opposite signs.

For example, $[45/-30/30/-45]$ is shown in Figure.

For anti-symmetric laminates the terms $A_{16} = A_{26} = D_{16} = D_{26} = 0$. The proof is left to the readers as an exercise.

Balanced Laminates:

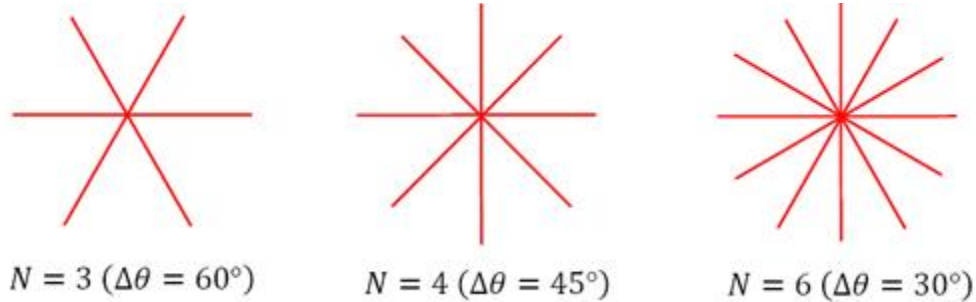
A laminate is called balanced laminate when it has pairs of plies with same thickness and material and the angles of plies are $+\theta$ and $-\theta$. However, the balanced laminate can also have layers oriented at 0° and 90° . For this laminate also $A_{16} = A_{26}$ are zero. It should be noted that angle-ply laminates are balanced laminates. For example, $[30/60/-45/-30/-60/45]$ is shown .

Quasi-Isotropic Laminates:

A laminate is called quasi-isotropic when its extensional stiffness matrix behaves like an isotropic material. This requires that $A_{11} = A_{22}$, $A_{16} = A_{26} = 0$ and $A_{66} = (A_{11} - A_{12})/2$. Further, this extensional stiffness matrix is independent of orientation of layers in laminate. This requires a laminate with $N \geq 3$ equal thickness layers and N equal angles between adjacent fibre orientations. The N equal angles, $\Delta\theta$ between the fibre orientations in this case can be given as

$$\Delta\theta = \frac{\pi}{N}$$

The quasi-isotropic laminate with this construction for $N=3, 4$ and 6 will have fibre orientations as shown .



Fibre orientations in a typical quasi-isotropic laminates

It should be noted that the isotropy in these laminates is in-plane only. The matrices B and D may not behave like an isotropic material. Hence, such laminates are quasi-isotropic in nature.

Some examples of quasi-isotropic laminate are: $[0/\pm 60]_5$, $[0/\pm 45/90]_5$

Example 1 Consider Example 5.3. Let this laminate be subjected to the forces $N_{xx} = 1000 \text{ N/mm}$, $N_{yy} = 500 \text{ N/mm}$ and $N_{xy} = 100 \text{ N/mm}$.

Calculate global strains and stresses in each ply.

Solution: The laminate in this example is a symmetric laminate. Hence, B matrix is zero. It means that there is no coupling between extension and bending actions. Thus, the applied stresses will produce only in-plane and shear strains and it will not produce any curvatures. Thus, it is easy to understand that the mid-plane strains will be the strains in each ply.

We can find the mid-plane strains as follows:

$$\begin{aligned}\{N\} &= [A]\{\epsilon^{(0)}\} + [B]\{\kappa\} \\ &= [A]\{\epsilon^{(0)}\}\end{aligned}$$

This gives

$$\{\epsilon^{(0)}\} = [A]^{-1}\{N\}$$

Thus,

$$\begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = 10^{-3} \begin{bmatrix} 0.01120 & -0.00773 & 0 \\ -0.00773 & 0.01120 & 0 \\ 0 & 0 & 0.00759 \end{bmatrix} \begin{Bmatrix} 1000 \\ 500 \\ 100 \end{Bmatrix} = 10^{-3} \begin{Bmatrix} 7.335 \\ -2.130 \\ 0.759 \end{Bmatrix}$$

The strains are same in all layers. However, the stresses in each layer will be different as their stiffnesses are different.

Stresses in $+45^\circ$ layer are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}_{(+45)} = 10^{-3} \begin{bmatrix} 42.63 & 29.43 & 28.94 \\ 29.43 & 42.63 & 28.94 \\ 28.94 & 28.94 & 32.93 \end{bmatrix} \begin{Bmatrix} 7.335 \\ -2.130 \\ 0.759 \end{Bmatrix} = \begin{Bmatrix} 0.2719 \\ 0.1471 \\ 0.1756 \end{Bmatrix} \text{ GPa}$$

And stresses in -45° layer are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}_{(-45)} = 10^{-3} \begin{bmatrix} 42.63 & 29.43 & -28.94 \\ 29.43 & 42.63 & -28.94 \\ -28.94 & -28.94 & 32.93 \end{bmatrix} \begin{Bmatrix} 7.335 \\ -2.130 \\ 0.759 \end{Bmatrix} = \begin{Bmatrix} 0.2281 \\ 0.1031 \\ -0.1256 \end{Bmatrix} \text{ GPa}$$

Now, let us find the strains and stresses in principal material directions as well for these laminae.

Let us transform the strains in $+45^\circ$ layer as

$$\{\epsilon\}_{12} = [T_2(+45)]\{\epsilon\}_{xy}$$

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1.0 & 1.0 & 0.0 \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} 0.00298 \\ 0.00222 \\ -0.00946 \end{Bmatrix}$$

Similarly, the strains in -45° layer in principal directions are

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} 0.00222 \\ 0.00298 \\ 0.00946 \end{Bmatrix}$$

Now, stresses in principal directions in $+45^\circ$ layer are

$$\{\sigma\}_{12} = [T_1(+45)]\{\sigma\}_{xy}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} 0.29728 \\ 0.12166 \\ -0.12500 \end{Bmatrix} \text{ GPa}$$

And stresses in principal material directions for -45° layer are

$$\{\sigma\}_{12} = [T_1(-45)]\{\sigma\}_{xy}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} 0.29114 \\ 0.03990 \\ 0.06250 \end{Bmatrix} \text{ GPa}$$

Hygrothermal Stresses

$$\sigma_{Al} = E_{Al} \epsilon_{Al}^M$$

$$\sigma_S = E_S \epsilon_S^M$$

$$\sigma_S = 2\sigma_{Al}$$

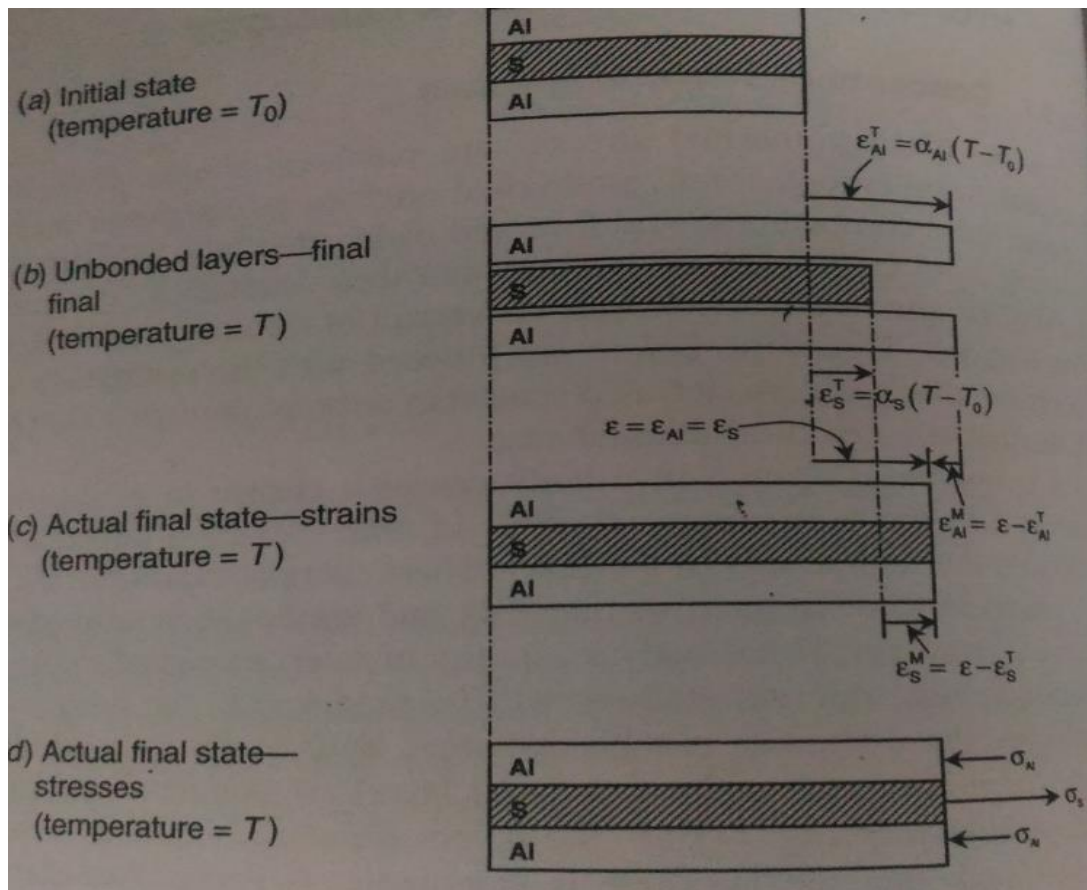


Fig: Thermal strain and stresses in a three-layered symmetric laminate

The strain due to change in temperature and moisture can be represented as

$$\epsilon^T = \alpha \Delta T$$

$$\epsilon^H = \beta \Delta C$$

Hygrothermal changes in longitudinal and transverse directions are

$$\epsilon_L^T = \alpha_L \Delta T$$

$$\epsilon_T^T = \alpha_T \Delta T$$

$$\epsilon_L^H = \beta_L \Delta C$$

$$\epsilon_T^H = \beta_T \Delta C$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} = [T_2]^{-1} \begin{Bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{Bmatrix} = [T_2]^{-1} \begin{Bmatrix} \beta_L \\ \beta_T \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{Bmatrix} = \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_{xy} \Delta T \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x^H \\ \varepsilon_y^H \\ \gamma_{xy}^H \end{Bmatrix} = \begin{Bmatrix} \beta_x \Delta C \\ \beta_y \Delta C \\ \beta_{xy} \Delta C \end{Bmatrix}$$

The mechanical strains then are given as

$$\begin{Bmatrix} \varepsilon_x^M \\ \varepsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{Bmatrix} - \begin{Bmatrix} \varepsilon_x^H \\ \varepsilon_y^H \\ \gamma_{xy}^H \end{Bmatrix}$$

Using the above equations the mechanical strain can be rewritten as

$$\begin{Bmatrix} \varepsilon_x^M \\ \varepsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 + zk_x \\ \varepsilon_y^0 + zk_y \\ \gamma_{xy}^0 + zk_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_{xy} \Delta T \end{Bmatrix} - \begin{Bmatrix} \beta_x \Delta C \\ \beta_y \Delta C \\ \beta_{xy} \Delta C \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 + zk_x - \alpha_x \Delta T - \beta_x \Delta C \\ \varepsilon_y^0 + zk_y - \alpha_y \Delta T - \beta_y \Delta C \\ \gamma_{xy}^0 + zk_{xy} - \alpha_{xy} \Delta T - \beta_{xy} \Delta C \end{Bmatrix}$$

Performing integration the resultant forces and moments are represented as below

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} + \begin{Bmatrix} N_x^H \\ N_y^H \\ N_{xy}^H \end{Bmatrix} \quad (6.63)$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} + \begin{Bmatrix} M_x^H \\ M_y^H \\ M_{xy}^H \end{Bmatrix}$$

where $\{N^T\}$, $\{M^T\}$, $\{N^H\}$, and $\{M^H\}$ are

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \Delta T \sum_{k=1}^n \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k (h_k - h_{k-1})$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = \frac{1}{2} \Delta T \sum_{k=1}^n \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k (h_k^2 - h_{k-1}^2)$$

$$\begin{Bmatrix} N_x^H \\ N_y^H \\ N_{xy}^H \end{Bmatrix} = \Delta C \sum_{k=1}^n \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{Bmatrix}_k (h_k - h_{k-1})$$

$$\begin{Bmatrix} M_x^H \\ M_y^H \\ M_{xy}^H \end{Bmatrix} = \frac{1}{2} \Delta C \sum_{k=1}^n \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{Bmatrix}_k (h_k^2 - h_{k-1}^2)$$

LAMINATE HYGROTHERMAL STRAINS

The changes in moisture concentration and temperature introduce expansional strains in each lamina. The stress-strain relation of an off-axis lamina is then modified as follows

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} + \begin{Bmatrix} \epsilon_1^e \\ \epsilon_2^e \\ \epsilon_6^e \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_1^e \\ \epsilon_2^e \\ \epsilon_6^e \end{Bmatrix} = \begin{Bmatrix} \epsilon_1^H \\ \epsilon_2^H \\ \epsilon_6^H \end{Bmatrix} + \begin{Bmatrix} \epsilon_1^T \\ \epsilon_2^T \\ \epsilon_6^T \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_1^H \\ \epsilon_2^H \\ \epsilon_6^H \end{Bmatrix} = \Delta \bar{C} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_6 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \epsilon_1^T \\ \epsilon_2^T \\ \epsilon_6^T \end{Bmatrix} = \Delta T \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_6 \end{Bmatrix}$$

where the superscripts e, H, T refer to expansion, moisture and temperature, respectively, ΔC and ΔT are the change in specific moisture concentration and temperature, respectively, and β 's and α 's are coefficients of moisture expansion and thermal expansion respectively.

Note that the spatial distributions of moisture concentration and temperature are determined from solution of moisture diffusion and heat transfer problems.

Expansional strains transform like mechanical strains i.e.,

$$\{\epsilon'\} = [T_\epsilon] \{\epsilon\}$$

Inversion of Eq. 6.56 yields (see also Eq. 6.26), at any distance z ,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_z = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_z \begin{Bmatrix} \epsilon_1 & - & \epsilon_1^e \\ \epsilon_2 & - & \epsilon_2^e \\ \epsilon_6 & - & \epsilon_6^e \end{Bmatrix}_z$$

Thus, for a general laminate

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^0 \\ \epsilon_2^0 \\ \epsilon_6^0 \\ k_1 \\ k_2 \\ k_6 \end{Bmatrix} + \begin{Bmatrix} N_1^e \\ N_2^e \\ N_6^e \\ M_1^e \\ M_2^e \\ M_6^e \end{Bmatrix}$$

where the expansional force resultants are

$$\begin{Bmatrix} N_1^e \\ N_2^e \\ N_6^e \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_z \begin{Bmatrix} \epsilon_1^e \\ \epsilon_2^e \\ \epsilon_6^e \end{Bmatrix}_z dz$$

and the expansional moments are

$$\begin{Bmatrix} M_1^e \\ M_2^e \\ M_6^e \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_z \begin{Bmatrix} \epsilon_1^e \\ \epsilon_2^e \\ \epsilon_6^e \end{Bmatrix}_z z dz$$

These expansional force resultants and moments may considerably influence the deformation behaviour of a laminate.