UNIT 3
INTRODUCTION TO ALGEBRAIC SYSTEMS

Q. Define an Algebraic Structure.
A. A non-empty set G equipped with one or more binary operations is called an algebraic structure or algebraic system. Formally denoted as (G, *).

Q. What is a Binary Operation?
A. A binary operation * on a non-empty set G is a mapping from the Cartesian product $G \times G$ into G. That is, $*: G \times G \to G$.

Q. State the Commutative Property of an algebraic system.
A. An algebraic system (G, *) is said to be commutative if for all elements $a, b \in G$, the condition $a * b = b * a$ is satisfied.

Q. Differentiate between Closure Property and Associative Property with an explicit counterexample.
A. An operation * on a set G is closed if $\forall a, b \in G$, $a * b \in G$. It is associative if $\forall a, b, c \in G$, $(a * b) * c = a * (b * c)$.
Example: Set of natural numbers N under subtraction is not closed ($2 - 5 = -3 \notin N$), and not associative.

Q. Define the Identity Property in an algebraic system. Is the identity element unique?
A. An element $e \in G$ is called an identity element with respect to * if for all $a \in G$, $a * e = e * a = a$. An identity element must be unique within an algebraic system.

Q. Define a Semigroup.
A. An algebraic system (S, *) is called a semigroup if the binary operation * satisfies the closure property and the associative property.

Q. What is a Monoid?
A. A monoid is a semigroup that contains an identity element. That is, (M, *) satisfies closure, associativity, and identity properties.

Q. What is an Idempotent Element?
A. An element $x \in S$ is called idempotent under * if $x * x = x$.

Q. Justify the statement: 'Every monoid is a semigroup, but every semigroup is not a monoid' using a concrete mathematical counterexample.
A. Every monoid is a semigroup because a monoid satisfies closure and associativity by definition. However, a semigroup is not necessarily a monoid.
Example: (2N, +), the set of even natural numbers {2, 4, 6...} under addition, is a semigroup but has no identity element ($0 \notin 2N$), so it is not a monoid.

Q. Define Semigroup Homomorphism.
A. Let (S, *) and (T, ∘) be two semigroups. A function $f: S \to T$ is called a semigroup homomorphism if for all $a, b \in S$, $f(a * b) = f(a) \circ f(b)$.

5-Mark Questions

Q. Analytical Problem Statement: Let Q+ be the set of all positive rational numbers. A binary operation * is defined on Q+ by $a * b = (a \times b) / 3$ for all $a, b \in Q+$.
a) Prove that * is associative.
b) Find the unique identity element in (Q+, *).
c) Determine the formula for the inverse of an arbitrary element $a \in Q+$.

Proof of Associativity:
Let $a, b, c \in Q+$. We must prove that $(a * b) * c = a * (b * c)$.
$LHS = (a * b) * c = [(a \times b) / 3] * c = [((a \times b) / 3) \times c] / 3 = (a \times b \times c) / 9$
$RHS = a * (b * c) = a * [(b \times c) / 3] = [a \times ((b \times c) / 3)] / 3 = (a \times b \times c) / 9$
Since LHS = RHS, the binary operation * is strictly associative over Q+.

Finding the Identity Element:
Let $e \in Q+$ be the identity element. By definition, $a * e = a$.
$(a \times e) / 3 = a$
Multiplying both sides by 3: $a \times e = 3a$
Since $a \in Q+$ (positive rational, hence $a \neq 0$), we can divide both sides by a:
$e = 3$
Verification: $3 * a = (3 \times a) / 3 = a$. Thus, the identity element is $e = 3$.

Finding the Inverse Formula:
Let a^[-1] be the inverse of an element $a \in Q+$. By definition, a * a^[-1] = e.
Since $e = 3$, we write: (a × a^[-1]) / 3 = 3
Multiplying both sides by 3: a × a^[-1] = 9
Solving for a^[-1]: a^[-1] = 9 / a
Since a is a positive rational number, 9/a is also guaranteed to be a positive rational number ($9/a \in Q+$). Thus, every element has a valid inverse.

> ***JNTU Examiner Note:*** *In JNTU examinations, always verify that the calculated identity 'e' and inverse element 'a^[-1]' actually belong to the given underlying set domain (here, Q+). If they do not, properties fail.*

LATTICES AS PARTIALLY ORDERED SETS (POSETs)

Q1. Define a Lattice as a POSET.
Ans: A poset $(L, \leq)$ is called a lattice if every pair of elements $a, b \in L$ has a unique Least Upper Bound (LUB, called Join, denoted $a \vee b$) and a unique Greatest Lower Bound (GLB, called Meet, denoted $a \cap b$ or $a \wedge b$).

Q2. What is meant by Join (LUB) in a lattice?
Ans: The Least Upper Bound (LUB) or Join of two elements a and b is the unique element x such that $a \leq x$, $b \leq x$, and for any other upper bound y, $x \leq y$.

Q3. Define a Bounded Lattice.
Ans: A lattice L is called a bounded lattice if it possesses both a unique least element (denoted 0 or ⊥) and a unique greatest element (denoted 1 or ⊤).

Q1. State the Absorption Laws governing lattices.
Ans: In any lattice L, for all elements a, b ∈ L:
1. Absorption Law 1: $a \wedge (a \vee b) = a$
2. Absorption Law 2: $a \vee (a \wedge b) = a$

Q2. Define a Distributive Lattice.
Ans: A lattice L is distributive if the meet operation distributes over join, and vice-versa, for all a, b, c ∈ L:
1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

SEMIGROUPS AND MONOIDS

5-Mark Questions

Q. Structural Proof Problem: Let S be a non-empty set and let P(S) be its power set. Prove that the algebraic system (P(S), ∩) where ∩ represents set intersection is a monoid. Determine its identity and examine if any elements have inverses.

Proof Matrix:
To establish that (P(S), ∩) is a monoid, we must rigorously verify three explicit structural properties:

1. Closure Property: Let A, B ∈ P(S). This implies A ⊆ S and B ⊆ S. By the foundational laws of set theory, their intersection A ∩ B is also a subset of S, meaning (A ∩ B) ∈ P(S). Thus, closure holds.

2. Associative Property: Let A, B, C ∈ P(S). We know that set intersection is universally associative: (A ∩ B) ∩ C = A ∩ (B ∩ C). Hence, associativity holds.

3. Identity Property: We must look for a unique subset E ∈ P(S) such that A ∩ E = A for all A ∈ P(S). Choosing E = S (the universal set for this power set context), we find A ∩ S = A. Since S ∈ P(S), S serves as the unique identity element. Thus, (P(S), ∩) is verified to be a monoid.

4. Inverse Analysis: For an element A ∈ P(S) to have an inverse A^[-1], it must satisfy A ∩ A^[-1] = E = S. The intersection of two sets equals the full universe S if and only if both sets are identical to S. Therefore, the only element in the entire monoid that possesses an inverse is the identity element S itself. No other subset has an inverse.

Q. Mapping & Isomorphism Problem: Let (Z, +) be the algebraic group of integers under addition and (E, +) be the algebraic structure of even integers under addition. Let a mapping f: Z → E be defined by f(x) = 2x for all x ∈ Z.
a) Show that f is a homomorphism.
b) Determine whether f is an isomorphism.

a) Homomorphism Proof:
Let $x, y \in Z$. We need to evaluate whether $f(x + y) = f(x) + f(y)$.
$LHS = f(x + y) = 2(x + y) = 2x + 2y$
$RHS = f(x) + f(y) = 2x + 2y$
Since LHS = RHS, f preserves the additive operation. Therefore, f is a valid homomorphism.

b) Isomorphism Verification:
To be an isomorphism, the homomorphism f must be bijective (both injective and surjective).
1. Injectivity Check: Let $f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$. Thus, f is one-to-one (injective).
2. Surjectivity Check: Let $y \in E$ (even integers). By definition of even numbers, y can be written as 2k for some integer $k \in Z$. Therefore, there exists a pre-image $x = k \in Z$ such that $f(x) = 2k = y$. Thus, f is onto (surjective).
Conclusion: Since f is a bijective homomorphism, it is a true Isomorphism. Hence, $(Z, +) \cong (E, +)$.

BOOLEAN ALGEBRA

Q. Define a Boolean Algebra.
A. A Boolean Algebra is a complemented, distributive lattice $(B, \wedge, \vee, ', 0, 1)$ containing distinct identity elements 0 and 1, equipped with a unary complement operation.

Q. What is a Complement Element in Boolean Algebra?
A. In a bounded lattice, an element a' is called the complement of a if $a \vee a' = 1$ and $a \wedge a' = 0$.

Q. State the Principle of Duality.
A. The literal dual of a Boolean expression is formed by swapping all ∨ with ∧, all ∧ with ∨, all 0s with 1s, and all 1s with 0s, leaving the variables unchanged.

Q. State De Morgan's Laws for Boolean Algebra.
A. De Morgan's Laws state that for any elements a, b in a Boolean Algebra:
1. $(a \vee b)' = a' \wedge b'$
2. $(a \wedge b)' = a' \cup b'$ (or $a' \vee b'$)

Q. State the Involution Law.
A. The Involution Law states that the complement of a complemented element returns the original element. Formally: $(a')' = a$ for all elements a in a Boolean Algebra.

5-Mark Questions

Q1. Discrete Poset Lattice Problem: Let D_30 = {1, 2, 3, 5, 6, 10, 15, 30} be the set of divisors of 30, ordered by the relation of partial order 'divides' (denoted by |).
a) Construct the structured operations table for Join (LCM) and Meet (GCD).
b) Prove whether (D_30, |) forms a lattice.

Lattice Proof:
For any pair a, b ∈ D_30 under the division relation, the Greatest Lower Bound is their Greatest Common Divisor, $GLB(a, b) = GCD(a, b) = a \wedge b$. The Least Upper Bound is their Least Common Multiple, $LUB(a, b) = LCM(a, b) = a \vee b$.
Since the GCD and LCM of any two positive integers are unique and always divide 30 if both inputs

divide 30, every pair in D_30 has a unique LUB and GLB belonging to D_30. Hence, (D_30, |) is a valid lattice.

Structured Operations Matrix for Meet / GLB (GCD) in D_30:

∧	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Q. Axiomatic Lattice Proofs: Prove analytically that in any lattice (L, ≤), the Idempotent laws hold true for all elements a ∈ L:
a) a ∨ a = a
b) a ∧ a = a

a) Proof for Join Idempotency (a ∨ a = a):
By foundational definition of partial ordering, the relation ≤ is reflexive, meaning a ≤ a for all a ∈ L.
This implies that 'a' is an upper bound for the set {a, a}.
Let x be any other upper bound for {a, a}, meaning a ≤ x.
Since 'a' itself is an upper bound and for any other upper bound x we have a ≤ x, 'a' must be the Least Upper Bound (LUB).
Thus, LUB(a, a) = a ∨ a = a. Q.E.D.

b) Proof for Meet Idempotency (a ∧ a = a):
Similarly, due to reflexivity, we know that a ≤ a for all a ∈ L.
This implies that 'a' is a lower bound for the set {a, a}.
Let y be any other lower bound for {a, a}, meaning y ≤ a.
Since 'a' itself is a lower bound and for any other lower bound y we have y ≤ a, 'a' must be the Greatest Lower Bound (GLB).
Thus, GLB(a, a) = a ∧ a = a. Q.E.D.

Q. Axiomatic Mathematical Proofs: Prove analytically using the axioms of Boolean Algebra that for any elements a and b, the following properties hold true:
a) Boundedness / Domination Law: a ∨ 1 = 1
b) Uniqueness of Complement Proof.

a) Proof of Domination Law ($a \vee 1 = 1$):
LHS = $a \vee 1$
$= (a \vee 1) \wedge 1$ (Identity Law: $x \wedge 1 = x$)
$= (a \vee 1) \wedge (a \vee a')$ (Complement Law: $x \vee x' = 1$)
$= a \vee (1 \wedge a')$ (Distributive Law in reverse: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$)
$= a \vee a'$ (Identity Law: $1 \wedge x = x$)
$= 1$ (Complement Law: $x \vee x' = 1$) = RHS
Therefore, $a \vee 1 = 1$. By the Principle of Duality, it follows that $a \wedge 0 = 0$.

b) Proof of Uniqueness of Complement:
Let an element 'a' have two complements, say x and y. By definition of complements, they must satisfy:
Condition 1: $a \vee x = 1$ and $a \wedge x = 0$
Condition 2: $a \vee y = 1$ and $a \wedge y = 0$
Let's evaluate the element x:
$x = x \wedge 1$ (Identity Law)
$= x \wedge (a \vee y)$ (From Condition 2: $a \vee y = 1$)
$= (x \wedge a) \vee (x \wedge y)$ (Distributive Law)
$= (a \wedge x) \vee (x \wedge y)$ (Commutative Law)
$= 0 \vee (x \wedge y)$ (From Condition 1: $a \wedge x = 0$)
$= (a \wedge y) \vee (x \wedge y)$ (From Condition 2: $0 = a \wedge y$)
$= (a \vee x) \wedge y$ (Distributive Law in reverse)
$= 1 \wedge y$ (From Condition 1: $a \vee x = 1$)
$= y$ (Identity Law)
Since $x = y$, the complement of any element in a Boolean algebra is strictly unique.

Q. Expression Minimization & Verification Problem: In a Boolean Algebra, simplify the structural Boolean function expression: $E(x, y, z) = (x \wedge y) \vee [x \wedge (y' \vee z)]$ using axiomatic identities, and construct the validation Truth Table.

Algebraic Minimization:
Given expression: $E = (x \wedge y) \vee [x \wedge (y' \vee z)]$
Step 1: Apply Distributive Law to expand the bracket: $= (x \wedge y) \vee (x \wedge y') \vee (x \wedge z)$
Step 2: Factor out '$x \wedge$' from the first two terms using reverse distribution: $= [x \wedge (y \vee y')] \vee (x \wedge z)$
Step 3: Apply Complement identity ($y \vee y' = 1$): $= (x \wedge 1) \vee (x \wedge z)$
Step 4: Apply Identity law ($x \wedge 1 = x$): $= x \vee (x \wedge z)$
Step 5: Apply Absorption Law ($x \vee (x \wedge z) = x$): $= x$
Hence, the minimized functional expression simplifies beautifully down to just: $E(x, y, z) = x$.

Truth Table Verification Matrix:

x	y	z	y'	$x \wedge (y' \vee z)$	Total Expression (E)
0	0	0	1	0	0
0	0	1	1	0	0

0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	1	1