UNIT 4

BASICS OF COUNTING, PERMUTATIONS & COMBINATIONS

2-Mark Questions

Q. State the Fundamental Sum Rule of counting.
A. The Sum Rule states that if a task can be done in m ways and a second task can be done in n ways, and the tasks cannot be done simultaneously, then there are $m + n$ ways to choose one of the tasks.

Q. State the Fundamental Product Rule of counting.
A. The Product Rule states that if a procedure can be broken down into two successive stages, where the first stage can be performed in m ways and the second stage can be performed in n ways, then there are $m \times n$ ways to complete the procedure.

Q. Define a Permutation.
A. A Permutation is an ordered arrangement of a set of distinct objects, where the specific sequence or position matters significantly.

Q. Define a Combination.
A. A Combination is a selection of elements from a collection of distinct objects where the order of selection does not matter.

Q. Define the Pigeonhole Principle.
A. The Pigeonhole Principle states that if n items are put into m containers and $n > m$, then at least one container must contain more than one item.

Q. Formally differentiate between a Permutation and a Combination, stating their mathematical formulas.
A. Permutation calculates ordered linear sequences using the formula $P(n, r) = n! / (n - r)!$. Combination calculates unordered sets or selections using the formula $C(n, r) = n! / [r! \times (n - r)!]$. The analytical link between them is given by the relation: $P(n, r) = r! \times C(n, r)$.

Q. State the Extended (Generalized) Pigeonhole Principle.
A. Let the total number of items be N and the number of pigeonholes be k. The Extended Pigeonhole Principle states that if N items are distributed among k holes, then there is at least one hole containing at least $\lceil N / k \rceil$ items (where $\lceil\rceil$ denotes the ceiling function).

Q. State the formula for permutations of objects that are not all distinct.
Ans: The number of linear permutations of n objects, where r1 objects are of type 1, r2 objects are of type 2, ..., and rk objects are of type k, is given by the formula: $n! / (r1! \times r2! \times ... \times rk!)$.

Q. What is the formula for combinations with unlimited repetitions allowed?
A. The total number of combinations of choosing r objects out of n distinct types with infinite repetition allowed is given by the formula: $C(n + r - 1, r)$.

Q. Explain the structural relationship between combinations with repetition and integer solutions to linear algebraic equations.
A. The problem of finding the number of combinations of n object types with r repetitions allowed is

equivalent to finding the number of non-negative integer solutions to the linear Diophantine equation $x1 + x2 + ... + xn = r$, where each $xi \geq 0$. The formula for both is $C(n + r - 1, r)$.

Q. What are the formulas for Circular Permutations under fixed vs. reversible conditions?
A. The number of ways to arrange n objects in a circle is $(n - 1)!$. If the orientation does not matter (e.g., a key ring or a bead necklace where it can be flipped), the formula becomes $(n - 1)! / 2$.

Q1. State the Binomial Theorem.
Ans: The Binomial Theorem states that for any positive integer n: $(x + y)^n = \sum$ (from k=0 to n) $C(n, k) \times x^{(n-k)} \times y^k$.

Q. State Pascal's Identity for binomial coefficients.
Ans: Pascal's Identity states that for any integers n and r with $n \geq r \geq 1$: $C(n, r) + C(n, r - 1) = C(n + 1, r)$.

Q. State Vandermonde's Identity.
Ans: Vandermonde's Identity states that for any non-negative integers m, n, r: $C(m + n, r) = \sum$ (from k=0 to r) $C(m, k) \times C(n, r - k)$.

Q. Define the Multinomial Theorem and provide its general expansion formula.
Ans: The Multinomial Theorem extends the Binomial Theorem to expand expressions with more than two terms raised to a power. Formula: $(x1 + x2 + ... + xm)^n = \sum [n! / (n1! \times n2! \times ... \times nm!)] \times x1^{n1} \times x2^{n2} \times ... \times xm^{nm}$, where the sum runs over all non-negative integers satisfying $n1 + n2 + ... + nm = n$.

Q. Explain the process for determining the coefficient of a specific power x^k in the expansion of (ax + b/x)^n.
Ans: The general term in a binomial expansion is $T_{(r+1)} = C(n, r) \times x^{(n-r)} \times y^r$. To find the coefficient of an x^k term, set the exponent equal to k, solve for r, and then compute the value of $C(n, r)$.

Q. Define the Principle of Inclusion-Exclusion.
A. The Principle of Inclusion-Exclusion (PIE) is a counting technique to find the number of elements in the union of multiple sets by systematically adding the sizes of individual sets, subtracting the sizes of their pairwise intersections, adding the sizes of triple intersections, and so on.

Q. What is a Derangement?
A. A Derangement is a permutation of a set of objects such that no object appears in its original natural position. It is denoted by D_n.

Q. State the Inclusion-Exclusion formula for three finite sets.
A. For three finite sets A, B, and C, the formal set theory formula is:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Q. State the explicit formula for calculating the total number of derangements (Dn) of n objects.
A. The formula to compute the number of derangements of n objects is:
$D_n = n! \times [1 - 1/1! + 1/2! - 1/3! + 1/4! - ... + ([-1]^n / n!)]$.

5-Mark Questions

Q. Selection Combinatorics Problem: A team of 6 software developers is to be selected from a pool of 8 Senior Engineers and 5 Junior Engineers. In how many unique ways can this team be formed if:
a) The team must contain exactly 4 Senior Engineers?
b) The team must contain at least 4 Senior Engineers?
c) The team must include at least 1 Junior Engineer?

a) Finding configurations for exactly 4 Senior Engineers:
We need to pick exactly 4 Senior Engineers out of 8, and the remaining vacancies (6 - 4 = 2) must be filled by Junior Engineers out of 5.
Ways to choose Senior Engineers = $C(8, 4) = 8! / (4! \times 4!) = (8 \times 7 \times 6 \times 5) / (4 \times 3 \times 2 \times 1) = 70$
Ways to choose Junior Engineers = $C(5, 2) = 5! / (2! \times 3!) = (5 \times 4) / (2 \times 1) = 10$
By the Product Rule, total unique ways = $70 \times 10 = 700$ ways.

b) Finding configurations for at least 4 Senior Engineers:
The phrase 'at least 4' allows three mutually exclusive cases:
Case 1: Exactly 4 Senior and 2 Junior Engineers = $C(8, 4) \times C(5, 2) = 70 \times 10 = 700$
Case 2: Exactly 5 Senior and 1 Junior Engineer = $C(8, 5) \times C(5, 1) = 56 \times 5 = 280$
Case 3: Exactly 6 Senior and 0 Junior Engineers = $C(8, 6) \times C(5, 0) = 28 \times 1 = 28$
By the Sum Rule, total unique ways = 700 + 280 + 28 = 1008 ways.

c) Finding configurations for at least 1 Junior Engineer:
Instead of adding positive combinations, we can use the complement method:
Total unconditional ways to select any 6 developers from the combined pool of 13 (8 + 5) is $C(13, 6) = 13! / (6! \times 7!) = 1716$.
The unfavorable scenario is selecting a team with 0 Junior Engineers (meaning all 6 are Senior Engineers): $C(8, 6) \times C(5, 0) = 28 \times 1 = 28$.
Ways with at least 1 Junior Engineer = Total Ways - Unfavorable Ways = 1716 - 28 = 1688 ways.

Q. Pigeonhole Application Problem: An engineering college department has 9 computer laboratories. Each laboratory is equipped with a distinct number of computer terminals. The total number of terminals across all 9 labs is 172. Show that at least one laboratory contains 20 or more computer terminals.

Proof: Let the 9 computer laboratories be treated as the 'pigeonholes' ($k = 9$).
Let the 172 computer terminals be treated as the objects distributed among the labs ($N = 172$).
By applying the Generalized Pigeonhole Principle, at least one lab must contain at least $\lceil N / k \rceil$ terminals.
Calculation: $\lceil 172 / 9 \rceil = \lceil 19.11 \rceil = 20$ terminals.
Hence, it is mathematically guaranteed that at least one computer laboratory contains 20 or more computer terminals.

ENUMERATION WITH REPETITIONS & CONSTRAINED REPETITIONS

Q. Indeterminate Equation Counting Problem: Find the total number of non-negative integer solutions to the equation $x1 + x2 + x3 + x4 = 18$ under the following independent conditions:
a) There are no constraints on the variables ($xi \geq 0$).
b) Each variable must be strictly greater than or equal to 2 ($xi \geq 2$).
c) $x1 \leq 5$, $x2 \geq 0$, $x3 \geq 0$, $x4 \geq 0$.

a) Finding solutions with no constraints ($xi \geq 0$):
Here, the number of distinct variable types is $n = 4$, and the required sum is $r = 18$.
Using the combinations with repetition formula $C(n + r - 1, r)$:
Total solutions = $C(4 + 18 - 1, 18) = C(21, 18) = C(21, 3)$
Calculation: $(21 \times 20 \times 19) / (3 \times 2 \times 1) = 1330$ solutions.

b) Finding solutions with constraints ($xi \geq 2$):
We introduce a variable substitution to eliminate the lower bounds. Let $yi = xi - 2$, so that $yi \geq 0$.

Substituting $x_i = y_i + 2$ into the original equation:
$(y_1 + 2) + (y_2 + 2) + (y_3 + 2) + (y_4 + 2) = 18$
$y_1 + y_2 + y_3 + y_4 + 8 = 18$
$y_1 + y_2 + y_3 + y_4 = 10$
Now, we find the non-negative solutions for this new equation where $n = 4$ and the new sum $r' = 10$:
Solutions = $C(4 + 10 - 1, 10) = C(13, 10) = C(13, 3) = (13 \times 12 \times 11) / (3 \times 2 \times 1) = 286$ solutions.

c) Finding solutions with an upper bound constraint ($x_1 \leq 5$):
We can solve this using the complement method: Total solutions without constraints minus the unfavorable solutions where $x_1 \geq 6$.
Total solutions = 1330 (from part a).
To find the unfavorable solutions where $x_1 \geq 6$, we use the substitution $z_1 = x_1 - 6$ (where $z_1 \geq 0$), and leave the other variables unchanged ($z_i = x_i$).
The equation becomes: $(z_1 + 6) + x_2 + x_3 + x_4 = 18$
$z_1 + x_2 + x_3 + x_4 = 12$
The number of solutions for this case with $n = 4$ and $r'' = 12$ is:
Unfavorable solutions = $C(4 + 12 - 1, 12) = C(15, 12) = C(15, 3) = (15 \times 14 \times 13) / (3 \times 2 \times 1) = 455$.
Valid solutions satisfying $x_1 \leq 5$ = Total solutions - Unfavorable solutions = $1330 - 455 = 875$ solutions.

Q. Constrained Permutation Problem: Determine the total number of distinct linear arrangements that can be formed using all the letters of the word 'ENGINEERING' such that:
a) There are no constraints on the arrangement.
b) All three 'E's must always be grouped together as a single block.
c) The arrangement must start with 'G' and end with 'R'.

a) Total arrangements with no constraints:
Count the frequency of each letter in the word 'ENGINEERING' (Total letters $n = 11$):
E: 3, N: 3, G: 2, I: 2, R: 1
Using the permutation formula for identical objects:
Total arrangements = $11! / (3! \times 3! \times 2! \times 2! \times 1!) = 39{,}916{,}800 / (6 \times 6 \times 2 \times 2) = 39{,}916{,}800 / 144 = 277{,}200$.

b) Arrangements where all 'E's are grouped together:
Treat the three 'E's as a single mega-letter block: [EEE].
Now, the items to arrange are the block [EEE] plus the remaining 8 letters (N:3, G:2, I:2, R:1). This gives a total of $1 + 8 = 9$ items.
Arrangements = $9! / (3! \times 2! \times 2! \times 1!) = 362{,}880 / (6 \times 2 \times 2) = 362{,}880 / 24 = 15{,}120$.

c) Arrangements starting with 'G' and ending with 'R':
Fix 'G' at the first position and 'R' at the last position. This leaves 9 middle positions to fill.
The remaining letters to arrange in these 9 positions are: E:3, N:3, G:1, I:2.
Arrangements = $9! / (3! \times 3! \times 1! \times 2!) = 362{,}880 / (6 \times 6 \times 1 \times 2) = 362{,}880 / 72 = 5{,}040$.

BINOMIAL COEFFICIENTS & THEOREMS

Q. Coefficient Extraction Problem: Using the theorems of combinatorics, compute the following:
a) Find the coefficient of x^5 in the binomial expansion of $(2x^2 - 3 / x)^{10}$.
b) Find the coefficient of the term $a^3 \times b^4 \times c^2$ in the multinomial expansion of $(2a - b + 3c)^9$.

a) Finding the coefficient of x^5:
The general term in the expansion of $(2x^2 - 3/x)^{10}$ is:
$T_{(r+1)} = C(10, r) \times (2x^2)^{(10-r)} \times (-3/x)^r$
Separate the coefficients and the variable x:
$T_{(r+1)} = C(10, r) \times 2^{(10-r)} \times (-3)^r \times (x^2)^{(10-r)} \times (x^{[-1]})^r$
Combine the powers of x: $T_{(r+1)} = C(10, r) \times 2^{(10-r)} \times (-3)^r \times x^{(20 - 2r - r)} = C(10, r) \times 2^{(10-r)} \times (-3)^r \times x^{(20 - 3r)}$
We want to find the coefficient of x^5, so we set the exponent of x equal to 5:
$20 - 3r = 5$
$3r = 15$
$r = 5$
Substitute r = 5 back into the coefficient part of the general term expression:
Coefficient $= C(10, 5) \times 2^{(10-5)} \times (-3)^5 = 252 \times 32 \times (-243) = -1{,}959{,}552$.

b) Finding the coefficient of $a^3 \times b^4 \times c^2$:
Here, the power is n = 9, and the terms are x1 = 2a, x2 = -b, and x3 = 3c. We want to find the coefficient for the exponents n1 = 3, n2 = 4, and n3 = 2.
Check that the exponents sum to the power: 3 + 4 + 2 = 9, which matches.
According to the Multinomial Theorem, the term is:
$[\, 9! / (3! \times 4! \times 2!) \,] \times (2a)^3 \times (-b)^4 \times (3c)^2$
Calculate the multinomial coefficient part: $362{,}880 / (6 \times 24 \times 2) = 362{,}880 / 288 = 1260$.
Expand the terms with variables: $(2a)^3 = 8a^3$, $(-b)^4 = 1b^4$, $(3c)^2 = 9c^2$.
Combine the numbers: $1260 \times 8 \times 1 \times 9 = 90{,}720$.
The complete term is $90{,}720 \times a^3 \times b^4 \times c^2$, so the coefficient is 90,720.

Q. Combinor-Algebraic Identity Proofs: Prove the following binomial identities analytically using algebraic definitions or combinatorial proofs:
a) $C(n, 0) + C(n, 1) + C(n, 2) + ... + C(n, n) = 2^n$
b) $C(n, 1) + 2\times C(n, 2) + 3\times C(n, 3) + ... + n\times C(n, n) = n \times 2^{(n-1)}$

a) Proof for the sum of coefficients equal to 2^n:
The Binomial Theorem states that: $(x + y)^n = \sum$ (from k=0 to n) $C(n, k) \times x^{(n-k)} \times y^k$.
Let's substitute x = 1 and y = 1 into this equation:
$(1 + 1)^n = C(n, 0)\times 1^n\times 1^0 + C(n, 1)\times 1^{(n-1)}\times 1^1 + ... + C(n, n)\times 1^0\times 1^n$
$2^n = C(n, 0) + C(n, 1) + C(n, 2) + ... + C(n, n)$.
Hence, the sum of all binomial coefficients for a given power n is equal to 2^n. Q.E.D.

b) Proof for the weighted sum of coefficients:
The general terms can be rewritten using the algebraic identity: $k \times C(n, k) = n \times C(n-1, k-1)$.
Let's verify this step: $k \times [\, n! / (k! \times (n-k)!) \,] = k \times [\, n \times (n-1)! / (k \times (k-1)! \times (n-k)!) \,] = n \times [\, (n-1)! / ((k-1)! \times ((n-1)-(k-1))!) \,] = n \times C(n-1, k-1)$.
Now substitute this relation back into our original summation series:
LHS $= \sum$ (from k=1 to n) $k \times C(n, k) = \sum$ (from k=1 to n) $n \times C(n-1, k-1)$
Factor out the common term n from the summation: $= n \times [\, C(n-1, 0) + C(n-1, 1) + ... + C(n-1, n-1) \,]$
Using the coefficient sum identity proved in part (a), the sum inside the brackets equals $2^{(n-1)}$:
LHS $= n \times 2^{(n-1)}$ = RHS. Q.E.D.

Alternative Combinatorial Trick: *Identity proofs are common on JNTU exams. Remember that substituting values like x=1, y=1 or x=1, y=-1 into the Binomial Theorem is a powerful method for solving* *series summation problems.*

THE PRINCIPLE OF INCLUSION-EXCLUSION (PIE)

Q. Advanced Number Theory Counting Problem: Find the number of integers between 1 and 250 (inclusive) that are not divisible by any of the integers 2, 3, 5, or 7.

Solution Steps:
Let the universal set U be the set of integers from 1 to 250. The total number of elements is $|U| = 250$.
Let P1, P2, P3, and P4 be the properties that an integer is divisible by 2, 3, 5, and 7 respectively.
Let A, B, C, and D be the subsets of integers divisible by 2, 3, 5, and 7. The size of these subsets can be found using the floor function:
$|A| = \lfloor 250 / 2 \rfloor = 125$, $|B| = \lfloor 250 / 3 \rfloor = 83$, $|C| = \lfloor 250 / 5 \rfloor = 50$, $|D| = \lfloor 250 / 7 \rfloor = 35$

Step 1: Calculate the sizes of pairwise intersections:
The elements in the intersections are divisible by the Least Common Multiple (LCM) of the divisors:
$|A \cap B|$ (divisible by 6) $= \lfloor 250 / 6 \rfloor = 41$
$|A \cap C|$ (divisible by 10) $= \lfloor 250 / 10 \rfloor = 25$
$|A \cap D|$ (divisible by 14) $= \lfloor 250 / 14 \rfloor = 17$
$|B \cap C|$ (divisible by 15) $= \lfloor 250 / 15 \rfloor = 16$
$|B \cap D|$ (divisible by 21) $= \lfloor 250 / 21 \rfloor = 11$
$|C \cap D|$ (divisible by 35) $= \lfloor 250 / 35 \rfloor = 7$

Step 2: Calculate the sizes of triple intersections:
Next, find the sizes of the intersections of three sets:
$|A \cap B \cap C|$ (divisible by $2\times3\times5 = 30$) $= \lfloor 250 / 30 \rfloor = 8$
$|A \cap B \cap D|$ (divisible by $2\times3\times7 = 42$) $= \lfloor 250 / 42 \rfloor = 5$
$|A \cap C \cap D|$ (divisible by $2\times5\times7 = 70$) $= \lfloor 250 / 70 \rfloor = 3$
$|B \cap C \cap D|$ (divisible by $3\times5\times7 = 105$) $= \lfloor 250 / 105 \rfloor = 2$

Step 3: Calculate the size of the four-way intersection:
Find the size of the intersection of all four sets:
$|A \cap B \cap C \cap D|$ (divisible by $2\times3\times5\times7 = 210$) $= \lfloor 250 / 210 \rfloor = 1$

Step 4: Combine the terms using the PIE formula:
By the Principle of Inclusion-Exclusion, the number of integers divisible by at least one of the numbers is:
$|A \cup B \cup C \cup D| = \sum|\text{Individual Sets}| - \sum|\text{Pairs}| + \sum|\text{Triples}| - |\text{All Four}|$
$|A \cup B \cup C \cup D| = (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7) + (8 + 5 + 3 + 2) - 1$
$|A \cup B \cup C \cup D| = 293 - 117 + 18 - 1 = 193$ integers.
The number of integers not divisible by 2, 3, 5, or 7 is the size of the complement set:
Complement count $= |U| - |A \cup B \cup C \cup D| = 250 - 193 = 57$ integers.

Q2. Applied Derangement Counting Problem: A group of 5 engineering professors attend a board meeting and leave their umbrellas in a coat rack. When leaving, each professor selects an umbrella completely at random. Find:
a) The total number of ways the umbrellas can be distributed.
b) The number of ways such that no professor gets their own umbrella (complete derangement).\c) The number of ways such that exactly 2 professors receive their own matching umbrellas.

a) Finding the total number of ways:
Since there are 5 professors and 5 distinct umbrellas, the total number of ways to distribute the umbrellas is a standard linear permutation:
Total arrangements = $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ unique ways.

b) Finding the number of complete derangements (D5):
This is a complete derangement problem for $n = 5$ objects. Using the derangement formula:
$D_5 = 5! \times [1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5!]$
$D_5 = 120 \times [1 - 1 + 1/2 - 1/6 + 1/24 - 1/120]$
Convert the terms to a common denominator of 120:
$D_5 = 120 \times [(60 - 20 + 5 - 1) / 120] = 120 \times [44 / 120] = 44$ ways.

c) Finding the number of ways where exactly 2 professors get their own umbrellas:
To find the number of ways where exactly 2 professors get their own umbrellas, we first choose which 2 professors get their own umbrellas, and then derange the remaining 3 professors.
Step 1: Choose 2 professors out of 5 = $C(5, 2) = 10$ ways.
Step 2: Calculate the number of derangements for the remaining 3 professors ($n = 3$):
$D_3 = 3! \times [1 - 1/1! + 1/2! - 1/3!] = 6 \times [1 - 1 + 1/2 - 1/6] = 6 \times [(3 - 1) / 6] = 2$ ways.
Step 3: Combine the choices using the Product Rule:
Total ways = $C(5, 2) \times D_3 = 10 \times 2 = 20$ ways.