Discrete Mathematics

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UNIT-III

ALGEBRAIC STRUCTURES

CONTENT..

Algebraic Structures:

Introduction

Algebraic Systems

Semi groups and Monoids

Lattices as Partially Ordered Sets

Boolean Algebra.

Uses in Engineering:

- Helps in **abstract thinking** and **problem modelling**
- Forms the **mathematical foundation** for:
- Computer Science
- Electronics & Communication
- Electrical Engineering
- Information Technology
- Used to **represent systems using sets and operations**
- Essential for understanding:
- Data structures
- Logic design
- Cryptography
- Control systems

Introduction to Algebraic Structures

An **algebraic structure** consists of a non- empty set together with one or more binary operations defined on it.

Examples:

(Z, +) – integers under addition

(N, ×) – natural numbers under multiplication

2. Algebraic Systems: An **algebraic system** is a pair (A, *) where A is a non- empty set and * is a binary operation on A.

Properties of Binary Operations

Closure
Associativity
Commutativity
Identity element
Inverse element

ALGEBRAIC STRUCTURES..

Semigroups: A **semigroup** is an algebraic structure (S, *) satisfying:
Closure: a * b ∈ S
Associativity: (a * b) * c = a * (b * c)
Example:
(N, +) is a semigroup
(Z, –) is not a semigroup (not associative)
4. Monoids: A **monoid** is a semigroup with an identity element.
Conditions:
Closure
Associativity
Identity element
Example:
(Z, +) is a monoid with identity 0
(N, ×) is a monoid with identity 1

ALGEBRAIC STRUCTURES...

5. Groups: A group is an algebraic structure (G, *) satisfying the following axioms:
Closure: $a * b \in G$
Associativity: $(a * b) * c = a * (b * c)$
Identity: There exists $e \in G$ such that $a * e = e * a = a$
Inverse: For every $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = e$.
Example:
(Z, +) is a group
(N, +) is not a group (no inverse)
6.Abelian Groups: A group (G, *) is called an Abelian group if it also satisfies: 5. Commutativity: $a * b = b * a$
Example:
(Z, +) is an Abelian group
$(R - \{0\}, \times)$ is an Abelian group

Properties of Lattices

Idempotent laws
Commutative laws
Associative laws
Absorption laws

Boolean Algebra

A **Boolean algebra** is an algebraic structure $(B, +, \cdot, ')$ satisfying:

Closure

Associativity

Commutativity

Distributive laws

Identity laws

Complement laws

Boolean Laws

$x + 0 = x$

$x \cdot 1 = x$

$x + x' = 1$

$x \cdot x' = 0$

De Morgan's Laws

PROBLEMS..

Q. Show that (Z, +) is a monoid.
Solution:
Closure: $a + b \in Z$
Associative: $(a + b) + c = a + (b + c)$
Identity: 0 is identity
Hence (Z, +) is a monoid.
Q. Verify De Morgan's laws in Boolean algebra.
$(x + y)' = x' \cdot y'$ $(x \cdot y)' = x' + y'$
Q. Prove the following results. (Theorems)
Identity element in a group is unique.
Inverse of each element in a group is unique.
$(a^{-1})^{-1} = a$ for all $a \in G$.
$(ab)^{-1} = b^{-1}a^{-1}$

PROBLEMS..

Q. **Prove that (Z, +) is an Abelian group**

Solution: Closure: $a + b \in Z$ Associativity: $(a + b) + c = a + (b + c)$ Identity: $0 \in Z$ such that $a + 0 = a$ Inverse: For $a \in Z$, $-a \in Z$ such that $a + (-a) = 0$

Commutativity: $a + b = b + a$

Hence (Z, +) is an Abelian group.

Q. Show that (G,) is a group, where G = {1, –1} under multiplication

Solution: Closure: Product of any two elements is in G Associativity: Multiplication is associative Identity: 1 Inverse: $1^{-1} = 1$, $(-1)^{-1} = -1$

Hence (G, *) is a group.

Q. **Prove that identity element in a group is unique**

Proof: Let e and e′ be two identity elements. Then $e * e' = e'$ (since e is identity) Also $e * e' = e$ (since e′ is identity) Hence $e = e'$.

Therefore identity element is unique.

PROBLEMS..

Q. **Prove that inverse of an element in a group is unique**

Proof: Let b and c be inverses of a. Then $a * b = e$ and $a * c = e$ $b = b * e = b * (a * c) = (b * a) * c = e * c = c$

Hence inverse is unique.

Q. **Prove that $(ab)^{-1} = b^{-1}a^{-1}$**

Proof: $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$ Similarly, $(b^{-1}a^{-1})(ab) = e$

Hence $(ab)^{-1} = b^{-1}a^{-1}$.