# UNIT-I

## MathematicalLogic

**Syllabus**

**Mathematical Logic**: Introduction, Statements and Notation, Connectives, Truthtables, Well-formed formulas, Tautology, Contradiction, Contingency, Logical equivalence, Normal Forms, TheoryofInference fortheStatement Calculus,ThePredicateCalculus,InferenceTheoryofthe Predicate Calculus

**IntroductiontoDiscreteMathematics**

- Discrete Mathematics is the study of discrete structures which mathematical models dealing with discrete objects and relationships between them.
- ExamplesofdiscreteobjectsarelikeSets,Permutations,graphsand etc.

**Why it is important for computer science**

- In the world of computers, all the information is stored in bits,units of information that can take the value of either 0 or 1.It's not like in nature, where something can take all the values in between 0 and 1 as well. Instead, everything is binary.
- Since the bits are the building blocks of everything that happens in computer software, every thing becomes discrete.Forinstance,the hard drive on the laptop I'm using right now can store 1 845 074 329 600 bits of information.
- The study of algorithms is also firmly in the discrete world.Analgorithm is a step-by-step list of instructions to the computer and it's whatmakes computer programs possible.When determining how much time analgorithm needs to run,you count the number of operations it needs to perform. Notice the word count. Again, discrete mathematics.

- In continuous mathematics (the opposite of discrete),the calculation would go like this:
- $\int_0^5 \square dx=[1/2*x2]_0^5 =5^2/2-0=12.5$
- In discrete mathematics,the equivalent calculation would go likethis: $\sum_0^4 \square_{\square}=0+1+2+3+4=10$

**Applicationsofdiscretemathematicswithcomputerapplications**

- 1)Computernetworks
- 2)Programminglanguages
- 3)Finite stateautomataor compilers
- 4)Databases

Symbolic logic is used in framing algorithms and their verification and in automatic theorem proving.Settheory, Graph Theory,treesetc areused in storage and retrieval of information(data structures), Algorithms and their complexity studies also uses tools from discrete mathematics. Formal Languages, Automata theory, Turing machines etc. are themselves part of discrete mathematics and so is Recursive Function Theory. Undecidability of many problems are established using Turing machines which is the Mathematical model for studying theoretical limitations of Computation. Lattices and Boolean Algebra are used in Computer Science as well as in communications and networking.

**Mathematical Logic**

**Logic**

It is the study of the principles and methods that distinguishes between valid and invalid argument.

1. **Proposition Logic**

**PropositionorStatement**

A proposition or a statementcan definedas adeclarativesentencetowhich we can assign one and onlyone ofthe truth values either true (or) false but not both is called a proposition.

Thetrueorfalse ofaproposition iscalledtruthvalue ofa proposition

Thesetwovaluestrueandfalsearedenotedby thesymbols*T*and*F*respectively.Sometimes these are also denoted by the symbols 1 and 0 respect.

| **Proposition** | **truth value** |
|---|---|
| Ex:1)IndiacapitalisnewDelhi | True |
| 2) 2*3=5 | false |
| 3) 5isaprimenumber | true |

Thesearepropositions(orstatements) becausetheyareeither trueorfalse. Next

consider the following sentences:

4) Howbeautifulareyou?
5) Wishyouahappynewyear
*6) x+y=z*
7) Takeonebook.

Thesearenotpropositionsastheyarenotdeclarativeinnature,thatis,theydonotdeclarea definite truth value *T* or *F*.

TypesofPropositions

1) Atomicproposition

2)CompoundProposition

1)Atomicproposition

- APropositionwhichcannotbedividedfurther iscalledanatomic proposition.
- **Examples:**
- 1)India'scapitalisNewDelhi
- 2)2*3=5

2) CompoundProposition

- Twoor more atomic propositions can be combined to forma compound proposition with help of Connectives. Compound Proposition also called as Propositional function.

- **ExamplesofCompoundstatements**
- —3+2=-1 andDelhiisthecapitalofIndia.
- grassisgreenor—itishottoday.
- Discretemathematicsisnotdifficulttome.

And, Or ,Not are called connectives.

**Notations**

- StatementsaresymbolicallyrepresentedasA,B,C,..P,Q,R,S…Thosearecalled propositional variables or notations.
- **Examples**:
- P→Delhi isthecapitalofIndia.
- Q→17isdivisibleby3.

**LogicalConnectives**

- Thewordsor phrasesor symbolswhichareusedto makeacompound proposition bytwo or moreatomic propositions are called **logical connectives** or **simply connectives.**
- There are five basic connectives called negation, conjunction, disjunction, conditional and biconditional.

| **Connectivity** | **Symbol** | Word |
|---|---|---|
| Negation | ¬ | Not |
| Disjunction | ∨ | OR |
| Conjunction | □ | AND |
| Conditional | → | IFANDTHEN |
| Biconditional | ↔ | Ifandonlyif |

**Negation(~)or(¬)**

The **negation** of a statement is generally formed by writing the word ‘not’ at a proper place in the statement (proposition) orbyprefixingthe statement withthe phrase (¬).It is not thecasethat.If*p*denotesastatementthenthenegationof*p*iswrittenas¬*p*andreadasnot *p*. Ifthe truthvalue of*p* is*T* thenthe truthvalue of¬*p* is *F*. Also ifthe truthvalue of*p* is *F*then the truth value of ¬*p* is *T*.

**TruthtableforNegation**

| P | ¬P |
|---|---|
| F | T |

**Disjunction(OR)(V)**

- IfPandQ areanytwopropositionsthen**‘POR Q’**symbolicallywrittenas**PVQ**.
- PVQisapropositionwhosetruthvaluesisfalseonlywhenbothPandQarefalse otherwise True.∧

**TruthtableforDisjunction**

| P | Q | PVQ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

P:Ishallgotothegame.

Q:Ishallwatchthe game ontelevision.

P VQ:Ishallgotothegame ORIshallwatchthe gameon television.

**Conjunction(AND(∧))**

- IFPandQareanytwo propositionsthenPandQSymbolicallywrittenasP∧Q.
- P∧Qisapropositionwhosetruthvalueistrueonlywhenbotharetrue. Whole truth
- value is false when either P or Q are false and both are false.

**TruthtableforConjunction**

| P | Q | P∧Q |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

P:Itisrainingtoday.

Q:Thereare10chairsintheroom.

P  Q: It is raining today AND There are 10 chairs
in the room.

**Conditional(or)Implication(→)**

IfPandQareanytwostatements(orpropositions)thenthestatementP→Qwhichisread as, If P, then Q_ is called a **conditional statement** (or **proposition**) or **implication** and the connective is the **conditional connective.**

Truthtableforconditional

| P | Q | P→Q |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

Inthisconditionalstatement, Piscalledthe **hypothesis**or **premise**or **antecedent** andQ is called the **consequence** or **conclusion.**

**Biconditional(Ifandonly if)**

- IfP andQare anytwopropositionsthenPifandonlyifQWrittenasP↔Q.
- It'struthvalueistrueonlywhenbothP&Q have same truthvalues.

| *P* | *q* | p↔q |
|---|---|---|
| *T* | *T* | *T* |
| *T* | *F* | *F* |
| *F* | *T* | *F* |
| *F* | *F* | *T* |

**TAUTOLOGYANDCONTRADICTION**

**Tautology:** Apropositionissaidto be atautologyifitstruthvalue is T for anyassignment of truth values to itscomponents.

Example: ThepropositionP V¬Pisatautology.

**Contradiction**

Aproposition is said to be a contradiction if its truth value is F for any assignment oftruth values to its components. Example: The proposition P ∧ ¬P is a contradiction.

**Contingency:** Astatement formulawhichisneitheratautologynoracontradictionisknownas a **contingency.**

**Example**:P->Q

**Tautology:** A statement formula which is true regardless of the truth values of the statementswhichreplacethevariablesinitiscalleda**universallyvalidformula**ora**logical truth** or a **tautology.**

**Howtoprovegivencompoundproposition is tautology**

1) **ByConstructingtruth table**

2) **Byusingsubstitution method**

1) Constructingtruthtable

Showthat followingfunctionistautology

**a)(PVQ)V¬P**

**Solution**

| **P** | **Q** | **¬P** | **PVQ** | **(PVQ)V¬P** |
|---|---|---|---|---|
| **F** | **F** | **T** | **F** | **T** |
| **F** | **T** | **T** | **T** | **T** |
| **T** | **F** | **F** | **T** | **T** |
| **T** | **T** | **F** | **T** | **T** |

Intheabove table**(PVQ)V¬P**isgivingalltruthvaluesaretruesoitisa tautology.

**Showthatgivenpropositionisatautology((P→Q)^(Q→R))→(P→R)**

| P | Q | R | P → Q | Q→R | ((P→Q)^(Q→R) | P→R | ((P→Q)^(Q→R))→(P→R) |
|---|---|---|---|---|---|---|---|
| F | F | F | T | T | T | T | T |
| F | F | T | T | T | T | T | T |
| F | T | F | T | F | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| T | F | T | F | T | F | T | T |
| T | T | F | T | F | F | F | T |
| T | T | T | T | T | T | T | T |

**Implication:**

- IfPand QareanytwopropositionsthenP->Qie,IfP thenQ
- P->QispropositionwhosetruthvalueisfalseonlywhenP istrueandQisfalse.
- HerePisantecedentandQisConsequent..

| P | Q | P→Q |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | T |
| T | T | T |

- WheneverP is falseP->Qis true

- Ie,falseantecedentPimpliesanypropositionQ
- WheneverQistrueP->Qisalsotrue
- Ie,atrueconsequent Qimpliedbyanypropositional_P‘.
  fromthe implicationstatement wecanwriteanotherthreestatementswhichare converse, inverse and contrapositive

**Converse, InverseandContrapositive**

If$P \rightarrow Q$isaconditionalstatement,then(1).

$Q \rightarrow P$iscalledits*converse*(2).

$\neg P \rightarrow \neg Q$iscalledits*inverse*

(3).$\neg Q \rightarrow \neg P$iscalledits*contrapositive.*

Example:

- P:**Todayis Sunday**
- Q: **Itisaholiday**
- **ConverseStatement:**Ifitisaholiday,thentodayisSunday**.**
- **InverseStatement:**IftodayisnotSunday, thenitisnot aholiday.
- **ContrapositiveStatement**-Ifitisnot aholiday,thentodayisnotSunday.

Here Implication and Contrapositive are equal.

Converseand inverseareoppositepropositions.

**Wellformedformulas(wff):**

Notallstringscanrepresent propositionsofthepredicatelogic.Thosewhichproducea propositionwhentheirsymbolsareinterpretedmustfollowtherulesgivenbelow,andtheyare called wffs (well-formed formulas) ofthe first order predicate logic.

**Rulesforconstructing Wffs**

ApredicatenamefollowedbyalistofvariablessuchasP(x,y),wherePispredicatename,andx and yare variables, is called an atomic formula.

Awellformedformulaofpredicatecalculus isobtainedbyusingthefollowing rules.

1. Anatomicformulaisawff.
2. IfAisawff,then7Aisalsoawff.
3. IfAandBarewffs,then(AVB),(A∧B),(A→B)and(ADB).
4. IfAisawffandxisanyvariable,then(x)Aand($x)Aarewffs.
5. Onlythoseformulasobtainedbyusing(1)to(4)arewffs.

Since we will be concerned with only wffs, we shall use the term formulas for wff. We shall followthesameconventionsregardingtheuseofparenthesesaswasdoneinthecaseofstatement formulas.

**Wffsareconstructed usingthefollowingrules:**

1. *True*and*False*arewffs.
2. Eachpropositionalconstant(i.e.specificproposition),andeachpropositionalvariable (i.e. a variable representing propositions) are wffs.
3. Eachatomicformula(i.e.aspecificpredicatewithvariables)isa wff.
4. If*A,B,*and *C*arewffs,thensoare¬*A*,(*A*∧*B*),(*A*∨*B),(A*→*B),*and *(A*↔*B).*
5. If *x* is a variable (representing objects of the universe of discourse), and *A*is a wff, thenso are $\forall x\ A$ and $\exists x\ A$ .For example, "The capital of Virginia is Richmond." is a specific proposition. Hence it is a wffby Rule 2.

LetBbeapredicatenamerepresenting"beingblue"andletxbeavariable.ThenB(x)isanatomic formula meaning "x is blue". Thus itis a wff by Rule 3. above. By applying Rule 5. toB(x), xB(x) is a wff and so is xB(x). Then by applying Rule 4. to them xB(x)xB(x) isseento bea wff.Similarly,ifRisapredicatenamerepresenting"beinground".ThenR(x)isanatomic formula. Hence it is a wff. By applying Rule 4 to B(x) and R(x), a wff B(x) R(x)isobtained.In this manner, larger and more complex wffs can be constructed following the rules given above. Note,however,thatstringsthatcannotbeconstructedbyusingthoserulesarenotwffs.For example, xB(x)R(x), and B(x) are NOT wffs, NOR are B( R(x) ), and B( x R(x) )

.

More examples: To express the fact that Tom is taller than John, we can use the atomic formula ***taller*(Tom, John)**, which is a wff. This wff can also be part of some compound statement such as***taller*(Tom,John)** ***taller*(John,Tom)**,whichisalsoawff.If*x* is a variable representing people in the world, then ***taller(x,*Tom),** ***x taller(x,*Tom),** ***x taller(x,*Tom),** ***x y taller(x,y)*** are all wffs among others. However, *taller*(*x*,John) and *taller*(Tom Mary, Jim), for example, are **NOT** wffs.

**LogicalEquivalence**

> Two formulas *A* and *B* are said to equivalent to each other if and only if *A*↔ *B*is a tautology. If *A*↔*B* is a tautology, we write *A* ⇔ *B* which is read as *A* is equivalent to *B*.

Note:1.⇔isonlysymbol,butnot connective.

*A* ↔ *B* is a tautology if and only if truth tables of *A*and *B*are the same.Equivalence relation is symmetric and transitive.

(or)

LetPandQaretwopropositionalfunctionsPisEquivalenttoQ. Symbolicallywritten

as P⇔ Q or P ≡ Q ifP and Q have same truthtable.

Ex:P->Q⇔~PVQ.

Method I. Truth Table Method: One method to determine whether any two statementformulasare equivalent is to construct their truth tables.

Example:Prove$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

| $P$ | $Q$ | $P \rightarrow Q$ | $\neg P$ | $\neg P \vee Q$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

Intheabovetableboth$P \rightarrow Q$and$\neg P \vee Q$arehavesametruthvalues. So that $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

**EquivalenceFormulas:**

1. Idempotentlaws:

   (a)$P \vee P \Leftrightarrow P$ (b)$P \wedge P \Leftrightarrow P$

2. Associativelaws:

   (a)$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (b)$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Commutative laws:

   (a)$P \vee Q \Leftrightarrow Q \vee P$ (b)$P \wedge Q \Leftrightarrow Q \wedge P$

4. Distributive laws:

   $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

   Identitylaws(or)

5. Domination Law

   (a)(i)$P \vee F \Leftrightarrow P$ (ii)$P \vee T \Leftrightarrow T$

(b)(i)$P \wedge T \Leftrightarrow P$ (ii)$P \wedge F \Leftrightarrow F$

6. Componentlaws:

(a)(i)$P \vee \neg P \Leftrightarrow T$ (ii)$P \wedge \neg P \Leftrightarrow F$ .

(b)(i)$\neg \neg P \Leftrightarrow P$ (ii)$\neg T \Leftrightarrow F, \neg F \Leftrightarrow T$

7. Absorptionlaws:

(a)$P \vee (P \wedge Q) \Leftrightarrow$ (b)$P \wedge (P \vee Q) \Leftrightarrow$

8. DeMorgan'sLaws

(a)$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ (b)$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg$

**TABLE 7 Logical Equivalences Involving Conditional Statements.**

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

S...

**TautologicalImplications.**

Astatement formula $A$ is said to *tautologically imply* a statement $B$ ifand onlyif$A \rightarrow B$ is a tautology. Inthis case we write $A \Rightarrow B$, which is read as _$A$ implies $B$_.

Note:⇒isnotaconnective,$A \Rightarrow B$isnotastatement formula.

$A \Rightarrow B$statesthat$A \rightarrow B$istautology.

Clearly $A \Rightarrow B$ guarantees that $B$ has a truth value $T$ whenever $A$ has the truth value $T$ . One can determine whether $A \Rightarrow B$ byconstructing the truth tables of $A$ and $B$ in the same manner as was done in the determination of $A \Leftrightarrow B$.

Example:Provethat$(P \rightarrow Q) \Rightarrow (\neg Q \rightarrow \neg P)$.

| $P$ | $Q$ | $\neg P$ | $\neg Q$ | $P \rightarrow Q$ | $\neg Q \rightarrow \neg P$ | $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

Sinceallthe entriesinthelastcolumnaretrue,$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$isa tautology.

Hence$(P \rightarrow Q)$ isTautologicalImplicationsto $(\neg Q \rightarrow \neg P)$. Sothat

$(P \rightarrow Q) \Rightarrow (\neg Q \rightarrow \neg P)$.

Inordertoshowanyofthegivenimplications,itissufficienttoshowthatanassignmentofthe truthvalue$T$to the antecedentofthecorrespondingconditionalleadstothetruthvalue$T$forthe

consequent. This procedure guarantees that the conditional becomes tautology, thereby proving the implication.

Example:Provethat$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$.

Solution:Assumethattheantecedent$\neg Q \wedge (P \rightarrow Q)$hasthetruthvalue$T$,thenboth$\neg Q$and$P \rightarrow Q$havethetruthvalue$T$,whichmeansthat$Q$hasthetruthvalue$F$,$P \rightarrow Q$hasthetruth value $T$ .

Hence$P$musthavethetruthvalue$F$.Therefore,theconsequent$\neg P$musthavethetruthvalue $T$.

$\neg Q \wedge (P \rightarrow Q) \nrightarrow \neg P$.

**Anothermethod**toshow$A \Rightarrow B$istoassumethattheconsequent $B$hasthetruthvalue$F$andthen show that this assumption leads to $A$ having the truth value $F$. Then $A \rightarrow B$ must have the truth value $T$.

Example:Showthat$\neg (P \rightarrow Q) \Rightarrow P$.

Solution:Assume that$P$ has the truth value$F$ . When $P$ has $F$ , $P \rightarrow Q$has $T$ , then $\neg (P \rightarrow Q)$ has $F$. Hence $\neg (P \rightarrow Q) \rightarrow P$ has $T$ .

Sothat$\neg (P \rightarrow Q) \Rightarrow P$.

**NormalForms**

If a given statement formula $A(p_1, p_2, ...p_n)$ involves $n$ atomic variables, we have $2^n$possible combinations of truth values of statements replacing the variables.

The formula $A$ is a tautology if $A$ has the truth value $T$ for all possible assignments of the truth values to the variables $p_1, p_2, ...p_n$and $A$ is called a contradiction if $A$ has the truthvalue $F$ for all possible assignments of the truth values of the $n$ variables. $A$ is said to be *satisfiable* if $A$ has the truth value $T$ for at least one combination of truth values assigned to $p_1, p_2, ...p_n$.



Theproblemofdeterminingwhetheragivenstatement formulaisaTautology,oraContradiction is called a decision problem.

The constructionoftruthtable involves a finite number ofsteps, but the construction maynot be practical. We therefore reduce the given statement formula to normal form and find whether a givenstatementformulaisaTautologyorContradictionoratleastsatisfiable.Itwillbeconvenient to use the word product in place of conjunction and sum in place of disjunction .

A product of the variables and their negations in a formula is called an *elementary Product.* Similarly, asumofthevariablesandtheir negationsinaformulaiscalledan*elementarysum.*Let *P* and *Q* be any atomic variables Then $P, \sim P \wedge Q, \neg Q \wedge P \wedge \sim P, Q \wedge \sim P$ are some example ofelementaryproducts.

Ontheotherhand$P, \neg P \vee Q, \neg Q \vee P \vee \neg P, Q \vee \neg P$.aresomeexamplesofelementarysums. Types of Normal forms

1) DisjunctiveNormalforms

2) ConjunctiveNormal forms.

**DisjunctiveNormalForm(DNF)**

Aformulawhichisequivalent to agiven formula andwhichconsistsofasumofelementary products is called a *disjunctive normal form* of the given formula.

Example:Obtaindisjunctive normalformsof

(a)P □(P→Q)

P□(P→Q)⇔P□(¬P∨Q) (applydistributivelaw$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

-(P□¬P)∨(P□Q)

***b)*** **$\neg(P \vee Q) \leftrightarrow (P \wedge Q)$**

$\neg(P \vee Q) \leftrightarrow (P \wedge Q) \Leftrightarrow (\neg(P \vee Q) \square (P \wedge Q)) \vee ((P \vee Q) \square \neg(P \wedge Q))$[using $R \leftrightarrow S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$]

$\Leftrightarrow((\neg P \wedge \neg Q) \square (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$.

$\Leftrightarrow(\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$.

$\Leftrightarrow(\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$.

Whichistherequireddisjunctive normalform.Note:TheDNFofagivenformulaisnotunique.

**ConjunctiveNormalForm (CNF)**

Aformulawhichisequivalent to agiven formula andwhichconsistsofaproductofelementary sums is called a *conjunctive normal form* of the given formula.

The methodforobtainingconjunctive normalformofagivenformula is similartotheone given for disjunctive normal form. Again, the conjunctive normal form is not unique.

**(a)** $P\wedge(P\rightarrow Q)$obtaintheconjunctivenormalform

$P\wedge(P\rightarrow Q)\Leftrightarrow P\wedge(\neg P\vee Q)$

**(b)** $\neg(P\vee Q)\leftrightarrow(P\wedge Q)$

$-(\neg(P\vee Q)\rightarrow(P\wedge Q))\square((P\wedge Q)\rightarrow\neg(P\vee Q))$

$-((P\vee Q)\vee(P\wedge Q))\wedge(\neg(P\wedge Q)\vee\neg(P\vee Q))$

$-[(P\vee Q\vee P)\wedge(P\vee Q\vee Q)]\wedge[(\neg P\vee\neg Q)\vee(\neg P\wedge\neg Q)]$

$-(P\vee Q\vee P)\wedge(P\vee Q\vee Q)\wedge(\neg P\vee\neg Q\vee\neg P)\wedge(\neg P\vee\neg Q\vee\neg Q)$

**PrincipalDisjunctiveNormalForm**

Inthissection,wewilldiscusstheconceptofprincipaldisjunctive normalform(PDNF).

**Minterm:** For a given number of variables, the minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

Let$P$and$Q$bethetwostatementvariables.Thenthereare$2^2$mintermsgivenby $P\wedge Q,P\wedge\neg Q,$

$\neg P\wedge Q,$and$\neg P\square\wedge\neg Q$

Mintermsforthreevariables$P$ ,$Q$and$R$ are$P\wedge Q\wedge R,P\wedge Q\wedge\neg R,P\wedge\neg Q\wedge R,P\wedge\neg Q\wedge\neg R,\neg P\wedge Q\wedge R,\neg P\wedge Q\wedge\neg R,\neg P\wedge\neg Q\wedge R$and$\neg P\wedge\neg Q\wedge\neg R.$

Fromthetruthtablesofthesemintermsof*P*and*Q*,itisclearthat.

| *P* | *Q* | $P \wedge Q$ | $P \wedge \neg Q$ | $\neg P \wedge Q$ | $\neg P \wedge \neg Q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | T | F | F |
| F | T | F | F | T | F |
| F | F | F | F | F | T |

(i). Notwomintermsareequivalent

(ii). Eachmintermhasthetruthvalue*T*forexactlyonecombinationof the truthvaluesof the variables *P* and *Q*.

**PDNF**

**Definition**: For a given formula, an equivalent formula consisting of disjunctions of minterms only is called the Principal disjunctive normal form of the given formula. The principle disjunctive normal formula is also called the sum-of-products canonical form.

MethodstoobtainthePDNFofagivenformula.

**(a). By Truth table:**

**(b). withoutconstructingthetruthtable**



**<u>(a). By Truthtable:</u>**

(i). Constructatruthtableofthe givenformula.

(ii). For everytruth value *T* in the truth table of the given formula, select the minterm which also has the value *T* for the same combination of the truth values of *P* and *Q*.

(iii). Thedisjunctionofthesemintermswillthenbeequivalenttothegivenformula

Example: Obtain the PDNF of $P \rightarrow Q$.

Solution:Fromthetruthtable of$P \rightarrow Q$

| $P$ | $Q$ | $P \rightarrow Q$ | Minterm |
|---|---|---|---|
| T | T | T | $P \wedge Q$ |
| T | F | F | $P \wedge \neg Q$ |
| F | T | T | $\neg P \wedge Q$ |
| F | F | T | $\neg P \wedge \neg Q$ |

ThePDNF of$P \rightarrow Q$is$(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

**ObtainthePDNFfor$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.**

Solution:

| $P$ | $Q$ | $R$ | Minterm | $P \wedge Q$ | $\neg P \wedge R$ | $Q \wedge R$ | $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | $P \wedge Q \wedge R$ | T | F | T | T |
| T | T | F | $P \wedge Q \wedge \neg R$ | T | F | F | T |
| T | F | T | $P \wedge \neg Q \wedge R$ | F | F | F | F |
| T | F | F | $P \wedge \neg Q \wedge \neg R$ | F | F | F | F |
| F | T | T | $\neg P \wedge Q \wedge R$ | F | T | T | T |
| F | T | F | $\neg P \wedge Q \wedge \neg R$ | F | F | F | F |
| F | F | T | $\neg P \wedge \neg Q \wedge R$ | F | T | F | T |
| F | F | F | $\neg P \wedge \neg Q \wedge \neg R$ | F | F | F | F |



ThePDNFof$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$is$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$.

**(b). Withoutconstructingthetruthtable:**

Inordertoobtaintheprincipaldisjunctivenormalformofagivenformulaiscon- structed as follows:

(1). Firstreplace→,bytheirequivalentformulacontainingonly□,∨and¬.

(2). Next,negationsareappliedtothevariablesbyDeMorgan_slawsfollowedbytheapplication of distributive laws.

(3). Any elementarily product which is a contradiction is dropped. Minterms are obtained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

**Example:Obtaintheprincipaldisjunctivenormalformof**

**(a) ¬*P*∨*Q***
**(b) (*P*∧*Q*)∨(¬*P*∧*R*)∨(*Q*∧*R*).**

**(a) ¬*P*∨*Q***

$\neg P \vee Q \Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T)$

– $(\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$ [$\because P \vee \neg P \Leftrightarrow T$]

– $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

– $(\neg P \square Q) \vee (\neg P \square \neg Q) \vee (P \square Q)$ [$\because P \vee P \Leftrightarrow P$]

**(b) (*P*∧*Q*)∨(¬*P*∧*R*)∨(*Q*∧*R*)**

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R) \Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$

$\text{-}(P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P))$

$\text{-}(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q) \vee$

$(Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$

$\text{-}(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

**PrincipalConjunctiveNormalForm**

ThedualofamintermiscalledaMaxterm.Foragivennumberofvariables,the *maxterm*consists ofdisjunctionsinwhicheachvariableor itsnegation,butnotboth,appearsonlyonce. Eachofthe maxterm has the truth value *F* for exactly one combination of the truth values of the variables. Now we define the principal conjunctive normal form.



For a given formula, an equivalent formula consisting of conjunctions of the max-terms only is known as its *principle conjunctive normal form*. This normal form is also called the *product-of-sums canonical form.*The method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

**Example:Obtaintheprincipalconjunctivenormalformoftheformula(¬P→R)∧ (Q↔P)**

Solution:

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P) \Leftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \rightarrow P) \wedge (P \rightarrow Q)]$

$\Leftrightarrow [(P \vee R) \wedge [(\neg Q \vee P) \wedge (\neg P \vee Q)]$

$\Leftrightarrow (P \vee R \vee F) \wedge [(\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)]$

$\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \square [\neg Q \vee P) \vee (R \wedge \neg R)] \square [(\neg P \vee Q) \vee (R \wedge \neg R)]$

$\Leftrightarrow$

$(P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

**Rulesofinference:**

Thetworulesofinferencearecalled rulesPand T.

RuleP:Apremisemaybeintroducedatanypointinthe derivation.

RuleT:AformulaS maybe introducedinaderivationifs is tautologicallyimpliedby any one ormore of the preceding formulas in the derivation.

Beforeproceedingtheactualprocessofderivation, some important list ofimplications and equivalences are given in the following tables.

***Implications***

| | | | |
|---|---|---|---|
| I1 | P^Q=>P | } | Simplification |
| I2 | PQ^=>Q | | |
| I3 | P=>PVQ | } | Addition |
| I4 | Q=>PVQ | | |
| I5 | 7P=>P→Q | | |
| I6 | Q =>P→Q | | |
| I7 | 7(P→Q)=>P | | |
| I8 | 7(P→Q)=>7Q | | |
| I9 | P,Q=>P ^Q | | |
| I10 | 7P,PVQ=>Q | | (disjunctivesyllogism) |
| I11 | P,P→Q=>Q | | (modusponens) |
| I12 | 7Q,P→Q=>7P | | (modustollens) |
| I13 | P→Q,Q →R=>P→R | | (hypotheticalsyllogism) |
| I14 | PVQ,P→Q,Q→R=>R | | (dilemma) |

***Equivalences***

| | | | |
|---|---|---|---|
| E1 | 77P<=>P | | |
| E2 | P^Q<=>Q^P | } | Commutativelaws |
| E3 | PV Q<=>Q VP | | |
| E4 | (P ^Q) ^R<=>P^(Q ^R) | } | Associative laws |
| E5 | (PVQ)VR <=>PV(QVR) | | |
| E6 | P^(QVR)<=>(P ^Q)V(P^R) | } | Distributive laws |
| E7 | PV(Q^R) <=>(PVQ)^(PVR) | | |
| E8 | 7(P^Q) <=>7PV7Q | | |
| E9 | 7(P VQ)<=>7P ^7Q | } | DeMorgan‘slaws |
| E10 | PVP<=>P | | |
| E11 | P^P<=>P | | |
| E12 | RV(P^7P) <=>R | | |
| E13 | R^(PV7P)<=>R | | |
| E14 | RV(PV7P)<=>T | | |
| E15 | R ^(P^7P)<=>F | | |
| E16 | P→Q<=>7PVQ | | |
| E17 | 7(P→Q)<=>P ^7Q | | |
| E18 | P→Q <=>7Q→7P | | |
| E19 | P →(Q→R) <=>(P^Q) → R | | |
| E20 | 7(PDQ)<=>PD7Q | | |
| E21 | PDQ <=>(P→Q)^(Q→P) | | |
| E22 | (PDQ) <=>(P^Q)V(7P^7Q) | | |

**Example1.ShowthatRislogicallyderivedfromP→ Q, Q→ R,andP**

| | | | | |
|---|---|---|---|---|
| Solution. | {1} | (1) | P→Q | RuleP |
| | {2} | (2) | P | RuleP |
| | {1,2} | (3) | Q | Rule(1),(2)and I11 |
| | {4} | (4) | Q→R | RuleP |
| | {1,2,4} | (5) | R | Rule(3),(4)and I11. |

**Example2.ShowthatSV Rtautologicallyimplied by(PV Q)^(P→R)^(Q→S)**.

| | | | | |
|---|---|---|---|---|
| Solution. | {1} | (1) | PVQ | RuleP |
| | {1} | (2) | 7P→Q | T,(1),E1andE16 |
| | {3} | (3) | Q→S | P |
| | {1,3} | (4) | 7P→S | T,(2),(3),andI13 |
| | {1,3} | (5) | 7S→P | T,(4),E13andE1 |
| | {6} | (6) | P→R | P |
| | {1,3,6} | (7) | 7S→R | T,(5),(6),andI13 |
| | {1,3,6) | (8) | SVR | T,(7),E16andE1 |

**Example3.Showthat7Q,P→Q=>7P**

| | | | | |
|---|---|---|---|---|
| **S**olution. | {1} | (1) | P→Q | RuleP |
| | {1} | (2) | 7P→7Q | T,and E18 |
| | {3} | (3) | 7Q | P |
| | {1,3} | (4) | 7P | T,(2),(3),andI11. |

**Example4.ProvethatR ^(PVQ )isavalidconclusionfromthepremisesPVQ,Q →R,P→Mand7M.**

| | | | | |
|---|---|---|---|---|
| Solution. | {1} | (1) | P→M | P |
| | {2} | (2) | 7M | P |
| | {1,2} | (3) | 7P | T,(1),(2),and I12 |
| | {4} | (4) | PVQ | P |

| | | | |
|---|---|---|---|
| {1,2,4} | (5) | Q | T,(3),(4),andI10. |
| {6} | (6) | Q→R | P |
| {1,2,4,6} | (7) | R | T,(5),(6)and I11 |
| {1,2,4,6} | (8) | R ^(PVQ) | T,(4),(7),and I9. |

**Thereisathirdinference rule,knownasruleCPorruleof*conditionalproof*.**

**Rule CP**: Ifwe can derive s fromR and a setofpremises, thenwe canderive R → S fromtheset of premises alone.

Note. 1.RuleCP followsfromtheequivalenceE10whichstatesthat ( P ^R ) → S óP → (R → S).

2. Let P denote the conjunction of the set of premises and let R be any formula Theaboveequivalencestatesthatif R isincluded asanadditionalpremiseand S isderived fromP ^RthenR→S canbederived fromthe premisesP alone.
3. Rule CP is also called the *deduction theorem* and is generally used if the conclusion isofthe formR→S. Insuchcases, R istakenasanadditional premise and S is derived from the given premises and R.

**Example5.ShowthatR→Scanbederivedfromthepremises P → (Q → S), 7R V P , and Q.**

Solution.

| | | | |
|---|---|---|---|
| {1} | (1) | 7R VP | P |
| {2} | (2) | R | P,assumedpremise |
| {1,2} | (3) | P | T,(1),(2),andI10 |
| {4} | (4) | P→(Q→S) | P |
| {1,2,4} | (5) | Q→S | T,(3),(4),andI11 |
| {6} | (6) | Q | P |
| {1, 2,4,6} | (7) | S | T,(5),(6),andI11 |
| {1,4,6} | (8) | R →S | CP. |

**Example 6 Show that P → S can be derived from the premises, 7P V Q, 7Q V R, and R → S .**

Solution.

| | | | |
|---|---|---|---|
| {1} | (1) | 7PVQ | P |
| {2} | (2) | P | P,assumedpremise |
| {1,2} | (3) | Q | T,(1),(2)andI11 |
| {4} | (4) | 7QVR | P |
| {1,2,4} | (5) | R | T,(3),(4)and I11 |
| {6} | (6) | R →S | P |
| {1, 2,4,6} | (7) | S | T,(5),(6)and I11 |
| {2,7} | (8) | P→S | CP |

**Example7. ”Iftherewasaballgame, thentravelingwasdifficult. Iftheyarrivedontime, thentraveling wasnotdifficult. They arrived on time. Therefore, there wasnoballgame”. Show that these statements constitute a valid argument.**

Solution.LetP:There wasaballgame

Q:Travelingwasdifficult. R:

They arrived on time.

Givenpremisesare:P→Q,R →7Q andRconclusionis:7P

| | | | |
|---|---|---|---|
| {1} | (1)P→Q | P | |
| {2} | (2)R →7Q | P | |
| {3} | (3) R | P | |
| {2,3} | (4)7Q | T,(2),(3),and I11 | |
| {1,2,3} | (5) 7P | T,(2),(4)andI12 | |

**Consistency ofpremises:**

**Consistency**

A set of formulas H1, H2, …, Hm is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to be atomic appearing in H1, H2, …, Hm.

**Inconsistency**

Ifforeveryassignmentofthetruthvaluestotheatomicvariables,atleastoneofthe formulas H1,H2,…Hmisfalse,sothattheirconjunctionisidenticallyfalse,thentheformulasH1,H2,…, Hm are called inconsistent.

AsetofformulasH1, H2, …, Hmis inconsistent, iftheir conjunctionimpliesa contradiction, that isH1 ^H2 ^… ^Hm => R ^7R

WhereR isanyformula. Notethat R^7Risacontradictionand it is necessaryand sufficient thatH1, H2, …,Hm are inconsistentthe formula.

**Indirectmethodof proof**

In order to show that a conclusion C follows logically from the premises H1, H2,…, Hm, weassumethat Cis falseandconsider 7Casanadditionalpremise. Ifthenew setofpremises is inconsistent, sothat theyimplya contradiction, thenthe assumptionthat 7C is true does not hold simultaneouslywith H1 ^H2 ^.….^Hmbeing true. Therefore, C is true wheneverH1^ H2^.…^Hmistrue.Thus,Cfollowslogicallyfromthe premisesH1,H2…..,Hm.

Example8Showthat7(P^Q)followsfrom7P^7Q.Solution.

We introduce77(P^Q) asanadditionalpremiseandshowthatthisadditional premise leads to a contradiction.

| | | |
|---|---|---|
| {1} | (1)77(P^Q) | Passumedpremise |
| {1} | (2)P^Q | T,(1)andE1 |
| {1} | (3)P | T,(2)and I1 |

| | | |
|---|---|---|
| {1} | {4)7P^7Q | P |
| {4} | (5) 7P | T,(4)and I1 |
| {1,4} | (6)P^7P | T,(3),(5) and I9 |

Here(6)P^7Pisacontradiction. Thus{1,4}viz.77(P^Q)and7P^7Qleadsto a contradiction P ^7P.

Example9Showthatthefollowingpremisesareinconsistent.

1. IfJackmissesmanyclassesthroughillness,thenhefailshighschool.
2. IfJackfails highschool,thenheisuneducated.
3. IfJackreadsalotofbooks,thenheisnotuneducated.
4. Jackmissesmanyclassesthroughillnessandreadsalotofbooks.

*Solution.*

P:Jackmissesmanyclasses. Q:

Jack fails high school.

R:Jackreadsalotofbooks. S:

Jack is uneducated.

ThepremisesareP→ Q,Q→ S,R→7SandP^R

| | | | |
|---|---|---|---|
| {1} | (1) | P→Q | P |
| {2} | (2) | Q→S | P |
| {1,2} | (3) | P→S | T,(1),(2)and I13 |
| {4} | (4) | R→ 7S | P |
| {4} | (5) | S→7R | T,(4),and E18 |
| {1,2,4} | (6) | P→7R | T,(3),(5)and I13 |
| {1,2,4} | (7) | 7PV7R | T,(6)andE16 |
| {1,2,4} | (8) | 7(P^R) | T,(7)andE8 |
| {9} | (9) | P ^R | P |
| {1,2,4,9)(10) | | (P ^R)^7(P^R) | T,(8),(9)and I9 |

Therulesabovecanbesummedup inthe followingtable. The"Tautology"columnshowshow to interpret the notation of a given rule.

| Ruleof inference | Tautology | Nam e |
|---|---|---|
| | | Addition |
| | | Simplification |
| | | Conjunction |
| $\begin{array}{r} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$ | | Modusponens |
| $\begin{array}{r} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$ | | Modustollens |
| $\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$ | | Hypothetical syllogism |
| | | Disjunctivesyllogism |
| | | Resolution |

**Predicativelogic:**

A predicate or propositional function is a statement containing variables. For instance ―x + 2 = 7‖,―XisAmerican‖,―x<y‖,―pisaprimenumber‖arepredicates.Thetruthvalueofthe predicatedependsonthevalueassignedtoitsvariables. For instance ifwereplace xwith1 inthe predicate―x+2=7‖weobtain―1+2=7‖,whichisfalse,butifwereplaceitwith5weget―5 + 2 = 7‖, which is true. We represent a predicate by a letter followed by the variables enclosed between parenthesis: P (x), Q(x, y), etc. An example for P (x) is a value of x for which P (x) is true. A counterexample is a value ofx for which P (x) is false. So, 5 is anexample for ―x + 2 = 7‖, while 1 is acounterexample. Eachvariable in apredicate is assumed to belongto auniverse (or domain) of discourse, for instance in the predicate ―n is an odd integer‖ ‘n‘ represents an integer,sotheuniverseofdiscourseofnisthesetofallintegers.In―Xis American‖ we may assume that X is a human being, so in this case the universe of discourse is the set of all human beings.

**Free&Boundvariables:**

Let's now turn to a rather important topic: the distinction between **free variables** and **bound variable**s.

Havealookatthefollowingformula:

$$\neg(\text{THERAPIST}(x) \vee \forall x(\text{MORON}(x) \wedge \forall y\text{PERSON}(y)))$$

The first occurrence of x is *free*, whereas the second and third occurrences of x are *bound*, namelybythefirstoccurrenceofthequantifier.Thefirstandsecondoccurrencesofthe variable y are also bound, namely by the second occurrence of the quantifier .

Informally,theconceptofa*boundvariable*canbeexplainedasfollows:Recallthatquantifications are generally of the form:

$\forall x\varphi$

or

$\exists x\varphi$

wherexmay be any variable. Generally, all occurrences of this variable within the quantification are bound. But we have to distinguish two cases. Look at the following formula to see why:

$$\exists x(\text{MAN}(x) \wedge (\forall x\text{WALKS}(x)) \wedge \text{HAPPY}(x))$$

1. $x$ may occur within another, embedded, quantification $\forall x \psi$ or $\exists x \psi$, such as the $x$ in in our $\text{WALKS}(x)$ example. Then we say that it is bound by the quantifier of this embedded quantification (and so on, if there's another embedded quantification over $x$ within $\psi$).
2. Otherwise, we say that it is bound by the top-level quantifier (like all other occurrences of in $x$ our example).

Here's a full formal simultaneous definition of *free* and *bound*:

1. Any occurrence of any variable is free in any atomic formula.
2. No occurrence of any variable is bound in any atomic formula.
3. If an occurrence of any variable is free in $\varphi$ or in $\psi$, then that same occurrence is free in $\neg\varphi$, $(\varphi \rightarrow \psi)$, $(\varphi \vee \psi)$, and $(\varphi \wedge \psi)$.
4. If an occurrence of any variable is bound in $\varphi$ or in $\psi$, then that same occurrence is bound in $\neg\varphi$ $(\varphi \rightarrow \psi)$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$. Moreover, that same occurrence is bound in $\forall y \varphi$ and $\exists y \varphi$ as well, for any choice of variable y.
5. In any formula of the form $\forall y \varphi$ or $\exists y \varphi$ (where y can be any variable at all in this case) the occurrence of y that immediately follows the initial quantifier symbol is bound.
6. If an occurrence of a variable x is free in $\varphi$, then that same occurrence is free in $\forall y \varphi$ and $\exists y \varphi$, for any variable y distinct from x. On the other hand, all occurrences of x that are free in $\varphi$, are bound in $\forall x \varphi$ and in $\exists x \varphi$.

If a formula contains no occurrences of free variables we call it a ***sentence***

## Quantifiers

Thevariableofpredicatesisquantifiedbyquantifiers.Therearetwotypesofquantifierin predicate logic − Universal Quantifier and Existential Quantifier.

## Universal Quantifier

Universalquantifierstatesthatthestatementswithinitsscopearetrueforevery valueofthe specific variable. It is denoted by the symbol ∀

∀xP(x)isread asforeveryvalueofx,P(x)is true..

**Example**− "Man is mortal" can be transformed into the propositionalform ∀xP(x)∀xP(x)whereP(x)isthepredicatewhichdenotesxismortalandtheuniverseofdiscourse is all men.

## ExistentialQuantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.

∃xP(x)isreadasforsomevaluesofx, P(x)istrue.

**Example** −"Somepeoplearedishonest"canbetransformedintothepropositionalform

∃xP(x) where P(x) is the predicate whichdenotesx is dishonest and the universe ofdiscourse is some people.

## UNIT-II

## Relations

**Syllabus**

Introduction, Basic ConceptsofSet Theory, RepresentationofDiscrete Structures, Relations, Types of relations, Partial order relation, POSET, External elements in POSET, Lattices, Functions, Types of functions,inverseoffunctions,invertiblefunctionsandCompositionoffunctions

### Introduction

The elements of a set may be related to one another. For example, in the set of natural numbers thereisthe‘lessthanorequalto’relationbetweentheelements. Theelementsofoneset mayalso be related to the elements another set.

### Binary Relation

Abinaryrelationbetweentwo setsAand B isaruleRwhichdecides, for anyelements, whether a is in relation R to b. If so, we write a R b.

IfaisnotinrelationRtob,thenwe shallwrite a~~R~~b.

Wecanalso consider aRbastheorderedpair (a,b) inwhichcasewecandefinea binary relation from A to B as a subset of A X B. This subset is denoted bythe relation R.

Ingeneral,**anysetoforderedpairsdefinesabinaryrelation.**

**Forexample**,the relationoffatherto hischildisF={(a,b) /aisthe fatherofb}

InthisrelationF, thefirst member isthenameofthe fatherandthesecond isthenameofthe child.

Thedefinitionofrelationpermitsanysetoforderedpairstodefinearelation.

**Forexample**,the setS given by

S ={(1,2),(3,a),(b,a),(b,Joe)}

**Definition**

The**domain** Dofa binaryrelationS isthe set ofallfirst elementsoftheordered pairs inthe relation. (i.e) D(S)= {a/ ∃ b for which (a, b) Є S}

The **range**RofabinaryrelationS isthe set ofall second elementsofthe ordered pairs in the relation.(i.e) R(S) = {b /∃a for which (a, b) ЄS}.

**Forexample**

FortherelationS={(1,2),(3,a),(b,a),(b,Joe)} D(S)= {1, 3, b, b} and

R(S)={2,a,a,Joe}

Let X and Y be anytwo sets. Asubset ofthe Cartesianproduct X * Y definesa relation, sayC. For anysuchrelationC, wehave D( C)Í X and R( C)Í Y, andthe relation C is saidto fromX to Y. IfY = X,thenC is said to be arelation formXto X. Insuchcase, c is called a relation in X. ThusanyrelationinX isasubset ofX*X.Theset X*Xiscalleda *universalrelation* inX, while the emptyset which is also a subset ofX * X is called a *void relation* in X.

**Forexample**

LetLdenotetherelation―lessthanorequalto‖andDdenotetherelation ―divides‖wherexDymeans― xdividesy‖.BothLandDaredefinedonthe set {1, 2, 3, 4}

L={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

D={(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

LÇD ={(1,1),(1,2),(1,3),(1,4), (2,2),(2,4),(3,3),(4,4)}

=D

**PropertiesofBinaryRelations:**

1. **reflexive**

**Definition:** AbinaryrelationR inaset Xis **reflexive** if, foreveryxЄX,xR x, That is (x, x) ЄR.

**Forexample**

- Therelation£isreflexiveinthesetofrealnumbers.
- Thesetinclusionisreflexiveinthefamilyofallsubsetsofauniversalset.
- Therelationequalityofsetisalsoreflexive.
- Therelationisparallelinthesetlines inaplane.

- Therelationofsimilarityinthe setoftrianglesin aplaneis reflexive.

Examples:(i).If*R*1 ={(1,1),(1,2),(2,2),(2,3),(3,3)}bearelationon*A*={1,2,3},then*R*1 is

- areflexiverelation,sinceforevery*x*∈*A*,(*x*,*x*)∈*R*1.

(ii). If*R*2={(1, 1),(1,2),(2,3),(3,3)}bearelationon*A*={1,2,3},then*R*2isnota reflexive relation, since for every 2 ∈ *A*, (2, 2) □ *R*2.

**Symmetric**

**Definition:**ArelationRinasetXis**symmetric**ifforeveryxandyinX,whenever x Ry,theny Rx.

**Forexample**

- Therelationequalityofsetis symmetric.
- Therelationofsimilarityinthe setoftrianglesin aplaneis symmetric.
- Therelationofbeingasisterisnotsymmetricinthe setofallpeople.
- However,inthesetfemalesitissymmetric.

Example.If*R*3 ={(1,1),(1,2),(1,3),(2,2),(2,1),(3,1)}bearelationon*A*={1,2,3},then *R*3isasymmetricrelation.

**Transitive**

**Definition:** ArelationRinaset Xis**transitiv**e if,foreveryx, y,andzare inX, whenever x R y and y R z , then x R z.

**Forexample**

- Therelations<and >aretransitivein set ofrealnumbers
- Therelationofsimilarityintheset oftrianglesinaplaneis transitive.
- **Definition:**ArelationRinasetxis **antisymmetric**ifforeveryxandyinX, whenever x R y and y R x, then x = y.

Example: If*R*4=*{*(1,2),(2,2),(2,3)*}*on*A=*{1,2,3*}*isanantisymmetricrelation.

**EquivalenceRelation:**

**Definition**Typeequationhere.**:**ArelationRinasetAiscalledan**equivalence**relationif

- aRaforeveryi.e.Ris reflexive
- aRb=> bRaforeverya,bЄA i.e.Ris symmetric
- aRbandbRc=>aRc foreverya,b,cЄA,i.e.Ris transitive.

**Forexample**

- Therelationequalityofnumbersonsetofrealnumbers.
- Therelation beingparallelonaset oflinesinaplane.

**Problem1:** Let usconsider thesetToftriangles inaplane. Let usdefinearelation R in T

as R= {(a, b) / (a, b Є T and a is similar to b}

Wehavetoshowthat relationRisanequivalence relation

**Solution**:

- Atriangleaissimilartoitself.aRa
- Ifthetrianglea issimilar tothetriangle b, thentriangle bissimilartothetriangleathen a R b => b R a
- Ifa issimilar to bandb issimilar to c,thenaissimilar to c(i.e)aRband bRc=>aR c.

HenceRisanequivalencerelation.

**Problem2: L**et x={1, 2, 3, …7}andR={(x, y) /x–yisdivisible by3} Show that

R is an equivalence relation.

**Solution:** ForanyaЄX, a- aisdivisible by3, Hence a

R a, R is reflexive

Foranya,bЄ X,ifa – b isdivisible by3,thenb– aisalsodivisibleby3,

Rissymmetric.

Foranya,b,c€, ifaRband bRc,thena– bisdivisibleby3and b– cisdivisible by3. Sothat (a –b) +(b–c) isalso divisible by3, hence a – c is also divisible by 3. Thus R is transitive.

HenceRis equivalence.

**Problem3.** Let Z be the set of all integers.Letmbeafixedinteger.Twointegersaandbaresaid to be congruent modulo m if and only if m divides a-b, in which casewe write a≡b (mod m). Thisrelationiscalledtherelationofcongruencemodulomandwecanshowthatisanequivalence relation.

**Solution**:

- a-a=0 and m divides a – a(i.e)aRa,(a,a)€R,Risreflexive.
- a R b=mdivides a-b

mdividesb-a

b ≡a (mod m)

b R a

thatisR issymmetric.

- aRbandbRc =>a≡b(modm)andb≡c(mod m)
  - mdividesa –bandmdividesb-c
  - a–b= kmandb–c= lmforsomek,l€z
  - (a –b)+(b–c)=km+lm
  - a– c=(k +l)m
  - a≡c(modm)
  - aR c
  - Ristransitive

Hencethecongruencerelation isanequivalencerelation.

**EquivalenceClasses:**

Let Rbeanequivalence relationona set A. Foranya∈A, the**equivalence class**generated bya isthesetofallelementsb∈AsuchaRbandisdenoted[a].ItisalsocalledtheR–equivalence class and denoted by a ∈ A.

i.e.,[a]={b∈A/bR a}

Let ZbethesetofintegerandRbetherelationcalled ―congruencemodulo 3

defined byR = {(x, y)/ x∈Z and y∈ Z , (x-y) is divisible by3}

Thentheequivalenceclassesare

[0]={…-6,-3,0,3,6,…}

[1] ={…, -5, -2,1,4,7,…}

[2] ={…, -4, -1,2,5,8,…}

**Composition ofbinaryrelations:**

**Definition:**LetRbe a relationfromX toY andSbe a relationfromYtoZ.Thenthe relationR o S is a relation from X toZ given by R oS = {(x, z) /x∈ □,□ ∈ □S)} is called the composite relation of R and S.

TheoperationofobtainingRo S iscalled the **compositionofrelations**.

**Example**:LetR={(1,2),(3,4),(2,2)} and

S ={(4,2),(2,5),(3,1),(1,3)}

ThenRoS={(1,5),(3,2),(2,5)}andSoR={(4,2),(3,2),(1,4)}

It is to benotedthat Ro S≠So R.

Also Ro(S o T) = (R o S) o T = R o S o T

**Note**: We write RoRasR$^2$;RoRoRasR$^3$andso on.

**Definition**

Let R be a relation fromX to Y, a relation R from Y to X is called the **converse** ofR, wheretheorderedpairsofŘareobtained byinterchangingthe numbers ineachofthe ordered pairs ofR. This means for x ∈ X and y ∈Y,that x R yóyŘ x.

ThentherelationŘisgivenbyR={(x,y) /(y,x)∈R}iscalledthe**converse**ofR

Example:

LetR={(1,2),(3,4),(2,2)}

ThenŘ={(2,1),(4,3),(2,2)}

**Note:**IfRisanequivalence relation,thenŘisalsoanequivalencerelation.

**PartialOrderingRelations:**

**Definition**

Abinaryrelation*R*inaset*P*iscalled a*partial orderrelation* ora*partial ordering*in *P*if*R*isreflexive,antsymmetric,andtransitive. i.e.,

- *aRa*forall*a*∈*P*
- *aRb*and*bRa*→*a*=*b*
- *aRb*and*bRc*→*aRc*

Aset*P*together withapartialordering *R*iscalleda*partialordered set*or *poset*.Therelation*R* isoftendenotedbythe symbol≤which is different fromtheusualless thanequalto symbol. Thus, if≤is a partialorder in *P* , thenthe orderedpair (*P*, ≤) is called a poset.

Example:Showthattherelation‖greaterthanorequalto‖isapartialorderingonthesetof integers.

Solution: Let*Z*bethesetofallintegersandtherelation*R*=≥

(i). Since*a*≥*a*foreveryinteger*a*,therelation ≥isreflexive.

(ii). Let*a*and*b*beanytwointegers. Let

*aRb*and*bRa*→*a*≥*b*and*b*≥*a*

→*a*= *b*

∴ The relation≥ is antisymmetric.

(iii).Let *a*,*b*and*c*beanythreeintegers. Let

*aRb* and *bRc* →*a* ≥ *b* and *b* ≥*c*

→*a*≥c

∴TherelationLet*aRb*and*bRc*→*a*≥*b*and*b*≥*c*

→*a*≥c

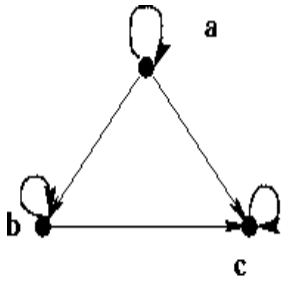
∴Therelation≥istransitive.

Since the relation≥is reflexive, antisymmetric and transitive,≥is partialorderingonthe setof integers. Therefore, (*Z*, ≥) is a poset.

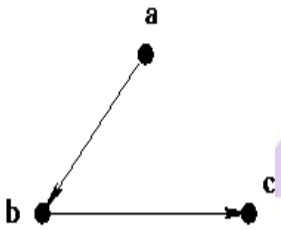
**HasseDiagram:**

AHassediagramis a digraphfor aposet whichdoesnot have loopsandarcs implied bythe transitivity.

Example 10: Fortherelation{<a, a>, <a, b>, <a, c >, <b, b>, <b, c >, <c,c>}onset {a, b,c}, the Hasse diagram has the arcs {< a, b >, < b, c >} as shown below.

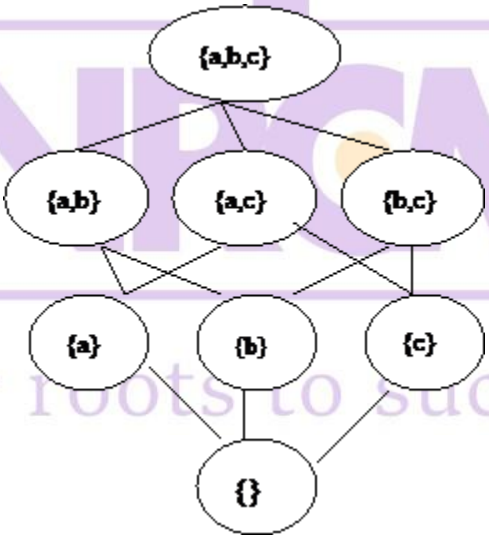



Digraph for Partial Order




Hasse Diagram

**Ex:** Let Abeagivenfiniteset andr(A) itspowerset.Let Íbethesubset relationontheelements of r(A). Draw Hasse diagram of (r(A), Í) for A = {a, b, c}

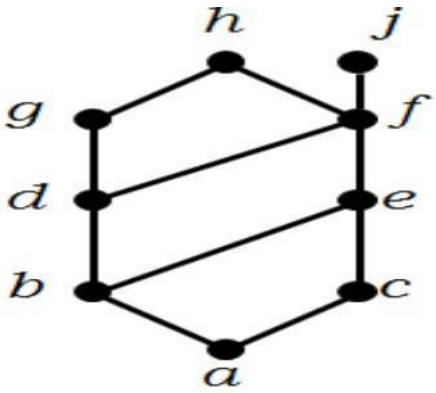


Qn.From the Hasse diagram, find the maximal, Minimal, Greatest element, Least element, Upper bound, Least upper bound, Lower bound and Greatest lower bound of {□,□,□}

order to qualify as a function.

The first condition is that everya Є Amust be relatedto some bЄ B, (i.e) the domainoffmust be Aand not merelysubset ofA. The second requirement ofuniquenesscanbe expressed as(a, b) Є f ^(b, c) Є f => b = c

IntuitivelyafunctionfromasetAtoasetBisarulewhichassignstoeveryelementofA,a




| | |
|---|---|
| **Maximal** | **: h, j** |
| **Minimal** | **: a** |
| **Greatest element** | **: None** |
| **Least element** | **: a** |
| **Upper bound of {a,b,c}** | **: e, f, j, h** |
| **Least upper bound of {a,b,c}** | **: e** |
| **Lower bound of {a,b,c}** | **: a** |
| **Greatest lower bound of {a,b,c}** | **: a** |

**Functions:**

**Introduction**

A function is a special type of relation. It may be consred as a relation in which each elementof the domain belongsto only one ordered pairin the relation.Thus a function from A to Bis a subset ofAX Bhaving the propertythatfor eacha ЄA, there is one and onlyone bЄ B such that (a, b) Î G.

**Definition**

LetA and B be any two sets. A relation f from A toB is calleda functionif for everya Є A there is a unique b Є B such that (a, b) Є f .

Notethatthedefinitionoffunctionrequires thatarelationmustsatisfytwo additional conditions in

unique element ofB. Ifa ЄA, then the unique element ofB assigned to a under f is denoted by f (a).The usual notation for a function f from Ato B is f: A® B defined by a ® f(a) where a Є A, f(a) is called the image of**a** under fand **a** is called pre image of f(a).

- LetX=Y=**R**andf(x) = $x^2$ +2.D(f)=RandR(f) ⊆R.
- LetXbethesetofallstatementsinlogicandletY={True,False}.

Amapping f:X®Y isafunction.

- Aprogramwritteninhighlevellanguageismappedintoamachinelanguagebya compiler. Similarly, the output from a compiler is a function of its input.
- LetX=Y=**R**andf(x)=$x^2$isafunctionfromX®Y,andg($x^2$)=xisnotafunction from X ® Y.

Amappingf:A®Biscalled **one-to-one**(injectiveor1–1)ifdistinct elements of A aremapped into distinct elements of B. (i.e) f is one-to-one if

a1=a2 =>f(a1)=f(a2)orequivalentlyf(a1)¹f(a2)=>a1 ¹ a2

***Forexample***,f:N® Ngivenbyf(x)=x is1-1whereNisthe set ofa naturalnumbers.

Amappingf:A®Biscalled**onto**(surjective)ifforeverybЄBthereisanaЄAsuchthat f(a) = B. i.e. if everyelement ofB has a pre-image in A.Otherwise it is called **into**.

***Forexample***,f:Z®Zgivenbyf(x)=x+1isanonto mapping.A mapping is both 1-1 and onto is called bijective

.

***Forexample*** f:R®Rgivenbyf(x)=X+1 is bijective

**Definition:** Amapping f: R® bis called a **constant mapping**if, foralla Є A, f(a)= b,afixed element.

***Forexample*** f: Z®Zgivenbyf(x)=0,forallxЄ Zisaconstantmapping.

.

**Definition**

Amapping f: A®Aiscalledthe **identitymapping** ofAiff(a) =a, for all a Є A. Usually it is denoted byIA or simply I.

**Composition offunctions:**

Iff: A®Band g: B®Caretwofunctions, thenthecompositionoffunctions fand g, denoted byg o f, is the function is given by g o f: A®C and is given by

go f={(a, c)/ aЄA^cЄ C^bЄB': f(a)=b^g(b) = c} and (g of)(a) = ((f(a))

**Example1:**ConsiderthesetsA={1,2,3},B={a,b}andC={x,y}.

Let f: A®Bbedefined byf(1) =a;f(2)=band f(3)=band Let g: B® C be defined by g(a) = x and g(b) = y

(i.e)f={(1,a),(2,b),(3,b)}andg={(a,x), (b,y)}. Thengof:A®Cisdefinedby

(gof)(1)= g(f(1))= g(a)=x

(gof)(2)=g(f(2))= g(b)=y

(go f)(3)=g(f(3))=g(b)= y i.e., g o f = {(1, x), (2, y),(3, y)}

Iff:A®Aand g:A®A,whereA={1,2, 3},aregiven by

f= {(1,2), (2,3),(3,1)} and g= {(1,3),(2,2),(3,1)}

Thengof={(1,2),(2,1),(3,3)},fog={(1,1),(2,3),(3,2)}

fof= {(1,3),(2,1),(3, 2)}andgog= {(1,1),(2,2),(3,3)}

**Example 2:** Let f(x) = x+2, g(x) = x – 2 and h(x) = 3x for x Î R, where R is the set of realnumbers.

Thenf o f = {(x, x+4)/x Є R}f

o g = {(x, x)/ x Є X}

gof= {(x,x)/xЄX}

gog={(x,x-4)/xЄX}

ho g={(x,3x-6)/xЄX} h o

f = {(x, 3x+6)/ x Î Є X}

**Inversefunctions:**

Let f: A® B be aone-to-one and onto mapping. Then, itsinverse, denotedbyf-1 isgivenbyf- 1 = {(b, a) / (a, b) Єf} Clearly $f^{-1}$: B® A is one-to-one and onto.

Also weobservethat fo $f^{-1}$=IBand $f^{-1}$o f=IA.If $f^{-1}$ exists then f is called **invertible**.

***Forexample:L***etf:R®Rbedefinedbyf(x)=x+2 Then f -1: R® R is defined by $f^{-1}(x) = x - 2$

**Theorem:** Let f:X®Yandg: Y®Zbetwoonetooneandontofunctions. Thengofisalsoone to one and onto function.

**Proof**

Letf:X ®Yg:Y ®Zbetwoonetooneandonto functions.Letx1,x2ЄX

- gof(x1)=gof(x2),
- g(f(x1))= g(f(x2)),
- g(x1)=g(x2)since[fis1-1]

x1 = x2 since [ g is 1-1}

so that gof is 1-1.

Bythe definitionofcomposition,gof:X ®Zisafunction.

We have to prove that everyelement ofz ЄZ an image element for some x Є Xunder gof.

Sinceg isonto$yЄ Y':g(y)= zand fisontofromX toY,

$ xЄ X':f(x)= y.

Now,gof(x)=g(f(x))

=g(y) [sincef(x)=y]

=z[since g(y)= z]

whichshowsthat gofisonto.

**Theorem** $(gof)^{-1} = f^{-1}og^{-1}$

(i.e) theinverseofacomposite functioncanbeexpressed intermsofthe composition of the inverses in the reverse order.



**Proof.**

f: A ® B is one to one and onto.

g: B® Cisonetooneand onto.

gof: A ® C is also one to one and

onto.Þ$(gof)^{-1}$:C®Aisonetooneandonto.

Letaє A,thenthereexistsanelement bє**B**suchthatf(a)=bÞa=$f^{-1}$(b). Nowb є B Þthere exists anelement c єC suchthat g(b) = cÞ b=$g^{-1}$ (c). Then (gof)(a) = g[f(a)] = g(b) = c Þ a = (gof) $^{-1}$(c) .....(1)

($f^{-1}og^{-1}$)(c)=$f^{-1}$($g^{-1}$(c))=$f^{-1}$(b)= a Þa=($f^{-1}$o $g^{-1}$)(c)….(2)

Combining(1)and (2),wehave

(gof)$^{-1}$ =$f^{-1}$ o$g^{-1}$

**Theorem:**Iff: A®Bisaninvertible mapping,then f o f $^{-1}$ = I B and $f^{-1}$ o f = IA

**Proof:** fis invertible, then f-1 is defined byf(a) = bó $f^{-1}$(b) =a where a Є A and b Є B .

Nowwehavetoprovethatfo$f^{-1}$=IB.

Let bє Band $f^{-1}$(b) = a,a ÎA

thenfo$f^{-1}$(b) = f($f^{-1}$(b))

=f(a)=b

therefore fo f-1 (b) = b " b Î B => fo f $^{-1}$ = IB

Now$f^{-1}$o f(a)= $f^{-1}$ (f(a))= $f^{-1}$ (b)=atherefore f -1 o f(a) = a " a єA => f $^{-1}$ o f = IA.

**LatticeanditsProperties:**

**Introduction**:

A lattice ispartiallyordered set (L, £) inwhicheverypair ofelementsa, bÎLhasagreatest lower bound and a least upper bound.

Theglbofasubset,{a,b}ÍLwillbedenotedbya*bandthelubbya Åb.

.

Usually, foranypaira,bÎL,GLB{a,b}=a*b, iscalledthe **meet**or**product**andLUB{a, b} = a Åb, is called the **join** or **sum** of a and b.

**Example**1Consideranon-emptysetSandletP(S)beitspowerset.TherelationÍ—contained in‖ is a partialordering onP(S). Foranytwo subsets A, BP(S)

GLB{A,B} and LUB{A,B} areevidentlyAÇBandAÈBrespectively.

**Example**2Let I+bethesetofpositive integers,andDdenotetherelationof‘division’in I+ such that for anya, b Є I+ , a D b iff a divides b. Then (I+, D) is a lattice in which thejoinofaandb isgivenbytheleastcommon multiple(LCM)ofaand b,thatis, a Å b = LCM ofa and b, and the meet ofa and b, that is , a * b is the greatest common divisor (GCD) of a and b.

Alattice can beconveniently represented byadiagram.

**Forexample**, let Snbethesetofalldivisorsofn, wherenisapositive integer. Let Ddenotethe relation—division‖ suchthat foranya, bcdi654p[]

**Fda21** **A**Sn,aDbiffadividesb.

Then(Sn,D)isalatticewitha*b=gcd(a,b)anda Åb= lcm(a,b).

Taken=6. ThenS6 ={1, 2,3,6}.It canberepresentedbyadiagramin Fig(1). Take n=8.

Then S8 = {1, 2, 4, 8}

Twolatticescanhavethesamediagram. For example ifS={1, 2, 3}then(p(s), Í) and(S6,D) have the same diagram viz. fig(1), but the nodes are differently labeled .

We observe that for any partial ordering relation £ on a set S the converserelation³isalso partialordering relation onS.If(S, £)isa lattice With meet a * b and join a Å b , then (S, ³ ) is the also a lattice with meet aÅband joina* bi.e., theGLB and LUBget interchanged. Thus wehave the principle of duality of lattice as follows.

Anystatement about latticesinvolvingtheoperations ^andVandtherelations£ and³remains true if ^, V, ³ and£ are replaced by V, ^,£ and³ respectively.

Theoperation^and Vare calleddualsofeachotherasaretherelations £and³..

Also,thelattice(L,£)and(L,³)arecalledthedualsofeachother.

**Propertiesoflattices**:

Let(L,£)bealatticewiththebinaryoperations* andÅthen for anya,b,cÎL,

- a*a=a aÅa = a (Idempotent)
- a*b=b*a, aÅb =bÅ a (Commutative)
- (a*b)*c=a*(b*c),(aÅ)Åc=aÅ(bÅc)

o(Associative)

- a*(aÅb)=a , aÅ(a*b)=a (absorption)

ForanyaÎL,a£a,a£LUB{a,b}=>a£a*(aÅb).Ontheotherhand,GLB {a, aÅb}£ ai.e.,(aÅb)Åa,hencea* (a Åb)= a

**Theorem1**

Let (L, £)bea latticewiththe binaryoperations*and Ådenotetheoperationsofmeet and join respectivelyFor any a, b Î L,

a£bóa*b=aóaÅb=b

**Proof**

Supposethat a£b. weknowthat a£a, a£GLB {a, b}, i.e., a£a*b. But from the definition of a * b, we get a * b £ a.

Hence a £ b => a * b = a…… (1)

Now we assume that a * b = a; but is possible only if a £ b,

that isa * b = a=> a £ b …… (2)

From (1) and (2), we get a £ bó a * b = a.

Supposea*b=a.

then b Å (a * b) = b Å a=a Å b ……. (3)

but b Å ( a * b) = b( by iv)…….. (4)

Hence a Å b = b, from (3) => (4)

SupposeaÅb=b,i.e.,LUB{a,b}=b,thisispossibleonlyifa£b,thus(3)=>(1)

(1)=>(2)=>(3)=>(1).Hencethesareequivalent.

Let us assume a * b = a.

Now(a*b)Åb=aÅb

Weknowthatbyabsorption law,(a*b)Åb= b

so that a Å b= b, therefore a * b = a Þ a Å b= b (5)

similarly, we can prove a Å b= b Þa * b = a (6)

From (5) and (6), we get

a*b=aÛaÅb=b Hence the

theorem.

**Theorem2** Foranya,b,cÎL,where(L,£)isa lattice. b £ c

=>{a * b £a * c and

{aÅb£aÅc

**Proof** Supposeb£c.wehaveprovedthatb£aób*c=b…..... (1)

Nowconsider

(a*b)*(a*c)=(a*a)*(b*c) (byIdempotent)

=a*(b*c)

=a*b (by(1))

Thus (a *b) * (a * c ) = a * bwhich => (a * b )£ (a * c)

Similarly(a Å b) Å ( a Å c) = (a Å a) Å (b Åc)

=aÅ(bÅc)

=aÅc

which=> (a Åb ) £ (a Åc )

note:Thesepropertiesareknownas**isotonicity.**

## UnitIII:
## AlgebraicStructures

**Syllabus:**

**Algebraic structures:** Algebraic systemswithexamplesandgeneralproperties, semigroupsand monoids,groups&itstypes,IntroductiontohomomorphismandIsomorphism(Proofoftheorems are not required)

### Algebraicsystems

**N={1,2,3,4,…..}=Setofallnaturalnumbers.**

**Z={0,±1,±2,±3,±4,…..}=Setofallintegers. Q= Set of all rational numbers.**

**R=Setofallrealnumbers.**

**BinaryOperation:** Thebinaryoperator*issaidto beabinaryoperation(closedoperation) on anon-emptysetA, ifa*b∈Aforalla, b ∈ A (Closure property).

Ex:ThesetNisclosedwithrespecttoadditionandmultiplicationbut not w.r.t subtraction and division.

Algebraic System: Aset Awithone or more binary(closed) operations defined on it is called an algebraic system.

Ex:(N,+),(Z,+,–),(R,+,.,–)arealgebraic systems.

### Properties

**Associativity:**Let*be abinaryoperationona set A.The operation * issaid to be associative inA .

if(a*b)*c=a*(b*c)foralla,b,cinA

**Identity:**Foran algebraicsystem(A,*)**,**an elementeinAissaidtobeanidentityelementof A if a * e = e * a = a for alla ∈ A.

**Note:**Foranalgebraic system(A,*),theidentityelement,ifexists,is unique.

**Inverse:**Let(A,*)beanalgebraicsystemwithidentity_e‘.LetabeanelementinA.An element b issaid to be inverse ofA.

ifa*b=b*a=e

## Semigroups

**SemiGroup:**Analgebraicsystem(A,*) issaid tobeasemigroupif

1. *isclosedoperationonA.
2. *isanassociativeoperation,foralla,b,cin

A.Ex.(N,+)isasemigroup.

Ex.(N,.)isasemigroup.

Ex.(N,–)isnot asemigroup.

## Monoid

Analgebraicsystem(A,*)issaid tobea**monoid**ifthefollowingconditionsare satisfied.

1) *isaclosedoperationinA.
2) *isanassociativeoperationinA.
3) Thereisan identityinA.

Ex.Showthattheset Nisamonoidwithrespectto multiplication. Solution: Here, N ={1,2,3,4,……}

Closureproperty:1Weknowthatproductoftwo naturalnumbersisagain anatural number.

i.e.,a.b=b.aforalla,b∈N

∴Multiplicationisaclosedoperation.

Associativity:Multiplicationofnaturalnumbersisassociative.

i.e.,(a.b).c=a.(b.c)foralla,b,c∈N Identity:

We have, 1 ∈ N suchthat

a.1=1.a= aforalla ∈N.

∴Identityelementexists,and1istheidentityelement.

Hence,Nisamonoidwithrespectto multiplication

## Examples

Ex. Let (Z,*)beanalgebraicstructure,whereZisthesetofintegers

andtheoperation*isdefinedby n*m=maximumof(n,m).

Showthat(Z,*)isasemigroup.Is(Z,*) amonoid?.Justifyyouranswer.

Solution:Leta,bandcareanythreeintegers.

Closureproperty:Now,a*b=maximumof(a,b)∈Z foralla,b∈Z

Associativity:(a*b)*c=maximumof{a,b,c}=a *(b*c)

∴(Z,*)isasemigroup.

Identity: Thereisnointeger xsuchthat

a*x=maximumof(a,x)=a foralla∈Z

∴Identityelementdoesnotexist.Hence,(Z,*)isnotamonoid.

Ex.ShowthatthesetofallstringsSisamonoidundertheoperationconcatenation of strings‘.

IsSagroupw.r.ttheaboveoperation?Justifyyouranswer.

Solution:Letusdenotetheoperation

_concatenationofstrings‘by+.

Lets$_1$,s$_2$,s$_3$arethree arbitrarystringsin S.

Closureproperty:Concatenationoftwostringsisagainastring.i.e., $s_1+s_2 \in S$

Associativity:Concatenationofstringsisassociative. ($s_1+s_2$) + $s_3$= $s_1$+ ($s_2$+ $s_3$)

Identity:Wehavenullstring,l∈Ssuchthats$_1$+l=S.

∴Sisamonoid.

Note: S is nota group, because the inverse of a nonempty string does not existunder concatenation ofstrings.

## Groups

**Group:**Analgebraicsystem(G,*)issaidtobea**group**ifthefollowingconditionsaresatisfied.

1) *isaclosedoperation.
2) *isanassociativeoperation.
3) ThereisanidentityinG.
4) Everyelement inGhasinverseinG.

**Abelian group (Commutative group):**Agroup (G, *)said to be***abelian*** (or ***commutative)***if

a*b=b*a foralla,b∈G.

Properties

InaGroup (G,*)thefollowingpropertieshold good

1. Identityelementisunique.
2. Inverseofanelementisunique.
3. Cancellationlawsholdgood

a*b=a*c=>b=c (leftcancellationlaw)

a*c=b*c=> a=b(Rightcancellation law)

4. $(a*b)^{-1}=b^{-1}*a^{-1}$

Inagroup,theidentityelementisitsowninverse.

***Orderof agroup***:The number ofelements inagroupiscalled order ofthe group.Finitegroup:IftheorderofagroupGisfinite,thenGiscalledafinite group.

Ex1.Showthat,theset ofallintegersisanabelian group withrespectto addition.

Solution:LetZ=setofallintegers.

Leta,b,careanythreeelementsofZ.

1. Closure property: We know that,Sum of two integersis again aninteger. i.e., a + b ∈ Zfor all a,b ∈ Z
2. Associativity:Weknowthatadditionofintegersis

associative.i.e.,(a+b)+c=a+(b+c)foralla,b,c∈

Z.

3. Identity:Wehave0∈Zanda+0=aforalla∈Z.

∴Identityelementexists,and‗0'istheidentityelement.

4. Inverse:Toeacha∈Z,wehave–a∈Zsuch that

a+(–a)=0

EachelementinZhasaninverse.

5. Commutativity:Weknowthatadditionofintegersis

commutative. i.e., a + b = b +a for alla,b∈ Z.

Hence, ( Z , + ) is an abelian group.

Ex2.Showthatsetofallnonzerorealnumbersisagroupwithrespecttomultiplication.

Solution:Let$R^*$=setofallnonzerorealnumbers.

Leta,b,care anythreeelementsof$R^*$.

1. Closure property:Weknowthat,productoftwononzerorealnumbersisagainanonzeroreal number.

i.e.,a.b∈$R^*$for alla,b∈$R^*$.

2. Associativity:Weknowthatmultiplicationofrealnumbersis

associative.

i.e.,(a.b).c=a.(b.c)foralla,b,c∈$R^*$.

3. Identity:Wehave1∈$R^*$anda.1=aforalla∈$R^*$.

∴Identityelementexists,and‗1'istheidentityelement.

4. Inverse:Toeacha∈$R^*$, wehave1/a∈$R^*$suchthat
a.(1/a)=1i.e.,EachelementinR*hasan inverse.

5. Commutativity: Weknowthatmultiplicationofreal

numbers iscommutative.

i.e., a . b = b . a foralla,b∈$R^*$.

Hence,$(R^*,.)$isanabeliangroup.

**Note:**Showthatsetofallrealnumbers_Risnotagroupwithrespecttomultiplication.

Solution:Wehave$0 \in R$.

Themultiplicativeinverseof0doesnotexist.

Hence.Risnotagroup.

**Example:** Let S be a finite set, and let F(S) be the collection of all functions f: S → S under the operation of composition of functions, then show that F(S) is a monoid.

Is S a group w.r.t the above operation? Justify your answer.

**Solution:** Let $f_1, f_2, f_3$ are three arbitrary functions on S.

Closure property: Composition of two functions on S is again a function on S.

i.e., $f_1 o f_2 \in F(S)$

Associativity: Composition of functions is associative.

i.e., $(f_1 o f_2) o f_3 = f_1 o (f_2 o f_3)$ Identity:

We have identity function I : S → S

such that $f_1 o I = f_1$.

∴ F(S) is a monoid.

Note: F(S) is not a group, because the inverse of an bijective function on S does not exist.

**Ex.** If M is set of all non singular matrices of order $n \times n$'. then show that M is a group w.r.t. matrix multiplication. Is (M,*) an abelian group?. Justify your answer.

Solution: Let A, B, C ∈ M.

1. Closure property: Product of two non-singular matrices is again a non-singular matrix, because $|AB| = |A||B| \neq 0$ (Since, A and B are non singular)

i.e., AB ∈ M for all A, B ∈ M .

2. Associativity: Matrix multiplication is associative.

i.e., (AB)C = A(BC) for all A, B, C ∈ M.

3. Identity: We have $I_n \in M$ and $AI_n = A$ for all $A \in M$.

∴ Identity element exists, and '$I_n$' is the identity element.

4. Inverse: To each $A \in M$, we have $A^{-1} \in M$ such that $AA^{-1} = I_n$ i.e., Each element in M has an inverse.

∴ M is a group w.r.t. matrix multiplication.

We know that, matrix multiplication is not commutative. Hence, M is not a abelian group.

Ex.Showthatthesetofallpositiverationalnumbersformsanabeliangroupunderthecomposition *definedby

a *b=(ab)/2.

Solution:LetA=set ofallpositiverationalnumbers.

Leta,b,cbe anythree elementsofA.

1. Closureproperty:Weknowthat,Productoftwopositiverationalnumbersisagaina rationalnumber.

i.e.,a*b∈Afor alla,b∈A.

2. Associativity: (a*b)*c= (ab/2)*c=(abc)/4

a*(b*c)=a*(bc/2)=(abc)/4

3. Identity: Letebethe identityelement.

We havea*e=(ae)/2…(1),Bythedefinitionofagain, a*e=a

…..(2),Since e is the identity.

From(1)and(2),(ae)/2=a ⇒e=2and2∈A

∴Identityelementexists,and2istheidentityelementinA.

4. Inverse:Leta∈Aletussupposebisinverseofa

. Now, a*b=(ab)/2….(1)(Bydefinitionofinverse.)

Again,a*b=e=2…..(2)(Bydefinitionofinverse) From (1) and (2),itfollows that(a b)/2 =2

=> b=(4/a)∈ A

∴(A,*)isagroup.

Commutativity: a*b=(ab/2)=(ba/2)=b*a

Hence,(A,*)isanabelian group.

Finite groups

Ex. ShowthatG= {1,-1} isanabeliangroupundermultiplication.

Solution:ThecompositiontableofGis

| * | 1 | -1 |
|---|---|---|
| 1 | 1 | -1 |
| -1 | -1 | 1 |

1. Closureproperty:Sincealltheentriesofthecompositiontablearetheelementsofthe givenset,theset Gisclosedunder multiplication.

2. Associativity:TheelementsofGarerealnumbers, andweknowthatmultiplication ofreal numbers is associative.

3. Identity:Here, 1istheidentityelementand1∈G.

4. Inverse:Fromthecompositiontable,weseethattheinverseelementsof

1 and– 1are1 and– 1 respectively.

Hence, G is a group w.r.t multiplication.

5. Commutativity:The corresponding rows and columns of the table are identical. Therefore the binary operation . is commutative.

Hence,G isanabeliangroup w.r.t. multiplication..

Ex.Showthat G={1, w,$w^2$}isanabeliangroupundermultiplication.
Where1,w,$w^2$arecuberootsofunity.

Solution:ThecompositiontableofGis

| * | 1 | w | $w^2$ |
|---|---|---|---|
| 1 | 1 | w | $w^2$ |
| w | w | $w^2$ | 1 |
| $w^2$ | $w^2$ | 1 | w |

1. Closureproperty:Sincealltheentriesofthecompositiontablearetheelementsofthegiven set,theset Gisclosed under multiplication.

2. Associativity:TheelementsofGarecomplexnumbers,andweknowthat multiplication of complex numbers is associative.

3. Identity:Here, 1istheidentityelementand1∈G.

4. Inverse:Fromthecompositiontable,weseethattheinverseelementsof1 w,$w^2$are 1, $w^2$,

w respectively.

Hence,Gisagroupw.r.tmultiplication.

5. Commutativity:Thecorrespondingrowsandcolumnsofthetableareidentical. Therefore the binary operation . is commutative.

Hence,G isanabeliangroup w.r.t. multiplication.

**Modulosystems**

**Additionmodulom($+_m$)**

letmisapositiveinteger.Foranytwopositiveintegersaandb

$a +_m b = a+b$ ifa+b<m

$a +_m b =$ r ifa+ b>mwhereristheremainderobtained bydividing(a+b)withm.

Multiplicationmodulop(*m)

letpisapositiveinteger.Foranytwopositiveintegersaandba

*mb=ab ifab<p

a*mb= r ifa b ³ p whereristheremainderobtained

by dividing (ab) with p.

Ex.$3*_5 4=2$ , $5*_5 4=0$ , $2*_5 2=4$

Example:ThesetG={0,1,2,3,4,5}isagroupwithrespecttoadditionmodulo6.

Solution:ThecompositiontableofGis

| $+_6$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 5 | 0 | 1 | 2 | 3 | 4 |

1. Closureproperty:Sincealltheentriesofthecompositiontablearetheelementsofthe givenset, theset Gisclosed under$+_6$.

2. Associativity:Thebinaryoperation$+_6$isassociativeinG.

forex.$(2+_63)+_64$ $=5+_64=3$ and

$2+_6(3+_64)=2+_61=3$

3. Identity:Here,Thefirstrowofthetablecoincideswiththetoprow.Theelement headingthat row,i.e., 0isthe identityelement.

4. .Inverse: From the composition table, we see that the inverse elements of 0,1,2,3,4.5 are0, 5,4, 3, 2, 1 respectively

5. Commutativity:Thecorrespondingrowsandcolumnsofthetableareidentical.Thereforethe binaryoperation $+_6$is commutative.

Hence,$(G,+_6)$isanabeliangroup.

Example:ThesetG={1,2,3,4,5,6}isagroupwithrespecttomultiplicationmodulo7.

Solution:ThecompositiontableofGis

| $*_7$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 4 | 1 | 5 | 2 | 6 | 3 |
| | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

1. Closureproperty:Sincealltheentriesofthecompositiontablearetheelementsofthegivenset, the set G isclosed under $*_7$.

2. Associativity:Thebinaryoperation$*_7$isassociativeinG.

forex.$(2*_73)*_74$ $=6*_74=3$ and

$2*_7(3*_74)=2*_75=3$

3. Identity:Here,Thefirstrowofthetablecoincideswiththetoprow.Theelement headingthat row,i.e., 1isthe identityelement.

4. .Inverse:Fromthecompositiontable,weseethattheinverseelementsof1,2,3,4.56are1,4, 5,2,5,6respectively.

5. Commutativity:Thecorrespondingrowsandcolumnsofthetableareidentical.Thereforethebinary operation $*_7$is commutative.

Hence,$(G,*_7)$isanabeliangroup. More on

finite groups

Inagroupwith2elements,eachelementisitsowninverse

Inagroupofevenordertherewillbeatleastoneelement(otherthanidentityelement) which is itsowninverse The

set G = {0,1,2,3,4,…..m-1} is a group with respect to addition modulo m.

The set G = {1,2,3,4,….p-1} is a group with respect to multiplicationmodulo p,wherep isa primenumber.

## Subgroups

Def.AnonemptysubsetHofagroup(G,*)is asubgroupofG,

if(H,*)isagroup.

Note:Foranygroup{G,*},{e,*}and(G,*)aretrivialsubgroups.

Ex.G ={1,-1,i,-i}isagroupw.r.tmultiplication.

$H_1$={1,-1 }isasubgroupofG.$H_2$={1}isatrivialsubgroupofG.

Ex.(Z,+)and(Q,+)aresubgroupsofthegroup(R+).Theorem:Anon-emptysubsetHofagroup

(G,*)isasubgroupofGiff

i) a*b∈H ∀a,b∈H

i) $a^{-1}$∈H ∀a∈H

## HomomorphismandIsomorphism

**Homomorphism:**Considerthegroups(G,*) and ($G^1$,⊕)

A functionf:G→$G^1$is calledahomomorphismiff(a*b)=f(a)⊕f(b)

**Isomorphism**:Ifa homomorphismf:G→$G^1$ isa bijectionthenfiscalledisomorphism betweenG and $G^1$ .

ThenwewriteG≡$G^1$

Example:LetRbeagroupofallrealnumbersunderadditionand$R^+$beagroupofallpositivereal numbersundermultiplication.Showthatthemapping f:R→$R^+$definedbyf(x)=$2^x$forallx∈ R is anisomorphism.

Solution:First,letusshowthatfisahomomorphism.Leta,b∈ R. Now,

$f(a+b) = 2^{a+b}$

$= 2^a 2^b$

=f(a).f(b)

∴fisanhomomorphism.

Next, let usprovethat fis a Bijection.For

any a , b ∈ R, Let, f(a) = f(b)

$=>2^a=2^b$

=>a=b

∴fisone.to-one.

Next,take anyc ∈$R^+$.

Then$\log_2 c$∈Randf($\log_2 c$)=$2^{\log 2c}$=c

.

⇒EveryelementinR$^+$hasapreimageinR.i.e.,fis onto.

∴fisabijection.

Hence,fisan isomorphism.

Ex.LetRbe agroupofallrealnumbers underadditionand$R^+$be a groupofallpositivereal Numbers under multiplication. Show that the mapping f:$R^+$→Rdefinedbyf(x)=$\log_{10}$xfor all x ∈ R is an isomorphism.

Solution: First, let usshowthat fisahomomorphism.Let a , b

∈ $R^+$ .

Now,f(a.b) =$\log_{10}$(a.b)

$=\log_{10}a+\log_{10}b$

=f(a) + f(b)

∴fisanhomomorphism.

Next,letusprove that fisa

Bijection.Forany $a,b \in R^+$ ,Let,

$f(a)=f(b)$

$\Rightarrow \log_{10} a=\log_{10} b$

$\Rightarrow a=b$

$\therefore$ fisone.to-one.

Next,take

any$c \in R$.hen$10^c \in R$and$f(10^c)=\log_{10} 10^c=c$.EveryelementinRhasaprei

mageinR$^+$. i.e., f is onto.

$\therefore$ fisabijection.

Hence,fisan isomorphism.

## UNIT-IV
## COMBINATORICS

**Basisofcounting:**

**If X is a set, let us use |X|to denote the numberof elements in X.**

**Two Basic Counting Principles**

Twoelementaryprinciplesactas—buildingblocks<forallcountingproblems.The first principle says that the whole is the sum of its parts; it is at once immediate and elementary.

**SumRule**: **Theprinciple ofdisjunctivecounting**:

IfasetXisthe unionofdisjointnonemptysubsetsS1,…..,Sn,then| X |=| S1|+ |S2|+…..+ |Sn|.

WeemphasizethatthesubsetsS1,S2,….,Snmusthavenoelementsincommon. Moreover, sinceX=S1US2U……USn, eachelement ofX is inexactlyoneofthe subsets Si. In other words, S1, S2, …., Sn is a partition of X.

If the subsets S1, S2, …., Sn were allowed to overlap, then a more profoundprinciplewillbeneeded--theprincipleofinclusionand exclusion.

Frequently,insteadofasking forthenumberofelements inaset*perse*,someproblems ask for how many ways a certain event can happen.

Thedifference is largelyinsemantics, forifAisanevent, wecanlet Xbethesetofways that Acan happenand count the number ofelements inX. Nevertheless, let us statethe sumrule for counting events.

IfE1,……,Enaremutuallyexclusiveevents,andE1canhappene1ways,E2happen e2 ways,…. ,En can happen en ways, E1or E2 or …. or En can happen e1 + e2 + ……..+enways.

Againweemphasizethat mutuallyexclusiveeventsE1andE2 meanthat E1orE2can happen but both cannot happen simultaneously.

Thesumrulecanalsobeformulatedintermsofchoices:Ifanobjectcanbe selected fromareservoirine1waysandanobjectcanbeselectedfromaseparatereservoirine2ways andanobject canbeselectedfromaseparatereservoir ine2ways, thenthe selectionofone object fromeither one reservoir or the other can be made in e1 + e2 ways.

**Product Rule:Theprincipleofsequencingcounting**

IfS1, …..,Snar**enonemptysets, thenthenumberofelementsin theCartesian product S1 x S2 x ….. x Sn is the product ∏in=1 |S i |. That is,**

|S1xS2x .....xSn|=**∏in=1|Si|.**

Observe that there are 5 branches in the first stage corresponding to the 5 elements of S1 and to each of these branches there are 3 branches in the second stage corresponding to the 3 elements ofS2 giving a totalof15 branches altogether. Moreover, the Cartesian product S1 x S2 can be partitioned as (a1 x S2) U (a2 x S2) U(a3 xS2) U (a4 x S2) U (a5 x S2), where (aix S2) ={(ai,b1),(aii,b2),(ai,b3)}.Thus, forexample,(a3xS2)correspondstothe thirdbranchin the first stage followed by each of the 3 branches in the second stage.

Moregenerally, ifa1,….., anarethendistinct elementsofS1and b1,….,bmarethe m distinct elements of S2, then S1 x S2 = Uin =1 (ai x S2).

For if x is an arbitrary element of S1 x S2 , then x = (a, b) where a Î S1 and b Î S2. Thus, a=aifor some iandb= bjfor some j. Thus, x=(ai, bj) Î(aixS2) and therefore x Î Uni =1(ai x S2).

Conversely, if xÎ Uin=1(aixS2), thenxÎ (aixS2) for some iand thus x=(ai, bj) where bj is some element of S2. Therefore, x Î S1 x S2.

Nextobserve that (aixS2)and(ajxS2)aredisjointifi≠jsinceif

xÎ (aixS2) ∩(ajxS2) thenx=( ai, bk) for somekand x=(aj, b1) for some l.

Butthen(ai,bk) =(aj,bl)impliesthat ai=ajand bk=bl.But sincei≠j,ai≠aj.

Thus, weconcludethatS1xS2isthedisjointunionofthesets(aix S2). Furthermore|aixS2|=|S2|sincethere isobviouslyaone-to-onecorrespondence between the sets ai x S2 and

S2,namely,(ai,bj)→bj.

Thenbythe sumrule |S1xS2|= ∑nni=1|aixS2|

7.(nsummands)|S2| +|S2| +…….....+ |S2|

8. n|S2|
9. nm.

Therefore, wehaveproventheproductrule fortwo sets. Thegeneralrule follows by mathematical induction.

Wecanreformulatetheproduct rule in termsofevents. IfeventsE1, E2, ….,En can happen e1, e2,…., and en ways, respectively, then the sequence of events E1 first,

followedbyE2,…., followed byEncanhappene1e2…enways.

Interms ofchoices, the product rule is statedthus: Ifa first object can be chosen e1ways, **a second e2 ways , …, and an nth object can be made in e1e2….en ways.**

**Combinations&Permutations**

**Definition**.

Acombinationofnobjectstakenr at atime (called anr-combinationofnobjects) is an unordered selection of r of the objects.

Apermutationofnobjectstakenr at atime (also called anr-permutationof n objects) is an ordered selection or arrangement of r of the objects.

Notethat wearesimplydefiningthetermsr-combinationsandr-permutations here and have not mentioned anything about the properties ofthe n objects.

Forexample,thesedefinitionssaynothingaboutwhetherornot agivenelement may appear more than once in the list of n objects.

Inother words, it maybethatthenobjectsdo notconstituteasetinthe normalusageof the word.

General formulas for enumerating combinations and permutations will now be presented. At this time, we will only list formulas for combinations and permutations without repetitionsor withunlimited repetitions. We willwait until laterto use generating functionsto give generaltechniques for enumerating combinationswhere otherrulesgovernttheselections.

LetP(n,r)denotethenumberofr-permutationsofnelements withoutrepetitions.

**Theorem5.3.1.**(Enumeratingr-permutationswithout repetitions).

P(n,r)=n(n-**1)…….(n**–r+1)=n!/(n-r)!

Proof.Sincetherearendistinct objects,the first positionofanr-permutationmaybe filled in nways.Thisdone,thesecondpositioncanbefilledinn-1wayssincenorepetitionsareallowed and there aren– 1objectsleft to choose from. Thethird canbe filled inn-2 ways. Byapplying the product rule, we conduct that

P (n, r) = n(n-1)(n-2)……. (n – r + 1).

Fromthedefinitionoffactorials,itfollows that

P(n,r)=n!/(n-r)!

Whenr =n,this formula becomes

P (n,n)=n!/ 0!= n!

Whenweexplicit referencetorisnot made, weassumethat alltheobjectsareto be arranged; thuswetalk about the permutations ofnobjects we meanthe case r=n. Corollary1. There are n! permutations of n distinct objects.

Number ofpermutationsthat can be formed froma collectionof'n'objectsofwhich $n_1$ are of onetype,$n_2$areofsecondtype…….$n_k$areofkthtypewith$n_1+n_2+……+n_k=n$.Thenthe numberofpermutationsoftheofthenobjectsis□!.

□1!+□2!+···+□□!

**Example 1**.

There are 3! = 6 permutations of {a, b, c}.

Thereare4!=24permutationsof(a,b,c,d).

Thenumberof2-permutations{a,b,c,d,e}isP(5,2)=5!/(5

-2)! =5x4=20.

Thenumberof5-letterwordsusingthe lettersa,b,c,d,andeatmostonce is P (5, 5)

= 120.

**Example 2** There are P (10, 4) = 5,040 4-digit numbers that contain no repeated digits since eachsuchnumber is just anarrangement offour ofthedigits0, 1, 2,3, ….,9(leadingzeroes are allowed). There are P (26, 3) P(10, 4) license plates formed by3 distinct letters followed by 4 distinct digits.

**Example3**. In how many ways can 7 women and 3 men be arranged in a row ifthe3 men must always stand next to each other?

There are 3! ways of arranging the 3 men. Since the 3 men always stand next to each other, we treat them as a single entity, which we denote by X. Then if W1, W2,…..,W7 represents the women,wenext areinterestedinthenumberofwaysofarranging{X,W1,W2, **W3,……,W7}.** There are 8! permutations these 8 objects. Hence there are (3!) (8!) permutations altogether. (of course, if there has to be a prescribed order of an arrangement on the

3menthentherareonly8!totalpermutations**).**

**Example4.** Inhowmanywayscanthe lettersoftheEnglishalphabet bearrangedsothatthere are exactly 5 letters between the letters a andb?

There are P(24, 5) ways to arrangethe 5 letters betweena and b, 2 ways to place a andb,and then 20! ways to arrange any 7-letter word treated as one unit along with the remaining 19 letters. The total is P (24, 5) (20!) (2).

Note: If instead ofarranging objects in a line, we arrange them in a circle, then the number of permutations decreases**.**

**Example5**.Inhowmanywayscan5childrenarrange themselvesinaring?

Solution:Here,the5childrenarenotassignedtoparticularplacesbutareonlyarrangedrelative tooneanother. Thus, thearrangements(seeFigure 2-3) areconsideredthesame ifthe children are inthe same orderclockwise. Hence, the positionofchild C1 is immaterialand itisonlythe position of the 4 other children relative to C1 that counts. Therefore, keeping C1 fixedin position, there are 4! arrangements of the remaining children.

**Example 6**. A certain question paper contains 3 parts A,B,C with 4 questions in part A, 5 questions in part B and 6 questions in part C. it is required to answer 7 questions selecting at least 2 question from each part. In how many different ways can a student select his seven question for answering?

Solution:Thedifferentpossiblewaysinwhichastudentcanmakeaselectionare

(1) 2questionsfrompartA,2frompartBand 3frompartC
(2) 2questionsfrompartA,3frompartBand 2frompartC
(3) 3questions frompart A, 2frompartBand2frompartC

Selection (1) can be made in C(4,2)xC(5,2)xC(6,3)=1200 ways
Selection (2) can be made in C(4,2)xC(5,3)xC(6,2)=900 ways
Selection (1) can be made in C(4,2)xC(5,2)xC(6,3)=600 ways
Therefore number ofpossible selection is1200+900+600=2700

1. **Howmanydifferentstrings(sequences)oflength4ncanbeformedusingthelettersofthewordFLOWER. Sol**:
   The given word FLOWER has 6 letters where all of which are distinct.
   Therequirednumberofstringsis$P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$
2. **Findthenumberofpermutationsofthelettersoftheword SUCCESS.**
   **Sol:** ThegivenwordSUCCESShas7 letters,ofwhich3areS's, 2areC'sand 1 each are U and E.
   Therequirednumber ofpermutationsis $\frac{7!}{3!2!1!1!} = 420$
3. **Findthenumberofpermutationsofthelettersoftheword MASSASAUGA. Inhowmany Of these, all four A's are together? How many of them begin with S?**
   **Sol:**ThegivenwordMASSASAUGAhas10lettersofwhich4 areA's,3 areS's and 1 eachareM,UandG.
   Requirednumberofpermutations is $\frac{10!}{4!3!1!1!1!} = 25,200$
   IfallA'sare together,wetreatallA'sasone singleletter.

Thenrequirednumberofpermutations is $\frac{7!}{1!3!1!1!1!} = 84$

Ifthewordbeginwithletter S,thereoccurs9openpositionsto fill, where2areS's, 4 are A's and one each are M,U,G

Thenrequired number ofpermutationsis $\frac{9!}{2!4!1!1!1!} = 7560$

4. **Howmanypositiveintegersncanweformusingthedigits3,4,4,5,5,6,7ifwewantntoexceed5,000,000 Sol:** Here n must be ofthe formn = $x_1x_2x_3x_4x_5x_6x_7$

Where$x_1,x_2,x_3,x_4,x_5,x_6,x_7$arethegivendigitswith$x_1$=5or6or7

Suppose wetake $x_1$= 5 then $x_2,x_3,x_4,x_5,x_6,x_7$is an arrangement of the remaining 6 digits which contains two4'sand one each of 3,5,6,7

Thenumberofsucharrangementsis $\frac{6!}{2!1!1!1!1!} = 36$

Next,Supposewe take$x_1$=6then$x_2,x_3,x_4,x_5,x_6,x_7$isanarrangementoftheremaining6digitswhich contains two each of 4 and 5and one each of 3,7

Thenumberofsucharrangementsis $\frac{6!}{2!2!1!1!} = 18$

Similarly,wetake$x_1$=7then$x_2,x_3,x_4,x_5,x_6,x_7$isanarrangement oftheremaining6digitswhichcontains two each of 4 and 5and one each of 3,6

Thenumberofsucharrangementsis $\frac{6!}{2!2!1!1!} = 18$

AccordingtoSumRule,

Thenumberofn'sofdesired typeis= 360+180+180 = 720

5. **Fourdifferent Mathematicsbooks, fivedifferentcomputersciencebooksandtwo Differentcontroltheorybooksaretobearrangedinashelf.Howmanydifferentarrangementsare possible if**
   a) **thebooksin eachparticularsubjectmust allbetogether?**
   b) **Only theMathematicsbooksmustbetogether?**

**Sol:**a) The Mathematics books can be arranged among themselves in 4! ways, the Computer sciencebookin5!ways, thecontroltheorybooksin2!waysandthethreegroups in3!Ways

thenumberofpossible arrangementsis4!X 5!X2!X3!= 34,560

b)Considerthefourmathematicsbooksasonesingle book.Thenwehave8bookswhich Can be arranged in 8! Ways. In all of these ways the mathematics books aretogether. But the mathematics books can be arranged among themselves in 4! Ways.

Hence,thenumberofarrangementsis8!X4!=967,680

6. **Findthevalueofnsuchthat2P(n,2)+50=P(2n,2) Sol**:

$2P(n,2) + 50 = P(2n,2)$

$2 \times \frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$

$2n(n-1)+50=2n(2n-1)$

$n^2 = 25$

$n = 5$ or $-5$

sincencannot benegative,thevalueofn =5

**THEPRINCIPLEOFINCLUSION-EXCLUSION:**

ConsiderafinitesetScontainingpnumberofelements.Here,thenumberpiscalled**order,** sizeor thecardinalityofthesetSandisdenotedby□(□),or□(□)or |□|.
Forexample,if□={1,2,6}and□={□,□,□,□}then□(□)=|□|=3and□(□)=|□|= 4
Itis obvious that |∅|=0, and |□|≥1 for everynon-emptyfinite set *S*.Further for anytwo finitesetsAandB, if□ ⊆□ then|□|≤ |□|and ifA□ ⊂ □then|□|<|□|

IfAisasubsetofa finiteuniversalset U,thenthenumberofelementsinthecomplement□̄(of A in U) is given by-

$$|\bar{□}|=|□|-|□| \qquad (1)$$

Supposeweconsider theunionoftwo finitesetsAandBandwishtodeterminethe number of elements in □∪□. Since, the elements of □∪□ consist ofall elements which are in Aor in B orbothAand B, thenumber ofelements in□∪□ isequalto thenumber ofelements inAplus thenumberofelementsin Bminusthenumberofelements(ifany)thatarecommon toA and B. Thatis,

$$|□∪□|=|□|+|□|-|□∩□| \qquad (2)$$

Amoreexplicit(visual)wayofobtainingthisresultisthroughtheuseofaVennDiagram.

Consider the Venn diagram shown above. In this diagram, the set A is made up of two parts □1and □2and thesetBismadeupoftwoparts□2and□3,where□2 = □∩□,and□∪□ ismadeupofparts□1,□2and□3. Therefore,

|□|=Numberofelementsin□1+Numberofelementsin□2
=|□1|+|□2|

Similarly, |□|=|□2|+|□3|, |A∪B|=|□1|+|□2|+|□3|

Fromthese,weget

|A∪B|=|□1|+|□2|+|□3|=(|□1|+|□2|)+(|□2|+|□3|)−|□2|
=|□|+|□|−|□∩□|

Thus,fordeterminingthenumberofelementsin □∪□,wefirst includeallelementsinAandallelementsin B, and then exclude all elements that are common to A and B.

IfUisafiniteuniversalsetofwhichAandBaresubsets,then,byvirtueofaDe'MorganLawandthe expression (1) above, we have-

$$|\bar{A}\cap\bar{B}| = |\overline{A\cup B}| = |U| - |A\cup B|$$

Withtheuseofformula(2) above, thisbecomes

$$|\bar{A}\cap\bar{B}| = |\overline{A\cup B}| = |U| - \{|A| + |B| - |A\cap B|\}$$

$$= |U| - |A| - |B| + |A\cap B| \qquad (3)$$

Expressions (2) and (3) are equivalent toone another. Either ofthese is referredto asthe AdditionPrinciple (Rule) or the Principle of inclusion-exclusion for two sets.

IntheparticularcasewhereAandBaredisjointsetssothat $A\cap B = \emptyset$,theadditionrule(2)becomes-

$$|A \cup B| = |A| + |B| - |\emptyset| = |A| + |B| \qquad (4)$$

This is known as the Principle of disjunctive counting for two sets.

**BinomialCoefficients:** Inmathematics,the**binomialcoefficient** $\binom{n}{\cdot}$ isthecoefficientofthe termx$^n$inthepolynomialexpansionofthebinomialpower(1+*x*).

In combinatorics, $\binom{n}{k}$ is interpreted asthe number of *k*-element subsets (the *k*-combinations) of an*n*-elementset,thatisthenumberofwaysthat*k*thingscanbe"chosen"froma setof*n*things.

Hence, isoftenread as"*n* choose *k*"and iscalledthechoosefunctionof*n* and *k*.The notation was introduced by Andreas von Ettingshausen in 182, although the numbers were alreadyknowncenturiesbeforethat(seePascal'striangle).Alternativenotations include$C(n,k)$, $_nC_k$, $C_k^n$, $C_n^k$, inallofwhichtheCstandsforcombinationsor*choices*.

Fornaturalnumbers(takentoinclude0)*n*and*k*,thebinomialcoefficient $\binom{n}{k}$ canbedefined as thecoefficient of$X^k$ themonomial$X^k$ intheexpansionof$(1+X)^n$.Thesamecoefficientalso

occurs(if $k \le n$)inthebinomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

(valid for anyelements *x*,*y*ofa commutative ring), which explains the name "binomial coefficient".

Another occurrence of this number is in combinatorics, where it gives the number of ways,disregarding order, that a *k* objects can be chosen from among *n* objects; moreformally,thenumber of *k*-element subsets (or *k*-combinations) of an *n*-element set. This number can be seen tobe equal to the one of the first definition, independently of any of the formulas below to compute*n*
it:ifineachofthe*n*factors ofthepower(1+*X*)onetemporarilylabelstheterm*X*withanindex*i* (runningfrom1to *n*),theneachsubsetof*k*indicesgivesafterexpansionacontribution
$k$
$X$ , and the coefficient of that monomial in the result will be the number of such subsets. This showsinparticularthat $\binom{n}{k}$ isanaturalnumberforanynaturalnumbers*n*and*k*.Therearemany other combinatorial interpretations of binomial coefficients (counting problems for which theanswer is given by a binomial coefficient expression), for instance the number of words formedof *n* bits (digits 0 or 1) whose sum is *k*, but most of these are easily seen to beequivalenttocounting *k*-combinations.

Severalmethodsexisttocomputethevalueof $\binom{n}{k}$ withoutactuallyexpandingabinomialpower or counting *k*-combinations.

**Methodofgenerating functionsforFirst-OrderRecurrenceRelations:-**

Supposetherecurrencerelationtobesolved isofthe form $a_n =$

$c\ a_{n-1} + F(n)$ for $n \ge 1$

ora$_{n+1}$=c a$_n$+□(□)forn≥0

generatingfunctionisf(x)= ∑$^{\infty}$ $_{□=0}$ □$_0$□$^{□}$

thenweget f(x)=$\frac{□_0+□□}{1-□□}$ $\frac{□h\ □□□□(□)}{(□)}$ =∑$^{\infty}$ $_{□=0}$ □(□)□$^{□}$

**Problems:**

1. **Findageneratingfunctionfortherecurrence relation$a_{n+1}-a_n=3^n$,n≥0 and $a_0$= 1. Hence solve the relation.**

**Sol:**Given$a_{n+1}-a_n=3^n$

=> $a_{n+1}=a_n+3^n$

this is ofthe forma$_{n+1}$=c a$_n$+□(□)for n≥ 0

Here c = 1 ,a$_0$=1 and □(□)= $3^n$

Generatingfunctionisf(x)=$\frac{a_0+(a_1-ca_0)x}{1-cx}$ $+h(x)$

$(1-cx)(1-x)=\sum_{n=0}^{\infty} g(n)x^n$

$=\sum_{n=0}^{\infty} 3^n x^n$

$=\sum_{n=0}^{\infty}(3x)^n$ $=(1-3x)^{-1}$

$f(x)=\frac{1+x(1-3x)^{-1}}{1-x}$

$=\frac{1+\frac{1}{1-3x}}{1-x}$ istherequiredG.F

$=\frac{1-2x}{(1-3x)(1-x)}$

Let $\frac{1-2x}{(1-3x)(1-x)}=\frac{A}{1-x}+\frac{B}{1-3x}$

1-2x =A(1-3x) + B(1-x)

OnsolvingA=B=½

$f(x)=\frac{1-2x}{(1-3x)(1-x)}=\frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1-3x}\right)$

$=\frac{1}{2}\left(\sum_{n=0}^{\infty} x^n+\sum_{n=0}^{\infty}(3x)^n\right)$

$=\frac{1}{2}\sum_{n=0}^{\infty}(1+3^n)x^n=\sum_{n=0}^{\infty} a_n x^n$

Hencerequiredsolutionisa$_n$=$\frac{1}{2}(1+3^n)$

**2. Findthegeneratingfunctionfoetherecurrencerelationa$_{n+1}$−a$_n$=n$^2$,n≥0anda$_0$=1 Hence solve it.**

**Sol:** Givena$_{n+1}$-a$_n$= n$^2$=>a$_{n+1}$= a$_n$+n$^2$, n≥0 this is ofthe forma$_{n+1}$=c a$_n$+$g(n)$for n≥ 0 Here c = 1 ,a$_0$=1 and $g(n)$= n$^2$

Generatingfunctionisf(x)=$\frac{a_0+(a_1-ca_0)x}{1-cx}$ $+h(x)$

$(1-x)(1-x)\sum_{n=0}^{\infty} g(n)x^n$

$=\sum_{n=0}^{\infty} n^2 x^n$

i.e.,g(x) istheG.F for <n$^2$>= 0$^2$,1$^2$,2$^2$,.....

theng(x)=$\frac{x(1+x)}{(1-x)^3}$

f(x) $=\frac{1}{1-x}\left(\frac{x^2(1+x)}{(1-x)^3}\right)$

f(x)=$\frac{1-3x+4x^2}{(1-x)^4}$ istherequiredG.F

f(x) =(1-3x+4x$^2$)(1-x)$^{-4}$

=(1-3x+4x$^2$)$\left[\sum_{n=0}^{\infty}\binom{4+n-1}{n}x^n\right]$

=(1-3x+4x$^2$)$\left[\sum_{n=0}^{\infty}\binom{3+n}{n}x^n\right]$

$=\sum_{n=0}^{\infty}\binom{3+n}{n}x^n-3\sum_{n=0}^{\infty}\binom{3+n}{n}x^{n+1}+4\sum_{n=0}^{\infty}\binom{3+n}{n}x^{n+2}$

Sincef(x)=$\sum_{n=0}^{\infty} a_n x^n$

Coeff.Ofx$^n$is

$a_n=\binom{n+3}{n+3}-3\binom{3+n-1}{n-1}+4\binom{3+n-2}{n-2}$

$=\binom{n+3}{2}-3\binom{n+2}{n-1}+4\binom{n+1}{n-2}$

$=\frac{(n+3)!}{n!3!}-3\frac{(n+2)!}{(n-1)!3!}+4\frac{(n+1)!}{(n-2)!3!}$

$=\frac{(n+1)}{6}\{(n^2+5n+6)-3(n^2+2n)+4(n^2-n)\}$

$=\frac{(n+1)}{6}(2n^2-5n+6)$ isthe required solution.

**Methodofgeneratingfunctionsforsecond-orderRecurrence relations:-**

Considerthesecondorderrecurrencerelation $a_n+$

A $a_{n-1}+$ $Ba_{n-2}=F(n)$ for $n\geq 2$

or $a_{n+2}+Aa_{n+1}+Ba_n=\phi(n)$ for $n\geq 0$

Generating functionis $f(x)=\sum_{n=0}^{\infty} a_n x^n$

then we get $f(x)=\frac{a_0+(a_1+a_0A)x+x^2\phi(x)}{1+Ax+Bx^2}$ $\phi(n)=\sum_{n=0}^{\infty}\phi(n)x^n$

**Note:** If $\phi(n)=0$ then generating function is $f(x)=\frac{a_0+(a_1+a_0A)x}{1+Ax+Bx^2}$

1. **Findageneratingfunctionfortherecurrence relation $a_n+a_{n-1}-6a_{n-2}=0$ for $n\geq 2$**
   **Given $a_0=-1$, $a_1=8$**

**Sol:** Given $a_n+a_{n-1}-6a_{n-2}=0$ for $n\geq 2$

$a_{n+2}+a_{n+1}-6a_n=0$ for $n\geq 0$

and $a_0=-1, a_1=8$, $A=1, B=-6, \phi(n)=0$

thengeneratingfunction $f(x)=\frac{a_0+(a_1+a_0A)x}{1+Ax+Bx^2}$

$f(x)=\frac{7x-1}{1+x-6x^2}$ isthe required generating function.

2. **Findageneratingfunctionfortherecurrencerelation $a_{n+2}-2a_{n+1}+a_n=2^n$ for $n\geq 0$ Given $a_0=1$, $a_1=2$ Hence solve it.**

**Sol:** Given $a_{n+2}-2a_{n+1}+a_n=2^n$ for $n\geq 0$

Here $a_0=1, a_1=2, A=-2, B=1$ and $\phi(n)=2^n$

thengeneratingfunction $f(x)=\frac{a_0+(a_1+a_0A)x+x^2\phi(x)}{1+Ax+Bx^2}$ $\phi(n)=\sum_{n=0}^{\infty}\phi(n)x^n$

$\phi(n)=\sum_{n=0}^{\infty}2^n x^n=\sum_{n=0}^{\infty}(2x)^n=(1-2x)^{-1}$

Thus, $\frac{f(x)}{1+x^2(1-2x)^{-1}(1-\square)^2} = \frac{1-2\square+\square^2}{(1-\square)^2(1-2\square)} = \frac{1}{1-2\square}$

$f(x)=(1-2x)^{-1}$

$=\sum_{0}^{\infty} (2\square)^{\square} = \sum^{\infty}_{\square=0} 2^{\square}\square^{\square}$

$a_n=2^n$ istherequiredsolution.

**3. Findageneratingfunctionfortherecurrence relation$a_{n+2}-5a_{n+1}+6a_n=2$forn≥0 Given $a_0= 3, a_1= 7$Hence solve it.**

**Sol:** Given $a_{n+2}-5a_{n+1}+6a_n=2$ forn≥0

Here $a_0=3, a_1=7, A=-5, B=6$ and $\square(n)=2$

thengeneratingfunction $f(x)=\frac{\square 0+(\square 1+\square 0\square)\square+\square(\square)}{1+\square\square+\square\square^2}$ $\square h\ \square\square\square$

$\square(\square) = \sum_{\square=0}^{\infty} \square(\square)\square^{\square}$

$=\sum_{0}^{\infty} 2\square^{\square}$

$=2(1-x)^{-1}$

Thus $f(x)=\frac{3-8\square+2\square^2(1-\square)^{-1}}{1-5\square+6\square^2}$

$= \frac{10\square^2-11\square+3}{(1-2\square)(1-3\square)(1-\square)}$ istherequiredgenerating function.

Now, Let $\frac{10\square^2-11\square+3}{(1-2\square)(1-3\square)(1-\square)} = \frac{\square}{1-\square} + \frac{\square}{1-2\square} + \frac{\square}{1-3\square}$

$10x^2-11x+3=A(1-2x)(1-3x)+B(1-x)(1-3x)+C(1-x)(1-2x)$

On solving $A=1, B=0, C=2$

$f(x)=\frac{10\square^2-11\square+3}{(1-2\square)(1-3\square)(1-\square)} = \frac{1}{1-\square} + \frac{2}{1-3\square}$

$=(1-x)^{-1}+2(1-3x)^{-1}$

$=\sum_{\square=0}^{\infty} \square^{\square} + 2\sum^{\infty}_{\square=0} (3\square)^{\square}$

$=\sum_{0}^{\infty} (1+2.3^{\square})\square^{\square}$

Hence $a_n=1+2.3^n$ istherequiredsolution.

**4. Find a generating function for the Fibonacci sequence <$F_n$> and hence obtain an expression for $F_n$**

**Sol:** RecurrencerelationforFibonaccisequenceis

$F_{n+2}=F_{n+1}+F_n$ forn≥0 with $F_0=0, F_1=1$

⟹ $F_{n+2}-F_{n+1}-$

$F_n = 0 \text{ for } n \geq 0$ Here $F_0 = 0, F_1 = 1, A = -1, B = -1$ and $\square(n) = 0$

thengeneratingfunctionf(x) = $\frac{□0+(□1+□0□)□}{1+□□+□□^2}$

$= \frac{-x}{x^2+x-1}$

Now, $f(x) = \frac{-x}{x^2+x-1} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$ where $x^2+x-1=(x-\alpha)(x-\beta)$

$\alpha, \beta = \frac{1}{2}(-1 \pm \sqrt{5})$

$-x = A(x-\beta)+B(x-\alpha)$

$A = \frac{1}{\alpha-\beta}$, $B = \frac{1}{\beta-\alpha}$

Now $f(x) = -\frac{A}{x-\alpha} - \frac{B}{x-\beta}$

$= -\frac{A}{\alpha}(1-\frac{x}{\alpha})^{-1} - \frac{B}{\beta}(1-\frac{x}{\beta})^{-1}$

$= -A\sum(\frac{x}{\alpha})^n - B\sum(\frac{x}{\beta})^n$

$= -\sum_{n=0}^{\infty}(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}})x^n$

Thus $F_n = -\frac{A}{\alpha^{n+1}} - \frac{B}{\beta^{n+1}}$

$= -\frac{1}{(\alpha\beta)^{n+1}}[\alpha^{n+1} - \beta^{n+1}]$

$= -\frac{1}{(-1)^{n+1}}[\frac{1}{\alpha-\beta}\alpha^{n+1} + \frac{1}{\beta-\alpha}\beta^{n+1}]$ where $\alpha = \frac{-1+\sqrt{5}}{2}$, $\beta = \frac{-1-\sqrt{5}}{2}$

$= -\frac{1}{(-1)^{n+1}}[\frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1}]$ where $\alpha+\beta=-1, \alpha-\beta=\sqrt{5}$

$= \frac{1}{(-1)^{n+1}\sqrt{5}}[\beta^{n+1} - \alpha^{n+1}]$

**Secondorderlinearhomogeneous Recurrencerelations BY CHARACTERESTIC ROOTS**

Considerthesecondorder RR

$C_n a_n + C_{n-1}a_{n-1} + C_{n-2}a_{n-2} = 0$ for $n \geq 2 \rightarrow (1)$

$C_n, c_{n-1}, c_{n-2}$ are constants with $c_n \neq 0$.

Wewantto getthesolutionof(1) inthe form $a_n = ck^n$ $C \neq 0$

& $k \neq 0$, put $a_n = ck^n$ in (1)

$C_n(Ck^n) + C_{n-1}Ck^{n-1} + C_{n-2}(Ck^{n-2}) = 0$

$Ck^{n-2}[C_n k^2 + C_{n-1}k + C_{n-2}] = 0$

$C_n k^2 + C_{n-1}k + C_{n-2} = 0 \rightarrow (2)$

$a_n = Ck^n$ isthe solutionof (1)if(2) istrue.Thisquadratic equation(2) ink iscalled auxiliary equation or Characteristic equation of RR (1). Then ∃ three cases

1) Thetworoots $k_1$ & $k_2$ arerealanddistinctthensolution is $a_n = Ak_1^n + Bk_2^n$.
2) The tworoots $k_1$ & $k_2$ arerealbutequalthensolutionis $a_n = (A+B_n)k^n$.
3) $K_1$ & $k_2$ arecomplexconjugateslike $k_1 = p+iq$ & $k2 = p-iq$

$A_n = r^n(A\cos n\theta + B\sin n\theta)$ where $r = \sqrt{\alpha^2 + \beta^2}$ & $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Q1: Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$, $a_0 = -1$, $a_1 = 8$

Sol: General form of second order RR is $c_n a_n + c_{n-1}a_{n-1} + c_{n-2}a_{n-2} = 0$

$\therefore C_n = 1, c_{n-1} = 1, c_{n-2} = -6$

Now the characteristic equation is $k^2 + k - 6 = 0$, on solving it $k_1 = -3, k_2 = 2$ (real and distinct).

$a_n = A(-3)^n + B(2)^n$ where A & B are constants.

Now using the initial condition find a & B $a_0 =$

$A + B \Rightarrow A + B = -1$

$a_1 = -3A + 2B \Rightarrow -3A + 2B = 8$ then $A = -2, B = 1$

$\therefore$ Solution is $a_n = -2(-3)^n + (2^n)$

Q2: Solve $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2 / a_0 = 1$ & $a_1 = 2$ Sol : $a_n - 2a_{n-1} + 2a_{n-2} = 0$, $c_n = 1$, $c_2 = -2$, $c_3 = 2$

characteristic equation is $k^2 - 2k + 2 = 0, k = 1 \pm i$

$\therefore$ General solution $a_n = r^n [A\cos n\theta + B \sin n\theta]$ r

$= \sqrt{\alpha^2 + \beta^2}, \alpha = 1 \beta = 1$

$= \sqrt{2}, \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}(1) \Rightarrow \tan^{-1}(1) \Rightarrow \theta = \frac{\pi}{4}$

$\therefore a_n = (\sqrt{2})^n [A\cos \frac{n\pi}{4} + B\sin \frac{n\pi}{4}]$

To get A & B using initial conditions, $a_0 = 1 \Rightarrow A\cos 0 + B\sin 0 \Rightarrow \alpha = 1$

$a_1 = 2 \Rightarrow (\sqrt{2})^1 [A\cos \frac{\pi}{4} + B\sin \frac{\pi}{4}]$

$= \sqrt{2}[A\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}}]$

$A + B = 2$ then $B = 1 \Rightarrow a_n = (\sqrt{2})^n [A\cos \frac{n\pi}{4} + B\sin \frac{n\pi}{4}]$

Q3: Solve $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0, F_0 = 0, F_1 = 1$ Sol: $F_{n+2} - F_{n+1} - F_n = 0$, $c_n = 1$, $c_{n-1} = -1$, $c_{n-2} = -1$

Characteristic equation $F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Using critical values, $A + B = 0 \rightarrow (1) \Rightarrow B = -A$

$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$= A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = 2 * A * \frac{\sqrt{5}}{2} \Rightarrow A = \frac{1}{\sqrt{5}}, B = \frac{-1}{\sqrt{5}}$

$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ is the solution.

**Third and Higher Order Linear Homogeneous Recurrence Relation:**

**Q1.** Solve $2a_{n+3}-a_{n+2}+2a_{n+1}-a_n$ for $n\geq 0$ given $a_0=0, a_1=1, a_2=2$

**Sol:** Let $2a_{n+3}-a_{n+2}+2a_{n+1}-a_n=0$ for $n\geq 0$

We have $C_n=2, C_{n-1}=-1, C_{n-2}=-2, C_{n-3}=1$

Characteristic equation is $2k^3-k^2-2k+1=0 \Rightarrow k_1=\frac{1}{2}, k_2=1, k_3=-1$

The roots are $_n$ real and distinct.

$$\therefore a_n=A\left(\frac{1}{2}\right)^n+B(1)^n+C(-1)^n$$

To find A, B & C:

$a_0=0=A+B+C$ ---------- (1)

$a_1=1=\frac{A}{2}+B-C$ ---------- (2)

$a_2=2=\frac{A}{4}+B+C$ ---------- (3)

Adding (1)+(2) gives $3A+4B=2$ ---------- (4)

Adding (2)+(3) gives $3A+8B=12$ ---------- (5)

Now, solving (4) & (5) we get $A=\frac{-8}{3}, B=\frac{5}{2}, C=\frac{1}{6}$

$$\therefore a_n=\frac{-8}{3}\left(\frac{1}{2}\right)^n+\frac{5}{2}(1)^n+\frac{1}{6}(-1)^n$$

This is the required solution.

**Q2.** Solve the recurrence relation $a_n+a_{n-1}-8a_{n-2}-12a_{n-3}=0$, given $a_0=1, a_1=5, a_2=1$.

**Sol:** Given $a_n+a_{n-1}-8a_{n-2}-12a_{n-3}=0$

We have $C_1=1, C_2=1, C_3=-8, C_4=-12$

Characteristic equation is $k^3+k^2-8k-12=0 \Rightarrow k=k_1=-2, k=k_2=3, k=k_3=3$

The roots are real and repeated.

Hence the solution is $a_n=(A+Bn)(-2)^n+C(3)^n$

$a_0=1=A+C$ ---------- (1)

$a_1=5=-2(A+B)+C$ ---------- (2)

$a_2=1=4(A+2B)+9C$ ---------- (3)



Solving (1), (2) and (3), we get $A=0, B=-1, C=1$

$\therefore a_n=(-n)(-2)^n+3^n$ is the required solution.

**Non-Homogeneous Recurrence Relation of Second and Higher Order:**

Recurrence relation is of the form,

$c_n a_n+c_{n-1}a_{n-1}+c_{n-2}a_{n-2}+\ldots=f(n)$ where $n\geq k\geq 2$ & $c_n\neq 0$ ---------- (1)

A general solution for this RR be $a_n=a_n^{(h)}+a_n^{(p)}$ where $a_n^{(h)}$ is the solution of the LHS part.

To get $a_n^{(p)}$ consider the following cases:

**Case(i):** Suppose f(n) is a polynomial of degree q & 1 is not a root of the characteristic equation of the homogeneous part (LHS) of relation in equation (1). In this case $a_n^{(p)}$ can be taken as $a_n^{(p)} = A_0 + An + An^2 + ... + An^q$. Then $A_0, A_1, A_2, ..., A_q$ should be evaluated.

**Case(ii):** Suppose f(n) is a polynomial of degree q & 1 is a root of multiplicity m, $a_n^{(p)}$ can be taken as $a_n^{(p)} = n^m\left(A_0 + An + An^2 + ... + An^q\right)$. Then $A_0, A_1, A_2, ..., A_q$ should be evaluated.

**Case(iii):** Suppose $f(n) = ab^n$ where a is constant and b is not a root of the characteristic equation then $a_n^{(p)} = Ab^n$

**Case(iv):** Suppose $f(n) = ab^n$ where a is constant and b is a root of multiplicity m of the characteristic equation then $a_n^{(p)} = An^m b^n$

**Problems:**

**Q1.** Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8$ for $n \geq 2$ and $a_0 = 1, a_1 = 2$

**Sol:** Given recurrence relation is $a_n + 4a_{n-1} + 4a_{n-2} = 8$ ---------- (1)

Consider the homogeneous part, $a_n + 4a_{n-1} + 4a_{n-2} = 0$ We have $C_n = 1, C_{n-1} = 4, C_{n-2} = 4$

The characteristic equation is $k^2 + 4k + 4 = 0 \Rightarrow k = k = -2$ ---------- (2)

$k_1 \quad k_2$

The roots are real and repeated, so the required solution is $a_n^{(h)} = (A + Bn)(-2)^n$ ---------- (3)

Consider the RHS, it is a constant 8, consider it as a polynomial with degree 0. (ie., n=0) and 1 is not a root for LHS.

By case(i), $a_n^{(p)} = A_0$ ---------- (4)

$A + 4A + 4A = 8 \Rightarrow A = \frac{8}{9}$ $a^{(p)} = A = \frac{8}{9}$

Put (4) in (1), we get . So we have

$0 \quad 0 \quad 0 \quad 0$ $n \quad 0$

Therefore, the general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$.

$\Rightarrow a_n = (A + Bn)(-2)^n + \frac{8}{9}$ ---------- (5)

Now, for finding values of A & B, $a_0 = 1 = A + \frac{8}{9} \Rightarrow A = 1 - \frac{8}{9} = \frac{1}{9}$ ---------- (6)

$a_1 = 2 = -2(A + B) + \frac{8}{9} \Rightarrow -2A - 2B = 2 - \frac{8}{9}$ ---------- (7)

Solving (6) & (7), we get $B = \frac{-6}{9}$

Substituting the values of A & B in (5), we have $a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$.



**Q2.** Solve the recurrence relation $a_{n+2} - 4a_{n+1} + 3a_n = -200$ for $n \geq 0$ and given $a_0 = 3000, a_1 = 3300$

**Sol:** Given $a_{n+2} - 4a_{n+1} + 3a_n = -200$ for $n \geq 0$ ---------- (1)

We have $C_n = 1, C_{n-1} = -4, C_{n-2} = 3$

Characteristicequationis$k^2-4k+3=0\Rightarrow k=1,k=3$

Therootsarerealanddistinct,sowehave $a_n^{(h)}=A(1)^n+B(3)^n$

Since1isthe root ofthecharacteristicequationwithmultiplicitym=1,wecanwrite $a_n^{(p)}=n^1(A)_0$, nowsubstitutingthis ineqn.(1)weget,

$(n+2)A_0-4(n+1)A_0+3nA_0=-200\Rightarrow -2A_0=-200\Rightarrow A_0=100$

Hence,wehave $a_n^{(p)}=100n$

Weknowthatthe generalsolutionisgivenby $a_n=a_n^{(h)}+a_n^{(p)}$

$\therefore a_n=A+B(3)^n+100n$ ---------- (2)

Also,wehave

$a_0=3000=A+B\Rightarrow B=3000-A$ ---------- (1)

$a_1=3300=A+B(3)^1+100(1)$ ---------- (2)

Solving(1)&(2)weget$A=2900,B=100$

Hencetheparticularsolutionisgivenby$a_n=2900+100(3)^n+100n$

**Q3.**Solvetherecurrencerelation$a_{n+2}-10a_{n+1}+21a_n=3n^2-2,n\geq 0.$

**Sol:**Given $a_{n+2}-10a_{n+1}+21a_n=3n^2-2,n\geq 0$ ---------- (1)

Wehave$C_n=1,C_{n-1}=-10,C_{n-2}=21$

Characteristic equationis$k^2-10k+21=0\Rightarrow k=3,k=7$

Therootsarerealanddistinct,sowehave $a_n^{(h)}=A(3)^n+B(7)^n$ ---------- (2)

SinceRHSisapolynomialofdegree2 and 1 isnot arootofthecharacteristicequation,wecanwrite

$a_n^{(p)}=A_0+An_1+An^2_2$ ---------- (3)

nowsubstituting(3)ineqn.(1) wehave,

$\lfloor A_0+A(n+2)_1+A(n+2)^2_2\rfloor-10\lfloor A_0+A(n+1)_1+A(n+1)^2_2\rfloor+21\lfloor A_0+An_1+An^2_2\rfloor=3n^2-2$

Onsolvingthis equationwe get $A_0=\frac{13}{72},A_1=\frac{1}{3},A_2=\frac{1}{4}$

Hence, wehave $a_n^{(p)}=\frac{13}{72}+\frac{1}{3}n+\frac{1}{4}n^2$

Weknowthatthegeneralsolutionisgivenby $a_n=a_n^{(h)}+a_n^{(p)}$

$\therefore a_n=A(3)^n+B(7)^n+\frac{13}{72}+\frac{1}{3}n+\frac{1}{4}n^2$ istherequired solution.

**Q4.**Solve$a_n+4a_{n-1}+4a_{n-2}=5\times(-2)^n$

**Sol:**Given $a_n+4a_{n-1}+4a_{n-2}=5\times(-2)^n$ ---------- (1)

Wehave$C_n=1,C_{n-1}=4,C_{n-2}=4$

Characteristicequationis$k^2+4k+4=0\Rightarrow k=k_1=k_2=-2$

The rootsarerealandrepeated,sowehave $a_n^{(h)}=(A+Bn)(-2)^n$ ---------- (2)

Here RHS is of the form $5\times(-2)^n = ab^n$ where $b=-2$ is a root of the characteristic equation with multiplicity, m=2.

So $a_n^{(p)}$ is of the form $a_n^{(p)} = A\,n^m b^n \Rightarrow a_0^{(p)} = A\,n^2(-2)^n$ ---------- (3)

Now substitute (3) in (1) to get $\lfloor An^2(-2)_0^n \rceil + 4\lfloor A(n-1)^2(-2)_0^{n-1} \rceil + 4\lfloor A(n-2)^2(-2)_0^{n-2} \rceil = 5\times(-2)^n$

$(-2)^{n-2}\left\{\lceil An_0^2(-2)^2 \rceil + 4\lfloor A(n-1_0)^2(-2)^1 \rceil + 4\lfloor A(n-2)^2 \rceil\right\}_0 = 5\times(-2)^n$

$\Rightarrow 4An_0^2 - 8A(n-_0 1)^2 + 4A(n-2)_0^2 = 5\times(-2)^2$

$\Rightarrow 4An_0^2 - 8A(n_0^2 - 2n+1) + 4A(n^2 - 4n_0 + 4) = 20$

$\Rightarrow 8A = 20 \Rightarrow A = \frac{5}{2}$

Substituting this value in (3), we get $a_n^{(p)} = \frac{5}{2}n^2(-2)^n$

We know that the general solution is given by $a_n = a_n^{(h)} + a_n^{(p)}$

$\therefore a_n = (A+Bn)(-2)^n + \frac{5}{2}n^2(-2)^n$ is the required solution.

**UNIT-V**

**GraphTheory**

**Syllabus**

**GraphTheory:**RepresentationofGraph,DFS,BFS,SpanningTrees,planarGraphs.

**RepresentationofGraphs:**

Therearetwodifferentsequentialrepresentationsofagraph.Theyare

- AdjacencyMatrixrepresentation
- PathMatrixrepresentation

**AdjacencyMatrixRepresentation**

SupposeGisasimpledirectedgraphwithmnodes, andsupposethenodesofGhavebeen ordered and arecalled v1, v2, . . ., vm. Thentheadjacency matrix A=(aij) ofthegraphG isthe m x m matrix defined as follows:

aij = 1,ifviis adjacenttoVj,thatis,ifthereisanedge(Vi,Vj)
0otherwise

SupposeGisanundirectedgraph. ThentheadjacencymatrixAofGwill beasymmetric matrix, i.e., one in which aij = aji; for every i and j.

**Drawbacks**

1. Itmaybedifficulttoinsertanddeletenodesin G.
2. Ifthenumberofedges is0(m) or0(mlog2 m), thenthe matrixAwillbesparse, hencea great deal of space will be wasted.



**PathMatrixRepresentation**

Let Gbeasimpledirectedgraphwithmnodes, v1,v2, ...,vm. ThepathmatrixofGisthe m-square matrix P = (pij) defined as follows:

Pij= 1ifthereis apathfromVitoVj

0otherwise

**Graphsand Multigraphs**

AgraphGconsistsoftwothings:

1.AsetVofelementscallednodes(orpoints or vertices)

2. AsetEofedgessuchthateachedgeeinE is identifiedwithaunique (unordered)pair[u,v]of nodesin V,denotedby e= [u,v]

Sometimes weindicatethepartsofagraphbywritingG =(V,E).

Suppose e=[u, v].Thenthe nodesu and vare called theendpointsofe, and u and vare said to be adjacent nodes or neighbors. The degree ofa node u, writtendeg(u), is the number ofedges containing u. If deg(u) = 0 — that is, if u does not belong to any edge—then u is called an isolated node.

**PathandCycle**

ApathP oflength n fromanodeu to anodev isdefined asasequenceofn+1 nodes. P = (v0, v1, v2, . . . , vn) such that u = v0; vi-1 is adjacent to vi for i= 1,2, . . ., n and vn = v.

**TypesofPath**

1. SimplePath
2. CyclePath

**(i) SimplePath**

Simplepathisapathinwhichfirstandlastvertexaredifferent(V0≠Vn)

**(ii) CyclePath**

Cyclepathisapathinwhichfirst and last vertexaresame(V0=Vn).It isalso calledas Closed path.

**ConnectedGraph**

AgraphGis said tobeconnectedifthereisapathbetweenanytwoofitsnodes.

**CompleteGraph**

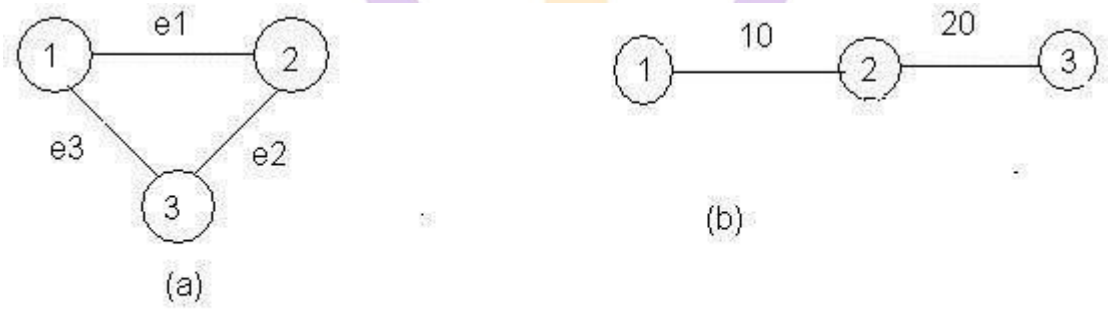
AgraphGissaid tobecompleteifeverynodeuin Gisadjacenttoeveryothernodevin G.

**Tree**

Aconnected graphTwithout anycyclesiscalledatreegraphorfreetreeor,simply,atree.

**Labeled orWeightedGraph**

Iftheweight isassignedto eachedgeofthegraphthenit iscalled as Weightedor Labeled graph.

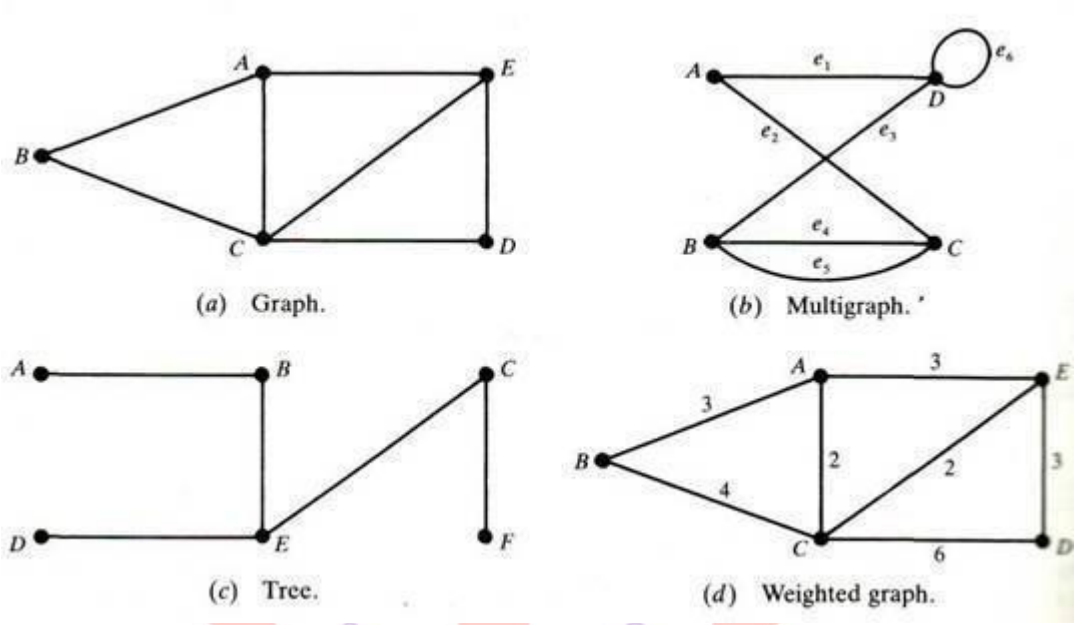



weighted or Labeled Graph

Thedefinitionofagraphmaybegeneralizedbypermittingthefollowing:

1. ***Multipleedges***:Distinctedgeseand e'arecalled multipleedges iftheyconnect thesame endpoints, that is, if e = [u, v] and e' = [u, v].
2. ***Loops*:**Anedgeeiscalledaloopifithas identicalendpoints,thatis,ife=[u, u].

3. ***Finite Graph***:A multigraphM issaid to be finite ifit hasa finite number ofnodes and afinite number of edges.




**Directed Graphs**

AdirectedgraphG,also calledadigraphorgraphisthesameasa multigraphexceptthat each edgee inG isassignedadirection, or inother words, eachedgee is identifiedwithanordered pair (u, v) of nodes in G.

**Outdegreeand Indegree**

**Indegree**:Theindegreeofavertexisthenumber ofedgesforwhichvishead

***Example***


Indegreeof1=1

Indegreepf2=2

**Outdegree**:Theoutdegreeofanodeorvertexisthenumberofedgesforwhich v istail.

***Example***


Outdegreeof1 =1

Outdegreeof2 =2

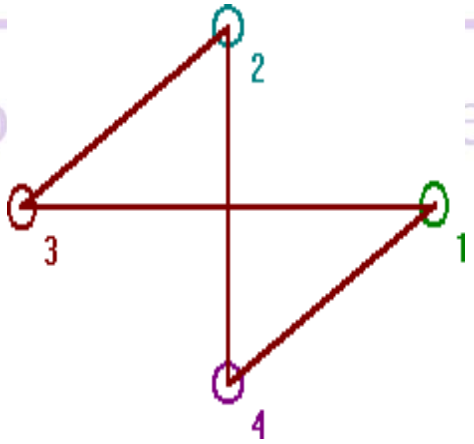
**SimpleDirectedGraph**

AdirectedgraphG issaidto besimple ifG has no paralleledges. AsimplegraphG may have loops, but it cannot have more than one loop at a given node.

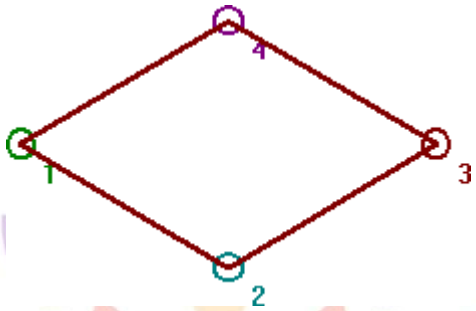
**Basic ConceptsIsomorphism**:

Let G1andG1betwo graphsand let fbea functionfromthe vertexset ofG1tothevertexset of G2.Supposethatfisone-to-oneandonto&f(v)isadjacenttof(w)inG2ifandonlyifvisadjacent to w in G1.

Thenwesay that the function f is an isomorphism and that the two graphs G1 and G2are isomorphic. So two graphs G1 and G2 are isomorphic if there is a one-to-one correspondence between vertices ofG1 and those of G2 with the propertythatif two vertices of G1 are adjacent thenso aretheir images inG2. Iftwo graphs are isomorphic thenas far as we concerned they are thesamegraphthoughthelocationoftheverticesmaybedifferent.Toshowyouhowthe program canbeusedto exploreisomorphismdrawthegraphinfigure4 withtheprogram(first getthenull graph on four vertices and then use the right mouse to add edges).



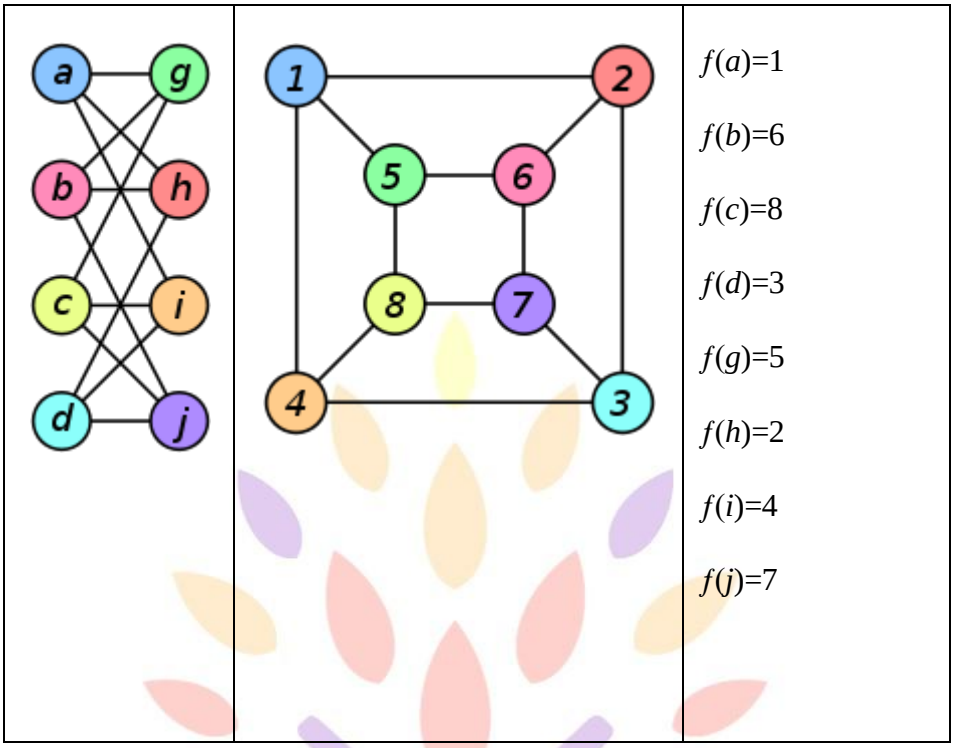
SavethisgraphasGraph1(youneedtoclickGraphthenSave). Nowgetthecircuit graphwith4 vertices. It looks like figure 5, and we shall call it C(4).




**Example:**

Thetwographsshownbelowareisomorphic, despitetheirdifferentlooking drawings.

| **GraphG** | **GraphH** | **Anisomorphism betweenGandH** |
|---|---|---|




**Subgraphs:**

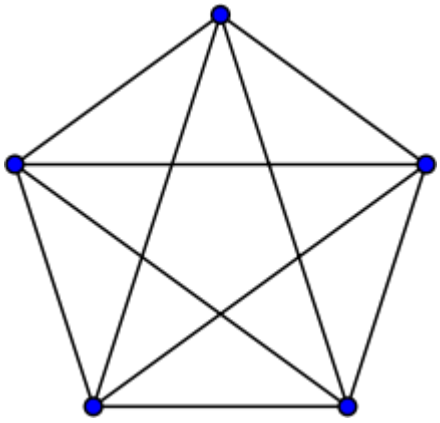
A**subgraph**ofagraph*G*isagraphwhosevertexsetisasubsetofthatof*G*,andwhoseadjacency relation isa subset ofthat of*G*restricted to this subset. Intheother direction, a**supergraph** ofa graph*G*isagraphofwhich*G*isasubgraph. Wesayagraph*G***contains**another graph*H*ifsome subgraph of *G* is *H* or is isomorphic to *H*.

Asubgraph*H* is a **spanningsubgraph**,or **factor**, ofa graph *G*if it has the same vertexset as *G*. We say *H* spans *G*.

Asubgraph *H*ofa graph *G* is said to be **induced** if, for anypair ofvertices *x* and *y*ofH, *xy* is an edge of *H* if and only if *xy*is an edge of G.In other words,*H* is an induced subgraph of *G* if it hasalltheedgesthat appear in *G* overthesame vertexset. Ifthevertexset of*H* isthesubset *S*of *V(G)*, then *H* can be written as *G*[*S*] and is said to be **induced by *S***.

Agraphthatdoes*not*contain*H*asaninduced subgraph issaid to be***H*-free**.

A **universal graph** in a class ***K*** of graphs is a simple graph in which every element in ***K*** can be embedded as a subgraph.



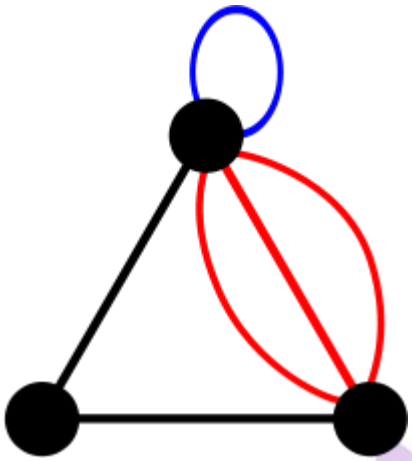
$K_5$, a complete graph. Ifa subgraph looks like this, the vertices in that subgraph forma clique of size 5.

**Multi graphs:**

In mathematics, a **multigraph** or **pseudograph** is a graph which is permitted to have multiple edges,(alsocalled"paralleledges"),thatis,edgesthathavethesameendnodes.Thustwo vertices maybe connected bymorethanone edge. Formally, a multigraph *G* is anordered pair *G*:=(*V*, *E*) with

- *V*asetof*vertices*or *nodes*,
- *E*amultisetofunordered pairsofvertices,called*edges*or*lines*.

Multigraphs might be used to modelthe possible flight connections offered by an airline. In this case the multigraph would be a directed graph with pairs of directed parallel edges connecting cities to show that it is possible to fly both *to* and *from* these locations.



A multigraph with multiple edges (red) and a loop (blue). Not all authors allow multigraphs to have loops.

**Eulercircuits:**

In graph theory, an **Eulerian trail** is a trail in a graph which visits every edge exactly once. Similarly, an**Eulerian circuit** isanEuleriantrailwhichstartsandendsonthesamevertex. They were first discussed by Leonhard Euler while solving the famous Seven Bridges of Königsberg problem in 1736. Mathematically the problem can be stated like this:

Given the graph on the right, is it possible to construct a path (or a cycle, i.e. a path starting and ending on the same vertex) which visits each edge exactly once?

Euler proved that anecessarycondition for theexistenceofEuleriancircuits isthat allvertices in the graph have an even degree, and stated withoutproof that connected graphs with all verticesof even degree have an Eulerian circuit. The first complete proof of this latter claim was published in 1873 by Carl Hierholzer.

The term **Eulerian graph** has two common meanings in graph theory. One meaning is a graph with an Eulerian circuit, and the other is a graph with every vertex of even degree. These definitions coincide for connected graphs.

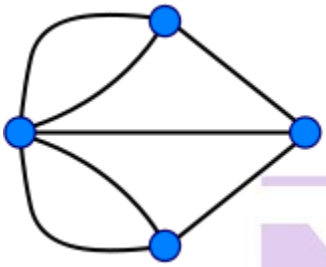
For the existence of Eulerian trails it is necessary that no more than two vertices have an odd degree;thismeanstheKönigsberggraphis*not*Eulerian.Iftherearenoverticesofodddegree,all Eulerian trails are circuits. Ifthere are exactly two vertices ofodd degree, all Eulerian trailsstart atoneofthemandendattheother.SometimesagraphthathasanEuleriantrailbutnotanEulerian circuit is called **semi-Eulerian**.

An **Eulerian trail**, **Eulerian trail** or **Euler walk** in an undirected graph is a path that uses each edge exactlyonce. If such a path exists, the graph is called **traversable** or **semi-Eulerian**.

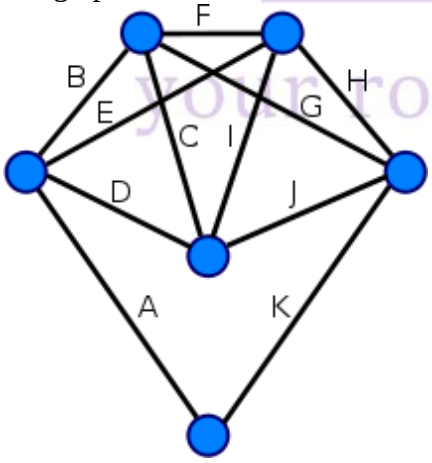
An**Euleriancycle**,**Euleriancircuit**or**Eulertour**inanundirectedgraphisacyclethatuseseach edge exactly once. If such a cycle exists, the graph is called **unicursal**. While such graphs are Eulerian graphs, not every Eulerian graph possesses a Eulerian cycle.

Fordirectedgraphs pathhastobe replaced withdirectedpathandcycle withdirectedcycle.

ThedefinitionandpropertiesofEuleriantrails,cyclesandgraphsarevalidformultigraphsaswell.



ThisgraphisnotEulerian,therefore,asolutiondoesnotexist.



Every vertex ofthis graph has an even degree, therefore this is an Eulerian graph. Following the edges in alphabetical order gives an Eulerian circuit/cycle.

**Hamiltonian graphs:**

In the mathematical field of graph theory, a **Hamiltonian path** (or **traceable path**) is a path inanundirectedgraphwhichvisitseachvertexexactlyonce.A**Hamiltoniancycle**(or**Hamiltonian circuit**) is a cycle inanundirected graphwhichvisits eachvertexexactlyonce andalso returnsto thestartingvertex. Determiningwhether suchpathsandcyclesexist ingraphs is theHamiltonian path problem which is NP-complete.

HamiltonianpathsandcyclesarenamedafterWilliamRowanHamiltonwho inventedtheIcosian game, now also known as *Hamilton's puzzle*, which involves finding a Hamiltonian cyclein the edge graph of the dodecahedron. Hamilton solved this problem using the Icosian Calculus,an algebraicstructurebasedonrootsofunitywithmanysimilaritiestothequaternions(alsoinvented by Hamilton). This solution does not generalize to arbitrary graphs.

A *Hamiltonian path* or *traceable path* is a paththat visits each vertex exactlyonce. A graph that contains a Hamiltonian path is called a **traceable graph**. A graph is **Hamilton-connected** if for everypair of vertices there is a Hamiltonian path between the two vertices.

A *Hamiltonian cycle*, *Hamiltonian circuit*, *vertex tour* or *graph cycle* is a cycle that visits each vertexexactlyonce (except the vertexwhich is boththe start and end, and so is visited twice). A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.

Similar notions may be defined for *directed graphs*, where eachedge (arc)ofa pathorcycle can onlybetracedinasingledirection(i.e.,theverticesareconnectedwitharrowsandtheedgestraced "tail-to-head").



A**Hamiltoniandecomposition**isanedgedecompositionofagraphinto Hamiltoniancircuits.

**Examples**

- acompletegraphwithmorethantwoverticesisHamiltonian
- everycycle graphisHamiltonian
- everytournamenthasanoddnumberofHamiltonianpaths
- everyplatonicsolid, consideredasagraph,isHamiltonian

**PlanarGraphs:**

Ingraphtheory,a**planargraph** isagraphthatcanbeembeddedintheplane, i.e.,itcanbedrawn onthe plane in such a waythat its edges intersect only at their endpoints.

A planar graph already drawn in the plane without edge intersections is called a **plane graph** or **planarembeddingofthegraph**.Aplanegraphcanbedefinedasaplanargraphwitha mapping fromeverynodetoapointin2Dspace,andfromeveryedgetoaplanecurve,suchthattheextreme points of each curve are the points mapped from its end nodes, and allcurves are disjoint except on their extreme points. Plane graphs can be encoded by combinatorial maps.

It is easily seen that a graph that can be drawn on the plane can be drawn on the sphere as well, and vice versa.

Theequivalenceclassoftopologicallyequivalentdrawingsonthesphereiscalleda**planarmap**.

Although a plane graph has an **external** or **unbounded** face, none of the faces of a planar map have a particular status.

**Applications**

- Telecommunications–e.g.spanningtrees
- Vehiclerouting–e.g.planning routesonroadswithoutunderpasses
- VLSI–e.g.layingoutcircuitsoncomputerchip.
- The puzzle game Planarity requires the player to "untangle" a planar graph so that noneof its edges intersect.

**Examplegraphs:**

| **Planar Graph** | **Nonplanar Graph** |
|---|---|

**ChromaticNumbers:**

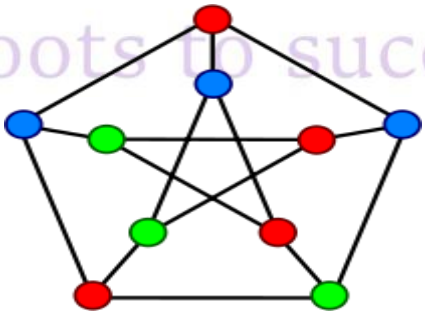
In graph theory, **graph coloring** is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a **vertex coloring**. Similarly, an **edge coloring** assigns a color to each edge so that no two adjacent edges share the same color, and a **face coloring** of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

Vertexcoloringisthestartingpointofthesubject,andothercoloringproblemscanbetransformed into avertexversion. For example, anedgecoloringofagraphis just avertex coloringofits line graph, and a face coloring ofa planar graph is just a vertexcoloring of its planar dual. However, non-vertexcoloringproblemsareoftenstatedandstudied *asis*.Thatispartlyforperspective,and partlybecausesomeproblemsarebeststudiedinnon-vertexform,asforinstanceisedgecoloring.

**GraphColouring:**

Graph coloring enjoys many practical applications as well as theoretical challenges. Beside the classicaltypesofproblems,different limitationscanalsobesetonthegraph,oronthewayacolor is assigned, or even on the color itself. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku. Graph coloring is stilla very activefield of research.





ApropervertexcoloringofthePetersengraphwith3colors, theminimumnumber possible.

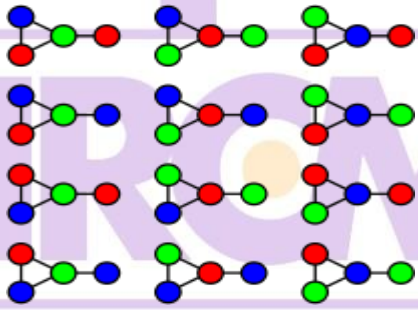
**Vertexcoloring**

When used without any qualification, a **coloring** of a graph is almost always a *proper vertex coloring*, namelya labelling ofthe graph‘s vertices with colors suchthat no two vertices sharing thesameedge havethesamecolor. Sincea vertexwitha loopcould never beproperlycolored, it is understood that graphs in this context are loop less.

The terminologyofusing *colors* for vertex labels goesback to map coloring. Labels like *red* and *blue*areonlyusedwhenthenumberofcolorsissmall,andnormallyitisunderstoodthatthe labels are drawn from the integers {1,2,3,...}.

A coloring using at most *k* colors is called a (proper) ***k*-coloring**. The smallest number of colors needed to color a graph *G* is called its **chromatic number**, $\chi(G)$. A graphthat can be assigned a (proper) *k*-coloring is ***k*-colorable**, and it is ***k*-chromatic** if its chromatic number is exactly *k*. A subsetofverticesassignedtothesamecoloriscalled*acolorclass*,every suchclassformsan

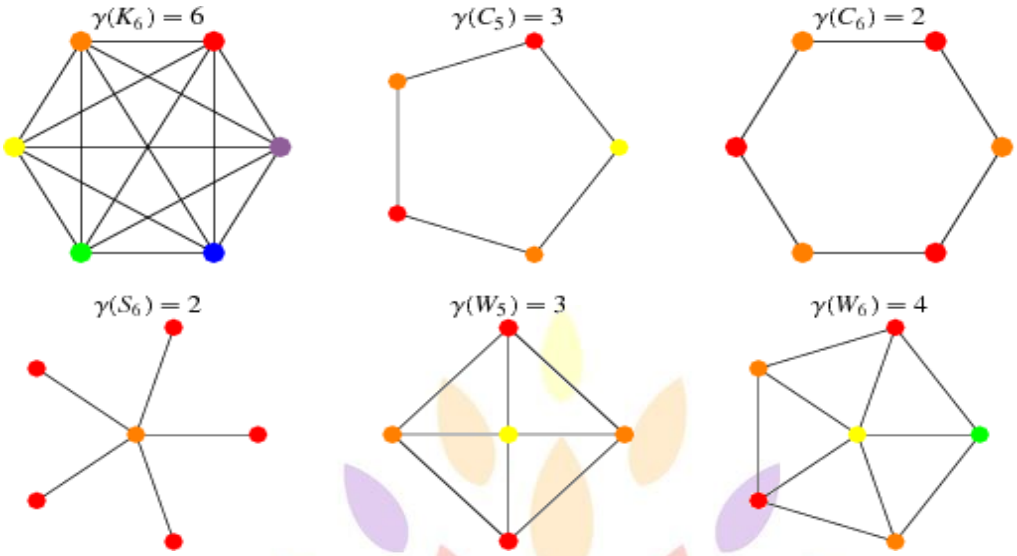
independent set. Thus, a *k*-coloring is the same as a partitionofthe vertex set into *k* independent sets, and the terms *k-partite* and *k-colorable* have the same meaning.



Thisgraphcanbe3-coloredin12differentways.

Thefollowingtablegivesthechromatic numberforfamiliar classesofgraphs.

| graph $G$ | $\gamma(G)$ |
|---|---|
| completegraph | $n$ |
| cyclegraph $C_n$, | $\begin{cases} 3 & \text{for n odd} \\ 2 & \text{for n even} \end{cases}$ |
| stargraph $S_n$, | 2 |




**Treesanditsbasic properties:**

AgraphGiscalledatreeifitis connectedandhas nocycles.
Apendantvertex(a vertexhasdegree one)ofa treeiscalled aleaf.

**Theorem 1:** Prove that of T is a tree if and only if there is one and only one path between every pair of vertices.

**Proof**:Let T beatree. ThenT isasimplegraph. SinceT isconnectedthere isat leastonepathbetweeneach pair ofvertices. Ifthere aretwo paths betweenone pair ofvertices, the unionofthese two paths willmake a cycle in T contradicting the fact that T is a tree. There for in a tree there is one and only one path between every pair of vertices.

Now conversely assume that in T there is one and only one path between every pair of vertices. Then it is connected. Since there is one and only one path between everypair of vertices in T, T cannot have a cycle. Therefore T is a tree.

**Theorem2:** Atreewithnverticeshasn-1edges.

**Proof**:Proofis doneby induction onnumber ofvertices n. Thetheoremis truefor n=1,n=2 andn=3(wecanseethis usinggraphs).Assumethetheoremistrueforn=kvertices.ConsideratreeT withk+1vertices.Let e=uvbeanedgein T. nowT-eis a disconnected graph(sinceT is tree). Thereexist 2 components of T-e, sayT1 and T2.T1 contains say k1 vertices andT2contains sayk2 vertices: k1+k2=k+1. SinceT1andT2has number of vertices less thanor equal to kbyourassumptionT1andT2containk1-1andk2-1edgesrespectively. ThenT-econtains (k1+k2)-2 edges,that is( k+1)-2 edges, k-1 edges. ThereforeT contains k-1+e edges, which implies k edges. Thetheorem is proved for n=k+1 vertices. Therefore the theoremis true for all positive integer n.

Result:Anyconnectedgraphwithnverticesandn-1edgesisatree.

**Problems:**

a. S.T the complete graph $K_n$ is not a tree for $n>2$.

**Solution**: Let, $v_1$, $v_2$, $v_3$ be any three vertices of $K_n$. Since, $K_n$ is a complete graph there exists an adjacency between each pair of vertices.

$\therefore v_1$ is adj $v_2$, $v_2$ will be adj to $v_3$ and $v_1$ is adj to $v_3$ also.

$\therefore v_1 v_2 v_3$ creates a cycle inside $K_n$ implies $K_n$ is not a cycle.

b. S.T the complete graph bipartite graph $K_{m,n}$ is not a tree when $m \geq 2$

**Solution**: Let, $v_1$, $v_2$ be any two vertices in the first vertex set & $v_1'$, $v_2'$ be two vertices of second set of the given complete bipartite graph $K_{m,n}$ with $m \geq 2$. Since, $K_{m,n}$ is complete there exists 1 adj 2 from $v_1 \longrightarrow v_1'$ & $v_1'$, $v_2 \longrightarrow v_1'$ & $v_2'$

$\therefore v_1 v_1' v_2 v_2' v_1$ is a cycle in $K_{m,n}$ which implies $K_{m,n}$ is not a tree.

**Minimally Connected Graph:**

A connected graph is said to be minimally connected if the removal of any one edge from it disconnects the graph.

*All trees are minimally connected.

**Theorem:** A connected graph is a tree iff it is minimally connected.

**Proof:** G is a connected graph which is not a tree implies there exists a cycle. From the cycle remove one edge 'e' implies G-e is still connected implies G is not minimally connected.

If a graph is not a tree, then it is not minimally connected. By contrapositivity we proved that, if a connected graph is minimally connected then it is a tree.

Conversely, let G be non-minimally connected graph then $\exists$ 'e' such that G-e is connected.

$\therefore$ e must be a part of a cycle i.e. G contains a cycle i.e. G is not a tree. By contrapositivity if G is a tree then it is minimally connected.

**Rooted Tree:**

A directed tree is a directed graph whose underlying graph is a tree. A directed tree T is called a rooted tree T contains -

1) a unique vertex, called the root (r) whose indegree is zero

2) the indegrees of all other vertices of T are 1

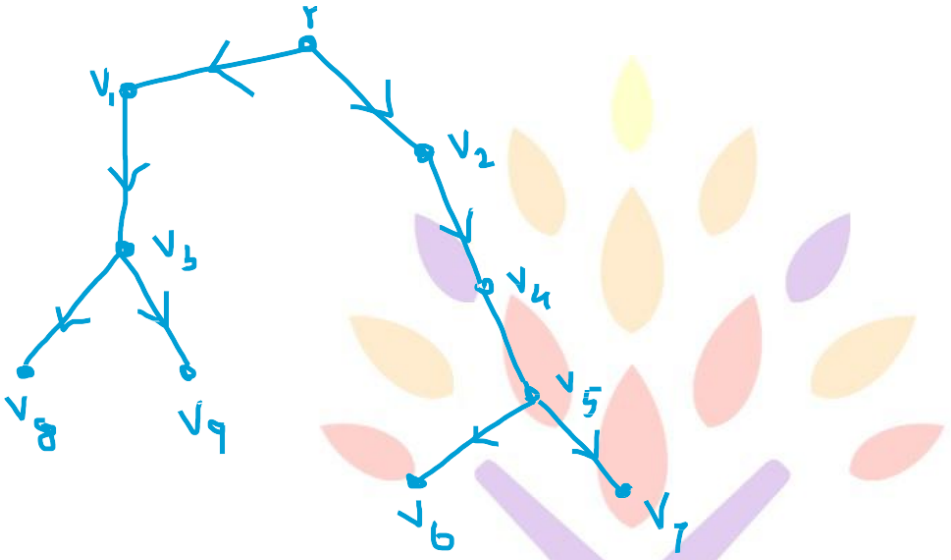
Eg.

A vertex $v$ of the rooted tree is said to be at the $h^{th}$ level if the path from $r \rightarrow v$ is of length $h$. In the above figure, $v$ is at the $3^{rd}$ level.

If $v_1$ & $v_2$ are two vertices: $v_1$ has a lower level than $v_2$ and there is path from $v_1$ to $v_2$ then we say that $v_1$ is an

ancestor of $□_2$and $□_2$is a descendant of$□_1$. If$□_1$&$□_2$areadjacent i.e. thereexists anedgefrom $□_1$→ $□_2$, then $□_1$is called parent of $□_2$and $□_2$is the child of $□_1$. Two vertices with a common parent are called siblings. A vertex whose outdegreezero is called a leaf. A vertex which is not a leaf is called as an internal vertex.

Eg



1) □istheroot.
2) □$_1$□□□□$_2$are in1$^{st}$level.
3) □$_3$□□□□$_4$are in2$^{nd}$level.
4) □$_1$istheancestorof□$_3$,□$_8$□□□□$_9$and□$_1$istheparentof□$_3$.
5) □$_8$□□□□$_9$are childrenof□$_3$andare calledsiblings.
6) □$_8$,□$_9$,□$_6$,□$_7$areleaves.

## BinaryTree

ArootedtreeT iscalleda binaryrootedtree, if everyinternal vertexis ofout degree1or 2.Thatiseveryvertex has at most 2 children. It is called a completebinary tree, if each vertex is of out degree2.

## SpanningTrees

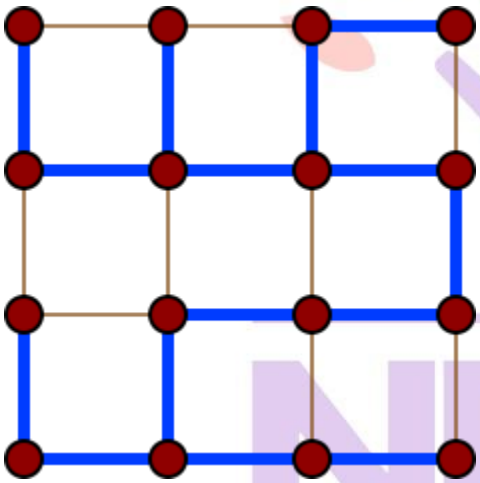
Let TG beaconnectedgraph.AsubgraphTofGis calledaspanningtreeofGif

(1) Tisatree

(2) Tcontainsallvertices ofG

Theedges ofthespanningtreearecalledbranches.

Example:



## ChordsandCotree

Let T beaspanningtreeofG.ThentheEdges ofGthat arenot inTarecalledchords ofGwithrespect toT.Theset ofallchordsofGiscalledachordsetoracotreeofTinGandisrepresentedby □̄. ThereforeG=T U □̄



**DFS(depthfirstsearch)Algorithmtofindthespanningtreeofagraph:**

Let□=(□,□)bea connectedgraphoforder 'n',withverticeslabelled□$_1$,□$_2$,....□$_□$insomespecifiedorder.

**StepI**:Assignthevertex '□$_1$'tothevariable □ andinitials T asthetreeconsistingofjust this vertex. (this vertex will become the root of the tree T)

**StepII**:Select thesmallest subscript k, for 2≤k≤n: {□,□$_□$}□□ &□$_□$has not alreadybeenselected inT. If no such subscript is found go to Step III, otherwise:

(i) Attachtheedge{□,□$_□$}tothetreeT

(ii) Assign□$_□$tov

(iii) ReturntoStepII.

**StepIII**:If□=□1,thetreeTisthespanningtree.

**StepIV**:For □≠□1, backtrackfrom□.If□istheparentof thevertexassignedto□inT,thenassign□to□ andreturntoStep II.

**BFS (breadthfirstsearch)Algorithmtofindthespanningtreeofagraph:**

Let G=(V,E)beaconnectedgraphof order nwithverticeslabelled$v_1,v_2,\ldots.v_n$.insomespecifiedorder. Werefer to an ordered list Q of vertices of G as a queue in G. Vertices areinserted in this list at one end (called therear of the queue) and deleted fromthe list at the other end (called the front of the queue). The BFS algorithmspecifies the following steps.

**Step I**: Assignthefirst vertex □1andinsert this vertex inthequeueandinitializeT as thetreemadeup ofthis one vertex □1.

**StepII**:Deletevfromthefrontof□.When □is deleted,consider□□foreach2≤□≤□.Iftheedge{□,□□} ∈□and □□has not been visited (considered) previously, attach this edge to T. If we examine all of the vertices previouslyvisitedandobtainnonewedge, thetreeT (generatedtothispoint)isthedesiredspanningtree.

**StepIII**:Insert thevertices adjacent to each □ (fromStep II)attherear ofthequeue □,accordingtotheorder in which they are (first) visited. Then return to Step II.

**ShowthataHamiltonianpathisaspanningtree.**

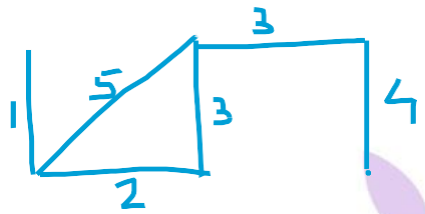
**Proof:**Bydefinitionnofa Hamiltonianpath,weknowthata Hamiltonianpathina graphGcontainsalltheverticesof G. Apathisatreesinceit doesn'tcontainacycle.

∴AHamiltonianpathisatreecontainsalltheverticesofthegivengraphG.AHamiltonianpathisa panning tree.

**WeightedGraph:**LetGbeagraphinwhichthereisapositivenumberassociatedwitheachedgeiscalledaweighted graph.

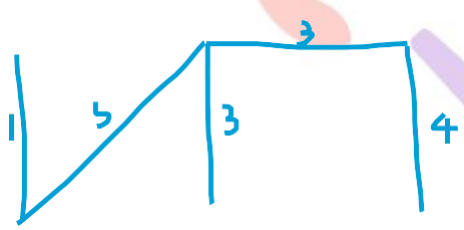
**Minimalspanningtree:**Thespanningtreeofa weightedgraphwhoseweight isleast iscalledtheminimalspanning tree

**Example:**

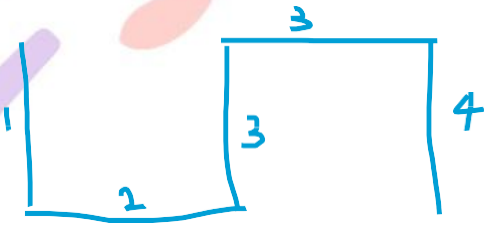



(G)

Thefollowingaresomespanningtreesof G




(T1)




(T2)

T1andT2arespanningtrees ofGbutSinceweightofT2 is 13anditis theminimalspanningtreeofG.

**Algorithmtofindminimalspanningtree:**

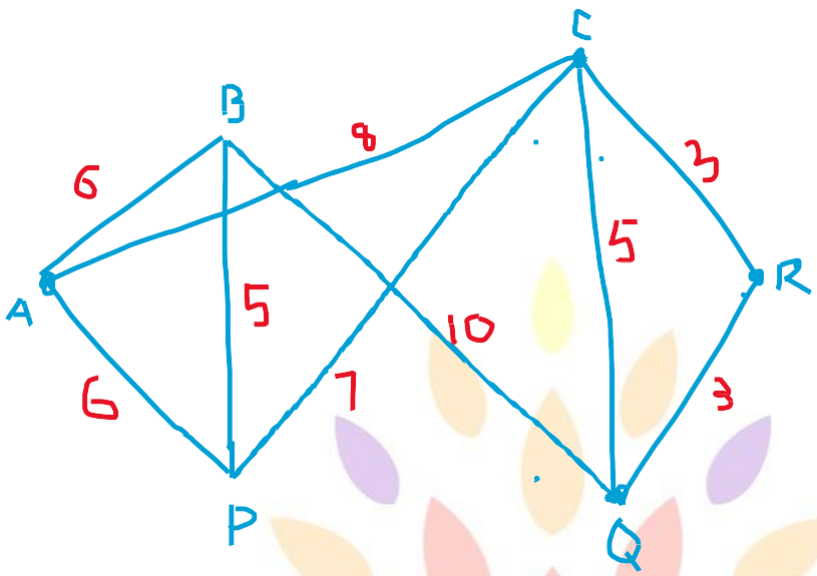
**1.Kruskal's Algorithm:**

**StepI:**LetGbeaconnectedweightedgraphGwithnvertices.ListedgesofGinincreasingweights.

**StepII**:Startingwitha smallestweightededge, proceedsequentiallybyselectingoneedgeatatime:nocycle is formed.

**StepIII**:StoptheprocessofStepIIwhen'n-1edgesareselectedasabove.Thesen-1edgesconstitutea minimal spanning tree G.

Example:




(G)

| Edge: | CR. | QR. | CQ. | BP. | AB. | AP. | CP. | AC. | BQ |
|---|---|---|---|---|---|---|---|---|---|
| Weight: | 3. | 3. | 5. | 5. | 6. | 6. | 7. | 8. | 10 |
| Selection: | Yes. | Yes | No. | Yes. | Yes. | No. | Yes. | - | - |

ByKruskal'sAlgorithm,thefollowingtreeTistheminimalspanningtreeoftheabovegraphwithweight 24

**PrimsAlgorithmtogetminimumspanningtree.**

**StepI:**Prepareanxntable inwhichweightsofalledgesareshown. Indicatethewightsof non-existing edges as ∞

**StepII**:Startfromvertex$v_1$andconnectittothenearestneighborwhichhasasmallerweightinthe$v_1$row say$v_k$.Nowconsideredgev1vkandconnectthisedgetoanewvertexwhichhasaminimumvalue in the $v_1$and $v_k$rows. Let this vertex be $v_m$.

**StepIII**:Start from$v_m$andrepeat stepII. Stoptheprocesswhenallthe nverticesareconnectedbyn-1 Edges.