DISCRETE MATHEMATICS

Subject Code:25CS301`

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Elementary Combinatorics

Introduction to Combinatorics

Definition: Combinatorics is the branch of mathematics dealing with counting, arrangement, and combination of objects.

Importance in Discrete Mathematics: Central to areas such as probability theory, graph theory, and computer science.

Basic Counting Principles:-

The Addition Principle: If a task can be done in **m** ways and another task can be done in **n** ways, and these tasks cannot be done at the same time, then the total number of ways to do either of the tasks is **m + n**.

The Multiplication Principle: If a task can be done in **m** ways and another task can be done in **n** ways, and these tasks are independent, then the total number of ways to do both tasks is **m × n**.

Factorials

Factorials

Definition: The factorial of a non-negative integer **n**, denoted by **n!**, is the product of all positive integers less than or equal to **n**.

$n! = n \times (n-1) \times \cdots \times 2 \times 1$

Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Permutations & Combinations

Permutation: A permutation is an arrangement of objects in a specific order.
Formula: The number of permutations of **n** objects taken **r** at a time is given by:
$P(n,r)=n!/(n-r)!$
Example: How many ways can 3 people be arranged in a row from a group of 5?
$P(5,3)=5!/(5-3)!=5\times4\times3=60$
Combination: A combination is a selection of objects without regard to the order.
Formula: The number of combinations of **n** objects taken **r** at a time is given by:
$C(n,r)=n!/r!(n-r)!$
Example: How many ways can 2 students be selected from a group of 4?
$C(4,2)=4!/2!(4-2)!=4\times3/2\times1$

Binomial Theorem

THE BINOMIAL THEOREM

Statement:

The Binomial Theorem describes the expansion of a binomial raised to a positive integer power.

$(x + y)^n = \Sigma(k=0 \text{ to } n)\ C(n,k)\ x^{(n-k)}\ y^k$

where

$C(n,k) = n! / [k!(n-k)!]$

Example: Expand $(x + y)^3$

$(x + y)^3 = C(3,0)x^3 + C(3,1)x^2y + C(3,2)xy^2 + C(3,3)y^3$

$= 1(x^3) + 3(x^2y) + 3(xy^2) + 1(y^3)$,

Therefore,

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Inclusion –Exclusion Principle

INCLUSION–EXCLUSION PRINCIPLE

Statement:

The Inclusion–Exclusion Principle is used to find the number of elements in the union of two or more sets.

For two sets:

$|A \cup B| = |A| + |B| - |A \cap B|$

Example:

10 students take Mathematics, 12 take Science, and 5 take both subjects. How many students take at least one subject?

Let A = students taking Mathematics

Let B = students taking Science

$|A| = 10$, $|B| = 12$, $|A \cap B| = 5$

Therefore, $|A \cup B| = |A| + |B| - |A \cap B| = 10 + 12 - 5 = 17$

Principle of Inclusion and Exclusion

Problem:
In a class of 60 students:
32 study Python
28 study Java
15 study both
Find students studying **neither**.
Solution:

$$n(P \cup J) = 32 + 28 - 15 = 45$$

Students studying neither:

$$60 - 45 = 15$$

✓ **Answer:** 15 students

Problems on Permutations

QUESTION 1: How many ways can 4 people be arranged in a row?

Solution:

The number of ways to arrange 4 people in a row is given by 4! (4 factorial).

$4! = 4 \times 3 \times 2 \times 1 \quad = 24$

Therefore, there are 24 ways to arrange 4 people in a row.

QUESTION 2: How many 3-letter words can be formed using the letters A, B, C, D, if repetition of letters is allowed?

Solution:

Since repetition is allowed, each of the 3 positions can be filled by any of the 4 letters.

Number of choices for the 1st position = 4

Number of choices for the 2nd position = 4

Number of choices for the 3rd position = 4

Therefore, Number of words $= 4 \times 4 \times 4 = 4^3 = 64$

Therefore, 64 possible 3-letter words can be formed.

Basics of Counting – Model Problem

Problem:
A username consists of **1 letter followed by 3 digits**. Digits may repeat.
Find the number of possible usernames.
Solution:
Number of letters = 26
Number of digits = 10
Using **Fundamental Principle of Counting**:

$$26 \times 10 \times 10 \times 10 = 26{,}000$$

✓ **Answer:** 26,000 usernames

Permutations

Problem:
Five different tasks are to be executed by a processor.
In how many ways can they be arranged?
Solution:

$$\text{Number } of\, permutations = 5! = 120$$

✓ **Answer:** 120 ways

More problems

QUESTION: How many ways can 3 people be chosen and arranged from a group of 7?
Solution:
This is a permutation problem because we are choosing and arranging 3 people out of 7.
The formula for permutations is:
$P(n, r) = n! / (n - r)!$
Here,
$n = 7$
$r = 3$
Therefore,
$P(7, 3) = 7! / (7 - 3)!$
$= 7! / 4!$
$= (7 \times 6 \times 5 \times 4!) / 4!$
$= 7 \times 6 \times 5$
$= 210$
Therefore, there are 210 ways to choose and arrange 3 people from a group of 7. ways to choose and arrange 3 people from a group of 7.

Permutations with Constrained Repetition

Problem:
How many 4-digit passwords can be formed using digits 0–9 if **no digit appears more than twice**?
Solution:
Total 4-digit numbers (repetition allowed):

$$10^4 = 10{,}000$$

Cases where **a digit repeats 3 or 4 times** must be excluded.
All digits same: 10 cases
One digit repeated 3 times and another digit once:

$$10 \times 9 \times \frac{4!}{3!} = 360$$

Valid passwords:

$$10{,}000 - (10 + 360) = 9{,}630$$

✓ **Answer:** 9,630 passwords

Permutations with identical items

PROBLEM 4: PERMUTATIONS WITH IDENTICAL ITEMS
Question: How many different ways can the letters of the word "BOOK" be arranged?
Solution:
The word "BOOK" has 4 letters, but the letter O is repeated twice.
Formula:
Number of permutations = $n! / (k_1! \, k_2! \ldots k_m!)$
For "BOOK":
Total letters, $n = 4$
Number of O's = 2
Therefore,
Number of arrangements = $4! / 2! = (4 \times 3 \times 2 \times 1) / (2 \times 1) = 24 / 2 = 12$
Therefore, there are 12 distinct arrangements of the letters in "BOOK".

Circular Permutation

QUESTION 1: How many ways can 6 people be seated around a round table?
Solution:
In a circular arrangement, the number of ways to arrange n objects is:$(n - 1)!$
For 6 people:
$(6 - 1)! = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
Therefore, there are 120 ways to arrange 6 people around a round table.
QUESTION 2: In how many ways can 4 red balls and 2 green balls be arranged in a row?
Solution:
There are 6 balls in total:
4 red balls (identical) and 2 green balls (identical).
The formula is: Number of arrangements $= n! / (k_1!\, k_2!)$
Here, $n = 6$, $k_1 = 4$, $k_2 = 2$
Therefore, Number of arrangements $= 6! / (4! \times 2!) = 720 / (24 \times 2) = 720 / 48 = 15$
Therefore, there are 15 ways to arrange the 4 red balls and 2 green balls in a row.

Permutation of Letters with Multiple Identical Letters

QUESTION: How many distinct arrangements can be made from the word "MISSISSIPPI"?

Solution:

In the word "MISSISSIPPI":

M appears 1 time

I appears 4 times

S appears 4 times

P appears 2 times

Total number of letters, $n = 11$

For arrangements with repeated letters:

Number of arrangements $= n! / (k_1! \, k_2! \ldots k_m!)$

Therefore, Number of arrangements $= 11! / (1! \times 4! \times 4! \times 2!)$

$= 39{,}916{,}800 / (1 \times 24 \times 24 \times 2) = 39{,}916{,}800 / 1{,}152 = 34{,}650$

Therefore, there are 34,650 distinct arrangements of the letters in "MISSISSIPPI".

Combinations with Repetition

QUESTION: How many ways can 5 identical apples be distributed among 3 children?

Solution: This is a combinations-with-repetition problem because the apples are identical.

Formula:

Number of ways = C(n + r - 1, r - 1)

where,

n = number of identical items

r = number of groups

Here,

n = 5 (apples)

r = 3 (children)

Therefore, Number of ways = C(5 + 3 - 1, 3 - 1) = C(7, 2) = 7! / [2!(7 - 2)!]

= 7! / (2! × 5!) = (7 × 6) / (2 × 1)

Combinations with Repetition

Problem:
Select 4 virtual machines from 3 types of servers. Repetition allowed.
Solution:
Formula:

$$^{n+r-1}C_r$$

Here, $n = 3, r = 4$

$$^{3+4-1}C_4 = \ ^6C_4 = 15$$

✓ **Answer:** 15 ways

Combinations from a Set

. QUESTION: How many ways can 2 red balls and 3 green balls be selected from a collection of 5 red balls and 4 green balls?

Solution:

Ways to select 2 red balls from 5:

$C(5, 2) = 5! / [2!(5 - 2)!] = (5 \times 4) / (2 \times 1) = 10$

Ways to select 3 green balls from 4:

$C(4, 3) = 4! / [3!(4 - 3)!] = 4! / (3! \times 1!) = 4$

Total number of ways $= C(5, 2) \times C(4, 3)$

$= 10 \times 4$

$= 40$

Team Selection

QUESTION: A committee of 5 members is to be formed from a group of 10 people. How many ways can the committee be chosen?

Solution:
Since the order of selection does not matter, this is a combination problem.

Formula:
$C(n, r) = n! / [r!(n - r)!]$

Here, $n = 10$ and $r = 5$.

$C(10, 5) = 10! / [5!(10 - 5)!]$
$= 10! / (5! \times 5!)$
$= (10 \times 9 \times 8 \times 7 \times 6) / (5 \times 4 \times 3 \times 2 \times 1)$

Group Selection with Constraints

QUESTION: How many ways can a team of 4 people be selected from a group of 6 men and 5 women if the team must include at least 2 men?

Solution:

Since the team must include at least 2 men, there are 2 possible cases.

CASE 1: 2 men and 2 women

Ways to choose 2 men from 6:

$C(6, 2) = 6! / [2!(6 - 2)!] = (6 \times 5) / (2 \times 1) = 15$

Ways to choose 2 women from 5:

$C(5, 2) = 5! / [2!(5 - 2)!] = (5 \times 4) / (2 \times 1) = 10$

Total = $15 \times 10 = 150$

CASE 2: 3 men and 1 woman

Ways to choose 3 men from 6:

$C(6, 3) = 6! / [3!(6 - 3)!] = (6 \times 5 \times 4) / (3 \times 2 \times 1) = 20$

Selecting a Committee with Restrictions

QUESTION: In how many ways can a committee of 3 men and 2 women be chosen from a group of 8 men and 6 women?

Solution:

We use combinations to select the required number of men and women.

Ways to choose 3 men from 8:

$C(8, 3) = 8! / [3!(8 - 3)!] = (8 \times 7 \times 6) / (3 \times 2 \times 1)$ 56

Ways to choose 2 women from 6:

$C(6, 2) = 6! / [2!(6 - 2)!] = (6 \times 5) / (2 \times 1) = 15$

Therefore,

Total number of ways $= C(8, 3) \times C(6, 2) = 56 \times 15 = 840$

Therefore, there are 840 ways to select the committee.

Combinations of Identical Objects

.QUESTION: How many ways can 6 identical balls be distributed into 4 distinct boxes?

Solution:

This is a combinations-with-repetition problem because the balls are identical and the boxes are distinct.

Formula:

Number of ways = $C(n + r - 1, r - 1)$

Here, $n = 6$ (balls) and $r = 4$ (boxes).

Number of ways = $C(6 + 4 - 1, 4 - 1) = C(9, 3)$

$C(9, 3) = 9! / [3!(9 - 3)!] = (9 \times 8 \times 7) / (3 \times 2 \times 1) = 84$

Therefore, there are 84 ways to distribute 6 identical balls into 4 distinct boxes.

QUESTION: What is the coefficient of x^3 in the expansion of $(2x + 3)^5$

Solution:

Using the Binomial Theorem,

$(a + b)^n = \Sigma(r = 0 \text{ to } n)\ C(n, r)\ a^{n-r}b^r$

Here, $a = 2x$, $b = 3$, $n = 5$.

General term:

$T(r + 1) = C(5, r)(2x)^{5-r}(3)^r$

For x^3, $5 - r = 3 \Rightarrow r = 2$.

$T_3 = C(5, 2)(2x)^3(3)^2$

$= 10 \times 8x^3 \times 9$

$= 720x^3$

BASIC MULTINOMIAL COEFFICIENT

QUESTION: Calculate the multinomial coefficient C(6; 2, 2, 2).

Solution:
The multinomial coefficient is given by:

$C(n; k_1, k_2, \ldots, k_m) = n! / (k_1!\, k_2! \ldots k_m!)$

Here, $n = 6$ and $k_1 = k_2 = k_3 = 2$.

Therefore,

$$
\begin{aligned}
C(6; 2, 2, 2) &= 6! / (2! \times 2! \times 2!) \\
&= 720 / (2 \times 2 \times 2) \\
&= 720 / 8
\end{aligned}
$$

Multinomial Theorem

Problem:

Find the coefficient of x^2y^2zin $(x + y + z(^5$.

Solution:

$$\frac{5!}{2!\,2!\,1!} = \frac{120}{4} = 30$$

✓ **Answer:** 30

PIGEONHOLE PRINCIPLE (PHP)

Problem (PHP):

Problem:

In a class of **31 engineering students**, each student is assigned one of the **12 months** as their birth month.

Show that **at least 3 students** must have been born in the same month.

Solution:

Number of students (pigeons) = **31**

Number of months (holes) = **12**

By **Pigeonhole Principle**,

$$\left\lceil \frac{31}{12} \right\rceil = \lceil 2.58 \rceil = 3$$

GENERALIZED PIGEONHOLE PRINCIPLE (GPHP)

Problem (GPHP):
Problem:
In a computer lab, **50 files** are stored across **7 servers**.
Prove that **at least one server** stores **at least 8 files**.

Solution:
Number of files (pigeons) = **50**
Number of servers (holes) = **7**
By **Generalized Pigeonhole Principle**,

$$\left\lceil \frac{50}{7} \right\rceil = \lceil 7.14 \rceil = 8$$

✓ **Conclusion:**
At least **one server contains at least 8 files.**