GRAPH THEORY

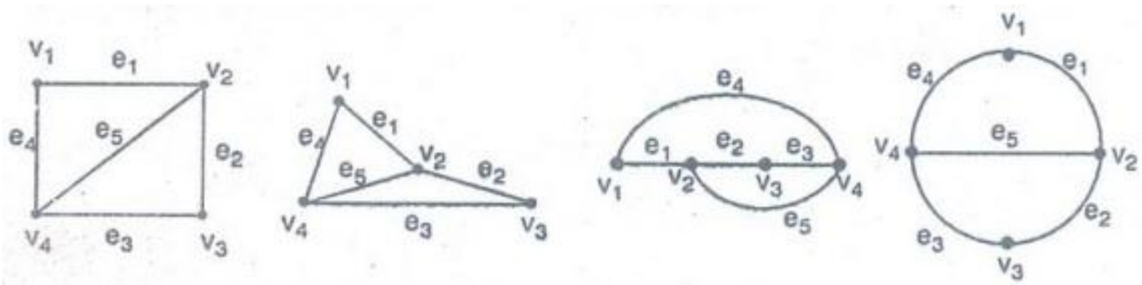
Definition: Graph

A graph G = (V, E) consists of V, a non-empty set of vertices (nodes or points) and E, a set of edges (also called lines).

i.e., A graph G is an ordered triple (V,E) consists of a non-empty set V called the set of vertices (nodes or points) of the graph G, E is said to be the set of edges of the graph G, and is a mapping from the set of edges E to a set of order or un ordered pairs of elements of V.

Example:

Let G = (V, E) where V (G)= $\{v_1, v_2, v_3, v_4\}$ E (G)=$(e_1, e_2, e_3, e_4, e_5)$



It should be noted that, in drawing a graph, it is immaterial whether the edges are drawn straight or curved, long or short, the important point is how the vertices are joined up.

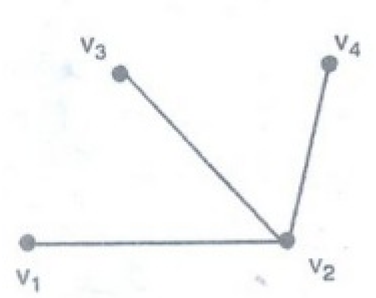
All above graphs are same.

Note: 1. We denote the graph G (V, E) simply as G.

2. If e ϵ E is an edge and its end vertices are $\{v_1, v_2\}$, then we say that e is an edge incident with v_1 and v_2.

Definition: Adjacent vertices

Any pair of vertices which are connected by an edge in a graph is called adjacent vertices.

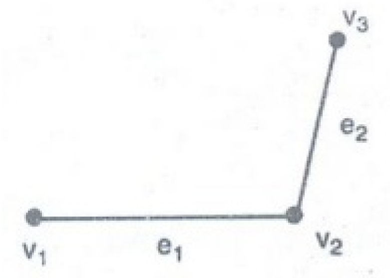


Here v_1, v_2 ; v_2, v_4 ; v_2, v_3 are adjacent vertices

v_1, v_3 ; v_3, v_4 ; v_1 v_4 are not adjacent.

Definition: Adjacent edges:

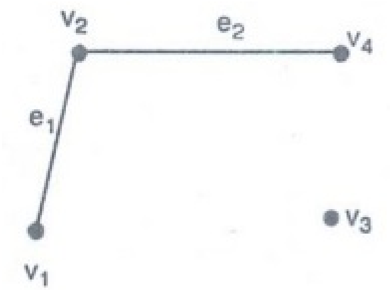
If two distinct edges are incident with a common vertex then they are called adjacent edges.



Here e_1 and e_2 are incident with a common vertex v_2.

Definition: Isolated vertex

In any graph, a vertex which is not adjacent to any other vertex is called an isolated vertex. Otherwise the vertex has no incident edge.



Here v_3 has no incident edge. Therefore, the vertex v_3 is called isolated vertex.

Note:

The set of edges in a null graph is empty.

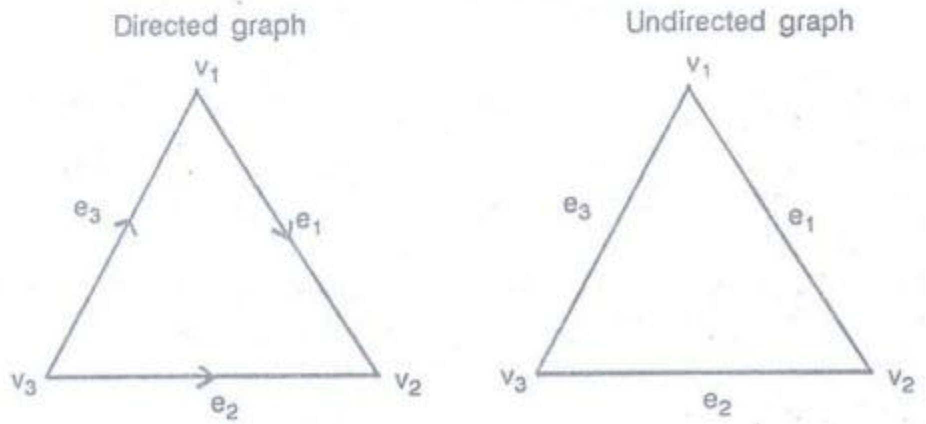
Definition: Directed graph and undirected graph

In a graph G (V, E), an edge which is associated with an ordered pair of vertices is called a directed edge of graph G, while an edge which is associated with an unordered pair of vertices is called an undirected edge.

A graph in which every edge is directed is called a directed graph simply a digraph.

A graph in which every edge is undirected is called an undirected graph.

The end vertices of an edge are said to be incident with the edge and vice versa.



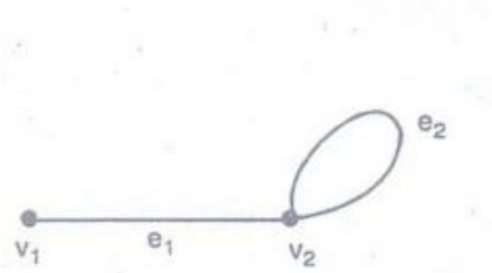
The edge e_1 is incident with the vertices v_1 and v_2 also the vertex v_1 is incident with e_1 and e_3.

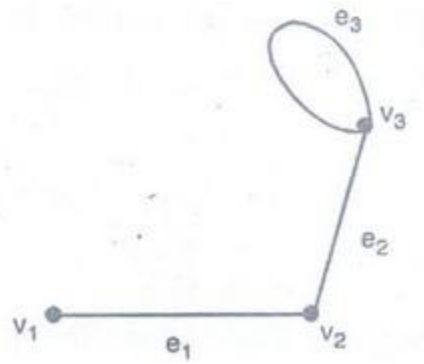
The vertices v_1 and v_2 are also called the initial and terminal vertices of the edge e_1.

Definition: Loop

A loop is an edge whose vertices are equal.

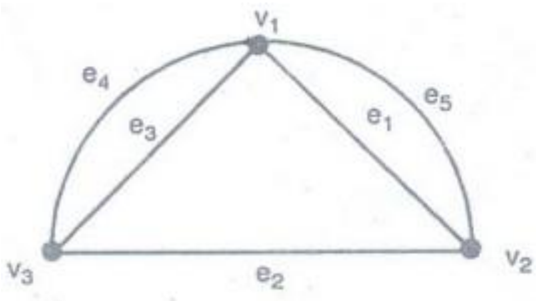
i.e., An edge of a graph which joins a vertex to itself is called a loop.





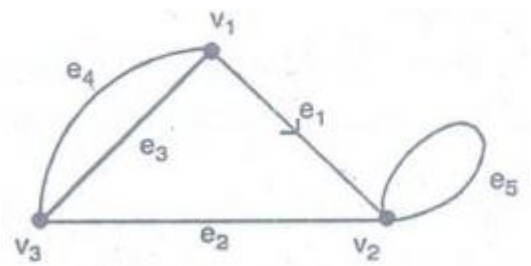
Definition: Parallel edges (Multiple edges)

Multiple edges are edges having the same pair of vertices.



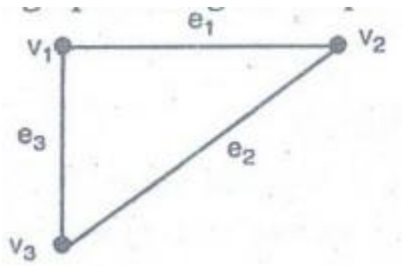
Definition: Multi graph

Any graph which contains some parallel edges and loops is called as multi graph.



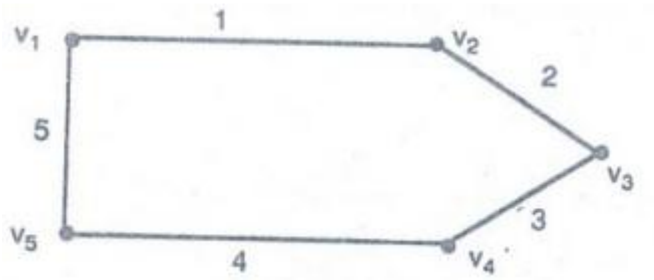
Definition: Simple graph

A simple graph is a graph having no loops or multiple edges.



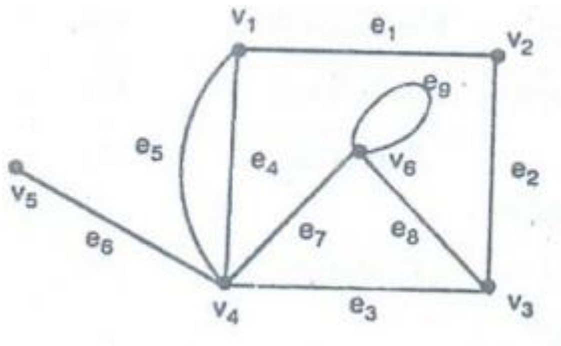
Definition: Weighted graph

A graph in which weights are assigned to every edge is called a weighted graph.



Note:

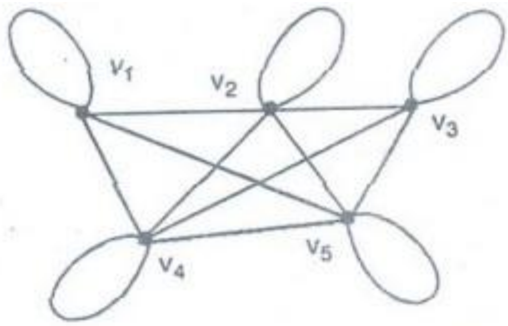
If the graph G is finite, |V| denotes the number of vertices of G known as order of G and |E| denotes the number of edges of G, known as size of G,



For this graph |V| = 6; |E| = 9

Definition Pseudographs :

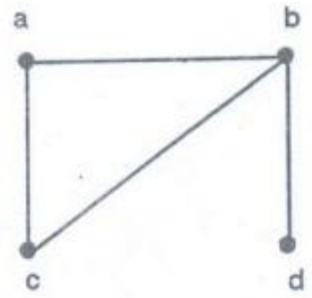
Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices, are sometimes called Pseudographs.



S.No.	Type	Edges	Multiple Edges	Loops
1.	Simple graph	Undirected	No	No
2.	Multigraph	Undirected	Yes	No
3.	*Pseudograph*	Undirected	Yes	Yes
4.	Simple *directed graph*	Directed	No	No
5.	Directed *multigraph*	Directed	Yes	Yes
6.	Mixed graph	Directed and *undirected*	Yes	Yes

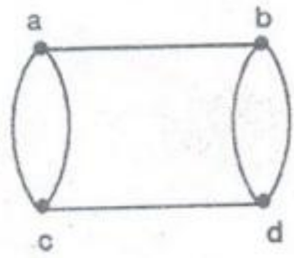
Question 1. What type of graphs are the following.

(a)



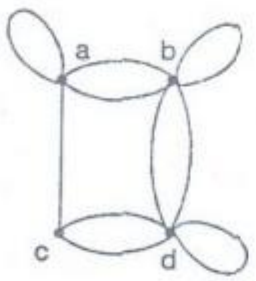
Ans. Simple graph

(b)



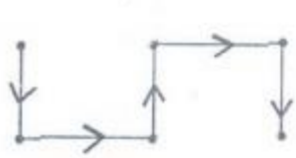
Ans. Multi graph

(c)



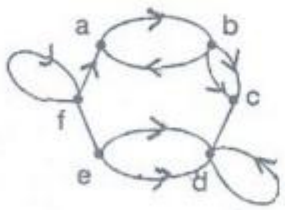
Ans. Pseudo graph

(d)



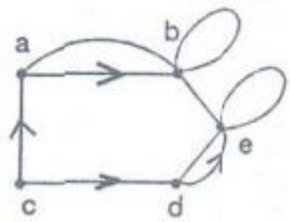
Ans. Simple directed graph

(e)



Ans. Directed multi graph

(f)



Ans. Mixed graph

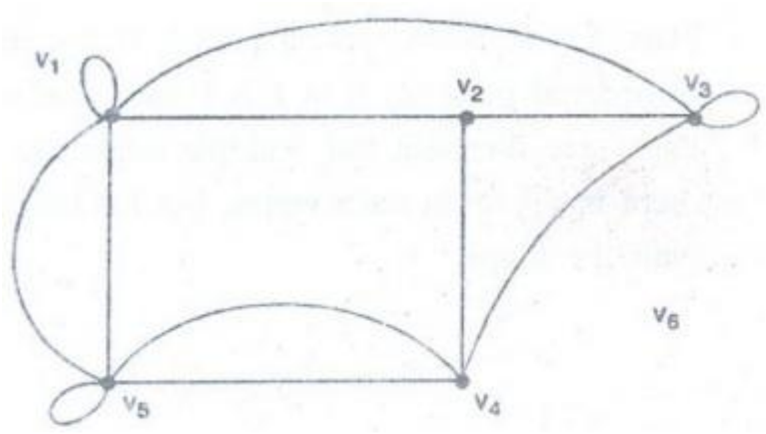
Definition: The degree of a vertex:

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The degree of the vertex is denoted by deg( ).

Example:

Let v be a vertex in a graph G. Then the degree of v is the number of edges incident with it (each loop is counted twice). The degree of a vertex can also be denoted by d(v).



deg (V_1) = 6

deg (V_2) = 3

deg (V_3) = 5

deg (V_4) = 4

deg (V_5) = 6

deg (V_6) = 0

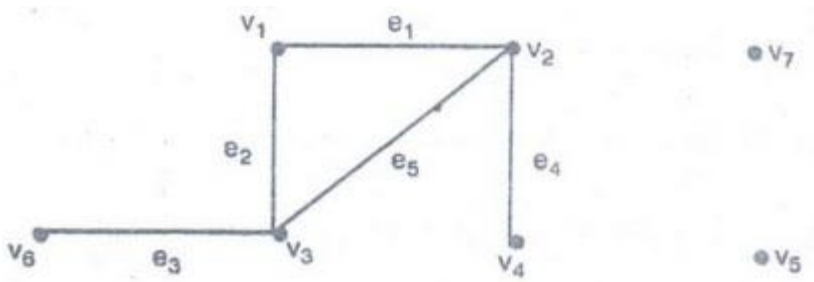
Note:

1. Let G be an undirected graph with |E| edges and |V| = n vertices, then $\Sigma^{n}_{i=1}$ deg (v_i) = 2 |E|.

2. In any graph, the number of vertices of odd degree is even.

3. A vertex of degree one is called a pendant or end vertex in G.

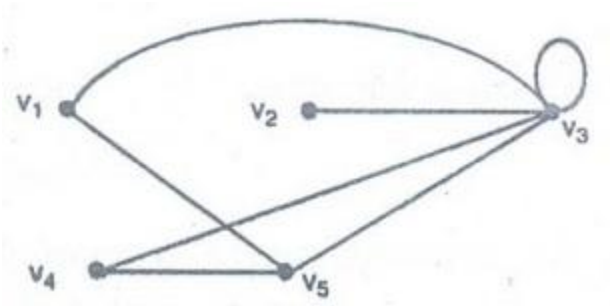
4. A vertex of degree zero is called an isolated vertex in G.



The vertices V_4, V_6 are pendant vertices.

The vertices V_5, V_7 are isolated vertices.

Question2. What are the degrees of the vertices in the graph G.



Solution:

deg (v_1) = 2, deg (v_2) = 1, deg (v_3) = 6

deg (v_4) = 2, deg (v_5) = 3.

[The Handshaking Theorem]

Theorem: For any graph G with E edges and V vertices $v_1, v_2, \ldots, v_n$, $\Sigma^n_{i=1}$ d (v_i) = 2 |E|

Proof:

Let G = G (V, E) be any graph, where V = {$v_1, v_2, \ldots, v_n$} and E = ($e_1, e_2, \ldots, e_n$}

Each edge e_i contributes two to the sum of degrees of all vertices in a graph.

Also, each edge e_i contributes one degree to each of the vertices on which it is incident.

Hence, if there are N edges in G, then

2**|E|** = d (v_1) + d (v_2) + …. + d (v_n)

Hence proved.

Note: RHS is even as LHS is always even.

Theorem: The number of odd degree vertices is always even.

Let G = {V, E} be any graph with 'n' number of vertices and '*e*' number of edges.

Let V_1, *2, …,*V_k be the vertices of odd degree and $V_1', V_2', \ldots, V_m'$ be the vertices of even degree.

To prove, k is even

We know that $\sum_{i=1}^{n} d(V_i) = 2|E| = 2e$

$$\Rightarrow \sum_{i=1}^{k} d(V_i) + \sum_{j=1}^{m} d(V_j') = 2e$$

Each of $d(V_j)$ is even $\Rightarrow \sum_{j=1}^{m} d(V_j')$ and $2e$

are even numbers (being the sum of even numbers)

$$\therefore \sum_{i=1}^{k} d(V_i) + \text{ an even number } = \text{ an even number}$$

$$\Rightarrow \sum_{i=1}^{k} d(V_i) = \text{ an even number.}$$

Since, each term d (V_i) is odd.

Therefore, the number of terms in the LHS sum must be even.

⇒ K is even.

Hence, the theorem.

Question3. How many edges are there in a graph with 10 vertices each of degree six?

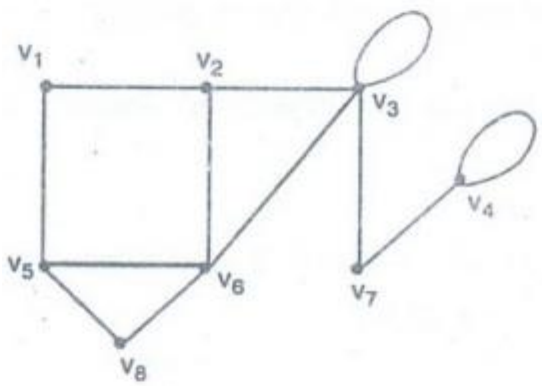
Solution: Sum of the degrees of the 10 vertices

is (6) (10) = 60

i.e., 2e = 60 (Since the sum of degrees is twice the number of edges.)

e = 30

Question4. Verify that the sum of the degree of all the vertices is even for the graph.



Solution: The sum of degree of all the vertices is

$= d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) + d(v_7) + d(v_8)$

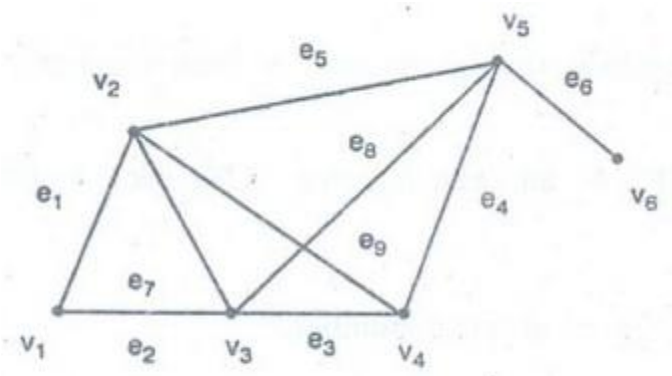
$= 2 + 3 + 5 + 3 + 3 + 4 + 2 + 2$

$= 24$ which is even.

Note: Odd vertices mean vertices of odd degree.

Even vertices mean vertices of even degree.

Question 5. Verify the handshaking theorem for the graph.



Solution: To prove $\Sigma \deg(v_i) = (2)$ (no. of edges)

i.e., $\Sigma \deg(v_i) = (2)(9) = 18$

$\Sigma \deg(v_i) = \deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4) + \deg(v_5) + \deg(v_6)$

$= 2 + 4 + 4 + 3 + 4 + 1$

$= 18$

Hence the theorem is true.

Question 6. Show that the degree of a vertex of a simple graph G on n vertices cannot exceed n-1.

Solution: Let v be a vertex of G because G is simple, no multiple edges or loops are allowed in G. Thus, v can be adjacent to at most all the remaining n - 1 vertices of G.

Hence v may have maximum degree n - 1 in G.

Then $0 \le \deg(v) \le n - 1 \ \forall\ v \in V(G)$

Question7. Is there a simple graph corresponding to the following degree sequences? (i) (1, 1, 2, 3) (ii) (2, 2, 4, 6)

Solution:

(i) There are odd number (3) of odd degree vertices, 1, 1 and 3. Which is not possible as there exist even number of odd vertices if they exist.

(ii) Number of vertices in the graph sequence is 4.

and the maximum degree of a vertex is 6 which is not possible as the maximum degree cannot exist one less than the number of vertices.

Question8. Show that the maximum number of edges in a simple graph with n vertices is n (n - 1) / 2

Solution: Use handshaking theorem.

i.e,. $\Sigma_{i=1}$ d (v_i) = 2 e,

where *e* is the number of edges with n vertices in the graph G.

i.e., d (v_1) + d (v_2) + ... d (v_n) = 2*e* ..... (1)

Since we know that the maximum degree of each vertex in the simple graph G can be (n - 1)

(1) ⇒ (n − 1) + (n − 1) + ... to n terms = 2*e*

⇒ n (n − 1) = 2*e*

e = n (n - 1) / 2

Hence the maximum number of edges in any simple graph with n vertices is n (n - 1) / 2.

Definition: In a digraph, the in-degree of a vertex v, denoted by $\deg^-$ (*v*), is the number of edges with v as their terminal vertex.

The out-degree of with v, denoted by $\deg^+$ (*v*), is the number of edges with *v* as their initial vertex.

Note: A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

In degree

In a directed graph G, the in degree of v denoted by $\deg^-$(v), is the number of edges incident from *v*.

Out degree

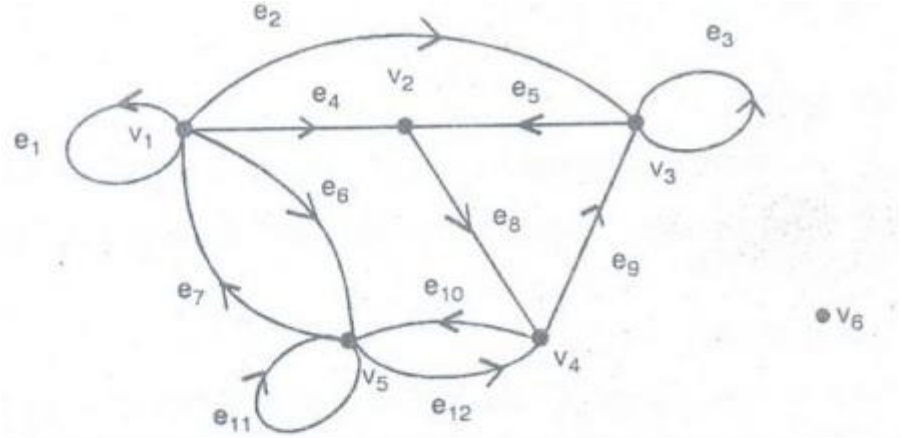
In a directed graph G, the out degree of vertex of v of G denoted by out $\deg^+$ (*v*), is the number of edges incident to *v*.

Note:

(1) The sum of the in degree and out degree of a vertex is called the total degree of the vertex.

(2) A verex with zero in degree is called a source and a vertex with zero out degree is called a sink.

Question9. Verify $\Sigma^n_{i=1}$ deg^+_G (v_i) = $\Sigma^n_{i=1}$ deg^- (v_i) = |E| = e in the following graph.



Solution :

deg	out degree : deg^-	indegree deg^+
v_1	4	2
v_2	1	2
v_3	2	3
v_4	2	2
v_5	3	9
v_6	0	0
Sum	12	12

$$\therefore \sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg^-(v_i) = e = 12$$

Regular graph: A graph in which all vertices are of equal degree is called a regular graph.

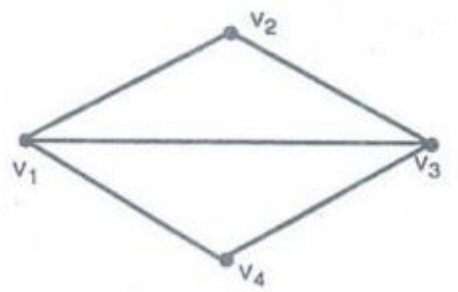
If the degree of each vertex is r, then the graph is called a regular graph of degree r.

d (V_1) = d (V_2) = d (V_3) = d (V_4) = 2

The given graph is 2-regular (or) regular graph of degree 2.

Sequence of degrees:

If V_1, V_2, V_3, …. V_n are the n vertices of G, then the sequence (d_1, d_2, …, d_n), where d_i = deg (v_i) is the degree sequence of G.

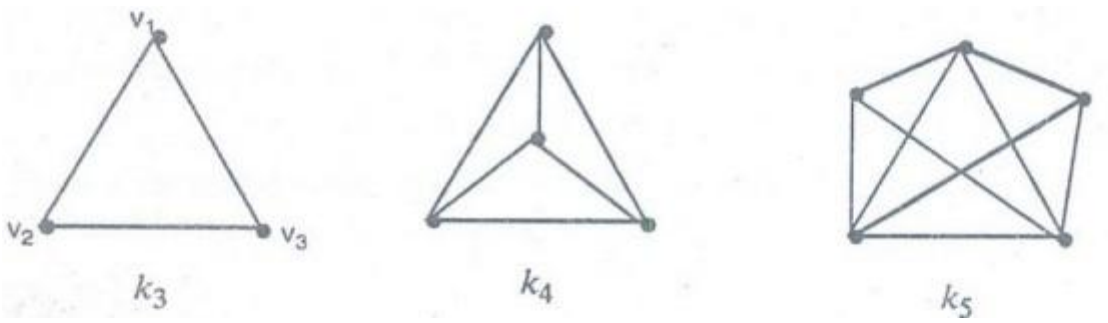


A degree sequence of G: (2, 2, 3, 5)

Question 10. Is there a graph with degree sequence (1, 3, 3, 3, 7, 6, 6).

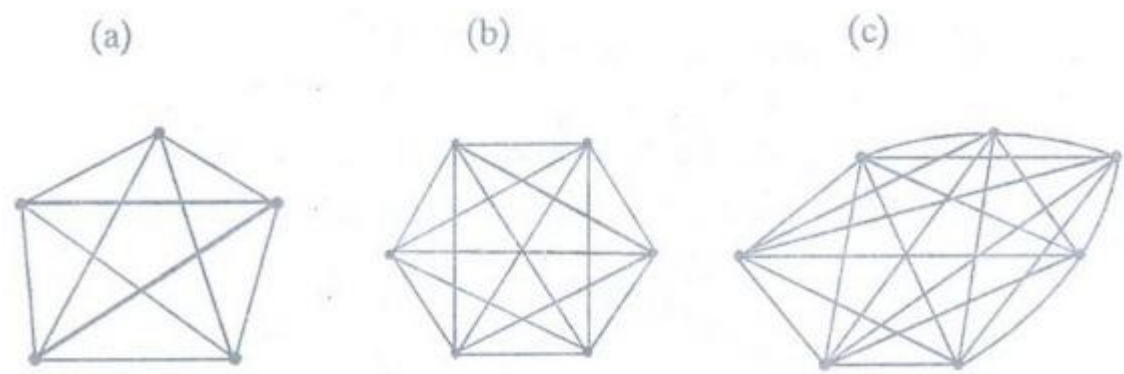
Ans. No, because the no. of vertices with odd degree is odd, a contradiction to corollary; the number of vertices of odd degree must be even.

Complete Graph: A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. If G has n vertices, then the complete graph will be denoted by K_n.



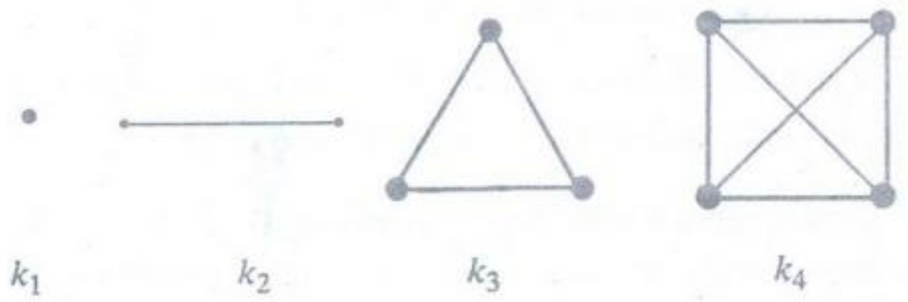
Question 11. Draw these graphs (a) K_5 (b) K_6 (c) K_7

Solution:



Question 12. Draw Graphs K_n for $1 \leq n \leq 4$

Solution:

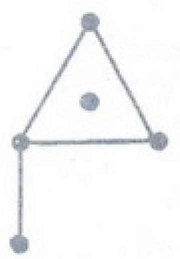


Question 13. What is the degree sequence of n-1, where n is a positive integer? Explain your Answer. Each of the n vertices is adjacent to each of the other n-1 vertices, so the degree sequence is n - 1, n - 1, ..., n - 1 (n terms)

Question 14. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence. (a) 5, 4, 3, 2, 1 (b) 3, 2, 2, 1, 0 (c) 1, 1, 1, 1, 1

Solution:

(a) No, (5+4+3+2+1=15) sum of degree is odd

(b) Yes 

(c) No, (1+1+1+1+1=5) sum of degrees is odd.

Question 14. How many vertices and how many edges the graph K_n has?

Ans. n vertices, n (n - 1)/2 edges.

Question 15. Find the degree sequence of each of the following graphs (a) K_4 (b) K_5 (c) K_2

Solution:

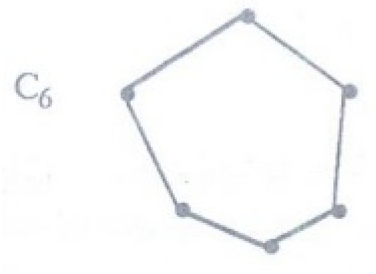
(a) 3, 3, 3, 3

(b) 4, 4, 4, 4, 4

(c) 1, 1

Definition: Cycle Graph

A cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n' and denoted by C_n.



Note:

(1) In a graph a cycle that is not a loop must have length atleast 3, but there may be cycles of length 2 in a multigraph.

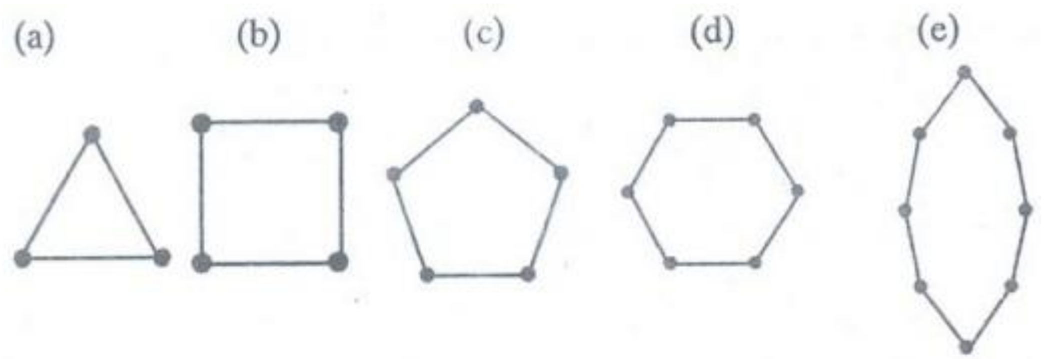
(2) A simple digraph having no cycles is called acyclic graph.

(3) A cyclic graph cannot have any loops.

(4) The cycle C_n, $n \geq 3$, consists of n vertices 1, 2, …, n and edges {1, 2}, {2, 3},... {n-1, n}

Question 15. Draw the graphs (a) C_3 (b) C_4 (c) C_5 (a) C_6, (e) C_8

Solution:



Question 16. How many vertices and how many edges do these graphs have (a) C_n (b) C_8 (c) Also find the degree sequence of C_4.

Solution:

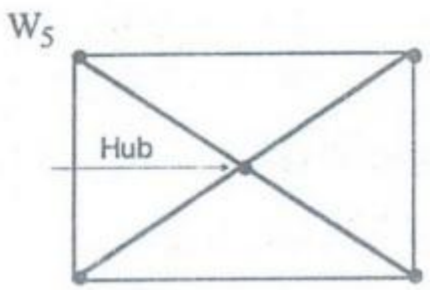
(a) n vertices, n edges

(b) 8 vertices, 8 edges.

(c) 2, 2, 2, 2

Definition: Wheel Graph:

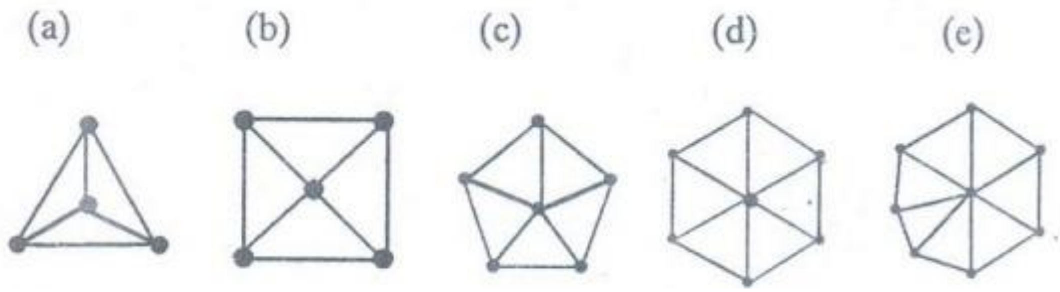
A wheel graph of order n is obtained by joining a new vertex called 'Hub' to each vertex of a cycle graph of order n 1, denoted by W_n.



Note: We obtain the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

Question 17. Draw the graphs (a) W_3, (b) W_4 (c) W_5, (d) W_6, (e) W_7

Solution :



Question 18. How many vertices and how many edges do these graph have (a) W_n (b) W_5 also find the degree sequence of W_4

Solution :

(a) n + 1 edges, 2n edges

(b) 5+1 = 6 vertices, (2)(5) = 10 edges

(c) 4, 3, 3, 3, 3

Note :

(1) Every null graph is regular of degree zero.

(2) The complete graph k_n of degree n - 1.

(3) If G has n vertices and is regular of degree r, then G has (½) r n edges.

Question19. Does there exists a 4-regular graph on 6 vertices if so construct a graph.

Solution: We know that Sum of degrees of vertices = 2e where e is the number of edges.

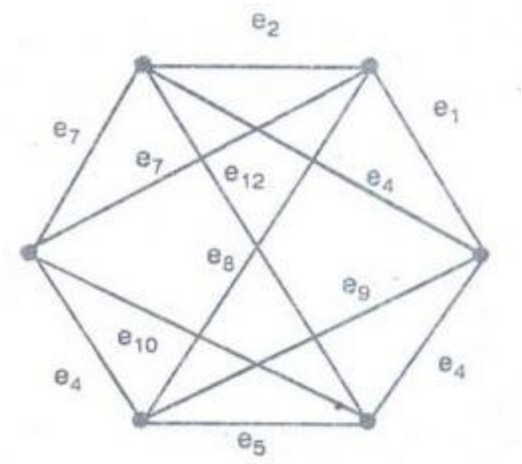
number of vertices in the given graph = 6

Degree of each vertex=4

So sum of degrees = (6)(4) =24=2e

e=24/2= 12

Four regular graph on 6 vertices is possible and it contains 12 edges.



Question20. For which values of n are these graphs regular? (a) K_n (b) C_n (c) W_n

Solution:

(a) For all $n \geq 1$

(b) For all $n \geq 3$

(c) For $n = 3$

Question21. How many vertices does a regular graph of degree four with 10 edges have?

Solution: Given e=10, d(v)=4

Let the number of vertices be n.

Then 4n=2(10)=20

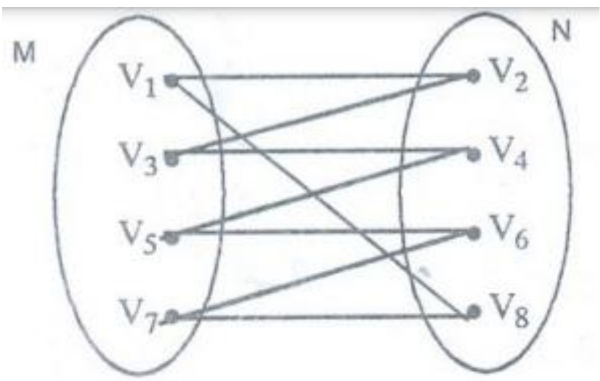
Therefore n = 5

Definition Bipartite graph

A bipartite graph is an undirected graph whose set vertices can be partitioned into two sets M and N is such a way that each edge joins a vertex in M to a vertex in N and no edge joins either two vertices in M or two vertices in N.

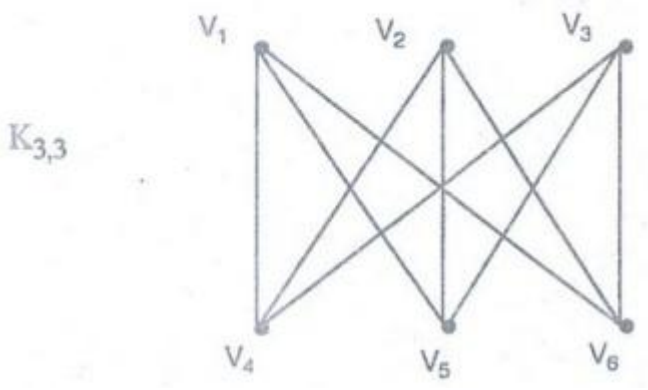
here $V = M \cup N$; $M \cap N = \phi$

$M = \{V_1, V_3, V_5, V_7\}$; $V = \{V_2, V_4, V_6, V_8\}$



Definition: Complete Bipartite graph

A complete bipartite graph is a bipartite graph in which every vertex of M is adjacent to every vertex of N. The complete bipartite graphs that may be partitioned into sets M and N as above s.t M = m and |N| = n are denoted by $K_{m, n}$

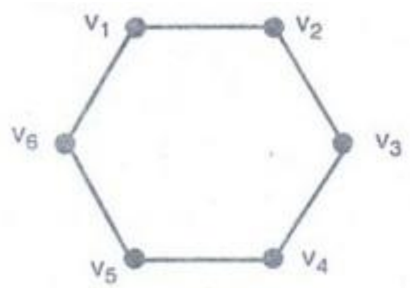


Note: Any graph that is $K_{1, n}$ is called a star graph.

Question22. Show that C_6 is a bipartite graph ?

Solution: The vertex set of C_6 can be partitioned into the two sets

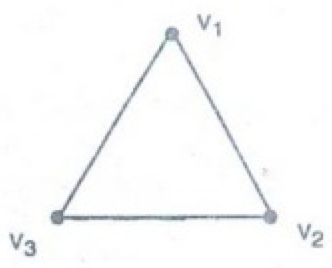
$V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2



Hence C_6 is a bipartite graph.

Question23. Is K_3 is bipartite?

Solution: No, the complete graph K_3 is not bipartite.



If we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices.

If the graph is bipartite, these two vertices should not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

K_3 is not bipartite.

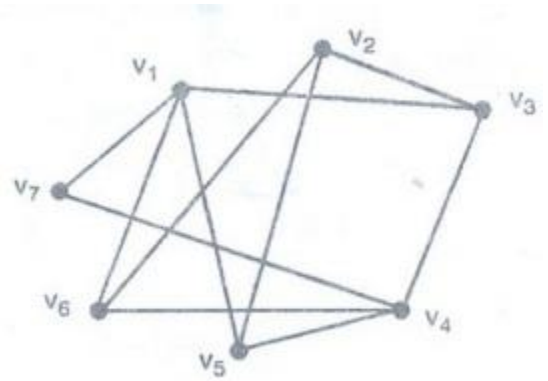
Question24. How many vertices and how many edges of $K_{m,n}$ graph have?

Solution: m + n vertices, mn edegs.

Question25. Find the degree sequence of the following graph $K_{2,3}$

Solution: 3, 3, 2, 2, 2

Question26. Show that the following graph G is bipartite.



Solution: Given graph G is bipartite since its vertex set is the union of two disjoint sets, {v_1, v_2, v_3} and {v_3, v_5, v_6, v_7}, and each edge connects a vertex in one of these subsets to a vertex in the other subset.

Question27. For which values of m and n is $K_{m,n}$ regular?

Solution: A complete bipartite graph $K_{m,n}$ is not a regular it m ≠ n

→ If m = n then $K_{m,n}$ is regular.

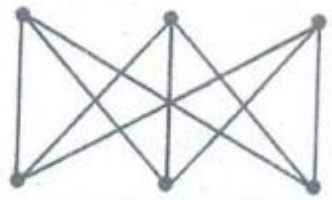
Question28. Prove that a graph which contains a triangle cannot be bipartite.

Proof: Atleast two of the three vertices must lie in one of the bipartite sets because there two are joined by edge, the graph cannot be bipartite.

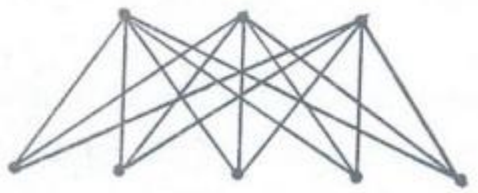
Question29. Draw the complete bipartite graphs $K_{2,3}$, $K_{3,3}$ $K_{3,5}$ and $K_{2,6}$

Solution :

$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

$K_{2,6}$

Graph colouring

The assign of colours to the vertices of G, one colour to each vertex, so that adjacent vertices are assigned different colours is called the proper colouring of G or simply vertex colouring.

If G has n colouring, then G is said to be n-colourable.

Chromatic Number: The chromatic number of a graph G (denoted $\chi(G)$) is the smallest k

such that there is a valid k-colouring of G

Theorem: A simple graph is bipartite if and only if it is possible to assign different colours to each vertex of the graph so that no two adjacent vertices are assigned the same colour.

Proof: Let G = (V,E) is a bipartite simple graph. Then $V = V_1UV_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.

If we assign one colour to each vertex in V_1 and a second colour to each vertex in V_2, then no two adjacent vertices are assigned the same colour.

Suppose that it is possible to assign colours to the vertices of the graph using just two colours.

→ No two adjacent vertices are assigned the same colour.

Let V_1 be the set of vertices assigned one colour and V_2 be the set of vertices assigned the other colour. Then, V_1 and V_2 are disjoint and $V = V_1UV_2$.

i.e., every edge connects a vertex in V_1 and a vertex in v_2 since no two adjacent vertices are either both in V_1 or both in V_2. Consequently, G is bipartite.

Question30.Job Assignment problem

A company receives 5 applications to fill 6 vacant positions. Applicant x_1 is qualified to fill positions P_2 and P_4. Applicant x_2 is qualified to fill positions P_1, P_2, P_3, P_4 and P_5.

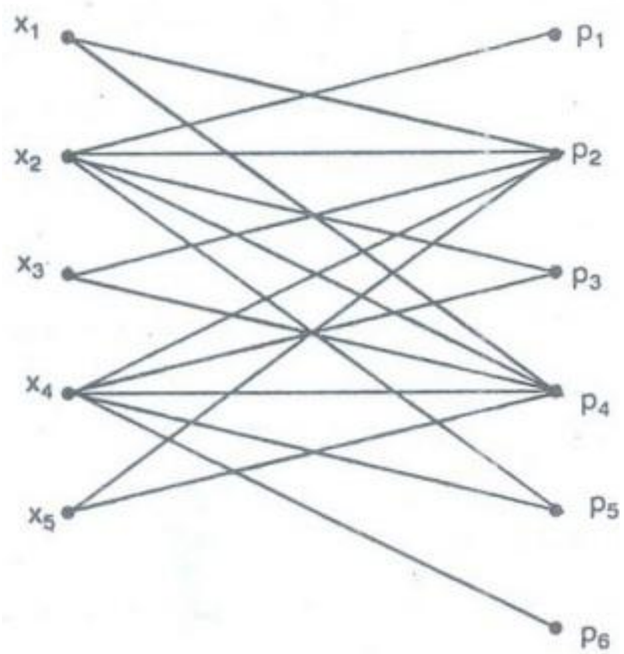
Applicant x_3 is qualified to fill positions P_2 and P_4. Applicant x_4 is qualified to fill positions. P_2, P_3, P_4, P_5 and P_6. Applicant x_5 is qualified to fill positions P_2 and P_4.

The problem is: Is it possible to recruit all the applicants and assign a job for which he is qualified?

We introduce a graph model for this with a vertex representing an applicant or a job and if an applicant is qualified for a job then we introduce an edge between the corresponding vertices.

Now, the problem becomes, finding a matching in this graph which is incident with all the x j s.

Such a matching does not exist in the graph, since the neighbour set of x_1, x_3 and x_5 is the same set $\{P_2, P_4\}$, that is, the applicants x_1, x_3 and x_5 are qualified only for 2 jobs, P_2 and P_4. Thus one of the 3 applicants cannot be matched.



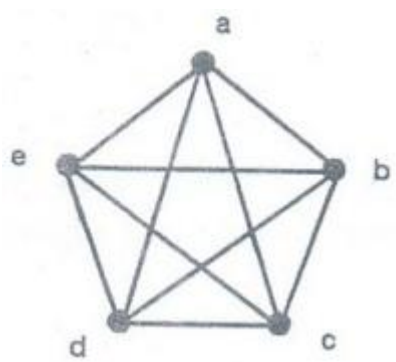
Hence, we cannot recruit all the five applicants and assign a job for which they are qualified.

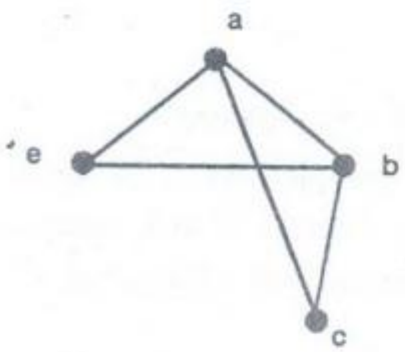
Definition: Subgraph

A subgraph of a graph G = (V,E) is a graph H = (W, F), where W⊆≤V and F⊆E. A subgroup H of G is a proper subgraph of G if H≠G.

Question31. Draw two subgraph of K_5.

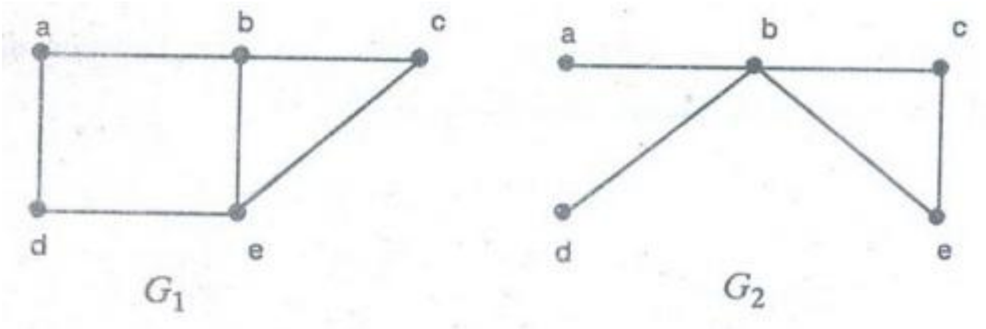
Solution:





Definition: The union two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

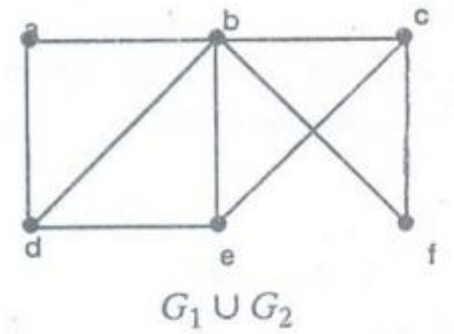
Question32. Find the union of the graphs G_1 and G_2



Solution: The vertex set of G_1 = {a, b, c, d, e}

The vertex set of G_2 = {a, b, c, d, f}

$G_1 \cup G_2$ = {a, b, c, d, e, f}



Definition: Complement

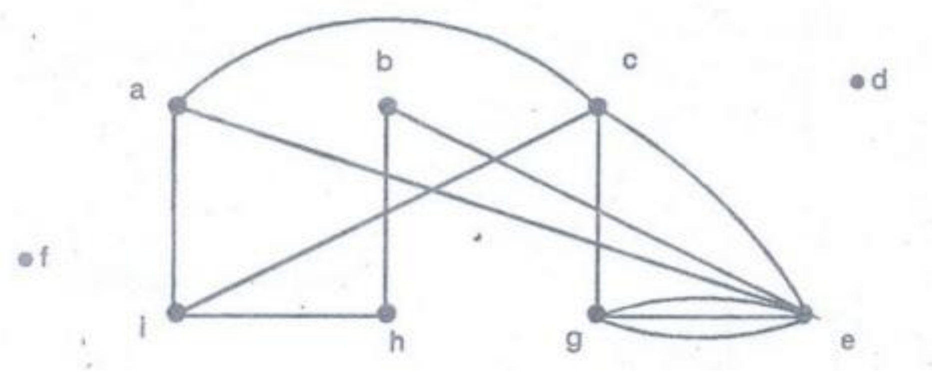
The complement of G is defined as a simple graph with the same vertex set as G and value two vertices u and v are adjacent only when then are not adjacent in G.

Practice Questions:

1. Can a simple graph exist with 15 vertices each of degree five?

Ans. No. because the sum of the degree of the vertices cannot be odd.

2. Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



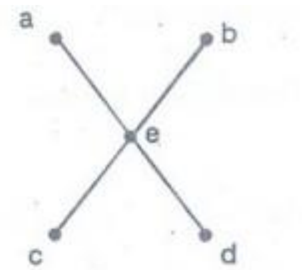
Ans. v = 9, e = 12; deg (a) = 3, deg (b) = 2, deg (c) = 4, deg (d) = 0, deg(e) 6, deg (f) = 0; deg (g) = 4; deg(h) = 2; deg(i) = 3, d and ƒ are isolated.

3. What does the degree of a vertex represent in a collaboration graph? What do isolate and pendant vertices represent?

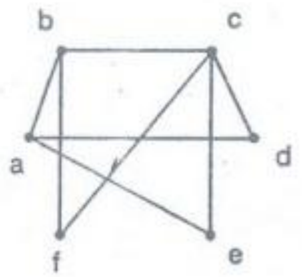
Ans. The degree of a vertex is a number and it represents the number of collaborators v has; isolated vertex means someone who has never collaborated; pendant vertex means someone who has just one collaborator.

4. Determine whether the graph is bipartite.

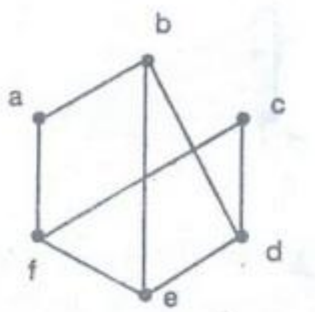
(a)



(b)



(c)



5. For which values of n are these graphs bipartite?

(a) K_n (b) C_n (c) W_n

Representing Graphs and Graph Isomorphism

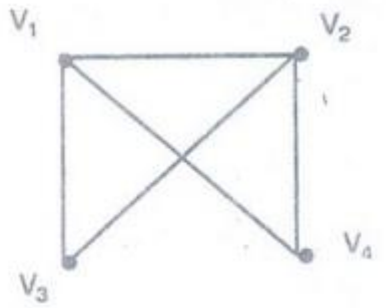
Definition: Matrix representation of graphs and Digraphs:

Definition: Adjacency matrix

Let G (V, E) be a simple graph with n. Vertices ordered from V_1 to V_n, then the adjacency matrix A = $[aij]_{n \cdot n}$ of G is an n^*n symmetric matrix defined by the elements.

$$a_{ij} = \begin{cases} 1 & \text{when } V_i \text{ is adjacent to } V_j \\ 0 & \text{Otherwise} \end{cases}$$

It is denoted by A (G) or A_G



$$A_G = \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{pmatrix} V_1 & V_2 & V_3 & V_4 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Properties of adjacency matrix

1. An adjacency matrix completely defines a simple graph

2. The adjacency matrix is symmetric

3. Any element of the adjacency matrix is either 0 or 1, therefore it is also called as, bit matrix or boolean matrix.

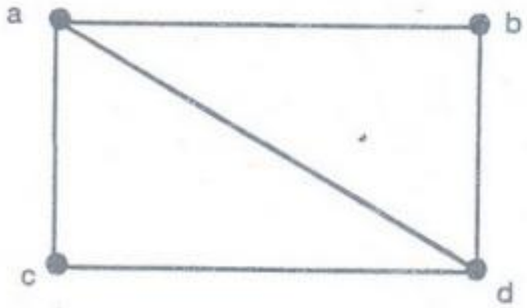
4. The i^{th} row in the adjacency matrix is determined by the edges which originate in the node V_i.

5. If the graph G is simple, the degree of the vertex V; equals the number of 1's in the i^{th} row (or i^{th} column) of A_G.

6. Given an n^*n symmetric boolean matrix A, we can find a simple graph G s.t A is the adjacency matrix of G.

7. G is null ↔ A(G) is the zero matrix of order n.

Question33.(i). Represent the following graph with an adjacency matrix.

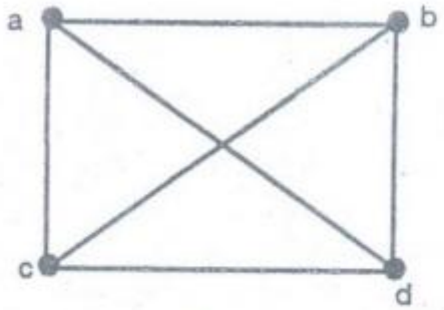


Solution :

$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

Question33.(ii).Write the adjacency matrix of K_4

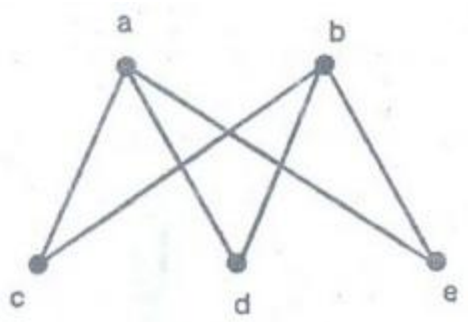
Solution: K_4 graph is



$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

Question33.(iii).Write the adjacency matrix of $K_{2,3}$

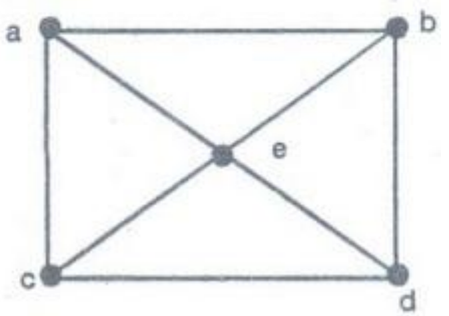
Solution: $K_{2,3}$ graph is



$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Question33.(iv).Write the adjaency matrix of W_4.

Solution: W_4 graph is



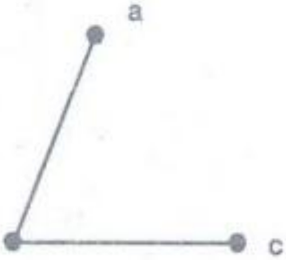
$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Question34.Draw a graph of the given adjacency matrix.

$$\text{(a)} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

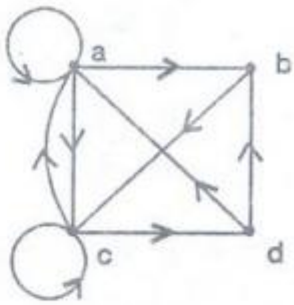
Solution : (a) Let

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

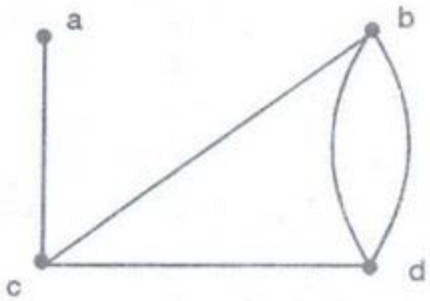


(b) Let

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



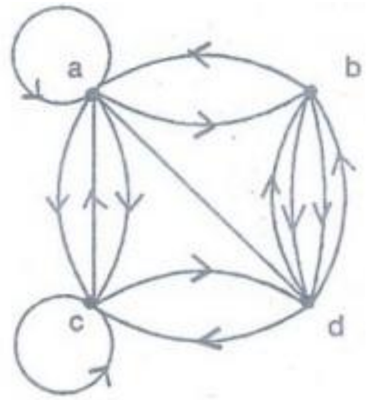
Question35. Represent the given graph using an adjacency matrix.



Solution :

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

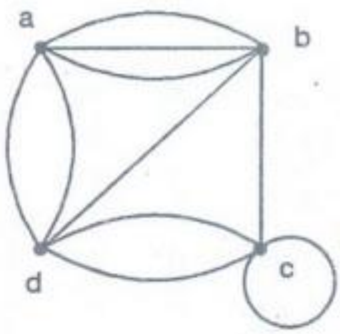
Question36. Find the adjacency matrix of the given directed multigraph.



Solution :

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}.$$

Question37. Use an adjacency matrix to represent the pseudograph shown in figure.



Solution: The adjacency matrix using the ordering of vertices a, b, c, d is

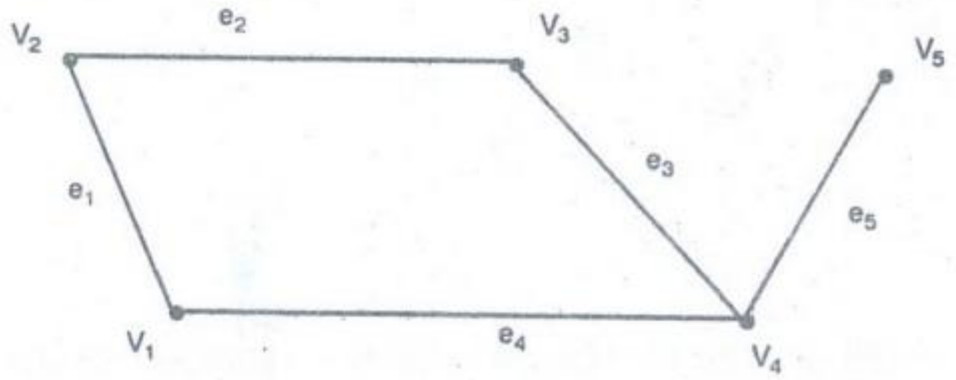
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Definition: Incidence matrix

Let G be a graph with n vertices, Let V = $\{V_1, V_2, \ldots V_n\}$ and E = $(e_1, e_2, \ldots e_m)$. Define n x m matrix

$I_G = [mij]_{n^{\cdot}m}$ where

$$m_{ij} = \begin{cases} 1 \text{ when } V_i \text{ is incident with } e_j \\ 0 \text{ Otherwise} \end{cases}$$



$$I_G = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Note :

1. The incidence matrix contains only 0 and 1.

2. The number of 1's in each row equals to the degree of the corresponding vertex.

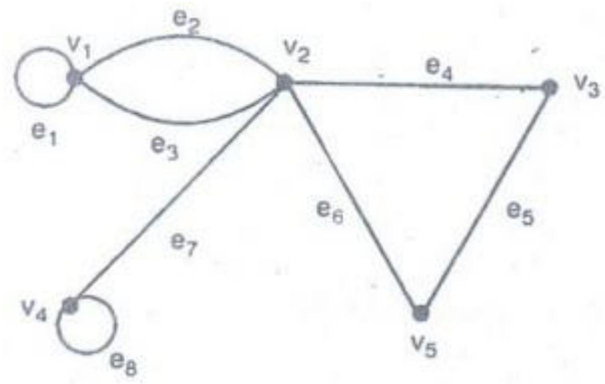
3. A row with all zeros represents an isolated vertex.

4. Every edge is incident on exactly two vertices, each column of the Incidence matrix has exactly two ones except the loop.

5. The parallel edges in a graph produce identical columns in its incidence matrix.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabelling the vertices and the edges of the same graph.

Question38.(i). Represent pseudograph shown in figure using an incidence matrix.



Solution: The incidence matrix for this graph is

$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

Example 13. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

Solution: Sum is 2 if e is not a loop, 1 if e is a loop.

Question39. Find the incidence matrices for the graphs (a) K_n (b) C_n (c) W_n

Solution:

(a) $\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$

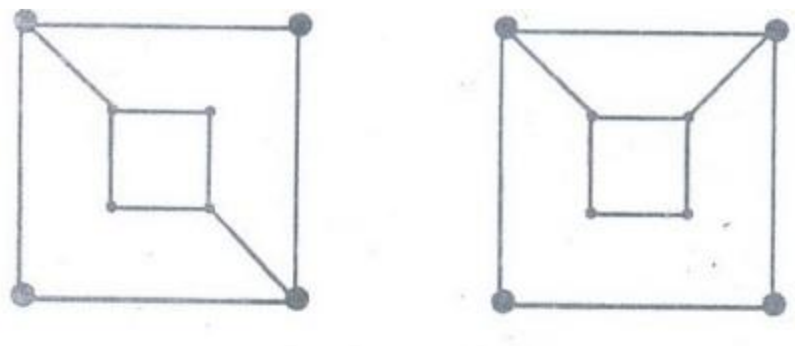
(c) $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & & & & 1 & 0 & \cdots & 0 \\ & & B & & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \vdots & & \vdots \\ & & & & 0 & 0 & \cdots & 1 \end{bmatrix}$

Where B is the answer to (b)

Definition: Isomorphic Graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if in both graphs (i). Number of vertices are same. (ii). Number of edges are same. (iii). The sequences of degrees of all vertices are same. (iv). there is a one-to-one correspondence between vertices (v). there is a one-to-one correspondence between edges and (vi). Adjacency matrices are same.

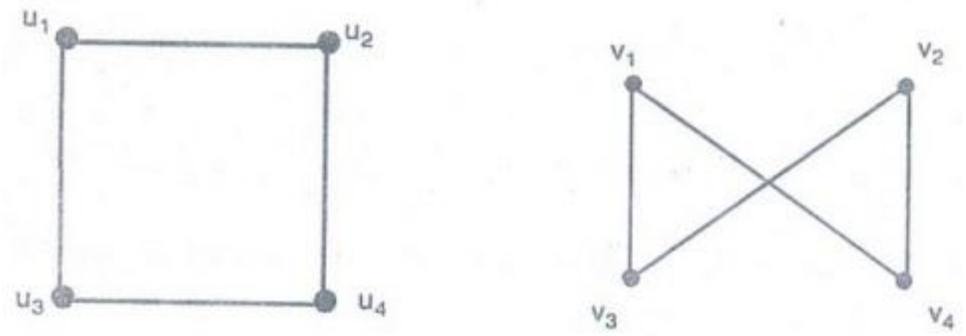
Question40.(i).Explain why the two graphs given below are not isomorphic.



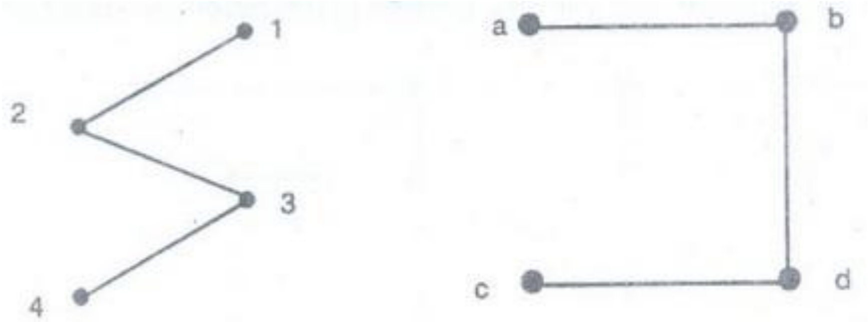
(a) (b)

Solution: In the graph (a), no vertices of degree two are adjacent while in the graph (b) vertices of degree two are adjacent. Because isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

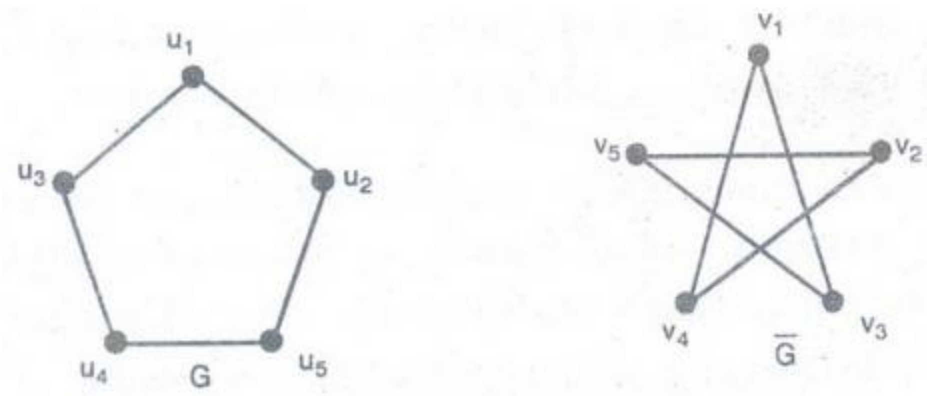
Question40.(ii). Show that the graphs G = (V, E) and H = (W, F), shown in figure are isomorphic.



Question40.(iii). Show that the two graphs shown in figure are isomorphic.



Question40.(iv).Prove that the graphs G and $\overline{G}$ given below are isomorphic

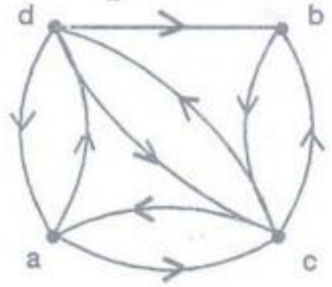


Practice Questions:

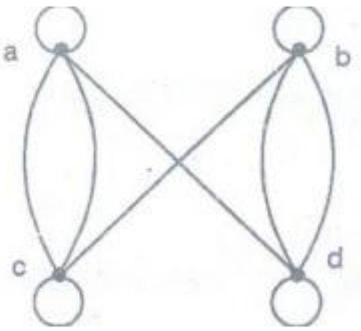
1. Draw a graph with the given adjacency matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

[Ans.



2. Represent the given graph using an adjacency matrix.

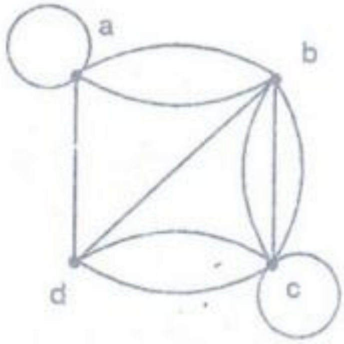


[Ans. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

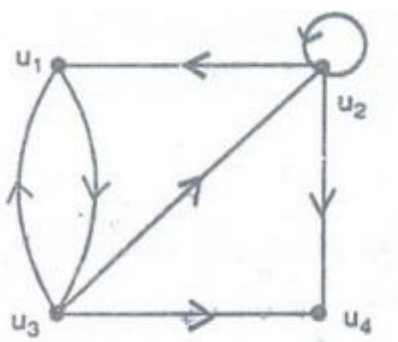
3. Draw an undirected graph represented by the given adjacency matrix.

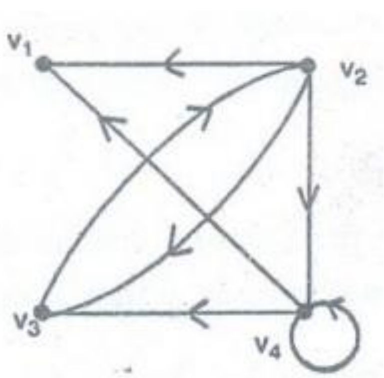
$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

[Ans.



4. Find the adjacency matrix of the given directed multigraph.



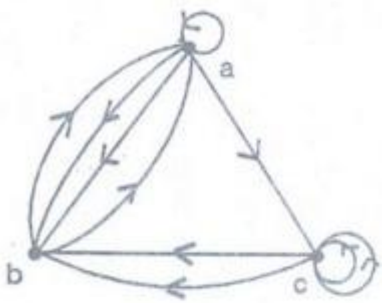


[Ans. Yes]

5. Draw the graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

[Ans.



6. Is every zero-one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

[Ans. Yes]

7. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertiex.

[Ans. Zeros]

8. Find a self-complementary simple graph with five vertices.

[Ans. C_5]

9. For which integers n is C_n self-complementary?

[Ans. for n = 5 only]

10. Are the simple graphs with the following adjacency matrices isomorphic?

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ [Ans. Yes]

(b) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ [Ans. No]

Definition: Path

A path in a multigraph G consists of an alternating sequence of vertices and edges of the form

$V_0, e_1 V_1, e_2, V_2, \ldots e_{n-1}, V_{n-1}, e_n, V_n$

where each edge e_i contains the vertices V_{i-1} and V_i

The number n of edges is called the length of the path.

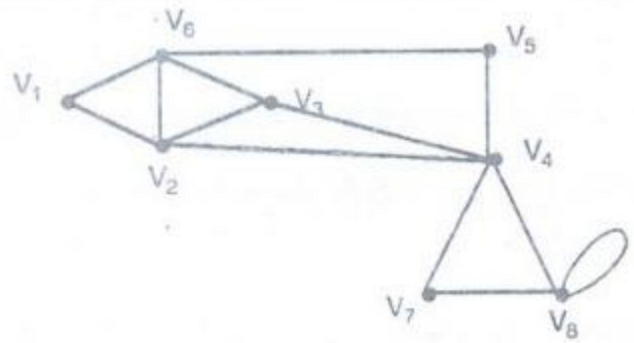
The path is said to be closed if $V_o = V_n$ we say the path is from V_o to V_n or between V_o and V_n or Connects V_o to V_n.

Note :

1. A simple path is a path in which all vertices are distinct. (A path in which all edges are distinct will be called a trail)

2. If $V_o = V_n$, then P is called a closed path. On the other hand if $V_o \neq V_n$, then P is an open path.

3. For the following graph



Path	Length	Simple path	Closed path	Circuit	Cycle
$V_1 - V_2 - V_6 - V_1$	3	Yes	Yes	Yes	Yes
$V_6 - V_2 - V_3 - V_6$	3	Yes	Yes	Yes	Yes
V_1	0	Yes	Yes	No	No
$V_8 - V_8$	1	Yes	Yes	Yes	Yes
$V_1 - V_2 - V_1$	2	No	Yes	No	No
$V_5 - V_4 - V_7 - V_8 - V_4 - V_3$	5	No	No	No	No

Definition: Circuit:

A path of length ≥ 1 with no repeated edges and whose end vertices are same is called a circuit.

Note :

4. A cycle is a circuit with no other repeated vertices except its end vertices.

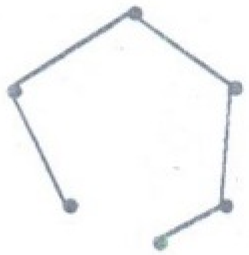
5. A cycle is a simple circuit, a loop is a cycle of length 1.

6. In a graph a cycle that is not a loop must have length atleast 3, but there may be cycles of length 2 in a multigraph.

7. Two paths in a graph are said to be edge-disjoint if they have no common edges, they are vertex - disjoint if they have no common vertices.

Definition: Path graph

A path graph of order 'n' is obtained by removing on edge from a C_n graph, denoted by P_n.



Definition: Trail

A trail from v to w is a path from v to w that does not contain a repeated edge.

Thus a trial from v to w is a path of the form

$v = v_0, e_1, v_1, e_2, \ldots v_{k-1}, e_k, v_k = w$ where all the e_i are distinct.

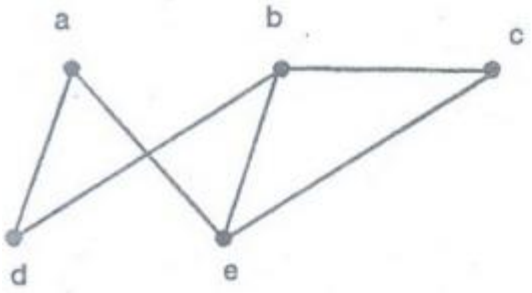
Note: Every simple path is also a trail, and every trail is also a path, but these inclusions do not reverse.

	Repeated edge	Repeated vertex	Starts and Ends at same point?
Path	allowed	allowed	allowed
Trail	no	allowed	allowed
Simple path	no	no	no
Closed path	allowed	allowed	yes
Cycle	no	allowed	yes
Simple cycle	no	first and last only	yes

Question42. Does each of these lists of vertices from a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

(a) a, e, b, c, b (b) e, b, a, d, b, e

(c) a, e, a, d, b, c, a (d) c, b, d, a, e, c



Solution:

(a) path of length 4, not a circuit, not simple.

(b) not a path

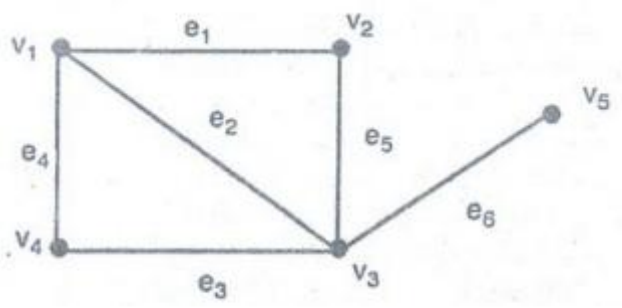
(c) not a path

(d) simple circuit of length 5.

Definition: Connected and disconnected graphs

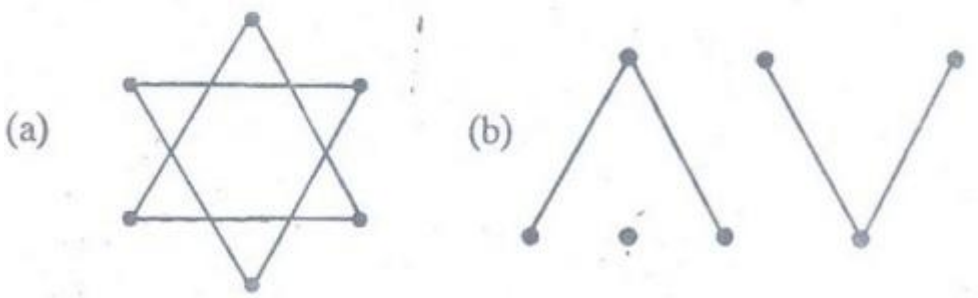
A graph G is a connected graph is there is atleast one path between every pair of vertices in G. Otherwise G is a disconnected graph.

Example:



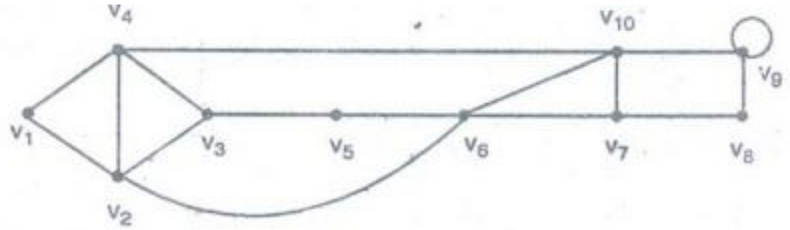
Note: A disconnected graph consists of two or more connected graphs. Each of these connected subgroups is called a component (or a block).

Example 2. Determine whether the given graph is connected.



Solution: (a) No (b) No

Example 3. For the following graph



we have

Path	Length	Simple path	Closed path	Circuit	Cycle
$v_1 - v_4 - v_3 - v_5 - v_6 - v_{10} - v_4 - v_1$	7	no	yes	no	no
$v_2 - v_3 - v_5 - v_6 - v_7 - v_{10} - v_6 - v_2$	7	no	yes	yes	no
$v_1 - v_2 - v_1$	2	no	yes	no	no
$v_1 - v_4 - v_3 - v_2 - v_1$	4	yes	yes	yes	yes
$v_9 - v_9$	1	yes	yes	yes	yes
v_1	0	yes	yes	no	no
$v_5 - v_6 - v_7 - v_{10} - v_6 - v_2$	5	no	no	no	no
$v_4 - v_2 - v_3 - v_4$	3	yes	yes	yes	yes

Euler and Hamilton paths

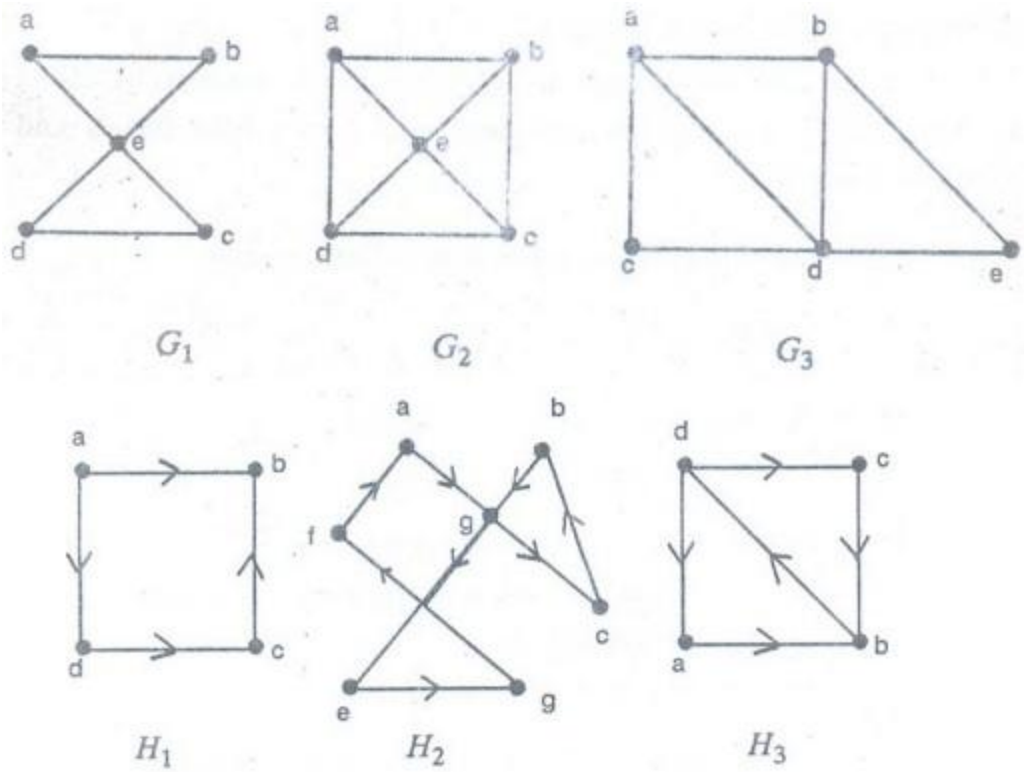
Definition: Euler circuit

A Euler circuit in a graph G is a simple circuit containing every edge of G.

Definition: Euler path

A Euler path in G is a simple path containing every edge of G.

Question43. Which of the graphs in Figures has a Euler circuit? Of those that do not, which have a Euler path?



Solution: G_1 has a Euler circuit, for example, a, e, c, d, e, b, a.

Neither G_2 or G_3 has a Euler circuit.

However, G_3 has a Euler path, namely, a, c, d, e, b, d, a, b. G_2 does not have a Euler path.

The graph H_2 has a Euler circuit, for example, a, g, c, b, g, e, d, f, a.

Neither H_1 nor H_3 has a Euler circuit.

H_3 has a Euler path, namely, c, a, b, c, d, b, but H_1 does not.

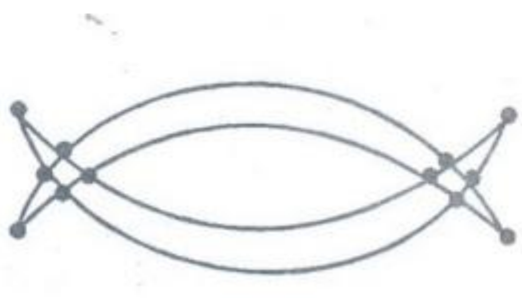
Definition: Euler graph

If a graph has a Euler circuit in it then it is said to be a Euler graph.

Example 2. Give two examples of Euler graph.

Solution :

Start of David



Mohammed's seimitans

Theorem: A given connected graph G is a Euler graph iff all vertices of G is of even degree.

Proof: Suppose, G is a Euler graph.

→ G contains a closed walk covering all edges.

To prove:

All vertices of G are of even degree.

In tracing the closed walk, every time the walk meets a vertex v, it goes through two new edges incident on V with one we 'entered' and other 'exited'. This is true, for all vertices, because it is a closed walk. Thus, the degree of every vertex is even.

Conversely, suppose that all vertices of G are of even degree.

To prove:

G is a Euler graph.

(i.e.,) to prove G contains a Euler circuit.

Construct a closed walk starting at an arbitrary vertex v and going through the edge of G such that no edge is repeated. Because, each vertex is of even degree, we can exit from each end, every vertex where we enter, the tracing can stop only at the vertex *v*. Name the closed walk as *h*.

Case (i)

If h covers all edges of G, then h becomes a Euler line, and hence, G is an Euler graph.

Case (ii)

If *h* does not cover all edges of G then remove all edges of *h* from g and obtain the remaining graph G'. Because both G and G' have all their vertex of even degree.

→ Every vertex in G' is also of even degree.

Since G is connected, h will touch G' atleast one vertex v'. Starting from v', we can again construct a new walk h' in G'. This will terminate only at v', because, every vertex in G' is also of even degree.

Now, this walk h' combined with h forms a closed walk starts and ends at v and has more edges than h. This process is repeated until we obtain a closed walk covering all edges of G. Thus, G is a Euler graph.

Theorem: A connected multigraph has a Euler path but not a Euler circuit if and only if it has exactly two vertices of odd degree.

Proof: First Part:

Given: A connected multigraph G has a Euler path but not a Euler circuit.

To prove: G has exactly two vertices of odd degree suppose that a connected multigraph does have a Euler path from a to b, but not a Euler circuit. The first edge of the path contributes one to the degree of a. A contribution of two to the degree of a is made every time the path passes through a. The last edge in the path contributes one to the degree of b.

Every time the path goes through b there is a contribution of two to its degree. Consequently, both a and b have odd degree. Every other vertex has even degree, because the path contributes two to the degree of a vertex whenever it passes through it.

Second Part:

Given The graph has exactly two vertices of odd degree.

To prove: G has a Euler path

Suppose that a graph has exactly two vertices of odd degree, say a and b. Consider the larger graph made up of the original graph with the addition of an edge {a, b}.

Every vertex of this larger graph has even degree, so there is a Euler circuit. The removal of the new edge produces a Euler path in the original graph.

Theorem

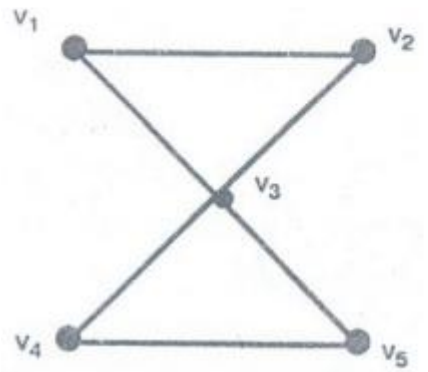
Show that a directed multigraph having no isolated vertices have a Euler path but not an Euler circuit if and only if the graph the in-degree and out degree of each vertex are equal for all but two vertices, one that has in-degree one larger than its out-degree and the other that has out-degree one larger than its in-degree.

Proof: If there is a Euler path, as we follow it each vertex except the starting and ending vertices must have equal in-degree and out-degree, since whenever we come to a vertex along an edge, we leave it along another edge. The starting vertex must have out-degree 1 larger than its in-degree, since we use one edge leading out of this vertex and whenever we visit it again, we use one edge leading into it and one leaving it.

Similarly, the ending vertex must have in-degree 1 greater than its out-degree. Since the Euler path with directions erased produces a path between any two vertices, in the underlying undirected graph, the graph is weakly connected.

Conversely, suppose the graph meets the degree conditions stated. If we add one more edge from the vertex of deficient out-degree to the vertex X with equal in-degree and out-degree. Because the graph is still weakly connected, by this new graph has a Euler circuit. Now delete the added edge to obtain the Euler path.

Question44. Let G be a graph, verify that G has a Eulerian circuit,

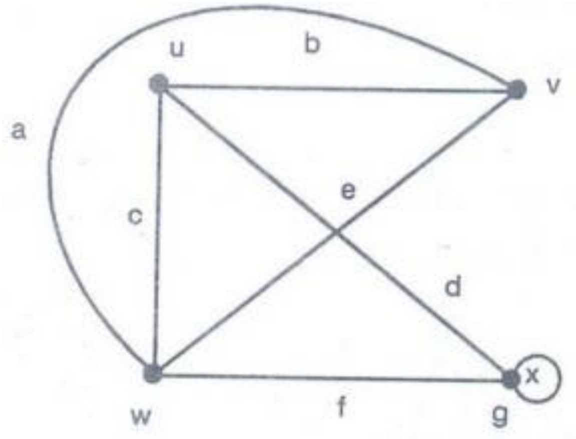


Solution: Here G is connected and all the vertices are having even degree deg (v_1) = deg (v_2) = deg (v_4) = deg (v_5) = 2

Thus G has a Eulerian circuit, we find the Eulerian circuit.

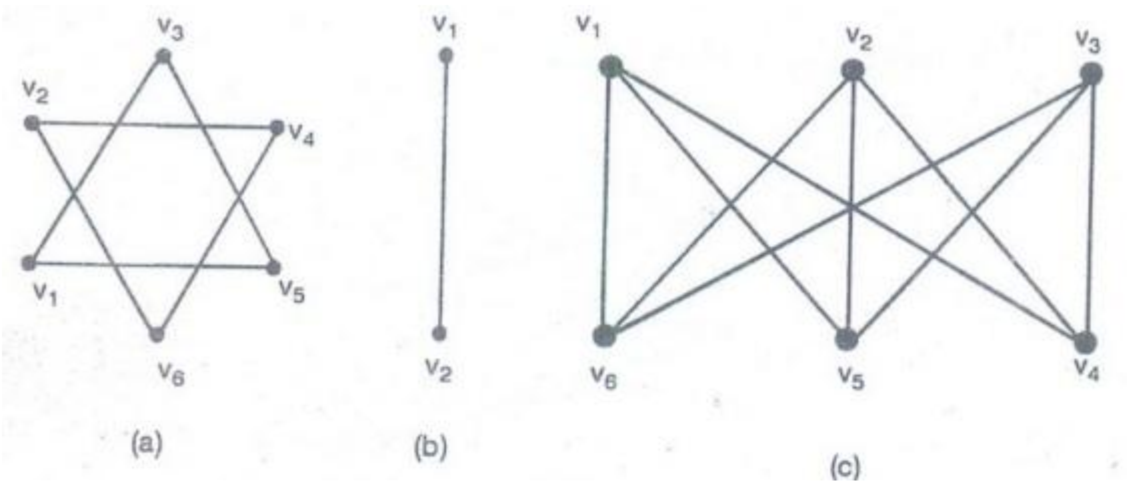
$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1$

Question45. Show that the graph, has no Eulerian circuit but has a Eulerian path.



Solution: Here deg (u) = deg (v) =3 and deg (w) = dég (x) = 4, because u and v have only two vertices of odd degree, the graph does not contain Eulerian circuit, but the path b – a – c – d – g – f – e is an Eulerian path.

Question46. Show that the graphs contain no Eulerian circuit.



Solution: The graph does not contain Eulerian circuit since it is not connected.

Question47. Which of the following graphs are Eulerian?

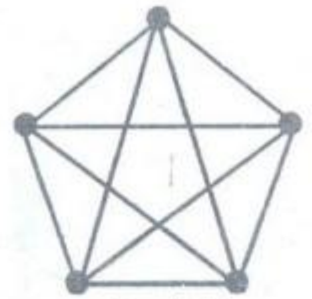
(i) the complete graph K_5

(ii) the complete bipartite graph $K_{2,3}$

(iii) the graph of the octahedron

Solution :

(i) We know that the degree of each vertex in the complete graph K_5 is 5 - 1 = 4 which is even.

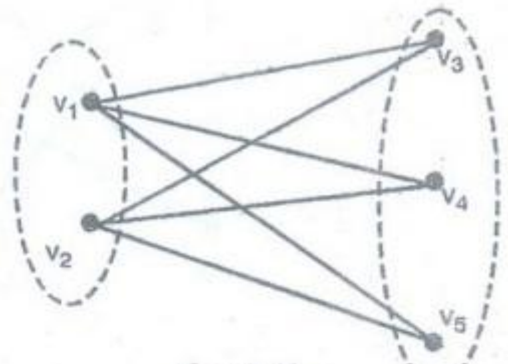


Graph K_5

Hence degree of each vertex in K_5 is even.

Because Ks is also a connected graph, therefore by theorem K_5 is an Euler graph.

(ii) The graph $K_{2,3}$ is given below :



Graph $K_{2,3}$

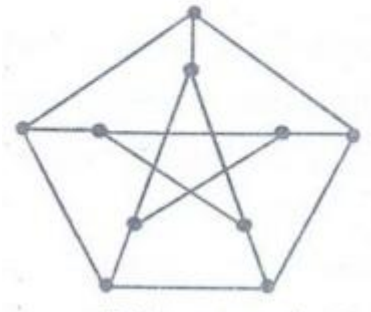
It is clear that each vertex in the graph $K_{2,3}$ is not of even degree. Hence the graph $K_{2,3}$ is not Euler.

(iii) We know from figure that each vertex in the graph of the octahedron is of degree 4 which is even.

Hence the graph of the octahedron is an Euler graph.

Octahedron

(iv) From figure, it is clear that the Petersen graph is not an Euler graph since each vertex of this graph is of degree 3 which is not even.



Petersen graph

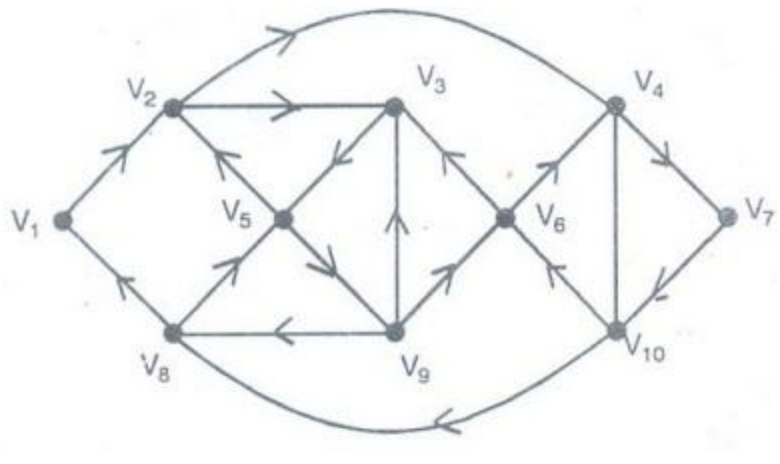
Question48(i). Fow that values of n is the graph K_n Eulerian?

Solution: We know that K_n, the complete graph of n vertices is a connected graph in which degree of each vertices is n-1. Because a graph is Eulerian if and only if it is connected and degree of each vertex is even, we conclude that K_n is an Euler graph if and only if n is odd.

Question48(ii). Which complete bipartite graphs $K_{m,n}$ are Euler graphs?

Solution: From the definition of $K_{m,n}$ graph that it is an Euler graph if and only if both m and n are even.

Question48(iii). The graph shown in figure is an Euler graph. Determine Euler circuit for this graph.

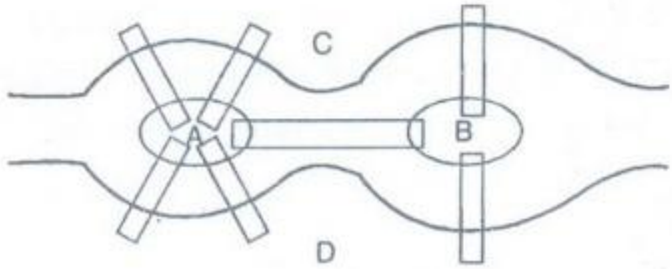


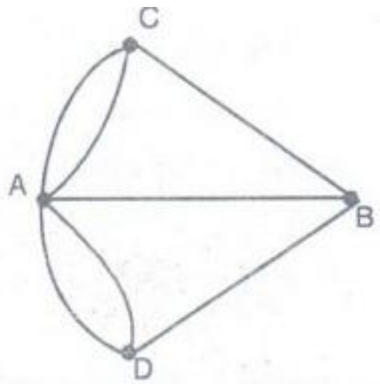
Solution: The Euler circuit for this graph is

V_1, V_2, V_3, V_5, V_2, V_4, V_7, V_{10}, V_6, V_3, V_9, V_6, V_4, V_{10}, V_8, V_5, V_9, V_8, V_1

Question48(iv). Explain Konisberg bridge problem. Represent the problem by means of graph. Does the problem have a solution?

Solution: There are two islands A and B formed by a river. They are connected to each other and to the river banks C and D by means of 7 bridges. The problem is to start from any one of the 4 land areas A, B, C, D, walk across each bridge exactly once and return to the starting point.





This problem is the famous Konisberg bridge problem.

When the situation is represented by a graph, with vertices representing the land areas and the edges and the bridges.

This problem is the same as that drawing the graph without lifting the pen from the paper and without retracing any line.

In other words, the problem is to find whether there is Eulerian circuit in the graph. But a connected graph has an Eulerian circuit if and only if each of its vertices is of even degree.

In the present case all the vertices are of odd degree. Hence, Konisberg bridge problem has no solution.

Hamilton path and Hamilton circuits

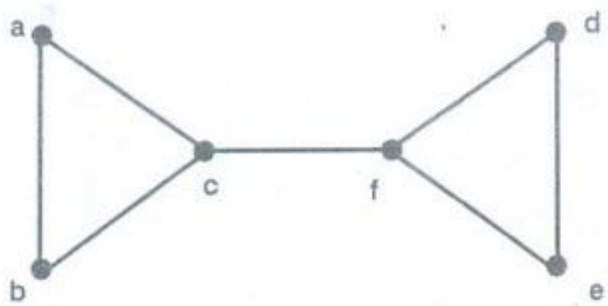
Definition: Hamilton path

A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path. That is, the simple path $x_0, x_1, ..., x_{n-1}, x_n$ in the graph G = (V,E) is a Hamilton path if V = {$x_0, x_1, ..., x_{n-1}, x$) and $x_i = x_j$ for $0 \leq i < j \leq n$

Definition: Hamilton cycle

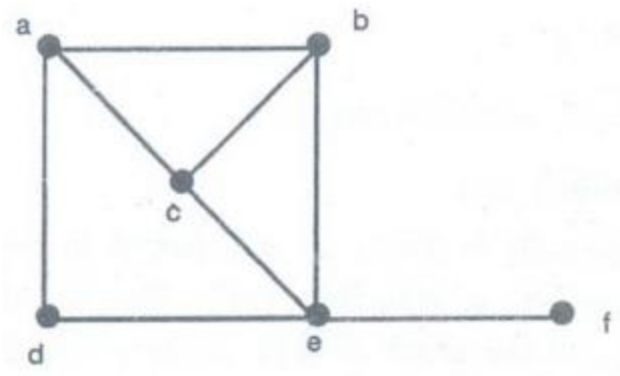
A cycle in a graph G that passes through every vertex exactly once is called a Hamilton cycle.

Example 1. Does the graph given below have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.



Solution: a, b, c, f, d, e is a Hamilton path.

Example 2. Does the graph given below have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

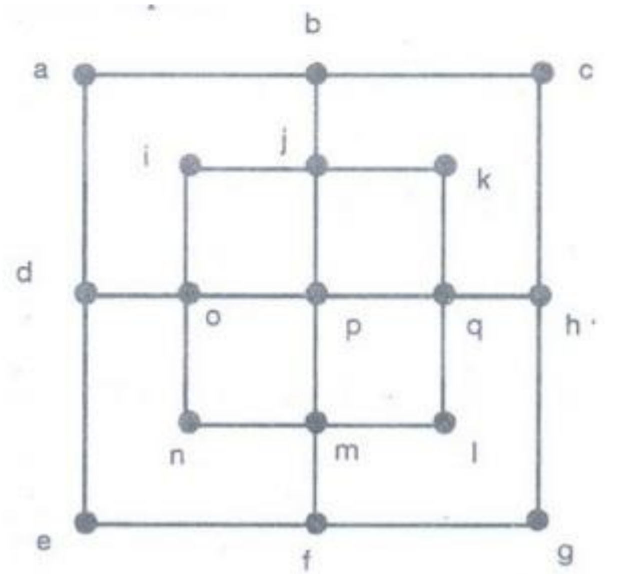


Solution: f, e, d, a, b, c is a Hamilton path.

Example 3. Does the graph given below have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

No Hamilton path exists.

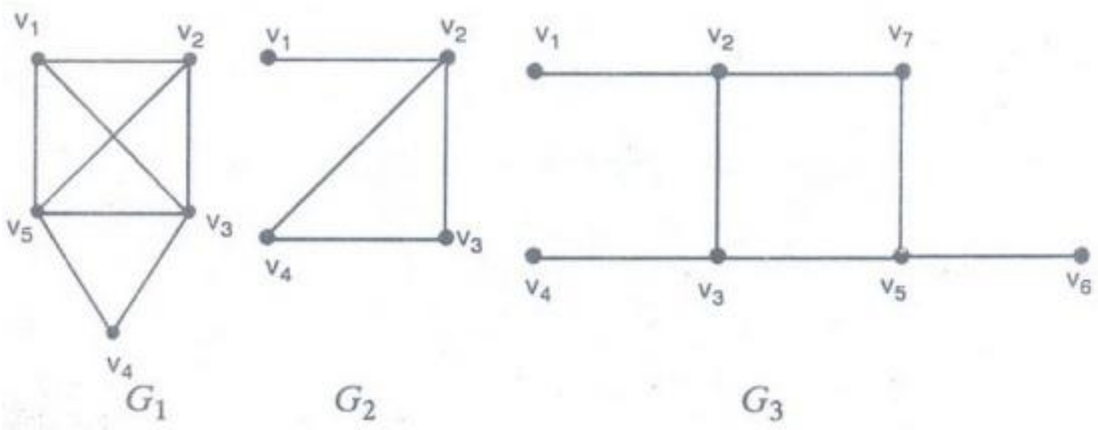
There are eight vertices of degree 2, and only two of them can be end vertices of a path.



For each of the other six, their two incident edges must be in the path.

It is not hard to see that if there is to be a Hamilton path, exactly one of the inside corner vertices must be an end, and that this is impossible.

Question 49. Which of the simple graphs given below have a Hamilton cycle or, if not, a Hamilton path?



Solution :

(i) G_1 has a Hamilton cycle.

$v_1, v_2, v_3, v_4, v_5, v_1$

(ii) G_2 has no Hamilton cycle.

but G_2 has Hamilton path.

v_1, v_2, v_3, v_4

(iii) G_3 has neither a Hamilton cycle nor a Hamilton path. Since any path containing all vertices must contain one of the edges {a,b}, {e,f} and {c, d} morethan once.

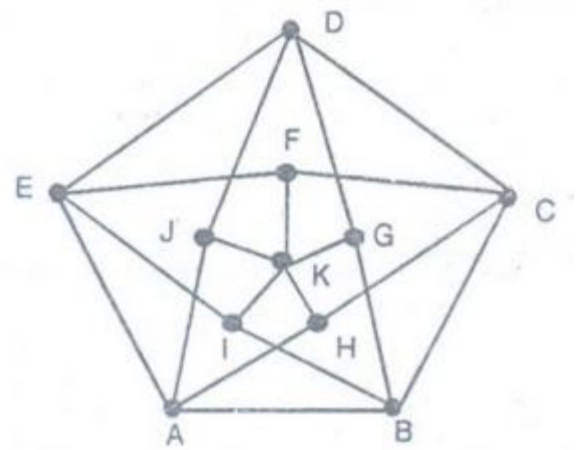
Question 50. Show that Kn has a Hamilton cycle whenever n ≥ 3.

Solution: Now we can form a Hamilton cycle in K_n beginning at any vertex.

Such a cycle can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once.

It is possible since there are edges in K_n between any two vertices.

Question 51(i). Show that the graph is Hamiltonian

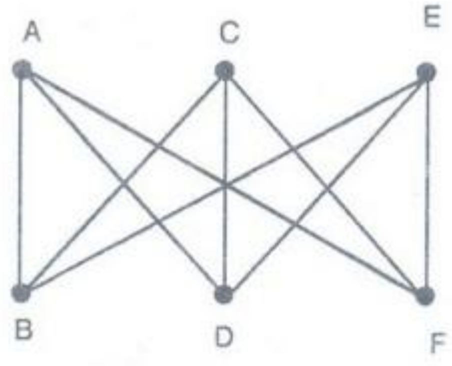


Solution: From figure we observe that the closed walk

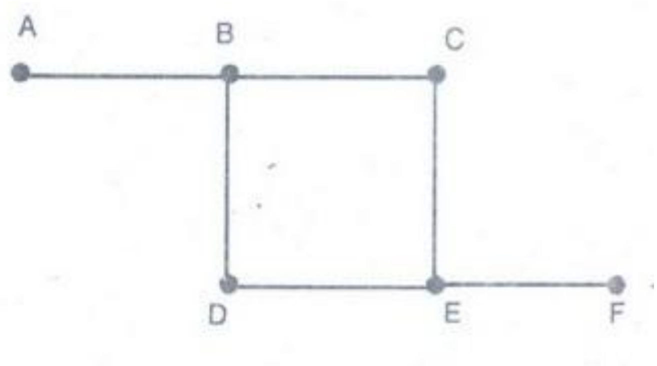
A → H → C → F → E → D → G → B → I → K→ J→ A is a Hamiltonian cycle.

Hence the graph is Hamiltonian.

Question 51(ii). Find a Hamiltonian path or a Hamiltonian cycle, if it exists in each of the three graphs given below. If it does not exist, explain why?



G_1



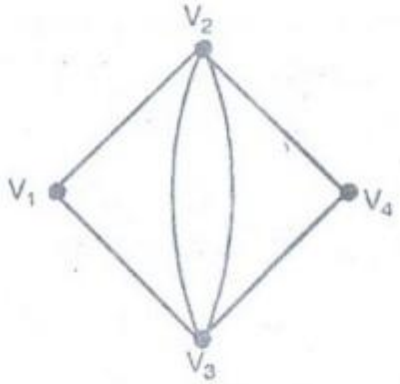
G_2

Solution: G_1 contains a Hamiltonian cycle, for example A-B-C-D-E-F-A. In fact there are 5 more Hamiltonian cycle in G_1, namely, A-B-C-F-E-D-A, A-B-E-D-C-F-A, A-B-E-F-C-D-A, A-D-C-B-E-F-A and A-D-E-B-C-F-A

G_2 contains neither a Hamiltonian path nor, a Hamiltonian cycle, since any path containing all the vertices must contain one of the edges A-B and E-F more than once.

Question 51(iii). Give an example of a graph that has an Euler circuit and a Hamiltonian cycle, which are distinct.

Solution: The graph having an Euler circuit and a Hamiltonian cycle which are distinct is shown in figure.

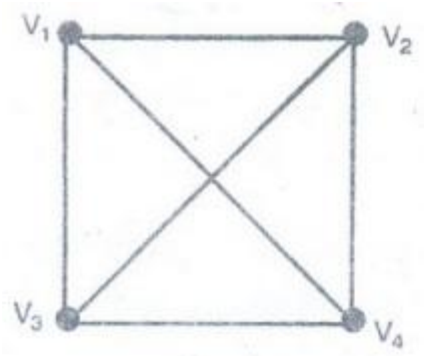


The Euler circuit is V_1, V_3, V_2, V_3, V_4, V_2, V_1

The Hamiltonian cycle is V_1, V_2, V_4, V_3, V_1

Question 51(iv). Give an example of a graph which has a Hamiltonian cycle but not an Euler circuit.

Solution: The graph having a Hamiltonian cycle but not an Euler circuit is shown in figure.

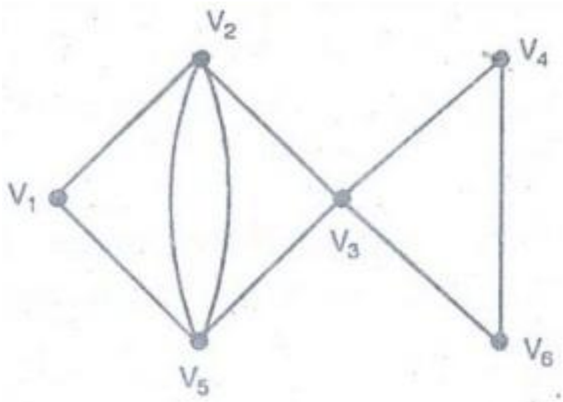


The Hamiltonian cycle is V_1, V_2, V_4, V_3, V_1.

There is no Euler circuit.

Question 51(v). Give an example of a graph which has a Euler circuit but not a Hamiltonian cycle.

Solution: The graph having a Euler circuit but not a Hamiltonian cycle is shown in figure.



The Euler circuit is V1, V5, V2, V5, V3, V4, V6, V3, V2, V1.

There is no Hamiltonian cycle.

Practice Question. For which values of n do these graphs have a Euler circuit? (a) K_n (b) C_n (c) W_n

Trees

Question 52. Define a tree. What are its key properties?
Tree: A tree is an undirected graph that is connected and has no cycles.
Properties: There is a unique path between every pair of vertices in G.
A tree with N number of vertices contains (N-1) number of edges.
The vertex which is of 0 degree is called root of the tree.
The vertex which is of 1 degree is called leaf node of the tree and the degree of an internal node is at least 2.

Question 53. What is a rooted tree? Give an example. Define ancestor, descendant, siblings, and subtree in a rooted tree.
Rooted Tree: A free tree has no organization of vertices and edges.
A rooted tree has a designated root at the top.
A free tree can be made rooted by choosing a root.
Depth of a vertex: Distance from the root to that vertex.

Height of a tree: maximum depth of any vertex.

Parent of a vertex: the node above that vertex (towards the root).

When u is the parent of v, then v is a child of u, Vertices with the same parent are called siblings.

Leaf: a vertex with no children.

The ancestors of a vertex (other than the root) are the vertices in the path from the root to the vertex (including the root, but excluding the vertex).

The root has no ancestors.

The descendants of a vertex, v, are all vertices that have v as an ancestor.

Question 54. Define Spanning tree and what are its properties. Explain with an example to find all spanning trees of a graph.

Spanning Trees: A spanning tree is a sub-graph of an undirected connected graph, which includes all the vertices of the graph with a minimum possible number of edges. If a vertex is missed, then it is not a spanning tree.

Properties of Spanning Tree

(i). A connected graph can have multiple spanning trees. A graph with n vertices can have an n^(n-2) number of spanning trees.

(ii). Spanning trees does not have any loop or cycle.

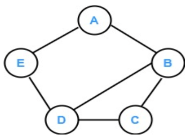
(iii). Spanning trees have n vertices and n-1 number of edges.

(iv). All spanning tree of a graph has equivalent vertices.

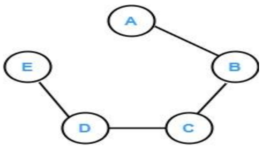
(v). Removing a single edge from the spanning tree will make the graph disconnected as the spanning tree is minimal connected.

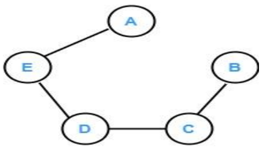
(vi). Adding any edge can create a loop or cycle in the spanning tree.

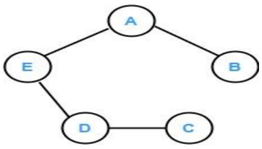
(vii). Spanning trees can be formed on connected graphs only, disconnected graphs cannot form spanning trees.

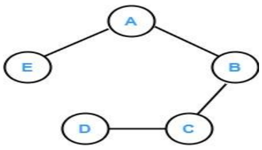
EX. 

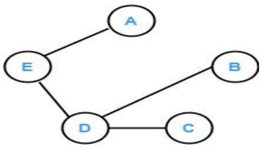
For this graph, Spanning trees are given as follows

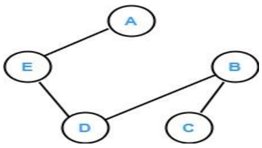












Question 54. Define a spanning tree. How many spanning trees can a complete graph have?

The total number of spanning trees with n vertices that can be created from a complete graph is equal to $n^{(n-2)}$.

If we have n = 4, the maximum number of possible spanning trees is equal to 4^{4-2} = 16. Thus, 16 spanning trees can be formed from a complete graph with 4 vertices.

Question 55. Write the algorithm for BFS and DFS. Illustrate BFS and DFS traversal on a given undirected graph. Or **Differentiate between Breadth-First Search (BFS) and Depth-First Search (DFS).**

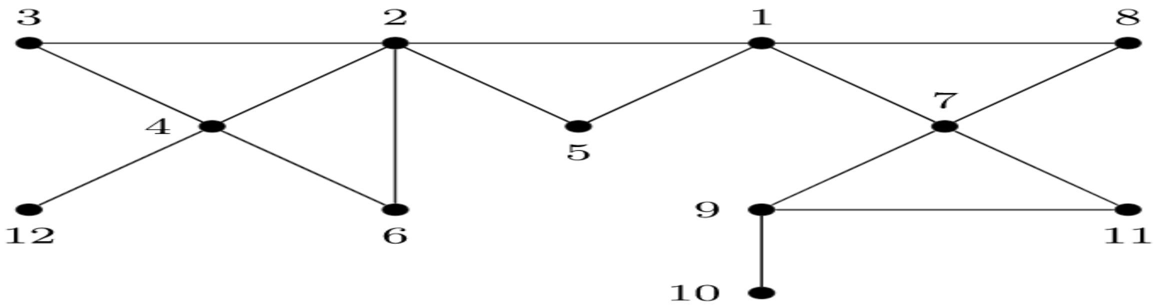
Answer. BFS (Breadth-First Search) is an algorithm that starts from the source vertex and explores all its neighboring vertices before moving on to the next level

DFS (Depth-First Search) is an algorithm that starts from the source vertex and goes deep along one path until it reaches the end before backtracking to explore other paths.

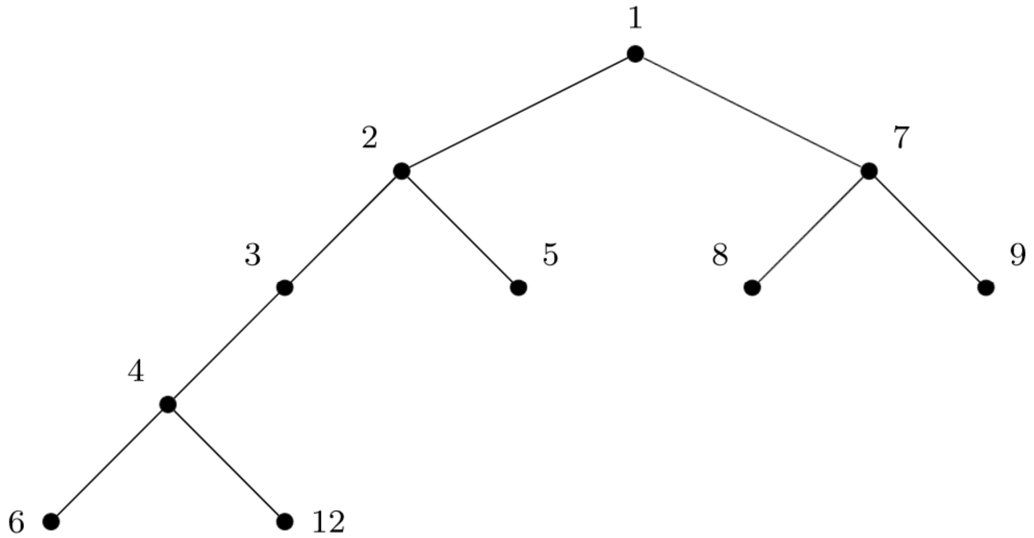
Question 56. What is a minimum spanning tree (MST)? Give an example with Kruskal's Algorithm.

It is a greedy algorithm used to find out the shortest path in a minimum-spanning tree. The algorithm aims to traverse the graph and detect the subset of edges with minimal value and cover all the vertices of the graph. At every step of the algorithm and analysis, it follows a greedy approach for an overall optimized result.

Kruskal's algorithm can be summed up as a minimum spanning tree algorithm taking a graph as input and forms a subset of the edges of the graph, which has a minimum sum of edge weight among all the trees that can be formed from that graph. that form a tree including each vertex of the graph without forming any cycle between the vertex. Example: Apply Kruskal's algorithm on a graph with edges and weights and track selected edges step-by-step.



We perform a *Depth-first search* on the graph



We perform a *Breadth-first search* on the graph