Unit-V

VIBRATIONS

1

When elastic bodies such as spring, a beam, & a shaft are displaced from equilibrium position by the applicatn of external forces & released, they execute a vibratory motion.

Terms:

1. Period of vibratn (or) time period: It is the time interval after which the motion is repeated itself. in 's'.

2. cycle: It is the motn completed after during one time period.

3. Frequency: It is no. of cycles described in one second. Hz.

* Types of Vibratory Motions:

1. Free (or) Natural Vibratns: When no external force acts on body, after giving it an initial displacement, then body is said to be under natural (or) free vibratn.
'Natural (or) free frequency'

2. Forced Vibratn: When the body vibrates under the influence of external force, then that vibratn is said to be forced vibratn.
natural Freq = Forced frequency Then resonance takes place.

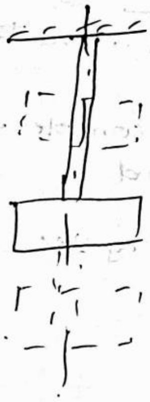
3. Damped Vibratn: When there is a reductn in amplitude over every cycle of vibratn, that vibratn is said to be Damped vibratn.

* Types of ~~Free~~ Vibrations

1. Longitudinal Vibrations
2. Transverse "
3. Torsional "

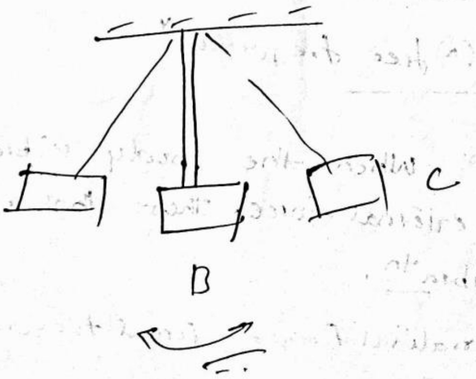
1. Longitudinal Vibrations:

When the particles of shaft (or) disc moves parallel to axis of shaft, that vibrations are known as Longitudinal.



2. Transverse Vibrations:

When the particles of shaft (or) disc move approximately $\perp$ or to axis of shaft



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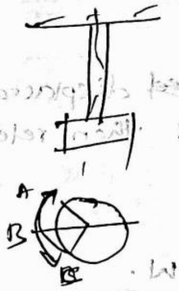
3. Torsional:

when the particles of a shaft moves in a circle about the axis of shaft.

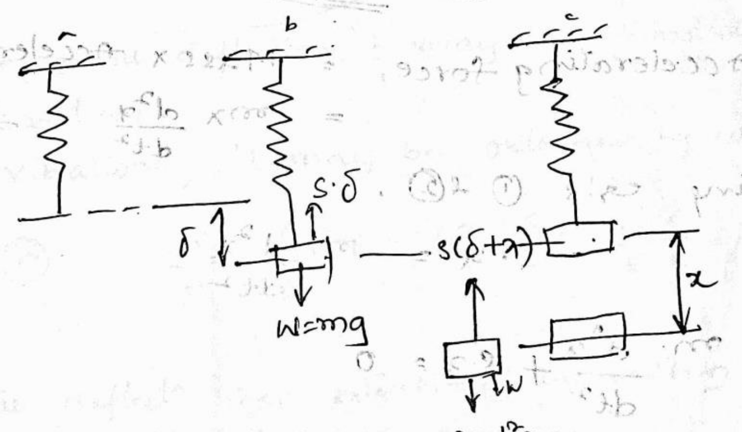


Natural frequency of Free Longitudinal Vibrations:

1. ~~Energy method~~ Equilibrium Method:

→ Accelerating force ↓

→ stiffness ↑



$mg = s\cdot\delta$

let,

$s \rightarrow$ stiffness of spring.

$m \rightarrow$ mass of body suspended from spring

$W \rightarrow$ wt. of body in N

$\delta \rightarrow$ static deflection of spring, m due to weight.

$x \rightarrow$ displacement given to body by external force.

$W = mg = s.\delta$ — (a)

In the equilibrium positn, $W = mg$;

$$\delta = \frac{mg}{s}$$

$\frac{\delta}{g} = \frac{m}{s}$

$\frac{s}{m} = g/\delta$

∴ the mass is now ~~replaced~~ displaced from its equilibrium positn by a distance of 'x' & then released after time 't'.

then Restoring force,

$\Rightarrow s(\delta + x) = W.$

$\cancel{s(\delta+x) - W}$; $W - s(\delta + x)$

$\Rightarrow W - s\delta - s.x.$

Restoring force.

$\Rightarrow \cancel{s(\delta + x) - W = 0}$; $s.\delta - s.\delta - s.x = -s.x$

$\cancel{s\delta + s.x - s\delta}$

$\cancel{\Rightarrow s.x}$ — (1)

$W - s(\delta + x)$

$- s.x$ { Upward force consider "−ve"

& Accelerating force, = Mass × Acceleration.

$$= m \times \frac{d^2x}{dt^2} \quad -(2)$$

Equating eq's (1) & (2).

$$-s.x = m.\frac{d^2x}{dt^2}$$

$$m.\frac{d^2x}{dt^2} + s.x = 0$$

$$\frac{d^2x}{dt^2} + \frac{s}{m}x = 0. \quad -(3)$$

W.K.T fundamental eq. of SHM is.

$$\frac{d^2x}{dt^2} + \omega^2 \times x = 0 \quad (4)$$

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Comparing eq (3) & (4)

$$\omega^2 = \frac{s}{m}$$

$$\omega = \sqrt{\frac{s}{m}}$$

∴ time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$

Natural frequency,

$$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\ Hz.$$

NOTE: The value of static deflection 'δ' may be found out from given conditions.

For longitudinal vibrations, it may be obtained by relatn:

$$\frac{\text{stress}}{\text{strain}} = E \quad (\text{or}) \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad (\text{or}) \quad \delta = \frac{W \cdot l}{E \cdot A}$$

$\delta \rightarrow$ static deflectn i.e. extension (or) compression.

$W \rightarrow$ load attached to free end of spring

$l \rightarrow$ length of spring

$E \rightarrow$ Young's modulus

$A \rightarrow$ c/s nal area of spring

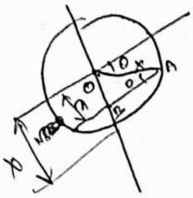
⑥

* Rayleigh's Method:

In this method, the max. K.E at mean position ($P.E = 0$) is equal to the maximum P.E at extreme position ($K.E = zero$). Assuming the motion executed by the vibration is SHM,

then, $OB = OA \sin\theta = X \sin\theta$

$$x = X \sin \omega . t \quad \text{— (a)}$$



$x \rightarrow$ displacement of body from mean position after time 't' secs.

$X \rightarrow$ max. displacement from mean position to extreme position.

Now, diff. eq (a)

$$\frac{dx}{dt} = \omega \times X \cos \omega . t$$

∵ at mean position, $t = 0$ ∴ Max. velocity at mean position,

$$v = \frac{dx}{dt} = \omega . X .$$

∴ Max. K.E at mean position,

$$= \frac{1}{2} m v^2 = \frac{1}{2} . m \omega^2 \times X^2 \quad \text{— (1)}$$

& Max. P.E at extreme position,

$$\left[\frac{0 + s . X}{2}\right] . X = \frac{1}{2} s X^2 \quad \text{— (2)}$$

∵ $P E$ = mean force × displacement

Equating eq's (1) & (2)

$$\cancel{\frac{1}{2}} m \omega^2 \cancel{X^2} = \cancel{\frac{1}{2}} s \cancel{X^2}$$

min + max.

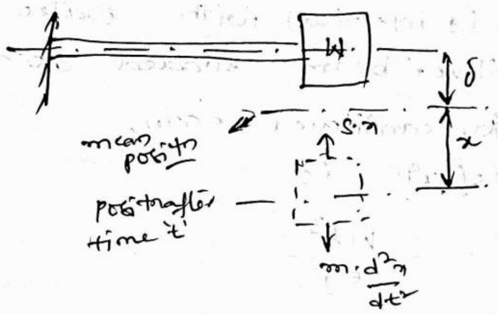
$$m \omega^2 = s$$

$$\omega = \sqrt{\frac{s}{m}}$$

∴ time period, $t_p = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{s}}$

$$f_n = \frac{1}{t_p} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

* Natural Frequency of Transverse Vibrations:



Consider a shaft of negligible mass, whose end is fixed & other end carries a body of weight W.

Let, $S \rightarrow$ stiffness of shaft.
$\delta \rightarrow$ static deflection due to weight of body
$x \rightarrow$ Displacement of body from mean position after time 't'.
$m \Rightarrow$ Mass of body $= W/g$.

W.k.t, Restoring force, $= -s \cdot x$. —ⓐ

& Accelerating force, $= m \times \frac{d^2x}{dt^2}$ —ⓑ

Equating ① & ② eq's.

$$\frac{d^2x}{dt^2} + \frac{s}{m} \cdot x = 0.$$

$$\therefore \boxed{t_p = 2\pi\sqrt{\frac{m}{s}}}$$

$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}}$$

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NOTE: The shape of the curve, in which the vibrating shaft deflects, is identical with static deflection curve of a cantilever beam loaded at end.

~~It has been~~ for cantilever beam,

static deflection is,

$$\delta = \frac{Wl^3}{3EI}$$

$W \rightarrow$ load at free end

$l \rightarrow$ length of shaft

$E \rightarrow$ Young's modulus ~~form~~ of shaft

$I \rightarrow$ Moment of inertia of shaft.

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Problem

1. A cantilever ~~beam~~ shaft of 50mm dia & 300mm long has a disc of mass 100kg at its free end. The Young's modulus for the shaft mtrl is 200GN/m². Determine the frequency of longitudinal & transverse vibration.

Sol: Given:

$d = 50mm = 0.05\ m$; $l = 300mm = 0.3m$

$m = 100kg$; $E = 200\ GN/m^2 = 200 \times 10^9\ N/m^2$.

W.K.T c/s nal area of shaft.

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times (0.05)^2 = 1.96 \times 10^{-3}\ m^2.$$

& Moment of inertia,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} (0.05)^4 = 0.3 \times 10^{-6}\ m^4.$$

Frequency of longitudinal vibration.

→ $\delta = \frac{W.l}{A.E}$; $\delta = 0.751 \times 10^{-6}\ m$.

∴ frequency of longitudinal vibration, $f_n = \frac{0.4965}{\sqrt{\delta}}$

$f_n = 575\ Hz$

→ **Frequency of transverse vibration:**

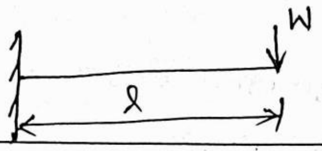
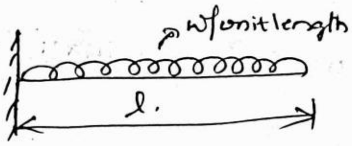
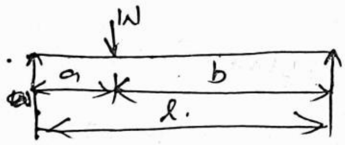
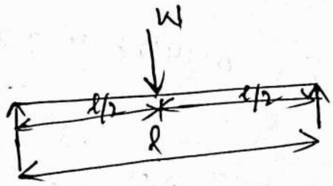
$\delta = \frac{W.l^3}{3.EI}$; $\delta = 0.147 \times 10^{-3}\ m$

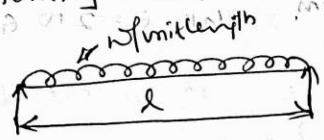
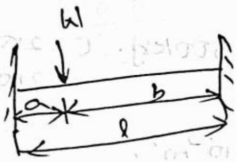
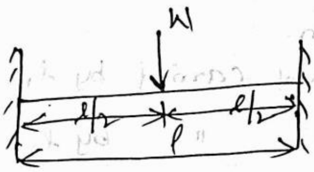
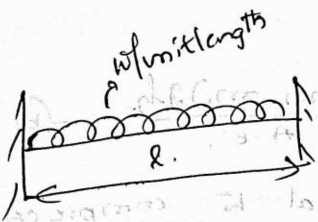
$$f_n = \frac{0.4965}{\sqrt{\delta}} = 41\ Hz$$

Pl. allow for xerox

(10)

* Values of static deflection for various types of beams & under various load conditions:

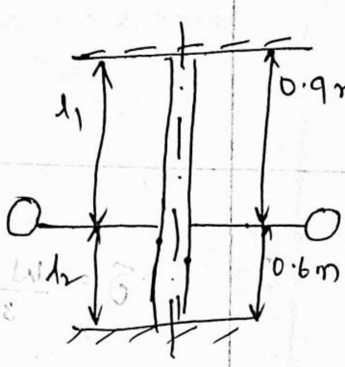
S.No.	Type of Beam	Deflection [δ]
1.	Cantilever Beam with pt. load W at free end. 	$\delta = \frac{Wl^3}{3EI}$ (at free end)
2.	Cantilever Beam with UDL w/unit length 	$\delta = \frac{wl^4}{8EI}$ (at free end)
3.	Simply supported Beam with an eccentric pt. load W. 	$\delta = \frac{Wa^2b^2}{3EIl}$ (at pt. load)
4.	Simply supported beam with central point load. 	$\delta = \frac{Wl^3}{48EI}$ (at centre

S.No	Type of Beam	Deflection [δ]
5.	Simply supported beam with uniformly distributed load 	$\delta = \frac{5}{384} \times \frac{Wl^4}{EI}$ (at centre)
6.	Fixed Beam with an eccentric pt. load W 	$\delta = \frac{W a^3 b^3}{3EI\, l^3}$ (at pt. load)
7.	Fixed Beam with central load W 	$\delta = \frac{Wl^3}{192EI}$ (at centre)
8.	Fixed Beam with UDL 	$\delta = \frac{Wl^4}{384EI}$ (at centre)

Problems

(12)

1. A flywheel is mounted on a vertical shaft as shown. The both ends of shaft is fixed & its dia is 50mm. The flywheel has a mass of 500 kg. Find the natural frequencies of longitudinal & transverse vibratn. Take $E = 210\ GN/m^2$.



Sol: Given, $d = 50mm = 0.05m$; $m = 500kg$; $E = 210 \times 10^9 N/m^2 = 210 \times 10^9 N/m^2$.

W.K.T, c/sectional area of shaft,

$A = \pi/4 d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3} m^2$.

& M.I of shaft, $I = \frac{\pi}{64} d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} m^4$.

* Natural frequency of longitudinal vibratn.

Let, $m_1 \rightarrow$ mass of the flywheel carried by l_1

$m - m_1 \rightarrow$ " " " " by l_2.

Extension of length l_1

$$= \frac{W_1 l_1}{A \cdot E} = \frac{m_1 g \cdot l_1}{A \cdot E} \quad \text{—(a)}$$

Compression of length l_2

$$= \frac{(W - W_1) l_2}{A \cdot E} = \frac{(m - m_1) g l_2}{A \cdot E} \quad \text{—(b)}$$

∵ Extension of length l_1 is equal to compression of length l_2

∴ Equating eq's (a) & (b)

$$m_1 l_1 = (m - m_1) \cdot l_2$$

$$m_1 \times 0.9 = (500 - m_1) \cdot 0.6$$

$$m_1 = 200 kg.$$

$$f_n = \frac{0.4965}{\sqrt{\delta}}$$

$$f_n = 235 Hz$$

∴ Extension of l_1

$$\delta = \frac{m_1 \cdot g \cdot l_1}{A \cdot E} = \frac{200 \times 9.81 \times 0.9}{1.96 \times 10^{-3} \times 210 \times 10^{9}}$$

$$\delta = 4.29 \times 10^{-6} m$$

* Natural frequency of transverse vibration.

$$\delta = \frac{Wa^3 b^3}{3EI l^3} = \frac{500 \times 9.81 (0.9)^3 (0.6)^3}{3 \cdot 210 \times 10^9 \times 0.307 \times 10^{-6} \times (1.5)^3}$$

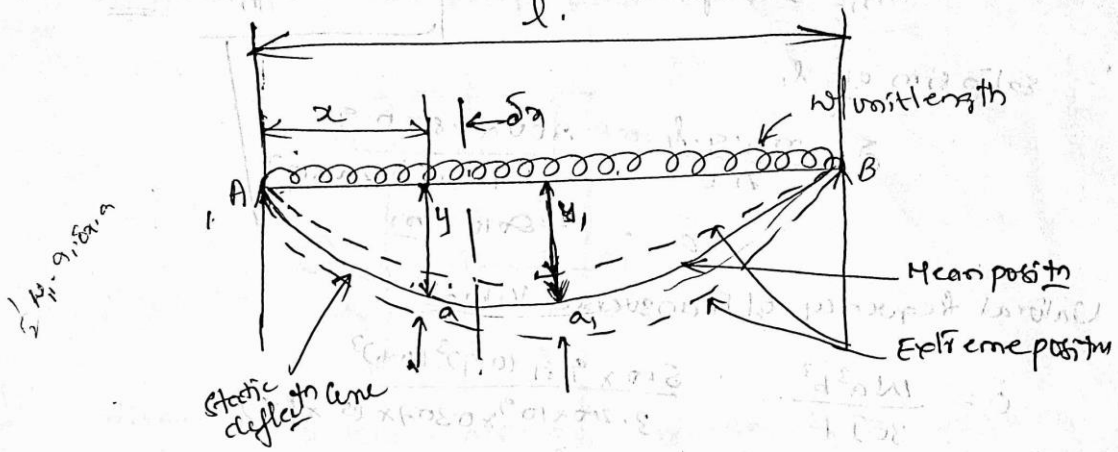
$$= 1.24 \times 10^{-3} m$$

$$f_n = \frac{0.4965}{\sqrt{\delta}} = 14.24 Hz$$

(14)

* Natural frequency of Free Transverse vibrations due to UDL acting over a simply supported shaft.

consider a shaft AB carrying UDL of w/unit length l.



Let, $y_1 \rightarrow$ static deflectn at middle of shaft

$a_1 \rightarrow$ Amplitude of vibratn at " " "

$w_1 \rightarrow$ UDL per unit static deflectn at middle of shaft $= w/y_1$

Let us consider an elementary sectn of shaft at a distance x from 'A' & length δx.

Let, $y \rightarrow$ static deflectn at distance x from 'A'

$a \rightarrow$ Amplitude of its vibratn.

$\therefore$ Work done on this small section.

$$= \frac{1}{2} \times w_1 \times a_1 \cdot \delta x \cdot a = \frac{1}{2} \cdot \frac{w}{y_1} \times a_1 \cdot \delta x \cdot a$$

$$= \frac{1}{2} \cdot w_1 \times \frac{a_1}{y_1} \cdot a \cdot \delta x$$

$\therefore$ the max. P.E at extreme positn is equal to amount of work done to move the shaft from mean position to one of its extreme positn.

$\therefore$ Max. P.E at extreme positn

$$\int_0^l \frac{1}{2} \times w_1 \times \frac{a_1}{y_1} \times a \cdot dx. \quad \text{--- (1)}$$

Assuming that the shape of curve of a vibrating shaft is similar to static deflection curve of a beam

$$\therefore \frac{a_1}{y_1} = \frac{a}{y} = \text{constant}, \ c \quad \text{(or)} \quad \frac{a_1}{y_1} = c$$

$$\& \ a_1 = y_1 c$$

sub. 'a' in eq (1).

$$\int_0^l \frac{1}{2} \cdot w \times c \cdot y \cdot c \cdot dx = \frac{1}{2} \cdot w \times c^2 \int_0^l y \cdot dx \quad \text{--- (2)}$$

$\because$ the max. velocity at mean position, $\omega \cdot a_1$, where ω is circular frequency of vibration, Therefore,

$$\text{Max. K E} = \int_0^l \frac{1}{2} \frac{w \cdot dx}{g} (\omega \cdot a)^2 = \frac{w}{2g} \times \omega^2 \cdot c^2 \int_0^l y^2 \cdot dx \quad \text{--- (3)}$$

Equating eq's (2) & (3)

$$\frac{1}{2} \cdot w \times c^2 \int_0^l y \cdot dx = \frac{w}{2g} \times \omega^2 \cdot c^2 \int_0^l y^2 \cdot dx$$

$$\therefore \omega^2 = \frac{g \int_0^l y \, dx}{\int_0^l y^2 \cdot dx} \quad \text{(or)} \quad \omega = \sqrt{\frac{g \int_0^l y \, dx}{\int_0^l y^2 \cdot dx}}$$

When shaft is simply supported, then static deflection from A to x is

$$\boxed{y = \frac{w}{24EI} \left(x^4 - 2lx^3 + l^3 x\right)}$$

Where $w \Rightarrow$ ~~on~~ UDL,

$E \Rightarrow$ Young's modulus

$I \Rightarrow$ M.I. of shaft

$$\cancel{\delta_s = \frac{5wl^4}{384EI}}$$

Natural frequency due to UDL,

$$f_n = \frac{\omega}{2\pi} = \frac{\pi^2}{2\pi}\sqrt{\frac{EIg}{wl^4}} = \frac{\pi}{2}\sqrt{\frac{EIg}{wl^4}} \quad \text{--- (1)}$$

W.K.T

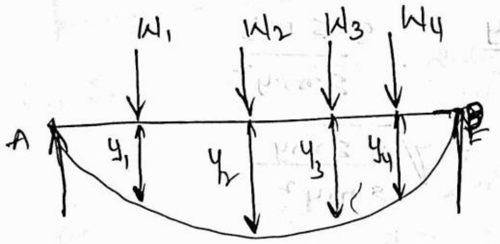
$$\delta_s = \frac{5wl^4}{384EI} \quad \Rightarrow \quad \frac{EI}{wl^4} = \frac{5}{384\delta_s}$$

$\therefore$ Eq (1) can be written as,

$$f_n = \frac{\pi}{2}\sqrt{\frac{5g}{384\delta_s}} = \frac{0.5615}{\sqrt{\delta_s}} \text{ Hz.}$$

(17)

* Natural frequency of free vibrations for a shaft subjected to no. of pt. loads:



Consider a shaft AB of negligible mass loaded with point loads, W_1, W_2, W_3, W_4 in N. let m_1, m_2, m_3 & m_4 be the corresponding masses.

(I) Rayleigh's Method:

let y_1, y_2, y_3 & y_4 ... be total deflection under loads $W_1, W_2, W_3, W_4, \ldots$

W.K.T Max. P.E

$$= \frac{1}{2} m_1 \cdot g \cdot y_1 + \frac{1}{2} m_2 g y_2 + \frac{1}{2} m_3 g y_3 + \frac{1}{2} m_4 g y_4 + \ldots$$

$$= \frac{1}{2} \Sigma m g y$$

& Min. K.E.

$$= \frac{1}{2} m_1 (\omega \cdot y_1)^2 + \frac{1}{2} m_2 (\omega \cdot y_2)^2 + \frac{1}{2} m_3 (\omega \cdot y_3)^2 + \frac{1}{2} m_4 (\omega \cdot y_4)^2 + \ldots$$

$$= \frac{1}{2} \times \omega^2 \cdot \left[m_1 \cdot y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2 + \ldots \right]$$

$$= \frac{1}{2} \cdot \omega^2 \cdot \Sigma m y^2$$

Where, $\omega \Rightarrow$ circular frequency of vibration

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Equating Max. P.E & min. K.E

$$\frac{1}{2}\omega^2 \Sigma m \cdot y^2 = \frac{1}{2}\Sigma m \cdot g \cdot y$$

$$\therefore \omega^2 = \frac{\Sigma mgy}{\Sigma my^2} = \frac{g\Sigma my}{\Sigma m \cdot y^2}$$

$$\omega = \sqrt{\frac{g\Sigma my}{\Sigma my^2}}$$

$$\therefore \cancel{f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\Sigma my^2}{g\Sigma my}}}$$

$$\therefore \boxed{f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g\Sigma my}{\Sigma my^2}}}$$

2. **Dunkerly's Method:**

The natural frequency of transverse vibratn for a shaft carrying no. of pt. loads & UDL is obtained by Dunkerly's empirical formula.

According to this,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \cdots + \frac{1}{(f_{ns})^2}$$

$f_n \rightarrow$ Natural frequency of transverse vibration of shaft carrying pt. load & UDL

$f_{n1}, f_{n2}, f_{n3} \ldots \rightarrow$ Natural frequency of T.V. of each pt. load.

$f_{ns} \rightarrow$ Natural frequency of T.V. of UDL or due to mass of shaft

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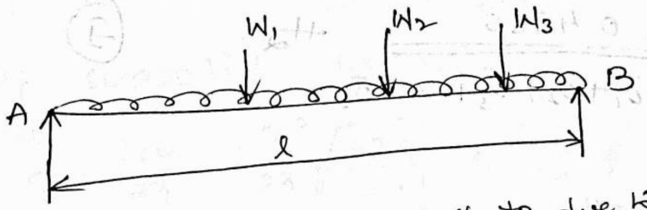

Let, $\delta_1, \delta_2, \delta_3 \ldots \Rightarrow$ static deflectn due to load $W_1, W_2, W_3, \ldots$ when considered separately

$\delta_s \rightarrow$ static deflectn due to UDL (or) due to mass of shaft.

W.K.T the natural frequency of transverse vibratn, due to load W_1;

$$\Rightarrow f_{n1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

||ly,

$$f_{n2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

$$f_{n3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

~~For sns. beam carry UDL~~

$\cancel{\delta_s = \frac{5Wl^4}{384EI}}$

$\cancel{\frac{EI}{Wl^4} = \frac{5}{384\delta_s}}$

$\cancel{f_n = \frac{\pi}{2}}$

Also Natural frequency of T.V due to UDL,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_s}} \text{ Hz}$$

Therefore, according to Dunkerly's method

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_n)^3} + \ldots + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \ldots + \frac{\delta_s}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_s}{1.27}\right]$$

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$$f_n = \frac{0.4985}{\sqrt{\delta_1+\delta_2+\delta_3+\cdots+\delta_s}} \ Hz \qquad (I)$$

NOTE:

(1). When there is no UDL (or) mass of shaft Then $\delta_s = 0$

$\therefore$ eq (I) Becomes,

$$f_n = \frac{0.4985}{\sqrt{\delta_1+\delta_2+\delta_3+\cdots}} \ Hz.$$

(2). The value of δ_1, δ_2, δ_3 etc. for a simply supported shaft may be obtained as

$$\delta = \frac{W a^2 b^2}{3EIl}.$$

$\delta \rightarrow$ static deflection due to load W,

a and b $\rightarrow$ Distances of loads from ends

$E \rightarrow$ Young's modulus for mtrl of shaft

$I \rightarrow$ Moment of inertia of shaft.

$l \rightarrow$ length of shaft

(21)

for Simply supported beam carrying UDL,

$$f_n = \frac{\omega}{2\pi} = \frac{\pi^2}{2\pi}\sqrt{\frac{EIg}{Wl^4}} = \frac{\pi}{2}\sqrt{\frac{EIg}{Wl^4}}$$

Static deflectn, $\delta_s = \frac{5Wl^4}{384EI}$

$$\cancel{\text{tat}}\ \frac{EI}{Wl^4} = \frac{5}{384\delta_s}$$

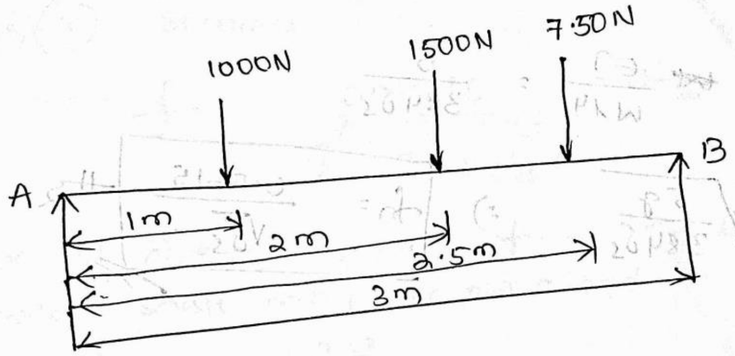
$$f_n = \frac{\pi}{2}\sqrt{\frac{5g}{384\delta_s}} \Rightarrow \boxed{f_n = \frac{0.5615}{\sqrt{\delta_s}}}\ Hz$$

Problem

22

1. A shaft 50mm diameter & 3metres long is simply supported at ends & carries three loads of 1000N, 1500N, 750N at 1m, 2m & 2.5m from left support. The Young's modulus of shaft mtl is 200 GN/m². Find the frequency of T.V.

Sol:



Given:

$d = 50mm = 0.05m$; $l = 3m$; $W_1 = 1000N$; $W_2 = 1500N$;

$W_3 = 750N$; $E = 200\ GN/m^2$

$= 200 \times 10^9\ N/m^2$;

W.K.T, M.I of shaft,

$$I = \pi/64 \times d^4 = \frac{\pi}{64} \times (0.05)^4$$

$$I = 0.307 \times 10^{-6}\ m^4.$$

& static deflection to pt. load W,

$$\delta = \frac{W a^2 b^2}{3EIl}$$

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∴ Static deflection due to 1000 N,

$$\delta_1 = \frac{1000 \times (1)^2 (2)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3}$$

$$\delta_1 = 7.24 \times 10^{-3} \, m$$

$\parallel^{ly}$,

$$\delta_2 = \frac{1000 \times (2)^2 (1)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3}$$

$$= \cancel{7.12 \times 10^{-3} \, m} \; 10.86 \times 10^{-3} \, m$$

&

$$\delta_3 = \frac{1000 \times (2.5)^2 (0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3}$$

$$= \cancel{0.41} \; 2.12 \times 10^{-3} \, m$$

W.K.T. frequency of T.V

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24 \times 10^{-3} + 10.86 \times 10^{-3} + 2.12 \times 10^{-3}}}$$

$$f_n = 3.5 \, Hz$$

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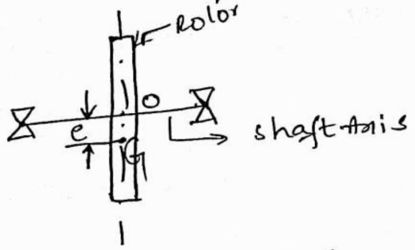
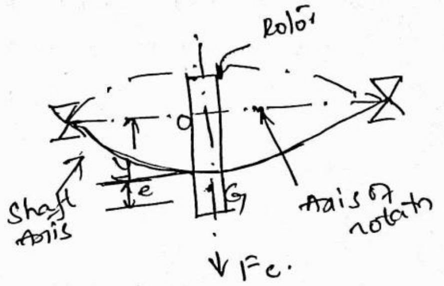
* Critical (or) Whirling Speed of a Shaft:

The speed at which the shaft runs so that the additional deflection of shaft from the axis of rotation becomes infinite, is known as critical (or) whirling speed.



When shaft is stationary

(a)



When shaft is rotating

(b).

Consider a shaft of negligible mass carrying rotor. The point 'O' is on the shaft axis & G is the C.G of rotor. When the shaft is stationary, the centre line of bearing & axis of shaft coincides. Fig (b) shows the shaft when rotating about axis of rotation at a uniform speed ω rad/sec.

Let,

$m \rightarrow$ Mass of rotor.

$e \rightarrow$ initial distance of C.G of rotor from centre line of bearing or shaft axis, when shaft is stationary

$y \rightarrow$ Additional deflection of centre of gravity of rotor when the shaft starts rotating at ω rad/sec

$s \rightarrow$ Stiffness of the shaft i.e; the load reqd. per unit deflection of shaft.

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Since the shaft is rotating at ω; ∴ centrifugal force acting radially outwards through G causing the shaft to deflect

$$F_c = m \cdot \omega^2 (y+e)$$

The shaft behaves like a spring, ∴ The force resisting the deflection y,

$$= s \cdot y$$

for equilibrium,

$$m \cdot \omega^2 (y+e) = s \cdot y$$

$$m \cdot \omega^2 y + m\omega^2 \cdot e = s \cdot y$$

$$y(s - m\omega^2) = m\omega^2 \cdot e$$

$$y = \frac{m \cdot \omega^2 \cdot e}{(s - m\omega^2)} = \frac{\omega^2 \cdot e}{\left[\frac{s}{m} - \omega^2\right]}$$

W.K.T, circular frequency

$$\omega_n = \sqrt{\frac{s}{m}}$$

$$\therefore y = \frac{\omega^2 \cdot e}{(\omega_n^2 - \omega^2)}$$

A little consideration show that when $\omega > \omega_n$, the value y will be negative, then shaft rotates in opposite dirn.

~~In order to have the value of y always +ve.~~

$$y = \pm \frac{\omega^2 \cdot e}{(\omega_n)^2 - (\omega^2)} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

$$\omega_n = \omega_c$$

$\omega_c \rightarrow$ critical (or) whirling speed.

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \ Hz$$

~~If ω_c is~~

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If N_c is the critical (or) whirling speed,

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad (or) \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$N_c = \frac{0.4985}{\sqrt{\delta}} \text{ rps.}$$

Natural frequency,

$$f_n = \frac{1}{t_p} = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$= \frac{0.4985}{\sqrt{\delta}}.$$

Hence, the critical (or) whirling speed is ~~the~~ same as natural frequency of transverse vibratn. unit: "rps"

NOTE:

1. When the c.G of the rotor lies b/w centre line of shaft & centre line of bearing, 'e' is taken -ve. On the other hand, if the c.G of rotor does not lie b/w the centre line of shaft & centre line of bearing. the value 'e' is +ve can be taken.

2. To determine the critical speed of a shaft which may be subjected to pt. loads, UDL (or) combinatn of both, find natural frequency of T.V which is equal to critical speed of a shaft in r.p.s. The D.M (Dunkerley's method) may be used for calculatg frequency.

3. A shaft supported is short bearings (or) ball bearings is assumed to simply supported shaft while shaft supported in long bearings is assumed to have both ends fixed.

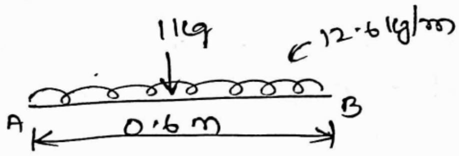
Problem (98)

1. calculate the whirling speed of a shaft 20mm dia. and 0.6m long carrying a mass of 1kg at its midpt. The density of shaft mtl is 40 Mg/m^3, Young's modulus 200 GN/m^2. Assume shaft to be freely supported.

Sol: Given:

$d = 20mm = 0.02m$; $l = 0.6m$; $m_1 = 1kg$; $\rho = 40\ Mg/m^3$
$= 40\times10^6\ g/m^3$.
$= 40\times10^3\ kg/m^3$.

$E = 200\ GN/m^2 = 200\times10^9\ N/m^2$.



W.K.T,

M.I, $I = \frac{\pi}{64}\times d^4 = \pi/64\,(0.02)^4$
$= 7.855\times10^{-9}\ m^4$

since the density of shaft mtl is 40×10^3 kg/m^3.

$\therefore$ Mass of shaft/mtr length

m_s = Area x length x density
$= \frac{\pi}{4}(0.02)^2\times(0.6)\times 40\times10^3 = 12.6$ kg/m

W.K.T; static deflectn due to 1kg of mass at centre,

$$\delta = \frac{Wl^3}{48EI} = \frac{1\times9.81\times(0.6)^3}{48\times200\times10^9\times7.855\times10^{-9}} = \underline{28\times10^{-6}\ m}$$

& static deflectn due to mass of shaft,

$$\delta_s = \frac{5Wl^4}{384EI} = \frac{5\times12.6\times9.81\,(0.6)^4}{384\times(200\times10^9)\times7.855\times10^{-9}}$$

$$= \underline{0.133\times10^{-3}\ m}$$

(29)

∴ Frequency of Transverse Vibratn.

$$f_n = \frac{0.4985}{\sqrt{\delta + \frac{\delta_s}{1.27}}}$$

$$= \frac{0.4985}{\sqrt{28\times10^{-6} + \frac{0.133\times10^{-3}}{1.27}}}$$

$f_n = 43.3\ Hz$

Let, $N_c \rightarrow$ whirling Speed.

∴ $N_c = 43.3$ r.p.s

$= 43.3 \times 60 = 2598$ rpm.

---

(2) A shaft 1.5m long, supported in flexible bearings at the ends carries two wheels each of 50kg mass. One wheel is situated at the centre of the shaft and other at a distance of 375mm from centre towards left. The shaft is hollow of external dia. 75mm & internal diameter 40mm. The density of shaft mtrl is 7700 kg/m³ & its modulus of elasticity is 210 GN/m². Find the lowest whirling speed of shaft, taking in to account the mass of shaft.

Sol: $l = 1.5m$; $m_1 = m_2 = 50kg$; $d_1 = 75mm = 0.075m$; $d_2 = 40mm = 0.04m$;

$\rho = 7700\ kg/m^3$; $E = 210\times10^9\ N/m^2$

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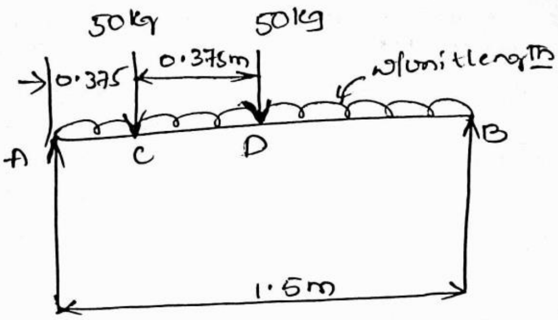

W.K.T. M.I of shaft, $I = \frac{\pi}{64}\left[d_1^4 - d_2^4\right]$

$$= \frac{\pi}{64}\left[(0.075)^4 - (0.04)^4\right] = 1.4\times10^{-6}\,m^4.$$

since the density of shaft 7700 kg/m³

∴ Mass per unit ~~inche~~ length:

$$m_s = \text{Area} \times \text{length} \times \text{density}$$

$$= \frac{\pi}{4}\left[(0.075)^2 - (0.04)^2\right]\times 1 \times 7700$$

$$m_s = 24.34\ kg/m$$

W.K.T, static deflectn due to load W,

$$= \frac{Wa^2b^2}{3EIl} = \frac{m\cdot g\cdot a^2b^2}{3EIl}$$

δ_1 at 'C' $= \frac{m_1\, g\cdot a^2b^2}{3EIl} = \frac{50\times9.81\times(0.375)^2\times(1.125)^2}{3\times210\times10^7\times1.4\times10^{-6}\times1.5}$

$$\delta_1 = 70\times10^{-6}\,m$$

δ_2 at D : $\frac{m_2\, g\, a^2b^2}{3EIl} = \frac{50\times9.81\times(0.75)^2(0.75)^2}{3\times200\times10^7\times1.4\times10^{-6}\times1.5}$

$$\delta_2 = 123\times10^{-6}\,m$$

static deflectn due to UDL,

$$\delta_s = \frac{5}{384}\times\frac{Wl^4}{EI} = \frac{5}{384}\times\frac{24.34\times9.81\times(1.5)^4}{210\times10^7\times1.4\times10^{-6}} = 56\times10^{-6}\,m$$

W.K.T for T.V $f = \frac{0.4985}{\sqrt{\delta_1+\delta_2+\frac{\delta_s}{1.27}}} = \frac{0.4985}{\sqrt{70\times10^{-6}+123\times10^{-6}+\frac{56\times10^{-6}}{1.27}}}$

$$= 32.4\ Hz$$

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9. A vertical shaft of 5mm dia is 200mm long & it is supported in long bearings at its ends. A disc of mass 50kg is attached to centre of shaft. Neglecting any increase in stiffness due to attachment of disc to shaft, find the critical speed of rotation & max. bending stress when shaft is rotating at 75% of critical speed. The centre of disc is 0.25 mm from geometric axis of shaft. $E = 200\, GN/m^2$.

Sol: Given:

$d = 5mm = 0.005\,m$; $l = 200mm = 0.2m$; $m = 50kg$;

$e = 0.25mm = 0.25\times10^{-3}\,m$ | $E = 200\,GN/m^2 = 200\times10^{9}\,N/m^2$.

Critical speed of Rotation:

W.K.T, M.I of shaft,

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64}(0.005)^4 = 30.7\times10^{-12}\,m^4$$

Since the shaft is supported in long bearings, it is assumed to be fixed at both ends.

W.K.T, static deflection at centre of shaft ~~which~~ is fixed at both ends,

$$\delta = \frac{Wl^3}{192EI} = \frac{(9.81)(50)(0.2)^3}{192\times200\times10^{9}\times30.7\times10^{-12}}$$

$$\delta = 3.33\times10^{-3}\,m$$

$$N_c = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{3.33\times10^{-3}}} \Rightarrow N_c = 8.64\,rps \Rightarrow 518.4\,rpm$$

(37)

* Maximum Bending stress:

Let, $\sigma \rightarrow$ Max. Bending stress
$N \rightarrow$ speed of shaft
$= 75\%$ of N_c
$= 0.75 N_c$

When the shaft starts rotating, the additional dynamic load (W_1) to which the shaft is subjected may be obtained by,

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{(or)} \quad M = \frac{\sigma \cdot I}{y_1} \quad \text{—(a)}.$$

W.K.T, a shaft fixed at both ends & carrying pt. load (W_1) at centre the max. bending moment,

$$M = \frac{W_1 \cdot l}{8} \quad \text{—(b)}$$

Equating eq's (a) & (b)

$$\frac{\sigma \cdot I}{y_1} = \frac{W_1 \cdot l}{8} \qquad \because (y_1 = d/2)$$

$$W_1 = \frac{\sigma \times 30.7 \times 10^{-12} \times 8}{\left(\frac{0.005}{2}\right) \times 0.2} \Rightarrow W_1 = 0.49 \times 10^{-6} \sigma . N$$

$\therefore$ Additional deflectn due to load W_1,

$$y = \frac{W_1}{W} \times \delta = \frac{0.49 \times 10^{-6} \cdot \sigma}{50 \times 9.81} \times 3.33 \times 10^{-3}$$

$$= 3.327 \times 10^{-12} \sigma .$$

W.K.T,

$$y = \frac{\pm e}{\left[\frac{\omega_c}{\omega}\right]^2 - 1} = \frac{\pm e}{\left[\frac{N_c}{N}\right]^2 - 1}$$

$$3.327 \times 10^{-12} \sigma = \frac{\pm 0.25 \times 10^{-3}}{\left[\frac{N_c}{0.75 N_c}\right]^2 - 1} = \pm 0.32 \times 10^{-3}$$

$$\sigma = 0.0962 \times 10^{9} \ N/m^2$$

$$= 96.2 \times 10^{6} \ N/m^2$$

$$\boxed{\sigma = 96.2 \times M \ N/m^2}$$

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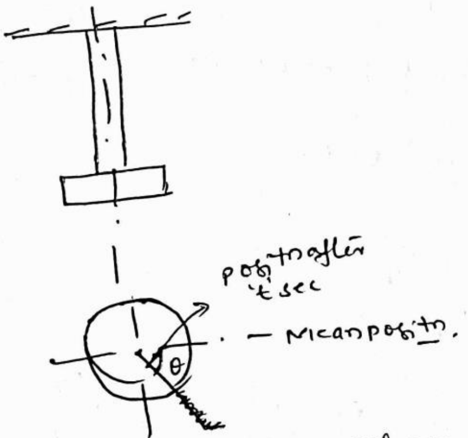
* **Torsional Vibrations:**

When the particles of a shaft or disc move in circle about axis of shaft, then the vibrations are known as "Torsional Vibrations"

* **Natural Frequency of Free Torsional Vibratns:**



Consider a shaft of negligible mass whose one end is fixed & other end carrying disc.

Let, $\theta \rightarrow$ Angular displacement of shaft from mean positn after 't' sec's

$m \rightarrow$ mass of disc, kg

$I \rightarrow$ Mass M.I of disc, kg-m²

$k \rightarrow$ radius of gyratn, m

$q \rightarrow$ Torsional stiffness of shaft in N-m

$\therefore$ Restoring force, $= q \cdot \theta$ —(i)

& accelerating force, $= I \times \dfrac{d^2\theta}{dt^2}$ —— (ii)

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Equating eq's (i) & (ii)

$$I \cdot \frac{d^2\theta}{dt^2} = -q \cdot \theta$$

$$I \cdot \frac{d^2\theta}{dt^2} + q \cdot \theta = 0.$$

$$\frac{d^2\theta}{dt^2} + \frac{q}{I} \cdot \theta = 0.$$

From fundamental eq of SHM

$$\frac{d^2\theta}{dt^2} + \omega^2 \cdot x = 0.$$

$$\omega^2 = \frac{q}{I}$$

$$\omega = \sqrt{q/I}$$

$\therefore$ time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{q}}$

Natural frequency, $\boxed{f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{q}{I}}}$

NOTE: The value of torsional stiffness 'q' may be obtained from torsion equatn.

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad (\text{or}) \quad \frac{T}{\theta} = \frac{C \cdot J}{l}.$$

$$\boxed{q = \frac{C \cdot J}{l}} \qquad \therefore \boxed{\frac{T}{\theta} = q}$$

where, $C \rightarrow$ Modulus of rigidity for shaft mtl.

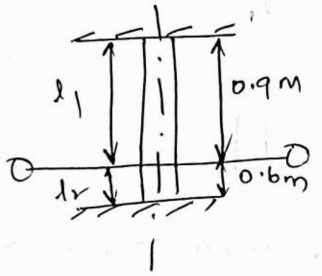
$J \rightarrow$ polar m.I of shaft c/s.

$= \frac{\pi}{32} d^4$; $d \rightarrow$ dia of shaft.

$l \rightarrow$ length of shaft.

Problem (35)

1. A flywheel is mounted on a vertical shaft as shown. The both ends of a shaft are fixed & its diameter is 50mm. The flywheel has a mass of 500kg & its radius of gyration is 0.5m. Find the natural frequency of torsional vibrations, if modulus of rigidity is 80 GN/m².



Sol: Given:

$d = 50mm = 0.05m$ | $m = 500kg$ | $k = 0.5m$, $G = 80\ GN/m^2 = 80\times10^9\ N/m^2$.

W.K.T, Polar moment of inertia,

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32}(0.05)^4 = 0.6\times10^{-6}\ m^4$$

∴ Torsional stiffness, for length l_1,

$$q_1 = \frac{C\cdot J}{l_1} = \frac{80\times10^9 \times 0.6\times10^{-6}}{0.9}$$

$$q_1 = 56\times10^3\ N\text{-}m$$

$$q_2 = \frac{C\cdot J}{l_2} = \frac{80\times10^9\times0.6\times10^{-6}}{0.6} = 84\times10^3\ N\text{-}m$$

$$q = q_1 + q_2$$

Mass m.I, $I = m\cdot k^2 = 500\times(0.5)^2 = 125\ kg\text{-}m^2$

Natural freq $f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{140\times10^3}{125}}$

$f_n = \underline{5.32\ Hz}$

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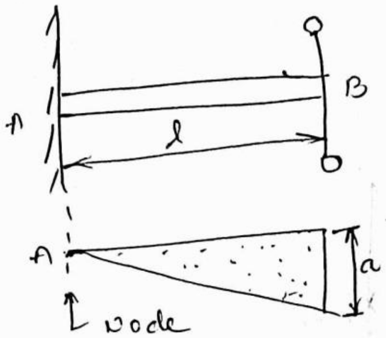
* Free Torsional Vibrations of Single Rotor System:



Natural frequency for a shaft fixed at one end at other end carrying a single rotor,

is. $f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{I \cdot l}}$.

$\because \left(q = \frac{C \cdot J}{l} \right)$.

Where, $C \to$ modulus of rigidity for shaft mtl.

$J \to$ Polar moment of inertia of a shaft

$= \pi/32\, d^4$

$d \to$ diameter of shaft

$l \to$ length of shaft.

$m \to$ mass of rotor.

$k \to$ radius of gyration.

$I \to$ Mass m.I of rotor $= m \cdot k^2$.

A little consideration will show that the amplitude of vibration is zero at 'A'. at max. 'B'. It may be noted that the point (or) section of shaft whose amplitude of torsional vibration is zero, is known as "node". In other words, at the node the shaft is unaffected due to vibration.

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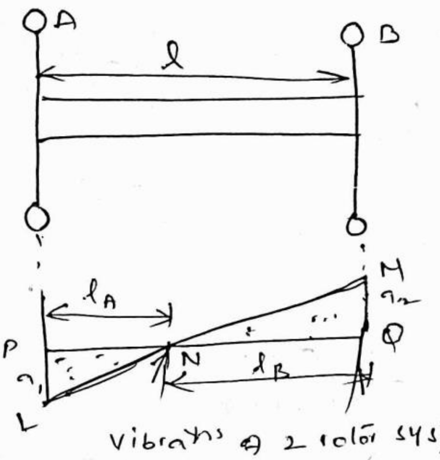
* Free Torsional Vibration of Two Rotor System:-



Vibrations of 2 rotor system.

Consider a 2 rotor system. It consists of a shaft with 2 rotors at its ends. In this system the torsional vibrations occurs only when the 2 rotors A & B move in opposite direction. If A moves ACW then B should CW & vice versa but with same frequency.

From the above fig. Node lies at pt 'N', & that pt. may be considered as a fixed end & shaft may be considered as 2 shafts NP & NQ each fixed to one of its end & carrying rotors at free ends.

Let,

$l \rightarrow$ length of shaft

$l_A \rightarrow$ length of part NP.

$l_B \rightarrow$ length of part NQ.

$I_A \rightarrow$ Mass MI of rotor A

$I_B \rightarrow$ " " " B

$d \rightarrow$ diameter of shaft

$J \rightarrow$ Polar MI of shaft &

$C \rightarrow$ modulus of rigidity.

∴

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$\therefore$ Natural frequency of Torsional vibratn for rotor A,

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_A \cdot I_A}} \quad \text{--- (i)}$$

IIly,

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_B \cdot I_B}} \quad \text{--- (ii)}$$

$\because f_{nA} = f_{nB}$

$$\therefore \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_A \cdot I_A}} = \frac{1}{2\pi}\sqrt{\frac{C \cdot J}{l_B \cdot I_B}}$$

(or)
$$l_A \cdot I_A = l_B \cdot I_B \quad \text{--- (iii)}$$

$$\therefore l_A = \frac{l_B \cdot I_B}{I_A}$$

We also know that,
$$l = l_A + l_B \quad \text{--- (iv)}$$

from eq's (iii) & (iv) we can find l_A & l_B & substituting l_A & l_B in eq's (i) & (ii) to find the corresponding f_n's

The line LNM is known as elastic line

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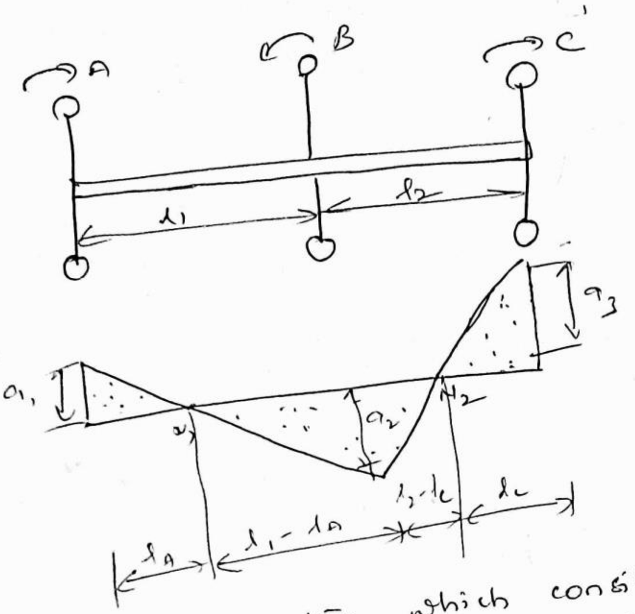
* Free Torsional vibratn of 3 rotor system:



consider 3 rotor system which consists of a shaft & 3 rotors A, B & C. where 2 rotors attached to the end of shaft and other is attached ~~to the~~ in b/w the shaft. Torsional vibratn may occur in 2 ways either 1 node (or) 2 node. In each case the 2 rotors rotate in one dirn & other rotate in opp. dirn. with same frequency.

$l_1 \rightarrow$ distance of

:

$J \rightarrow$ polar moment

$C \rightarrow$ Modulus of rigidity for a shaft mtl.

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{CJ}{l_A \cdot I_A}}$$

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_B}\left[\frac{1}{(l_1-l_A)} + \frac{1}{(l_2-l_C)}\right]}$$

$$f_{nC} = \frac{1}{2\pi}\sqrt{\frac{CJ}{l_C \cdot I_C}}$$

$\therefore f_{nA} = f_{nB} = f_{nC}$

$$\frac{1}{2\pi}\sqrt{\frac{CJ}{l_A \cdot I_A}} = \frac{1}{2\pi}\sqrt{\frac{CJ}{l_C \cdot I_C}}$$

$$l_A \cdot I_A = l_C \cdot I_C$$

(40)

When amplitude of vibration of rotor A (a_1) is known then amplitude of a_2 of rotor B

$$\therefore a_2 = \frac{l_A - l_1}{l_A} \cdot a_1$$

& amplitude of rotor 'C'

$$a_3 = \frac{l_C}{l_C - l_2} \times a_2$$

→ A steel shaft 1.5 m long is 95 mm in dia for first 0.6 m of its length & 60 mm in dia 0.5 m of length & 50 mm of dia for remaining 0.4 m of its length. The shaft carries 2 flywheels at 2 ends the first having a mass of 900 kg & 0.85 m 'K' located at 95 mm dia end & second having a mass of 700 kg & 0.55 m 'K' located at other end. Determine the location of node & f_n of free torsional vibration of system. C = 80 GN/m²

Sol:

Given: L = 1.5 m

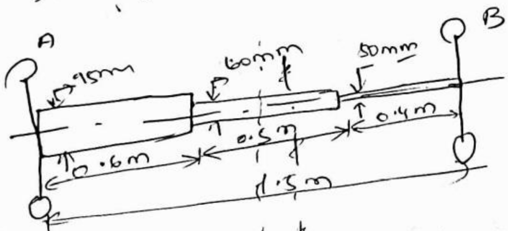
$l_1 = 0.6$ m, $d_1 = 95$ mm

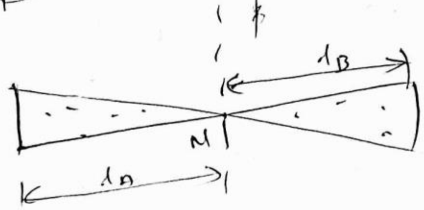
$l_2 = 0.5$ m, $d_2 = 60$ mm

$l_3 = 0.4$ m, $d_3 = 50$ mm

$m_A = 900$ kg; $m_B = 700$ kg

$C = 80 \times 10^9$ N/m²





W.K.T., length of equivalent shaft $l = l_1 + l_2\left[\frac{d_1}{d_2}\right]^4 + l_3\left[\frac{d_1}{d_3}\right]^4$, $l = 8.95$ m

(41)

Location of node:

$l_A \rightarrow$ Distance of node from flywheel A

$l_B \rightarrow$ " " " " B.

$I_A = m_A k^2 = 650\ kg\text{-}m^2$

$I_B = m_B k^2 = 212\ kg\text{-}m^2$

W.K.T, $l_A \cdot I_A = l_B \cdot I_B$ ⇒ $l_A = \dfrac{l_B \cdot I_B}{I_A}$

$l_A = 0.326\, l_B$

$l_A + l_B = l.$

$0.326\, l_B + l_B = 8.95\ m$

$l_B = 6.75\ m$. | $l_A = 2.2\ m$.

Hence node lies at a distance of 2.2 m from flywheel A & 6.75 m from flywheel B.

Original position of shaft from flywheel A,

$\therefore\ l_1 + (l_A - l_1)\left[\dfrac{d_1}{d_2}\right]^4 = 0.655\ m$.

Natural frequency:

$J = \pi/32\,(d_1)^4 = 8\times10^{-6}\ m^4$

$f_n = f_{nA} = f_{nB}$

$= \dfrac{1}{2\pi}\sqrt{\dfrac{CJ}{I_A \cdot l_A}} = \dfrac{1}{2\pi}\sqrt{\dfrac{80\times10^{9}\times8\times10^{-6}}{2.2\times650}} \Rightarrow f_n = 3.37\ Hz.$